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Removal of instabilities of the higher derivative theories in the light of antilinearity

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Abstract Theories with higher derivatives involve linear instabilities in the Hamiltonian commonly known as Ostrogradski ghosts and can be viewed as a very serious problem during quantization. To cure *this*, we have considered the properties of antilinearity that can be found inherently in the non-Hermitian Hamiltonians. Owing to the existence of antilinearity, we can construct an operator, called the V-operator, which acts as an intertwining operator between the Hamiltonian and its Hermitian conjugate. We have used this V-operator to remove the linear momentum term from the higher derivative Hamiltonian by making it non-Hermitian in the first place via an isospectral similarity transformation. The final form of the Hamiltonian is free from the Ostrogradski ghosts under some restriction on the mass term.

1 Introduction

From the usual quantum mechanics, it is well known that the Hamiltonian must be Hermitian, i.e. $H = H^\dagger$, in order to obtain a real energy spectrum. Very recently Bender et al. proposed that parity ($\mathcal{P}$) and time ($\mathcal{T}$) symmetry can serve as a better condition for obtaining the real energies of the system as it includes non-Herimitian [1–4] Hamiltonians too. $\mathcal{PT}$-symmetry is actually a physical condition obeyed by almost every phenomenon. In this regard, we may consider a non-Hermitian system *by* employing the conditions of $\mathcal{PT}$-symmetry one can obtain real energy spectrum [5]. While working along the same line of considerations of the physical interpretations of the non-Hermitian Hamiltonians, one may consider the obvious nature of these non-Hermitian Hamiltonians, i.e. the antilinear property. Evidently, the Hamiltonian that is non-Hermitian can be written in an antilinear form and the corresponding antilinearity operator can be obtained [6,7]. The reality of the energy eigenvalues and the unitarity of the system was *found* to be more subtle in this case.

A common source of these non-Hermitian Hamiltonians is the Higher Derivative (HD) theories. By higher derivative, we refer here to the theories that have time derivatives of the fields numbering more than two in the Lagrangian. The higher derivative terms are added to the Lagrangian as a correction term and these may lead to *avoid* the ultraviolet divergences appearing in the theory. Due to this nature, the HD theories are actively under consideration in various branches of physics, e.g. string theory [8,9], cosmology [10,11], and general relativity [12–14]. In the HD theories, the Hamiltonian actually consists of the momenta of the higher derivative fields multiplied by the momenta of other fields and not corresponding to its own momenta. These linear momenta, on quantization, can lead to instabilities as the spectrum become infinite. This is a very classic problem in physics and the corresponding field is known to be 'ghost state' or Ostrogradskian instability.

To remove these ghost states, there have been different attempts made by many authors. In [15–17], the authors have tried to remove the ghost states by incorporating new constraints in the phase space, which is applicable only if the phase space is reduced. Very recently, the authors in [18] considered the inclusion of velocity dependent constraints to remove the Ostrogradskian instability. The Ostrogradski ghosts can also be removed by the introduction of new variables which can be *obtained* by a combination of primary and secondary constraints [19,20]. By considering a degenerate Lagrangian which has a non-invertible kinetic matrix, the theory can be made ghost-free as shown in [21]. In the case of analytic mechanics, the ghosts appeared as usual [22] and they were removed by different degeneracy conditions [23]. It is thus seen that there are multiple attempts to address the issue of Ostrogradski instability with limited applicability.

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In this paper, we shall consider the properties of antilinearity in order to remove the instabilities from the theory. Due to antilinearity, the Hamiltonian of these kinds of models can be put into a form so that they remain invariant under a suitable choice of similarity transformation. It was later shown that the similarity transformation mentioned here, in general, can be found in any real Hamiltonian and the corresponding operator can be identified [6,24]. This, however, is not the case when one has a non-Hermitian Hamiltonian $H \neq H^{\dagger}$. As the Ostrogradski ghosts are inherent in any HD theory, we can look into the matter by transforming it into a complex form and analyzing them using the properties due to antilinearity. As an example, we have considered is the Galilean invariant Chern–Simon model. This is a toy model which manifestly involves the characteristics of Chern–Simon's model with a mass term. Lukiersky et al. have shown that the model can be quantized in the noncommutative plane [25]. This is an interesting model as one can see that the model has been used in different fields like quantum gravity [26], Newton–Hooke symmetry [27,28], and anyons [28].

The plan of the paper is as follows. In Sect. 2 we consider a very brief discussion on higher derivative models and how one can construct the non-Hermitian Hamiltonian as a necessary transformation. Section 3 deals with the properties of antilinear Hamiltonians. Here, we also show how HD theories have an inherent property of antilinearity. In Sect. 4 we have considered an example, the Galilean invariant Chern–Simon model, to illustrate the above discussion of the removal of the ghosts. Finally, we conclude in Sect. 5.

2 Higher derivative models

We may write a general HD lagrangian in the form of $\mathcal{L}(q, \dot{q}, \ddot{q} \ldots q^{(n)})$ which represents a theory with nth order derivatives in time. Now we can write the Lagrangian with respect to some new variable Q defined as

$$Q_0 = q,\ Q_1 = \dot{q},\ Q_2 = \ddot{q}, \ldots Q_n = \frac{d^n}{dt^n} q. \tag{1}$$

This redefinition of the space variables has expanded the configuration space, thereby increasing the number of degrees of freedom. In the newly defined configuration space, we can see the new constraints which are given by $\Phi_n = Q_n - \dot{Q}_{n-1}$. Hence the Lagrangian can be redefined by incorporating these constraints in the Lagrangian via Lagrange's undetermined multipliers as

$$\mathcal{L}' = L(Q, Q_1, Q_2 \ldots Q_n) + \sum_{i=1}^{n} \lambda_i \Phi_i. \tag{2}$$

The above Lagrangian is in the first order form and apparently free from the higher order derivatives. Thus, when written in this form we can easily find the appropriate phase space (Q_i, P_i). Unlike the Ostrogradski way [29] of defining the momenta, in the first order formalism, the momenta are defined in the usual way as $P_i = \frac{\partial \mathcal{L}'}{\partial \dot{Q}_i}$. Immediately we write the canonical Hamiltonian, which is given by

$$\mathcal{H}_{\text{can}} = \sum_{i=0}^{n} P_i \dot{Q}_i - \mathcal{L}'. \tag{3}$$

The definition of the canonical momenta contains variables from the phase space which, in this case, may contain the non-invertible momenta. Usually, the higher derivative theories contain at least one constraint resulting due to the definition of momenta of higher derivative fields irrespective of the Ostrogradskian or the first order formalism. These, in the final form of the canonical Hamiltonian, always appear in terms involving the product of fields in first order and linearly coupled canonical momenta. Interestingly, this momenta is not the one corresponding to the field which it is linearly coupled with. In the corresponding quantum picture, the linear momentum terms give rise to infinities, and hence a series of unstable states, known as ghost states, are being created. Next, we write the canonical momenta (3) in a generic form showing the involvement of the linear momentum term:

$$\mathcal{H}_{\text{can}} = \sum_{i=1}^{n} P_i Q_{i-1} + \bar{\mathcal{H}}. \tag{4}$$

Here, $\bar{\mathcal{H}}$ contains terms that do not include any linear momentum. One should remember that the term $\bar{\mathcal{H}}$ here is not general, as one can always find the remaining part in the canonical Hamiltonian as a term arising due to the higher derivative nature of the theory. Collecting all the primary constraints, the total Hamiltonian of this system can be written as

$$H_T = \mathcal{H}_{\text{can}} + \sum_{i=1}^{m} \Lambda_i \Phi_i. \tag{5}$$

Here, m represents the number of primary constraints arising in the theory. Till now our discussions were purely based on the classical picture. In the next section, we shall consider the corresponding quantum version of the system. For the transition from classical to quantum, we shall replace the variables with their quantum counterpart and also replace all the Poisson brackets with corresponding commutators.

3 Antilinearity in general quantum theories

The wave function $|\Psi(r,t)\rangle$ when acted upon by the total Hamiltonian gives the energy of the system as

$$\hat{H}_T |\Psi(r,t)\rangle = E|\Psi(r,t)\rangle. \tag{6}$$

If we replace t by $-t$ and consider an antilinear operator A we can rewrite the above expression as

$$A\hat{H}_T A^{-1} A|\Psi(r,-t)\rangle = E^* A|\Psi(r,-t)\rangle. \tag{7}$$

Thus, for the state $|\Psi(r,-t)\rangle = A|\Psi(r,t)\rangle$, we can consider under the action of operator A the Hamiltonian to be

$$\hat{\tilde{H}} = A\hat{H}_T A^{-1}. \tag{8}$$

Evidently, this similarity transformed Hamiltonian also has the energy eigen value E^* corresponding to the eigen state $|\Psi(r,-t)\rangle$:

$$\hat{\tilde{H}}|\Psi(r,-t)\rangle = E^*|\Psi(r,-t)\rangle. \tag{9}$$

Energy eigenvalues can be real or imaginary if the Hamiltonian $\hat{H}_T$ of the system shows antilinear symmetry as shown in [6,24]. On the other hand, the reality of energy eigenvalues was also considered by Wigner in [30], which deals with the necessity of the time reversal symmetry of the system. Thus, for the non-Hermitian Hamiltonians, it is possible to possess real energy spectra in the presence of a time reversal symmetry.

In order to find an appropriate form of the $\hat{A}$ operator discussed here, we can look into the theories having $\mathcal{PT}$ symmetries. In $\mathcal{PT}$ symmetries, the theory may not be invariant under individual transformations of $\mathcal{P}$ (which works as $\hat{x} \to -\hat{x}, \hat{p} \to -\hat{p}$) and $\mathcal{T}$ (which works as $\hat{x} \to \hat{x}, \hat{p} \to -\hat{p}, i \to -i$), but, under the collective effect of $\mathcal{PT}$, the theory should remain invariant. Apart from the $\mathcal{PT}$ operator, Bender et al. also pointed out another operator, called the $\mathcal{C}$ operator [31], which can be used to remove the ghost states as shown in [32,33]. The existence and completeness of the theory demand that the $\mathcal{C}$-operator should be governed by the three conditions

$$\mathcal{C}^2 = \mathcal{I}, \quad [\mathcal{C}, \hat{H}] = 0, \quad [\mathcal{C}, \mathcal{PT}] = 0. \tag{10}$$

The existence of the $\mathcal{C}$ operator also guarantees that the unitarity of the theory will be preserved.

In the case the Hamiltonian is not Hermtian, it should obey the following relation:

$$\hat{V}\hat{H}\hat{V}^{-1} = \hat{H}^\dagger, \tag{11}$$

which is obtained by defining the new operator [24]

$$\hat{V} = \hat{A}^\dagger \hat{A}. \tag{12}$$

These two properties in (11,12) also confirm that the transformation under these operators is unitarily equivalent. Equations (8) and (11) both refer to connecting two different transformations of the Hamiltonian, and for this reason they are called intertwining operators. In the present case of higher derivative systems, the canonical Hamiltonian $\hat{H}_{\text{can}}$ has a non-Hermitian form and the above equation will serve a very important role in eliminating the ghost states. For that purpose, we will consider the similarity transformations of the fields i.e. $\hat{V}Q\hat{V}^{-1}$ and a proper choice of the $\hat{V}$ will give us the Hamiltonian which is free from the Ostrogradski instability appearing in (4).

We note **the** properties arising due to the antilinearity present in the theory:

- If the Hamiltonian has antilinear symmetry then the energies may be real or complex. The complex energy eigen values must appear in conjugate pairs,
- If the parity operator obeys $\mathcal{P}\hat{H}\mathcal{P}^{-1} = \hat{H}^\dagger$ then we can write $[\mathcal{P}\hat{V}, \hat{H}] = 0$.
- One can write the $\mathcal{C}$-operator as $\mathcal{C} = \mathcal{P}\hat{V}$ which should obey the relations of $\mathcal{C}$ mentioned in (10).
- The unitary evolution of the system is governed by the condition $\langle\Psi_i|\hat{V}|\Psi_j\rangle = \delta_{ij}$ and we may treat this as the new definition of the inner product. This will arise only if the Hamiltonian is not Hermitian and consequently the basis states will not be orthonormal in the Dirac sense [34].
- The completeness relation in this case becomes $\Sigma|\Psi_i\rangle\langle\Psi_j|\hat{V} = \Sigma\hat{V}^\dagger|\Psi_j\rangle\langle\Psi_i| = \mathcal{I}$.

3.1 Ghost states and antilinearity

The Ostrogradski ghost problem appears due to the reason that there is nolower bound to the potential and consequently we cannot define a true minimum which we call the vacuum. This concept of vacuum is totally a quantum concept and does not exist at the classical level. As soon as one puts $\hbar \to 0$, the quantum phenomena no longer exist. Therefore, the quantum concepts are very much required to discuss the Ostrogradski ghost problem.

The phase space of HD theories is spanned by the momenta of the usual and the HD fields. In this subsection, we show the connection between HD theories and antilinearity. The existence of the higher derivative linear momentum terms makes the Schrodinger equation complex and therefore, to solve them, one requires complex planes. However, while quantizing, as the HD momenta span the entire imaginary plane, it is seen that the solution does not give a well behaved wave function [5]. For the well behaved wave function, one of the conditions says that it must vanish at infinity. In the complex plane, the wavefunctions must also vanish asymptotically along lines that are centered about the positive-real and negative-real axes. In complex geometry, these lines are known as Stokes wedges. The angular opening of the Stokes wedges depends upon the type of eigenfunctions. To disallow the solutions spanning the whole phase space, we can restrict them within Stoke wedges of 90°. For avoiding the imaginary axis, we can write the Hamiltonian (4) after an isospectral similar-

ity transformation, of the usual field (not the HD one's), defined as $\hat{Q}_0 = e^{-\pi \hat{P}_0 \hat{Z}_0/2} \hat{Z}_0 e^{\pi \hat{P}_0 \hat{Z}_0/2} = i\hat{Z}_0$, $\hat{P}_0 = e^{-\pi \hat{P}_0 \hat{Z}_0/2} \hat{P}_0 e^{\pi \hat{P}_0 \hat{Z}_0/2} = -i\hat{\Pi}_0$. Hence the transformed Hamiltonian is given by

$$\hat{H}_{can} = i\hat{\Pi}_1 \hat{Z}_0 + \sum_{i=2}^{n} \hat{\Pi}_i \hat{Z}_{i-1} + \hat{\tilde{\mathcal{H}}}. \tag{13}$$

The above equation shows that all HD theories can be brought to this general form where antilinearity emerges once the canonical Hamiltonian is defined in terms of the newly transformed Hamiltonian. This transformation has become a requirement for the HD theories due to the existence of momenta corresponding to the HD fields [32].

4 The Galilean invariant Chern–Simons model

In this section we consider a specific model to show the efficacy of the discussions of the earlier sections. We consider the Galilean invariant Chern–Simon model which is given by

$$\mathcal{L} = \frac{1}{2} m \dot{x}_i^2 - k\epsilon_{ij} \dot{x}_i \ddot{x}_j. \tag{14}$$

This is a nonrelativistic model in two spatial dimensions and k has the physical dimension of $[M][T]$. Being a higher derivative model, we convert this into a first order Lagrangian owing to the transformations

$$q_{1i} = x_i, \tag{15}$$

$$q_{2i} = \dot{x}_i. \tag{16}$$

With these new variables the first order Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2} m q_{1i}^2 - k\epsilon_{ij} q_{2i} \dot{q}_{2j} + \lambda_i (\dot{q}_{1i} - q_{2i}). \tag{17}$$

Here the λ_i are the Lagrange multipliers incorporated to account for the constraints $\dot{q}_{1i} - q_{2i}$ arising due to the redefinition of the fields. If $\{p_{1i}, p_{2i}, p_{\lambda_i}\}$ are the momenta corresponding to the fields $\{q_{1i}, q_{2i}, \lambda_i\}$ then we get a set of primary constraints given by

$$\Phi_i = p_{1i} - \lambda_i \approx 0, \tag{18}$$

$$\psi_i = p_{2i} - k\epsilon_{ij} q_{2j} \approx 0, \tag{19}$$

$$\Xi_i = p_{\lambda_i} \approx 0. \tag{20}$$

All these three primary constraints are of second class in nature due to their non-zero Poisson brackets, which are given by

$$\{\psi_i, \psi_j\} = -2k\epsilon_{ij}, \tag{21}$$

$$\{\Phi_i, \Xi_j\} = -\delta_{ij}. \tag{22}$$

The second class constraints are removed by setting them zero. This also replaces all the Poisson brackets in the theory by Dirac brackets defined as

$$\{\xi_i, \xi_j\}_D = \{\xi_i, \xi_j\} - \{\xi_i, \psi_m\} \Delta^{-1}_{mn} \{\psi_n, \xi_j\}. \tag{23}$$

In the present case, the set of phase space variables ξ_i is $\{q_{1i}, q_{2i}, p_{1i}, p_{2i}\}$ and the Poission brackets between the second class constraints are defined by $\Delta_{mn} = \{\psi_m, \psi_n\}$. The non-zero Dirac brackets are given by

$$\begin{aligned} \{q_{1i}, p_{1j}\}_D &= \delta_{ij}, \\ \{q_{2i}, q_{2j}\}_D &= \frac{1}{2k}\epsilon_{ij}, \\ \{q_{2i}, p_{2j}\}_D &= \frac{3}{2}\delta_{ij}, \\ \{p_{2i}, p_{2j}\}_D &= -\frac{k}{2}\epsilon_{ij}. \end{aligned} \tag{24}$$

Using the usual definition of the canonical Hamiltonian we can write for the present model

$$\mathcal{H}_{\text{can}} = -\frac{m}{2} q_{2i}^2 + \lambda_i q_{2i}. \tag{25}$$

Since the system has constraints we should consider the total Hamiltonian instead, which is obtained by adding the primary constraints linearly to the canonical Hamiltonian as

$$H_T = \mathcal{H}_{\text{can}} + \Lambda_{1i}\Phi_i + \Lambda_{2i}\psi_i + \Lambda_{3i}\Xi_i. \tag{26}$$

The Hamiltonian written thus involves the undetermined Lagrange multipliers $\{\Lambda_{1i}, \Lambda_{2i}, \Lambda_{3i}\}$ and it can be determined by considering the time evolution of the constraints. The time evolution of the constraints may give rise to secondary and tertiary constraints. We consider the brackets between the constraints and the total Hamiltonian and after equating them to zero, the following values of Lagrange's multipliers are obtained:

$$\Lambda_{3i} = 0, \tag{27}$$

$$\Lambda_{2i} = \frac{1}{2k}(m q_{2j} + \lambda_j)\epsilon_{ji}, \tag{28}$$

$$\Lambda_{1i} = -q_{2i}. \tag{29}$$

We can remove the second class constraints by setting them zero and considering the Dirac brackets in place of Poisson brackets. Consequently, in this case, the total Hamiltonian becomes equal to the canonical Hamiltonian,

$$H_T = \mathcal{H}_{\text{can}}. \tag{30}$$

So far, we have discussed the classical views of this Galilean invariant Chern–Simon's model. In the corresponding quantum version, we want to see the antilinear symmetry is present in the system. For that purpose, we should analyze the $\mathcal{PT}$-symmetries of this Hamiltonian under the changes of the space-time coordinates. The model is truely $\mathcal{PT}$ symmetric, which can be seen from

$$\mathcal{P}\hat{H}_T\mathcal{P}^{-1} = \hat{H}_T^{\dagger}, \tag{31}$$

$$(\mathcal{PT})\hat{H}_T(\mathcal{PT})^{-1} = \hat{H}_T. \tag{32}$$

As discussed in Sect. 2 this Hamiltonian contains linear momentum terms and hence the transition to the quantum picture is not possible as the states will be unstable. Due to the higher derivative nature, for removal of these linear fields, we may consider a similarity transformation in the form of the change of variables $\hat{q}_{1i} = i\hat{z}_i$, $\hat{p}_{1i} = -i\hat{p}_{zi}$ as suggested in [5] and obtain the Hamiltonian:

$$\hat{\tilde{H}}_T = -\frac{m}{2}\hat{q}_{2i}^2 - i\hat{p}_{zi}\hat{q}_{2i}. \tag{33}$$

The total Hamiltonian in (33) is clearly non-Hermitian in nature and $\mathcal{PT}$ symmetric. Being a non-Hermitian Hamiltonian, it does not necessarily has complex energy eigenvalues . As discussed in (33), owing to the $\mathcal{PT}$ symmetric nature, the model has real energy eigenvalues. The existence of antilinearity in the model is confirmed since the Hamiltonian obeys $\mathcal{P}\hat{H}_T\mathcal{P}^{-1} = \hat{H}_T^\dagger$. Hence, as a requirement of (11,12), for the present model, we can find the corresponding intertwining operators which can be written as

$$\hat{V} = e^{-\hat{Q}}, \tag{34}$$

$$\hat{A} = e^{\hat{Q}/2}. \tag{35}$$

These two operators are unitarily equivalent. To check the antilinearity of the Hamiltonian, we have considered

$$\hat{Q} = \alpha\hat{p}_{zi}\hat{p}_{2i} + \beta\hat{z}_i\hat{q}_{2i}. \tag{36}$$

Now, we calculate the similarity transformation of the total Hamiltonian. This is done by calculating the transformation of the individual fields and replacing their values. After some algebraic calculations, we obtain

$$\hat{A}\hat{H}_T\hat{A}^{-1} = e^{\hat{Q}/2}\hat{H}_T e^{-\hat{Q}/2} = \hat{\tilde{H}}_T, \tag{37}$$

where $\hat{\tilde{H}}_T$ is given by

$$\begin{aligned}\hat{\tilde{H}}_T = &-\frac{m}{2}\left(\hat{q}_{2i}^2\cosh 2\sqrt{\alpha\beta} - \frac{\hat{p}_{zi}^2}{2}D^2(\cosh 2\sqrt{\alpha\beta} - 1)\right)\\ &+\left(\frac{\hat{q}_{2i}^2}{D}\sinh 2\sqrt{\alpha\beta} - \frac{\hat{p}_{zi}^2}{2}D\sinh 2\sqrt{\alpha\beta}\right)\\ &-i\hat{p}_{zi}\hat{q}_{2i}\left(\frac{m}{2}D\sinh 2\sqrt{\alpha\beta} + \cosh 2\sqrt{\alpha\beta}\right).\end{aligned} \tag{38}$$

Here we have taken $D = \sqrt{\alpha/\beta}$. We can make this Hamiltonian free from the linear momentum terms, so that the states will be free from the Ostrogradski instability, owing to the condition

$$\frac{m}{2}D\sinh 2\sqrt{\alpha\beta} + \cosh 2\sqrt{\alpha\beta} = 0. \tag{39}$$

The relation between α and β can be found from the above equation. However, there remains arbitrariness in one of the variables, either α or β. The solutions will differ from model to model depending on the mass term m. The states $|\psi\rangle$ which correspond to the Hamiltonian H_T are related to $|\tilde{\psi}\rangle$ as

$$|\tilde{\Psi}\rangle = e^{-Q/2}|\Psi\rangle. \tag{40}$$

Here $|\tilde{\psi}\rangle$ are the states for the systems free from the ghosts. Thus being a higher derivative model and possessing antilinearity, we have successfully removed the Ostrogradski ghost from the system.

5 Conclusion

The higher derivative theories have been very useful in gravity [35], cosmology [36,37], and fractals [38] and have been used by many authors despite the existence of the Ostrogradski instability. To remove this instability, various attempts were made but none of them were a complete success due to the inherent conditions while developing the theory. In the present paper, we have considered another aspect of this problem of Ostrogradski ghosts using the antilinearity property of the HD theories. Due to this, we can employ the ideas developed earlier using $\mathcal{PT}$-symmetries and see if the Ostrogradski instability is curable or not.

We have seen that, in the HD theories, the presence of the antilinear symmetry in the Hamiltonian was not clear until a proper similarity transformation was made. After this transformation, the HD Hamiltonian became non-Hermitian and the antilinear symmetry also emerged. The operator connecting the antilinear symmetry of the Hamiltonian, in this case, was identified by comparing the form of the $\mathcal{C}$-operator of the $\mathcal{PT}$-symmetric theories. To illustrate the efficacy of this approach, we have considered the Galilean invariant Chern–Simon model in 2+1 dimensions. The model contains a higher derivative Chern–Simon term which, upon quantization, shows a connection with the noncommutative theory [25]. The Ostrogradski instability was still prevailing in the Hamiltonian. On the contrary, in the present case, the higher derivative Lagrangian was transformed into a first order Lagrangian and thus the canonical Hamiltonian was obtained. After a similarity transformation, using the newly defined operator, of the non-Hermitian Hamiltonian all the linear momentum terms vanished under a suitable condition that must be obeyed. This condition put some restriction on the mass term m of the particle.

Thus we have successfully removed the linear momentum (Ostrogradski ghost) term from the HD theory employing properties due to antilinearity. However, since this was a constrained system, the Hamiltonian was considered in the reduced phase space. This approach should be applicable to

other higher derivative theories also where the Ostrogradski ghost will plague the Hamiltonian.

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