



# Complexity growth of massive black hole with a probe string

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**Abstract** In this work, we study the computational complexity of massive gravity theory via the “Complexity = Action” conjecture. Our system contains a particle moving on the boundary of the black hole spacetime. It is dual to inserting a fundamental string in the bulk background. Then this string would contribute a Nambu–Goto term, such that the total action is composed of the Einstein–Hilbert term, Nambu–Goto term and the boundary term. We shall investigate the time development of this system, and mainly discuss the features of the Nambu–Goto term affected by the graviton mass and the horizon curvature in different dimensions. Our study could contribute interesting properties of complexity.

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## 1 Introduction

Along with the development of fundamental physics and information theory, some very remarkable tools were pro-

posed to reveal the mystery of nature. One of them is in the areas of general relativity and quantum field theory. In 1990s, the AdS/CFT duality which connects the gravity theory and its boundary conformal field theory was proposed [1–3] and it has opened a new avenue for us to investigate the gravity in the framework of holography.

Thanks to holography, the entanglement entropy (EE), which measures the degrees of freedom in a strongly coupled system originally, has a holographic description, stating that the EE for a subregion on the dual boundary is proportional to the minimal Hubeny–Rangamani–Takayanagi surface in the bulk geometry [4, 5]. The other significant concept in theoretical physics is the computational complexity which originally comes from the quantum information theory [6–12]. The computational complexity measures the difficulty of turning a quantum state into another state. However, from the side of the field theory, it is extremely difficult to evaluate it when the degrees of freedom of the system becomes large. It is also not clear on how to precisely define the initial state and reference state. Though there are plenty of works on this theme from quantum field theory [13–17] as well as works on geometric way to define the complexity [18–24], a well-defined theory of complexity is still unknown.

On the other hand, in gravity side, only considering the outside properties of black hole is not enough for us to understand the interior black hole. Susskind et al. suggested that the computational complexity can measure the size of wormhole which anchored at the two sides of boundary times of AdS black hole [25–29]. Accordingly, there are two conjectures have emerged. One is “Complexity=Volume”(CV) conjecture.  $V$  denotes the volume of Einstein–Rosen (ER) bridge that connects the two side of boundary times of AdS black hole. The other reliable candidate proposal is the “Complexity=Action” (CA) conjecture.  $A$  denotes the classical action of a spacetime region which was enclosed by the bulk Cauchy slice that anchored at the boundaries, and this domain was also called “Wheeler–Dewitt patch” [30, 31].

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Especially, the application of CA conjecture have been extensively studied in [32–53] and therein. However, the aforementioned work focused on the stationary system. An important direction is to generalize the CA conjecture into a time-dependent process such as a system moved by a drag force, which was motivated by the work of jet-quenching phenomena in ion collisions where the system is moved by a drag force [54]. When a charged particle passes through a quark–gluon plasma, it would lose energy because of the effect of shear viscosity. Thus, the most useful method to analyze this dissipative system is to use a probe. Many important work were inspired by this clue. For instance, in [55] the growth of complexity with the probe brane was studied. In [56] the non-local operator was studied in BTZ black hole. More recently, Nagasaki investigated the complexity of AdS<sub>5</sub> black hole via CA by inserting a probe string which moves on a circle in the spacial part of the AdS spacetime [57]. This study was also extended in rotating black holes [58, 59] and the black holes in Horndeski gravity [60, 61].

In this work, we also use this method and investigate the complexity growth in massive gravity with a probe string. Massive gravity theory is a theory beyond Einstein theory of gravity where the graviton is massless. Recently, significant progress has been made towards constructing massive gravity theories that avoid instability, see for example [62–68]. More importantly, it was addressed in [69] that the massive terms in the gravitational action break the diffeomorphism symmetry in the bulk, which corresponds to momentum dissipation in the dual boundary field theory. Moreover, the complexity growth for the stationary massive black hole has been studied in [32, 33], in which the complexity growth stems from the contribution from Einstein–Hilbert action and the boundary term.

The aim of this work is to study the effect of probe string on the complexity growth in massive gravity. This means that we shall consider a Wilson line operator in the theory, which is dual to a dynamical system. The novelty of this work is that the non-local operator could correspond to a particle moving on the boundary gauge theory with momentum relaxation. We shall focus on this effect of Wilson line operator on the Nambu–Goto action. In details, we shall study the influence of horizon curvatures, the graviton mass and the dimension of spacetime on the velocity dependent complexity growth, which is dual to Nambu–Goto action growth in massive black hole with a probe string.

This paper is organized as follows. In Sect. 2 we briefly review the massive gravity and then derive the Nambu–Goto action in massive gravity with probe string. In Sects. 3 and 4, we investigate the Nambu–Goto action growth in the three and higher dimensional gravity, respectively. We analyze the effect of horizon curvature and graviton mass on the features of the Nambu–Goto action growth. We summarize in Sect. 5. We shall use natural units with  $G = \hbar = c = 1$ .

## 2 Massive gravity and the Nambu–Goto action

### 2.1 Review of massive black hole

We consider  $(n + 2)$ -dimensional action of massive gravity [70, 71],

$$S = \frac{1}{16\pi G} \int d^{n+2}x \sqrt{-g} \left[ R + \frac{n(n+1)}{l^2} + m^2 \sum_i^4 c_i \mathcal{U}_i(g, f) \right], \quad (1)$$

where  $R$  is the scalar curvature,  $l$  is the AdS radius and  $f$  is a fixed symmetric tensor. In addition,  $c_i$  is constant and  $\mathcal{U}_i$  is the symmetric polynomial of the eigenvalue of the  $(n + 2) \times (n + 2)$  matrix  $\mathcal{K}_\nu^\mu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$  which can be written as follows

$$\begin{aligned} \mathcal{U}_1 &= [\mathcal{K}], & \mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], \\ \mathcal{U}_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] \\ &\quad + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4], \end{aligned} \quad (2)$$

where  $[\mathcal{K}] = \mathcal{K}_\mu^\mu$  and the square root in  $\mathcal{K}$  can be interpreted as  $(\sqrt{\mathcal{K}})_\nu^\mu (\sqrt{\mathcal{K}})_\lambda^\nu = \mathcal{K}_\lambda^\mu$ . As known that the presence of impurities in realistic materials leads that the momentum is not conserved, so that the system gives finite DC conductivity. Modeling systems via translationally invariant quantum field theories always comes across problems unless the effects of momentum dissipation is incorporated. In holographic framework, several models have been proposed to involve momentum dissipation, which brings in finite DC conductivity. Massive gravity is a complete candidate in which the momentum dissipation is involved, and it is an effective bulk theory that does not conserve momentum without borrowing additional fields. In the action (1), the last terms represent massive potentials associated with the graviton mass which breaks the diffeomorphism invariance in the bulk, which produces momentum relaxation in the dual boundary theory.

Varying the action, we can obtain the equations of motion

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{n(n+1)}{2l^2}g_{\mu\nu} \\ - \frac{1}{2} \left( F_{\mu\sigma} F_\nu^\sigma - \frac{1}{4}g_{\mu\nu} F^2 \right) + m^2 \chi_{\mu\nu} = 0, \end{aligned} \quad (3)$$

where

$$\begin{aligned} \chi_{\mu\nu} = & -\frac{c_1}{2} (\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) \\ & -\frac{c_2}{2} (\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) \\ & -\frac{c_3}{2} (\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 \mathcal{K}_{\mu\nu} \\ & + 6\mathcal{U}_1 \mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) - \frac{c_4}{2} (\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 \mathcal{K}_{\mu\nu} \\ & + 12\mathcal{U}_2 \mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1 \mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4) \end{aligned} \tag{4}$$

The static black hole solution of the above action yields

$$\begin{aligned} ds^2 = & -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 h_{ij} dx^i dx^j, \quad i, \\ & j = 1, 2, 3, \dots, n, \end{aligned} \tag{5}$$

where  $h_{ij} dx^i dx^j$  is a line element of Einstein space with constant curvature  $n(n - 1)k$  and  $k = 1, 0, -1$  corresponds to a spherical, Ricci flat, and hyperbolic horizon for black hole. We follow the ansatz in [71] and set the reference metric  $f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij})$ . Then  $\mathcal{U}_i$  can be computed as

$$\begin{aligned} \mathcal{U}_1 = & nc_0/r, \quad \mathcal{U}_2 = n(n - 1)c_0^2/r^2, \\ \mathcal{U}_3 = & n(n - 1)(n - 2)c_0^3/r^3, \\ \mathcal{U}_4 = & n(n - 1)(n - 2)(n - 3)c_0^4/r^4. \end{aligned} \tag{6}$$

Putting them back to the Einstein equation, we have the metric function  $f(r)$

$$\begin{aligned} f(r) = & k + \frac{r^2}{l^2} - \frac{m_0}{r^{n-1}} + \frac{c_0 c_1 m^2}{n} r + c_0^2 c_2 m^2 \\ & + \frac{(n - 1)c_0^3 c_3 m^2}{r} + \frac{(n - 1)(n - 2)c_0^4 c_4 m^2}{r^2}, \end{aligned} \tag{7}$$

where the integral constant  $m_0$  is the black hole mass parameter.

### 2.2 Nambu–Goto action

Following the approach in [57, 58], we consider a Wilson line operator in the spacetime by inserting a fundamental string in the massive gravity. This corresponds to a test particle moving on the boundary gauge theory, which is described by a non-local operator. Inserting the Wilson loop is described by adding a Nambu–Goto (NG) term, so the total action consists of the Einstein–Hilbert term, the Nambu–Goto term and the boundary term. Since the contribution of Einstein–Hilbert action and boundary term to the complexity growth in massive black hole has been studied in [32, 33], here we shall investigate the effect of the Nambu–Goto action.

Setup To this end, we assume that a probe string moves in a subspace with different topologies. Then the induced metric was  $h_{ij} dx^i dx^j = d\phi^2$ . We take the worldsheet parameter as

$$t = \tau, \quad r = \sigma, \quad \phi = v\tau + \xi(\sigma). \tag{8}$$

where  $v$  denotes the velocity of a string relative to the black hole. For simplicity, we will set  $l=1$ , then the induced metric is

$$\begin{aligned} ds_{n+2}^2(\text{ind}) = & (-f(\sigma) + \sigma^2 v^2) d\tau^2 \\ & + \left( \frac{1}{f(\sigma)} + \sigma^2 \xi'(\sigma) \right) d\sigma^2 + 2\sigma^2 v \xi' d\tau d\sigma, \end{aligned} \tag{9}$$

$$\begin{aligned} f(\sigma) = & k + \sigma^2 - \frac{m_0}{\sigma^{n-1}} + \frac{c_0 c_1 m^2}{n} \sigma + c_0^2 c_2 m^2 \\ & + \frac{(n - 1)c_0^3 c_3 m^2}{\sigma} + \frac{(n - 1)(n - 2)c_0^4 c_4 m^2}{\sigma^2}. \end{aligned}$$

The NG action is achieved by integrating over the WDW patch,

$$\begin{aligned} \frac{dS_{NG}}{dt} = & T_s \int_0^{r_h} d\sigma \sqrt{-g_{\text{ind}}(\sigma)} \\ = & T_s \int_0^{r_h} d\sigma \sqrt{1 - \frac{v^2 \sigma^2}{f(\sigma)} + \sigma^2 f(\sigma) \xi'(\sigma)^2} \\ \equiv & \int_0^{r_h} d\sigma \mathcal{L}_{n+2} \end{aligned} \tag{10}$$

where the horizon  $r_h$  is determined by  $f(r_h) = 0$ , and  $T_s$  is the tension of fundamental string.

Varying the above ‘action’, we obtain the equation of motion for  $\xi$

$$\frac{d}{d\sigma} \left( \frac{\sigma^2 f(\sigma) \xi'(\sigma)}{\sqrt{1 - v^2 \sigma^2 / f(\sigma) + \sigma^2 f(\sigma) \xi'(\sigma)^2}} \right) = 0, \tag{11}$$

from which we could define the constant  $c_\xi$  as

$$c_\xi \equiv \frac{\sigma^2 f(\sigma) \xi'(\sigma)}{\sqrt{1 - v^2 \sigma^2 / f(\sigma) + \sigma^2 f(\sigma) \xi'(\sigma)^2}}. \tag{12}$$

Subsequently,  $\xi'$  can be solved by

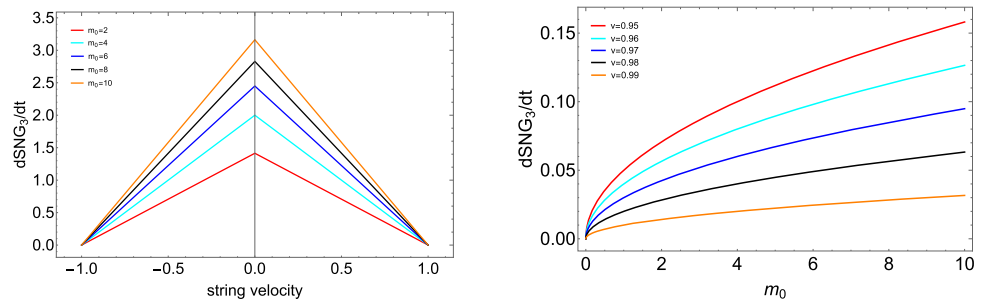
$$\xi'(\sigma) = \frac{c_\xi}{\sigma f(\sigma)} \sqrt{\frac{f(\sigma) - v^2 \sigma^2}{\sigma^2 f(\sigma) - c_\xi^2}}. \tag{13}$$

This expression must give real values, so the denominator and numerator in the square root should coincidentally be positive, negative or zero. It is noted that we shall set  $c_0 = c_1 = c_2 = c_3 = c_4 = 1$  without loss of generality. Then regarding the numerator,  $f(\sigma) - v^2 \sigma^2$ , as a function of  $\sigma$ , we find it is a monotonically increasing function and has a negative value for  $\sigma = 0$ . Therefore, there is only one solution  $\sigma = \sigma_H$  to  $f(\sigma) - v^2 \sigma^2 = 0$ . Subsequently, the constant  $c_\xi$  is determined by  $\sigma_H^2 f(\sigma_H) - c_\xi^2 = 0$  as

$$c_\xi = v \sigma_H^2. \tag{14}$$

Inserting the expressions (9), (13) and (14) into (10), we could rewrite the NG action as

**Fig. 1** Left: action growth v.s. string velocity in BTZ black hole for different  $m_0$ . Right: action growth v.s. black hole mass in BTZ black hole for different string velocities



$$\frac{1}{T_s} \frac{dS_{NG}}{dt} = \int_0^{r_h} d\sigma \sqrt{\frac{g(\sigma) - \sigma^4 v^2}{g(\sigma) - \sigma_H^4 v^2}} \quad \text{with}$$

$$g(\sigma) = k\sigma^2 + \sigma^4 - m_0\sigma^{3-n} + \frac{c_0 c_1 m^2}{n} \sigma^3 + c_0^2 c_2 m^2 \sigma^2 + (n-1)c_0^3 c_3 m^2 \sigma + (n-1)(n-2)c_0^4 c_4 m^2. \tag{15}$$

In the following we shall study the effect of horizon curvature and the graviton mass on the NG action in different dimensions.

### 3 Three dimensional case

For  $n = 1$ , i.e., the 3-dimensional massive gravity, the terms related with  $c_2, c_3$ , and  $c_4$  in (15) vanish, and the NG action would reduce to

$$\frac{1}{T_s} \frac{dS_{NG}}{dt} = \int_0^{r_h} d\sigma \sqrt{\frac{\sigma^4 - \sigma^2 m_0 + c_0 c_1 m^2 \sigma^3 - v^2 \sigma^4}{\sigma^4 - \sigma^2 m_0 + c_0 c_1 m^2 \sigma^3 - v^2 \sigma_H^4}}, \tag{16}$$

where  $r_h = \frac{-c_0 c_1 m^2 + \sqrt{c_0^2 c_1^2 m^4 + 4m_0}}{2}$  and  $\sigma_H = \frac{-c_0 c_1 m^2 + \sqrt{c_0^2 c_1^2 m^4 + 4(1-v^2)m_0}}{2(1-v^2)}$ . This integral for all parameter

ranges can be directly worked out by numeric. We first turn off the graviton mass and consider the static BTZ black hole, and then we move onto the massive BTZ case. The numerical results and some analytical study are shown as follows.

#### 3.1 BTZ black hole

First we set the graviton mass as  $m = 0$  and study the effect of BTZ black hole mass on the NG action. The explicit behaviors with different parameters are plotted in Fig. 1. In the left plot we show the relation between action growth and string velocity with different black hole masses. We see that the complexity growth takes the maximum value when the string is stationary, and as the string moves faster, the action growth becomes smaller. Moreover, the peak value of action growth is larger for larger black hole mass. These features are similar

as the observe for static black holes [57,58]. Additionally, the action growth will vanish as the velocity approaches to light speed, which was also observed in the rotating BTZ black holes [58] even though the peak is shifted.

The right plot of Fig. 1 shows the relation between action growth and black hole mass with different string velocities. In this case, the faster string gives the smaller action growth, which is consistent with that in the left plot. It behaves as a monotonically increasing function of the black hole mass, which is similar to the observes in rotating BTZ black hole [57]. A possible interpretation is that a larger system may have larger information and so it is more complex.

The above numerical results can be analytically confirmed. With tricks, the growth of NG action (16) could be analytically integrated as

$$\frac{1}{T_s} \frac{dS_{NG}}{dt} = \sqrt{m_0} - |v| \sqrt{m_0}, \tag{17}$$

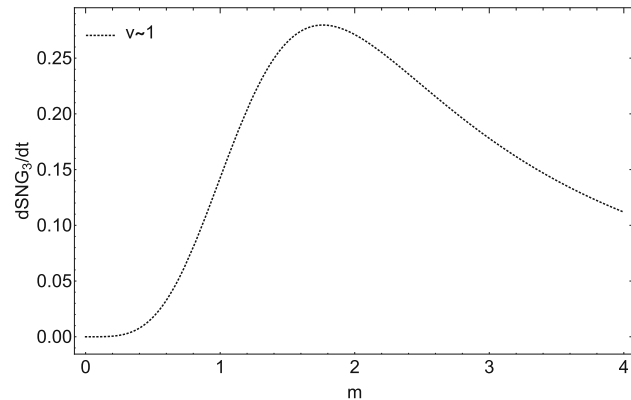
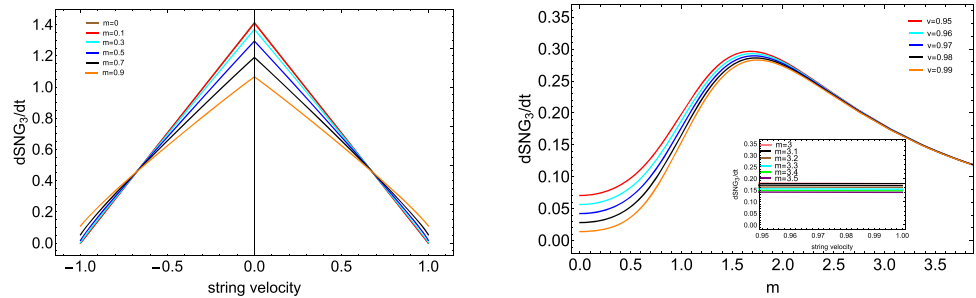
which depends on the black hole mass and string velocity in a simple way. It indicates that with fixed  $m_0$ , the action growth is maximum when the string is stationary with  $v = 0$  and the maximal value is  $\sqrt{m_0}$ . As  $|v|$  increases, it decreases and could vanish when the speed approaches to the light speed with  $|v| = 1$ . Moreover, the action growth could increase monotonously as  $m_0$  when  $v$  is fixed.

#### 3.2 Massive BTZ black hole

Then we turn on the graviton mass and study its effect on the NG action growth. So we fix the black hole mass as  $m_0 = 2$ . The results are shown in Fig. 2.

In left plot of Fig. 2, we show the relation between action growth and string velocity for massive black hole. As usual, the maximal complexity growth appears when the string velocity is equal to zero for different  $m$ . The interesting point is that the maximal value in massive BTZ black hole is smaller than that in BTZ black hole, and as the graviton mass increases, the maximal value becomes smaller. For all chosen  $m$ , as the string moves faster, the action growth becomes smaller. Moreover, comparing to the BTZ case, the action growth will not vanish when the string moves with light speed. In addition, It is also obvious that the effect of  $m$  on the action growth depends on the velocity.

**Fig. 2** Left: action growth v.s. string velocity in massive BTZ black hole. Right: action growth v.s. graviton mass in massive BTZ black hole. Here we fix the black hole mass  $m_0 = 2$



**Fig. 3** Action growth v.s. graviton mass in massive BTZ black hole. Here we fix the black hole mass  $m_0 = 2$  and  $v \sim 1$

Then in the right plot of Fig. 2, we present the relation between action growth and the graviton mass. We focus on the vicinity of the light speed. In all cases, as  $m$  increases, the action growth increases into a peak and then decreases, i.e., it is not a monotonically increasing function. Moreover, the velocity dependence is significant for small graviton mass, but it seems to be slight for large  $m$ , which is also explicit in the inserted plot. Thus, on the vicinity of the light speed, the graviton mass suppresses the velocity dependence of the action growth, which is very different from the effect of black hole mass that enhances it in all cases as studied in [57,58].

It is noted that in this case, it is difficult to integrate the action (16) analytically. However, we could estimate the behavior in some limit regimes to further check the numerical analysis. In the limit  $v \rightarrow 0$ , we can integrate (16) as

$$\frac{1}{T_s} \frac{dS_{NG}}{dt} \Big|_{v=0} = \frac{\sqrt{m^4 + 4m_0} - m^2}{2}, \tag{18}$$

which is a monotone decreasing function of positive  $m$ . So the value of the action growth at  $v = 0$  is smaller as the graviton mass increases, which is consistent with that shown in the left plot of Fig. 2.

In the speed light limit with  $v \rightarrow 1$ , a semi-analytic integration is obtained as

$$\frac{1}{T_s} \frac{dS_{NG}}{dt} \Big|_{v=1} \sim m \int_0^{r_h} \frac{\sigma d\sigma}{\sqrt{(\sigma + \frac{m_0}{m^2})(\sigma^2 + (\frac{m_0}{m^2})^2) + m^2\sigma^2}}, \tag{19}$$

which is shown in Fig. 3. It shows that in this limit, as  $m$  increases, the action growth increases into a peak and then decreases, i.e., it is not a monotonic function, which matches the numerical study shown in the right plot of Fig. 2.

### 4 Higher dimensional case

When the dimension is higher than three, the black hole horizon could have hyperbolic, plane and spherical topologies corresponding to the horizon curvatures  $k = -1, 0$  and  $1$ , respectively. In higher dimensions, the exact integration results of (15) is difficult, so we shall numerically integrate it in different dimensions and study the effect the horizon curvature and graviton mass on the NG action growth.

#### 4.1 AdS black hole

We first study the effect of horizon curvature. So we focus on the AdS black hole by setting the graviton mass as  $m = 0$ .

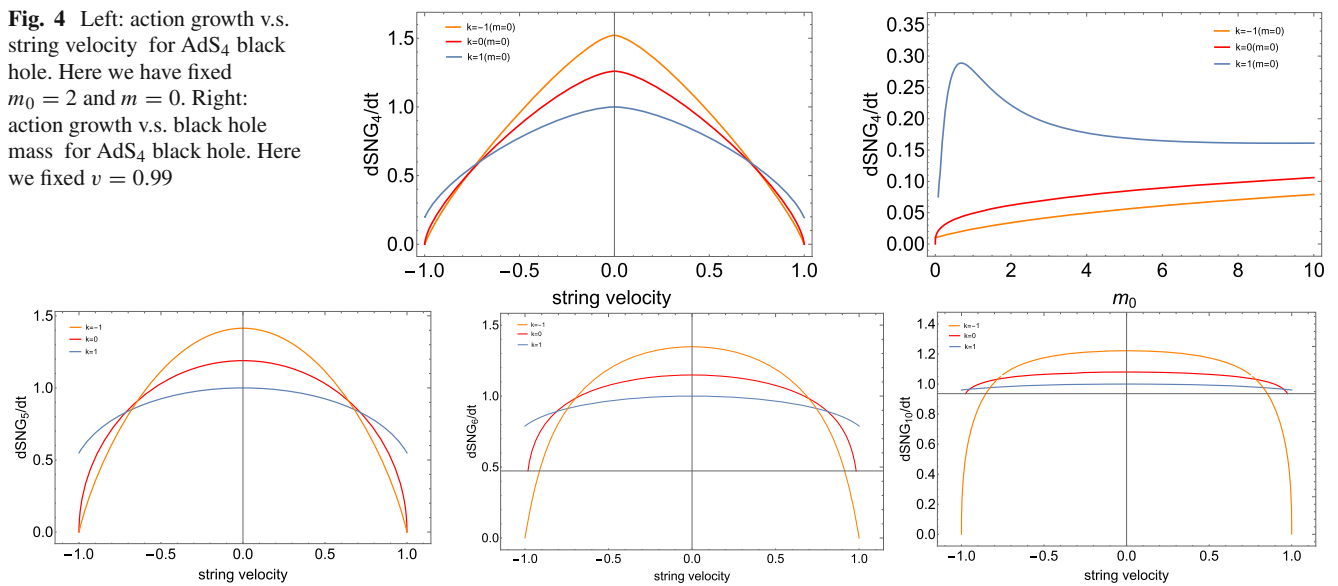
The velocity dependence and black hole mass dependence in four dimensional case are shown in Fig. 4. In the left plot with the black hole mass  $m_0 = 2$ , the action growth for  $k = 1$  reproduces the curve with black hole mass  $M = 1$  in Fig. 1 of [58]. In all cases, the maximal value appears when the string does not move, and as the velocity increases, the action growth decreases. It is obvious that the maximal values in the case with  $k = -1$  is the largest while it is larger in  $k = 0$  than in  $k = 1$  case. This phenomena can be understood in an analytical way. In the limit  $v \rightarrow 0$ , the integration (15) with four dimension ( $n = 2$ ) is

$$\frac{1}{T_s} \frac{dS_{NG}}{dt} = r_h. \tag{20}$$

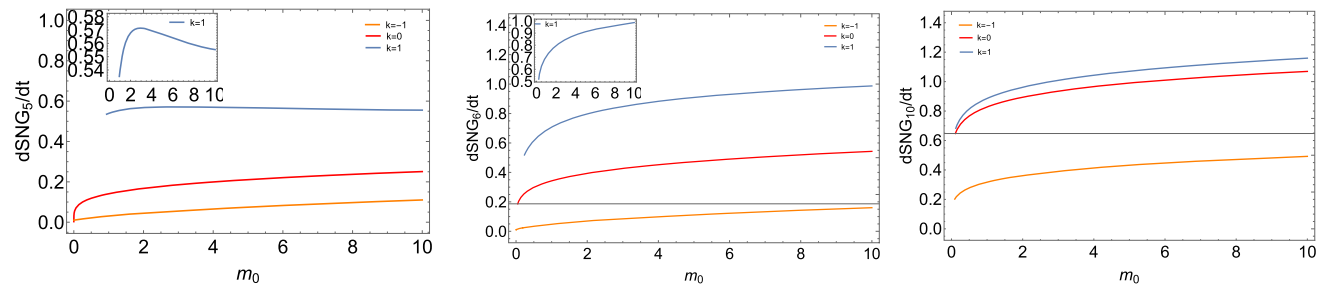
Then recalling the event horizon satisfying  $f(r_h) = 0$ , which gives us

$$k = \frac{m_0}{r_h} - r_h^2 - m^2 \left( \frac{r_h}{2} + 1 \right). \tag{21}$$

**Fig. 4** Left: action growth v.s. string velocity for AdS<sub>4</sub> black hole. Here we have fixed  $m_0 = 2$  and  $m = 0$ . Right: action growth v.s. black hole mass for AdS<sub>4</sub> black hole. Here we fixed  $v = 0.99$



**Fig. 5** Action growth v.s. string velocity for AdS black hole. From left to right, the dimension is 5, 6 and 10. In all cases, we have set the black hole mass  $m_0 = 2$



**Fig. 6** Action growth v.s. black hole mass from left to right, the dimension is 5, 6 and 10. In all cases, we have set the  $m = 0, v = 0.99$

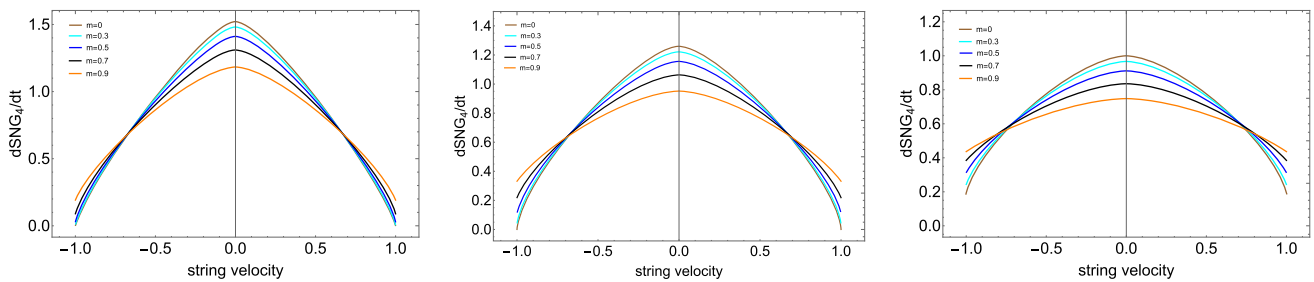
It is obvious that for AdS case with  $m = 0$  and  $m_0 = 2$ ,  $k$  decreases as  $r_h$  increases, which indicates that  $k = -1$  corresponds to largest  $r_h$ , so does the largest action growth while  $k = 1$  corresponds to the smallest action growth. It is easy to see that the conclusion is also hold in massive gravity with  $m \neq 0$ , which will match the numerical results as we will see soon. Noted that similar analysis could also be done in higher dimension cases.

But the action growth decays fastest when  $k = -1$ . As the string velocity increases, there exists a intersection and the effect of  $k$  becomes opposite to that in small velocity. Another notable feature is that for  $k = 0$  and  $k = -1$ , the action growth with light speed goes to zero, namely the contribution from NG action disappears such that the effect of string is zero in these cases. This phenomena is in contrast to the nonzero value for  $k = 1$ . As we increase the dimension (see Fig. 5 for 5D, 6D and 10D cases), the action growth from left to right becomes gentler and gentler as usual, and the intersection occurs at larger velocity. Moreover, similar as  $k = 1$ , the action growth near light speed for  $k = 0$  also increases to be

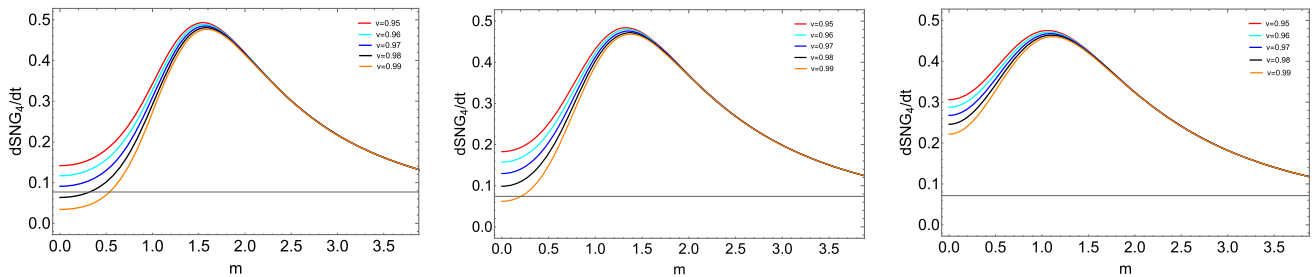
nonzero for higher dimensions, while for  $k = -1$ , it keeps zero.

In the right plot of Fig. 4, we show how the action growth depends on the black hole mass near the light speed. Again the blue curve for  $k = 1$  reproduces the curve with the same velocity in Fig. 1 of [58], and there exists a peak at small black hole mass. In a contrast, it behaves as a monotonically increasing function of the black hole mass for  $k = -1$  and  $k = 0$ . Moreover, it shows that in the vicinity of light speed, the action growth for  $k = 1$  is the largest, which is consistent with that in the left plot.

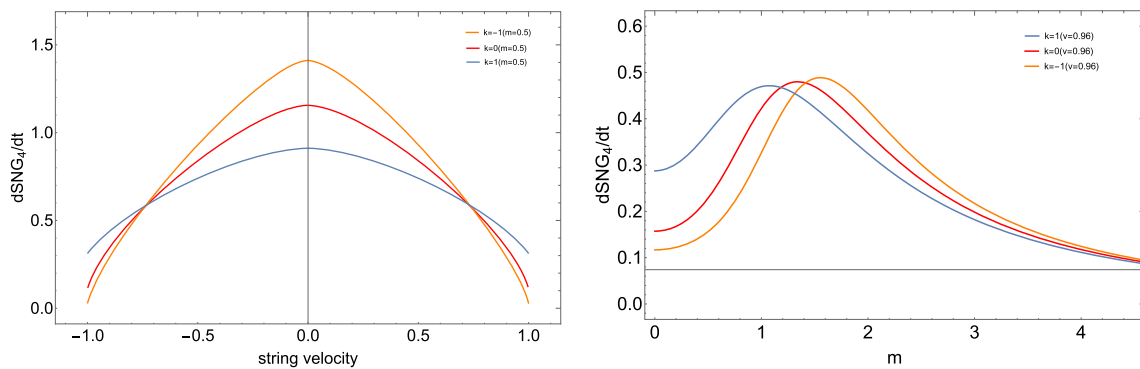
As the dimension increases (see Fig. 6 for 5D, 6D and 10D cases), the peak for  $k = 1$  could disappear and the black hole mass dependence becomes monotonically increasing function, which reproduces the outcome of [58]. Moreover, it also shows that as the dimension increases, the gap between  $k = 0$  and  $k = -1$  is wider while the gap between  $k = 0$  and  $k = -1$  is narrower. Unfortunately, it is difficult to find analytical insight of the features in the limit  $v \rightarrow 1$  even in four dimensional case.



**Fig. 7** Action growth v.s. string velocity for massive AdS<sub>4</sub> black hole with  $m_0 = 2$ . From left to right, the horizon curvature is  $k = -1$ ,  $k = 0$  and  $k = 1$ , respectively



**Fig. 8** Action growth v.s. graviton mass for massive AdS<sub>4</sub> black hole with  $m_0 = 2$ . From left to right, the horizon curvature is  $k = -1$ ,  $k = 0$  and  $k = 1$ , respectively



**Fig. 9** Action growth for massive AdS<sub>4</sub> black hole. Left: action growth v.s. string velocity with  $m = 0.5$ . Right: action growth v.s. graviton mass with  $v = 0.96$ . Here we have set black hole mass  $m_0 = 2$

### 4.2 Massive AdS black hole

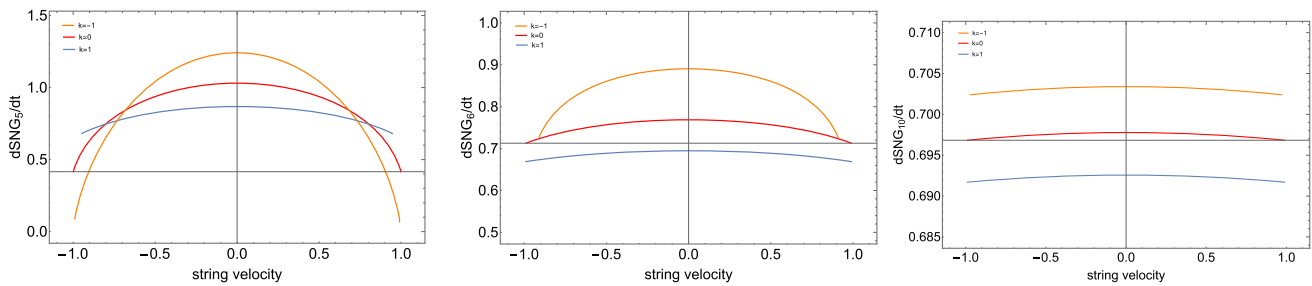
Then, we turn on  $m$  to study the effect of the graviton mass and horizon curvature.

We first study in detail the effect of graviton mass on the string velocity dependence as well as graviton mass dependence in the vicinity of light speed. We show the results for four dimensional case in Figs. 7 and 8, respectively. In all plots of Fig. 7, larger  $m$  gives the lower maximal values of action growth. This feature could be expected from (20) in  $v \rightarrow 0$  limit if we reform (21) as  $m^2 = (m_0/r_h - r_h^2 - k)/(r_h/2 + 1)$ , which indicates that with fixed  $m_0$  and  $k$ , larger  $m$  corresponds to smaller  $r_h$ . There exists a intersection where the effect of  $m$  on the string velocity dependence will change. This observe is consistent with that

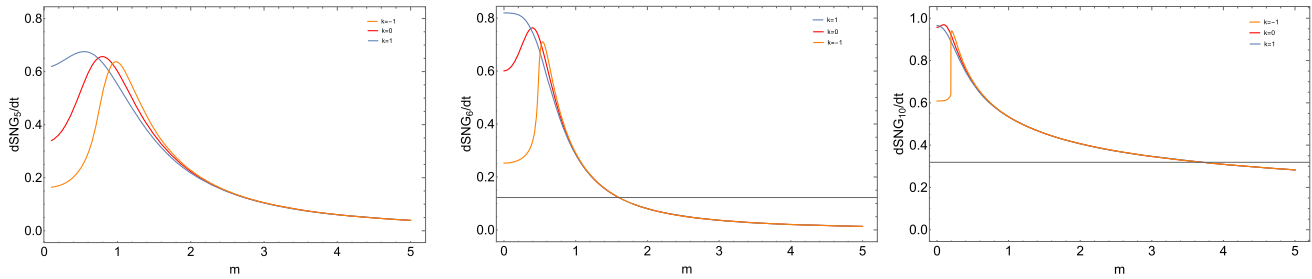
in three dimensional case (see Fig. 2), even though here the curves are more gentler.

Figure 8 shows the  $m$  dependence of action growth in the vicinity of light speed. In all cases, the smaller string velocity corresponds to larger action growth when the graviton mass is small. As the increase of  $m$ , it approaches to a maximal value and then decreases. The  $m$  dependence is slightly affected by the string velocity for large enough  $m$ . Though the behavior is similar to that for 3D case studied in previous section, we see that the location of peak tends to smaller  $m$ .

Then we fix  $m = 0.5$  and study the effect of horizon curvature. The results in four dimensional massive gravity are shown in Fig. 9. Comparing the left plots in Figs. 4 and 9, we see that in massive gravity, the effect of horizon curvature on the velocity dependent action growth is similar to that in



**Fig. 10** Action growth v.s. string velocity in massive AdS black hole with  $m = 0.5$ . From left to right, the dimension is 5, 6 and 10. In all cases, we have set the black hole mass  $m_0 = 2$



**Fig. 11** Action growth v.s. graviton mass for massive AdS black hole From left to right, the dimension is 5, 6 and 10. In all cases, we have set the black hole mass  $m_0 = 2$  and the string speed is  $v = 0.96$

Einstein gravity. As in the AdS case, the maximal value for stationary string with  $k = -1$  is the largest, the reason of which has been stated below Eq. (21). There also exists an intersection. As the dimension increases (see Fig. 10), the velocity dependent action growth is more gentler. One new feature in massive gravity is that in all cases, the action growth in light speed limit becomes non-vanished. Comparing with that in AdS case, the intersection in massive case for different curvatures could separate in lower dimension.

In the right plot of Fig. 9, we show the effect of  $k$  on the graviton mass dependence, which is more significant for small  $m$  than for large  $m$ . It also shows a peak at certain critical  $m_c$  as in three dimensional case. Here the location value of  $m$  for the peak decreases as  $k$  increases. Furthermore, in Fig. 11 we show the graviton mass dependence for higher dimensions. As the dimension increases,  $m$  for the location of peak decreases. Then for  $k = 1$ , the peak shows at  $m = 0$ , meaning the action grow monotonically decreases as  $m$  increases (see the middle plot for six dimension), which is also explicit in the ten dimensional case. Further increasing the dimension, the action grow could also become monotonically decreasing function of  $m$  for the massive black hole with  $k = -1$  and  $k = 0$ . This is a new phenomena.

### 5 Summary

We studied the complexity growth in the dynamical system with a Wilson line operator, which is holographically dual to

the massive black holes with a probe string. We focused on the Nambu–Goto action growth and employed the CA conjecture to explore the effect of string on the complexity growth in a boundary gauge theory with momentum relaxation. We mainly analyzed the effect of horizon curvatures, the graviton mass and the dimension on the velocity dependent complexity growth. Our study is mainly presented in numerical way accompanying with some analytic estimation. The results are summarized as follows.

In three dimension, the maximal complexity growth appears when the string is motionless, and as the graviton mass increases, the maximal value becomes smaller. As the string moves faster, the action growth becomes smaller. When the string moves with light speed, the action growth in massive BTZ black hole will not vanish which is different from that in BTZ black hole. In the vicinity of the light speed, as the graviton mass increases, the action growth first increases and then decreases. Moreover, the velocity dependence is significant for small graviton mass, while it is slight for large  $m$ . It means that in the vicinity of light speed, the graviton mass suppresses the velocity dependence of the action growth, which is very different from the effect of BTZ black hole mass that enhances it.

In higher dimensions, we could study the effect of horizon curvatures  $k$ . We first studied the effect of  $k$  on the velocity dependent action growth. Our results shows that the maximal value for the stationary string for  $k = -1$  is the largest, then for  $k = 0$ , the value for  $k = 1$  is the smallest. There is an intersection in the velocity dependence curves for different



$k$ . As the dimension increases, the velocity dependent action growth is more gentler. Comparing with AdS black hole, in massive black hole, the action growth in light speed limit becomes non-vanishing with different  $k$ , and the intersection for different curvatures could separate in lower dimensions.

We then studied the effect of  $k$  on the graviton mass dependence. It is more significant for small  $m$  than for large  $m$ . It also shows a peak at certain graviton mass as in three dimensional case, and the location value of the peak decreases as  $k$  increases. As the dimension increases, the critical graviton mass for the peak decreases. The peak for  $k = 1$  in high enough dimension could first appears at  $m = 0$ , meaning the action growth turns to monotonically decrease as  $m$  increases. Then further increasing the dimension, the action growth could also become monotonically decreasing function of  $m$  for the massive black hole with  $k = -1$  and  $k = 0$ .

Inspired by Lloyd's state on the quantum complexity growth rate [72], the authors of [30] addressed an analogous "Lloyd bound" via CA conjecture as

$$\frac{dS}{dt} \leq \frac{2M}{\pi}, \quad (22)$$

where  $S$  is the total action and  $M$  is the total mass or energy of the system. In the previous literatures [32,33] where the action consists of Einstein–Hilbert term and boundary term in massive gravity, it was shown that the bound was saturated. In our work with a probe string, the total action consists of Einstein–Hilbert term, boundary term and the NG term, while  $M$  contain the contributions of the black hole mass and the mass of string. Then the bound could be satisfied because the length of string is infinity stretching to the infinity of the space. It is noticed that in CA conjecture, the bound (22) may not be always satisfied, see [43,73–77] as examples for the violated cases.

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