



LREE of an unstable dressed-dynamical Dp -brane: superstring calculations

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Abstract We obtain the left-right entanglement entropy (LREE) for a Dp -brane with tangential motion in the presence of a $U(1)$ gauge potential, the Kalb–Ramond field and an open string tachyon field. Thus, at first we extract the Rényi entropy and then by taking a special limit of it we acquire the entanglement entropy. We shall investigate the behavior of the LREE under the tachyon condensation phenomenon. We observe that the deformation of the LREE, through this process, reveals the collapse of the brane. Besides, we examine the second law of thermodynamics for the LREE under tachyon condensation, and we extract the imposed constraints. Note that our calculations will be in the context of the type IIA/IIB superstring theories.

1 Introduction

Entanglement is one of the essential features of quantum mechanics. It correlates subsystems of a composite quantum system such that the quantum state of each subsystem cannot be described independently of the quantum states of the other subsystems. For a composite quantum system in a pure state, an applicable tool for measuring the entanglement between the subsystems is the entanglement entropy. This adequate quantity has been extensively studied, for example, in the context of the AdS/CFT there have been evidences for relations between the entanglement entropy and gravity [1,2]. In addition, a connection between the black hole entropy and entanglement entropy has been shown [3,4]. Besides, the entanglement entropy has been employed in condensed matter and the many-body quantum systems [5–7].

Traditionally, the entangled subsystems are separated geometrically which leads to a separation in the Hilbert space. However, in this paper the division of the subsystems occurs only in the Hilbert space. That is, the left- and right-moving

modes of closed superstring form the subspaces. The entropy of the entanglement between the left- and right-moving modes is called the left-right entanglement entropy (LREE) [8–12].

The crucial role of the D-branes in string theory has been highly remarked in the literature. Various areas such as the AdS/CFT, black holes and string phenomenology prominently depend on the D-brane dynamics. Since the boundary states accurately elaborate all properties of the associated D-branes, they have been commonly used in the brane analysis [13–35]. In this paper, we shall investigate the LREE of a special Dp -brane via the associated boundary state to it.

The early works were done by Zayas and Quiroz. They studied the LREE, associated with a one-dimensional boundary state, in a 2D CFT [8]. Then, they developed their analysis to a bare-static Dp -brane [9]. Their works motivated us to extend the LREE calculations for a dressed-dynamical Dp -brane [36], and, afterward, for an unstable dressed-dynamical Dp -brane [37]. Our papers have been written in the context of the bosonic string theory.

The current study will be in the context of the type IIA/IIB superstring theories. Therefore, we shall derive the LREE of a Dp -brane with the tangential rotation and tangential linear motion, in the presence of an internal $U(1)$ gauge potential, the Kalb–Ramond field and an open string tachyon field. In fact, there are some evidences for connection between the entanglement entropy and black hole entropy [3,4]. Hence, the LREE of our configuration may find a relation with the Bekenstein–Hawking entropy of the rotating-charged black holes.

Note that adding the open string tachyon to a D-brane gives rise to instability. Consequently, after condensing the tachyon, the brane loses its dimension, and one receives a lower-dimensional unstable brane [38–46]. Accordingly, presence of the open string tachyon on our brane enforces the brane to collapse. We shall examine the behavior of the LREE under this experience. We shall see that the LREE of the Dp -

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brane is decomposed to the LREE of a $D(p-1)$ -brane and an extra contribution which might be associated with the emitted closed superstrings via the brane collapse. In comparison with the bosonic case [37] a D-brane in the superstring theory is more stable than its counterpart in the bosonic string theory. Moreover, we shall see that the thermal entropy of the setup exactly is equivalent to its LREE. Because of the resemblance between the thermal and entanglement entropies [47–51], we investigate the second law of thermodynamics for the LREE through tachyon condensation process. We find that the survival of the second law imposes some conditions on the parameters of the configuration.

The paper is organized as follows. In Sect. 2, we introduce the boundary state, corresponding to the dressed-dynamical Dp -brane, and subsequently the interaction amplitude between two such Dp -branes will be written. This amplitude will be required for computing the left-right Rényi entropy. In Sect. 3, we obtain the LREE of our setup. In addition, we derive the thermodynamic entropy, which is equivalent to our LREE. In Sect. 4, the evolution of the LREE under the tachyon condensation phenomenon will be investigated. The second law of thermodynamics on the change of the LREE will be examined. Section 5 will be devoted to the results and conclusions.

2 The interaction amplitudes via the boundary states

2.1 The bosonic part of the boundary state

At first, we obtain the bosonic part of the boundary state, associated with a dynamical Dp -brane in the presence of the Kalb–Ramond field $B_{\mu\nu}$, the $U(1)$ gauge potential $A_\alpha(X)$ and the open string tachyon field $T(X)$. Therefore, we start with the following string action

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(\sqrt{-g} g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \varepsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) + \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} d\sigma \left(A_\alpha \partial_\sigma X^\alpha + \omega_{\alpha\beta} J_\tau^{\alpha\beta} + T(X^\alpha) \right), \tag{2.1}$$

where the sets $\{\sigma^a | a = 0, 1\}$ and $\{x^\alpha | \alpha = 0, 1, \dots, p\}$ represent the worldsheet coordinates and the parallel directions to the brane worldvolume, respectively. The set $\{x^i | i = p + 1, \dots, 9\}$ will be used for the perpendicular directions to the brane worldvolume. We shall take the flat worldsheet and spacetime with the signature $G_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$. Besides, we apply a constant antisymmetric tensor $B_{\mu\nu}$. The spacetime angular velocity $\omega_{\alpha\beta}$ includes the tangential rotation and tangential linear motion of the brane, and the angular momentum density is denoted

by $J_\tau^{\alpha\beta} = X^\alpha \partial_\tau X^\beta - X^\beta \partial_\tau X^\alpha$. For the gauge potential we use the profitable gauge $A_\alpha = -\frac{1}{2} F_{\alpha\beta} X^\beta$ with the constant field strength $F_{\alpha\beta}$, and the tachyon profile is adopted as $T = \frac{1}{2} U_{\alpha\beta} X^\alpha X^\beta$, with $U_{\alpha\beta}$ as a constant and symmetric matrix. We should mention that due to the presence of the various fields on the brane worldvolume the Lorentz symmetry has been manifestly lost. This clarifies that the tangential dynamics along the brane worldvolume is meaningful.

By varying the action with respect to X^μ we find the equation of motion and the flowing equations for the boundary state

$$\begin{aligned} (\Delta_{\alpha\beta} \partial_\tau X^\beta + \mathcal{F}_{\alpha\beta} \partial_\sigma X^\beta + B_{\alpha i} \partial_\sigma X^i + U_{\alpha\beta} X^\beta)_{\tau=0} |B_x\rangle &= 0, \\ (X^i - y^i)_{\tau=0} |B_x\rangle &= 0, \end{aligned} \tag{2.2}$$

where $\mathcal{F}_{\alpha\beta} \equiv B_{\alpha\beta} - F_{\alpha\beta}$ and $\Delta_{\alpha\beta} \equiv \eta_{\alpha\beta} + 4\omega_{\alpha\beta}$. The parameters $\{y^i\}$ exhibit the position of the brane. Applying the mode expansion of X^μ , we can conveniently express the above equations in terms of the closed string oscillators

$$\begin{aligned} \left[\left(\Delta_{\alpha\beta} - \mathcal{F}_{\alpha\beta} + \frac{i}{2m} U_{\alpha\beta} \right) \alpha_m^\beta + \left(\Delta_{\alpha\beta} + \mathcal{F}_{\alpha\beta} - \frac{i}{2m} U_{\alpha\beta} \right) \tilde{\alpha}_{-m}^\beta \right] |B_x^{(\text{osc})}\rangle &= 0, \\ (2\alpha' \Delta_{\alpha\beta} p^\beta + U_{\alpha\beta} x^\beta) |B_x^{(0)}\rangle &= 0, \end{aligned} \tag{2.3}$$

for the parallel directions to the brane worldvolume, and

$$\begin{aligned} (\alpha_m^i - \tilde{\alpha}_{-m}^i) |B_x^{(\text{osc})}\rangle &= 0, \\ (x^i - y^i) |B_x^{(0)}\rangle &= 0, \end{aligned} \tag{2.4}$$

for the normal directions. Note that we applied the decomposition $|B_x\rangle = |B_x^{(\text{osc})}\rangle \otimes |B_x^{(0)}\rangle$.

By employing the coherent state method and quantum mechanical techniques, specially the commutation relations among the string oscillators, we receive

$$\begin{aligned} |B_x^{(0)}\rangle &= \frac{T_p}{2\sqrt{\det(U/4\pi\alpha')}} \int_{-\infty}^{\infty} \prod_{\alpha=0}^p \exp \\ &\times \left[i\alpha' \sum_{\beta \neq \alpha} (U^{-1} \Delta + \Delta^T U^{-1})_{\alpha\beta} p^\alpha p^\beta + \frac{i\alpha'}{2} (U^{-1} \Delta + \Delta^T U^{-1})_{\alpha\alpha} (p^\alpha)^2 \right] |p^\alpha\rangle dp^\alpha \\ &\times \prod_{i=p+1}^9 \left[\delta(x^i - y^i) |p^i = 0\rangle \right], \end{aligned} \tag{2.5}$$

$$\begin{aligned} |B_x^{(\text{osc})}\rangle &= \prod_{n=1}^{\infty} [-\det M_{(n)}]^{-1} \exp \\ &\times \left[-\sum_{m=1}^{\infty} \left(\frac{1}{m} \alpha_{-m}^\mu S_{(m)\mu\nu} \tilde{\alpha}_{-m}^\nu \right) \right] |0\rangle_\alpha |0\rangle_{\tilde{\alpha}}, \end{aligned} \tag{2.6}$$

where the brane tension is T_p , and we defined $S_{(m)\mu\nu} = (Q_{(m)\alpha\beta}, -\delta_{ij})$, in which

$$\begin{aligned}
 Q_{(m)\alpha\beta} &\equiv (M_{(m)}^{-1}N_{(m)})_{\alpha\beta}, \\
 M_{(m)\alpha\beta} &= \Delta_{\alpha\beta} - \mathcal{F}_{\alpha\beta} + \frac{i}{2m}U_{\alpha\beta}, \\
 N_{(m)\alpha\beta} &= \Delta_{\alpha\beta} + \mathcal{F}_{\alpha\beta} - \frac{i}{2m}U_{\alpha\beta}.
 \end{aligned}
 \tag{2.7}$$

The prefactors of both parts of $|B_x\rangle$ originate from the normalization of the disk partition function. For more details see Refs. [36,37]. In fact, the boundary state $|B_x\rangle$ is not normalizable, i.e., the inner product $\langle B_x|B_x\rangle$ is divergent. In Sect. 3, we will introduce the regularization factor $e^{-\epsilon H}/\sqrt{\mathcal{N}_B}$, with a finite correlation length ϵ and a suitable normalization factor \mathcal{N}_B , to fix this problem.

The first equation in Eq. (2.3) tells us that applying the coherent state method on the set $\{\alpha_m^\alpha, \tilde{\alpha}_{-m}^\alpha|m \in \mathbb{N}\}$ gives a boundary state with the matrix $Q_{(m)\alpha\beta}$, while employing that method on the set $\{\tilde{\alpha}_m^\alpha, \alpha_{-m}^\alpha|m \in \mathbb{N}\}$ yields a boundary state which includes the matrix $\left(\left[Q_{(-m)}^{-1}\right]^\dagger\right)_{\alpha\beta}$. Equality of the resultant states imposes the following conditions on the parameters of the setup

$$\begin{aligned}
 \Delta U &= U \Delta^T, \\
 \Delta \mathcal{F} &= \mathcal{F} \Delta^T.
 \end{aligned}
 \tag{2.8}$$

The conformal ghosts also contribute to the bosonic part of the boundary state as in the following

$$\begin{aligned}
 |B_{\text{gh}}\rangle &= \exp\left[\sum_{n=1}^{\infty}(c_{-n}\tilde{b}_{-n} - b_{-n}\tilde{c}_{-n})\right] \\
 &\quad \frac{c_0 + \tilde{c}_0}{2} |q = 1\rangle |\tilde{q} = 1\rangle.
 \end{aligned}
 \tag{2.9}$$

2.2 The fermionic part of the boundary state

The unstable Dp -brane in our setup carries an open string tachyonic mode. In fact, survival of the open string tachyon after the GSO projection requires our D-brane to be a non-BPS D-brane with the wrong dimension, i.e., odd (even) dimension in the type IIA (IIB) theories. Therefore, the brane worldvolume does not couple to the R-R form fields of the type II theories, and hence it cannot carry any R-R charges. The corresponding boundary state to a non-BPS brane merely possesses the NS–NS sector $|B\rangle = |B\rangle_{\text{NS–NS}}$ [46,52,53], also see Ref. [9]. Thus, in this paper we apply only the NS–NS sector of the type II theories.

Due to the worldsheet supersymmetry, we can perform the following replacements on the bosonic boundary state equations (2.2) to obtain their fermionic counterparts

$$\begin{aligned}
 \partial_+ X^\mu(\sigma, \tau) &\rightarrow -i\eta\psi_+^\mu(\tau + \sigma), \\
 \partial_- X^\mu(\sigma, \tau) &\rightarrow -\psi_-^\mu(\tau - \sigma),
 \end{aligned}
 \tag{2.10}$$

in which $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/2$. The factor $\eta = \pm 1$ originates from the boundary conditions on the fermionic coordinates and will be used in the GSO projection on the boundary state. Because of the presence of the tachyonic field, a replacement for X^μ is also needed. Employing the above replacements and the mode expansions for ψ_\pm^μ , we acquire

$$X^\mu(\sigma, \tau) \rightarrow \sum_t \frac{1}{2t} \left(i\psi_t^\mu e^{-2it(\sigma-\tau)} + \eta\tilde{\psi}_t^\mu e^{-2it(\sigma+\tau)} \right),
 \tag{2.11}$$

where the index “ t ” is half-integer for the NS–NS sector.

Applying the replacements (2.10) and (2.11) into Eq. (2.2), and also using the mode expansion of ψ_\pm^μ , we obtain

$$\begin{aligned}
 &\left[\left(\Delta_{\alpha\beta} - \mathcal{F}_{\alpha\beta} + \frac{i}{2t}U_{\alpha\beta} \right) \psi_t^\beta - i\eta \left(\Delta_{\alpha\beta} + \mathcal{F}_{\alpha\beta} - \frac{i}{2t}U_{\alpha\beta} \right) \tilde{\psi}_{-t}^\beta \right] |B_\psi, \eta\rangle = 0, \\
 &(\psi_t^i + i\eta\tilde{\psi}_{-t}^i) |B_\psi, \eta\rangle = 0.
 \end{aligned}
 \tag{2.12}$$

Eqs. (2.12) can be combined as

$$(\psi_t^\mu - i\eta S_{(t)\nu}^\mu \tilde{\psi}_{-t}^\nu) |B_\psi, \eta\rangle = 0.
 \tag{2.13}$$

Again by making use of the coherent state method, the fermionic boundary state takes the feature

$$|B_\psi, \eta\rangle = \prod_t [\det M_{(t)}] \exp \left[i\eta \sum_t (\psi_{-t}^\mu S_{(t)\mu\nu} \tilde{\psi}_{-t}^\nu) \right] |0\rangle.
 \tag{2.14}$$

The total boundary state is given by

$$|B, \eta\rangle_{\text{NS}} = |B_x\rangle \otimes |B_\psi, \eta\rangle_{\text{NS}} \otimes |B_{\text{gh}}\rangle \otimes |B_{\text{sgh}}, \eta\rangle_{\text{NS}},
 \tag{2.15}$$

where the contribution of the superconformal ghosts is given by

$$\begin{aligned}
 |B_{\text{sgh}}, \eta\rangle_{\text{NS}} &= \exp \left[i\eta \sum_{t=1/2}^{\infty} (\gamma_{-t}\tilde{\beta}_{-t} - \beta_{-t}\tilde{\gamma}_{-t}) \right] \\
 &\quad \times |P = -1\rangle |\tilde{P} = -1\rangle.
 \end{aligned}
 \tag{2.16}$$

By employing the GSO projection the applicable boundary state is written as a combination of the total boundary states with $\eta = \pm 1$,

$$|B\rangle_{\text{NS}} = \frac{1}{2} (|B, +\rangle_{\text{NS}} - |B, -\rangle_{\text{NS}}).
 \tag{2.17}$$

2.3 The interaction in the NS–NS sector

For computing the LREE we need the partition function. Hence, we first introduce the interaction amplitude between

two identical and parallel Dp -branes. The branes have been dressed by the fields, and they have tangential dynamics. One can obtain this amplitude from the overlap of the GSO-projected boundary states, associated with the two Dp -branes, via the propagator “ D ” of the exchanged closed string

$$\begin{aligned}
 \mathcal{A} &= \langle B_1 | D | B_2 \rangle, \\
 D &= 2\alpha' \int_0^\infty dt e^{-tH}, \tag{2.18}
 \end{aligned}$$

in which “ H ” stands for the total Hamiltonian of the propagating closed superstring. It consists of the matter and ghost parts. Therefore, one receives

$$\begin{aligned}
 \mathcal{A}_{\text{NS-NS}} &= \frac{T_p^2 V_{p+1} \alpha'}{4(2\pi)^{9-p}} \frac{1}{\sqrt{\det(U_1/4\pi\alpha') \det(U_2/4\pi\alpha')}} \\
 &\times \prod_{m=1}^\infty \frac{\det[M_{(m-1/2)1}^\dagger M_{(m-1/2)2}]}{\det[M_{(m)1}^\dagger M_{(m)2}]} \\
 &\times \int_0^\infty dt \left\{ \left(\sqrt{\frac{1}{\alpha't}} \right)^{9-p} \right. \\
 &\times \exp \left(-\frac{1}{4\pi\alpha't} \sum_{i=p+1}^9 (y_2^i - y_1^i)^2 \right) \\
 &\times \frac{1}{q} \left(\prod_{m=1}^\infty \left[\left(\frac{1+q^{2m-1}}{1-q^{2m}} \right)^{7-p} \right. \right. \\
 &\times \left. \left. \frac{\det(\mathbf{1} + Q_{(m-1/2)1}^\dagger Q_{(m-1/2)2} q^{2m-1})}{\det(\mathbf{1} - Q_{(m)1}^\dagger Q_{(m)2} q^{2m})} \right] \right. \\
 &- \left. \prod_{m=1}^\infty \left[\left(\frac{1-q^{2m-1}}{1-q^{2m}} \right)^{7-p} \right. \right. \\
 &\times \left. \left. \frac{\det(\mathbf{1} - Q_{(m-1/2)1}^\dagger Q_{(m-1/2)2} q^{2m-1})}{\det(\mathbf{1} - Q_{(m)1}^\dagger Q_{(m)2} q^{2m})} \right] \right) \left. \right\}, \tag{2.19}
 \end{aligned}$$

where $q = e^{-2\pi t}$, and V_{p+1} indicates the Dp -brane world-volume. The first two factors in the integral originate from the zero-modes and the factor q^{-1} is related to the zero-point energy. For the first factors inside the infinite products we have the power $7-p = [10-(p+1)]-2$, where $10-(p+1)$ and -2 correspond to the contributions by the Dirichlet oscillators and ghosts-superghosts, respectively. The numerators determinants represent the contributions of the fermions Neumann oscillators, and that in the denominators is associated with the Neumann oscillators of the bosons.

The integer (half-integer) modes exhibit the bosons (fermions) contribution. The tension of a non-BPS brane includes an extra $\sqrt{2}$ factor, which in the above amplitude it has been considered.

3 The LREE corresponding to the unstable dressed-dynamical Dp -brane

Imagine a bipartite system which comprises only two subsystems A and B. Let the pure state of the composite system be $|\psi\rangle$. Thus, the density operator, associated with this state, is defined by $\rho = |\psi\rangle\langle\psi|$. The conservation of probability requires that $\text{Tr}\rho = 1$. The reduced density operator due to the subsystem A is defined as $\rho_A = \text{Tr}_B\rho$, where the Tr_B represents the partial trace with respect to the subsystem B.

The entanglement and Rényi entropies are the most desirable tools among the other quantities for measuring entanglement. The first quantity can be obtained by the von Neumann formula $S = -\text{Tr}(\rho_A \ln \rho_A)$ [54], while the second one is derived from $S_n = \frac{1}{1-n} \ln \text{Tr}\rho_A^n$, where $n \geq 0$ and $n \neq 1$. By taking the special limit, i.e. $n \rightarrow 1$, the Rényi entropy tends to the entanglement entropy [55].

3.1 The density operator of the system

The Hilbert space of closed superstring theory has the factorized form $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. The left- and right-moving oscillating modes of closed superstring form the bases of the subsystems “L” and “R”. For receiving the physical Hilbert space we should exert the Virasoro constraints. Precisely, a general state of closed superstring is given by $|\psi\rangle = |\psi\rangle_L \otimes |\psi\rangle_R$, where

$$\begin{aligned}
 |\psi\rangle_L &= \prod_{k=1}^\infty \prod_t \frac{1}{\sqrt{n_k!}} \left(\frac{\alpha_{-k}^{\mu_k}}{\sqrt{k}} \right)^{n_k} (\psi_{-t}^{\mu_t})^{n_t} |0\rangle, \\
 |\psi\rangle_R &= \prod_{k=1}^\infty \prod_t \frac{1}{\sqrt{m_k!}} \left(\frac{\tilde{\alpha}_{-k}^{\nu_k}}{\sqrt{k}} \right)^{m_k} (\tilde{\psi}_{-t}^{\nu_t})^{m_t} |0\rangle,
 \end{aligned}$$

where for the NS-NS sector the mode numbers “ t ” are positive half integers. Since ψ_{-t}^μ and $\tilde{\psi}_{-t}^\nu$ are Grassmannian variables we have $m_t, n_t \in \{0, 1\}$. The sets $\{n_t, n_k | k \in \mathbb{N}\}$ and $\{m_t, m_k | k \in \mathbb{N}\}$ are independent up to the condition

$$\sum_{k=1}^\infty kn_k + \sum_t tn_t = \sum_{k=1}^\infty km_k + \sum_t tm_t.$$

The quantity in the left-hand side (right-hand side) represents the total mode number, i.e., the summation of all mode numbers in the state $|\psi\rangle_L$ ($|\psi\rangle_R$). Thus, the Virasoro conditions at most impose only the equality of the total mode numbers of the states $|\psi\rangle_L$ and $|\psi\rangle_R$. This condition weakly relates the left- and right-moving string modes. Hence, the left- and right-sectors essentially remain independent. Therefore, the physical Hilbert space possesses the factorized form.

The boundary state, which is a coherent state of closed superstring, is also decomposed to the left- and right-moving modes by the Schmidt decomposition method [56,57]. In other words, the expansion of the exponential parts of Eqs.

(2.6) and (2.14) gives a series which manifestly illustrates entanglement between the two parts of the Hilbert space. Hence, similar to the non-geometric prescription of Refs. [8, 9], we take the GSO-projected boundary state as the composite system and the left- and right-moving modes of closed superstring as its subsystems.

The density operator, corresponding to a given boundary state, might be considered as $\rho = |B\rangle\langle B|$. In fact, the inner product $\langle B|B\rangle$ is divergent. To see this, according to Eq. (2.18), in the amplitude (2.19) remove the integral over “ t ” and apply $t \rightarrow 0$. A consequence of this divergence is violation of the condition $\text{Tr}\rho = 1$. Thus, we consider the regularized state $|B\rangle = (e^{-\epsilon H}/\sqrt{\mathcal{N}_B})|B\rangle_{\text{NS-NS}}$, where ϵ is a finite correlation length. Hence, the density operator is defined as

$$\rho = \frac{1}{\mathcal{N}_B} \left(e^{-\epsilon H} |B\rangle_{\text{NS-NS}} \right) \left({}_{\text{NS-NS}}\langle B| e^{-\epsilon H} \right), \tag{3.1}$$

where the normalization factor \mathcal{N}_B is fixed by the probability conservation condition $\text{Tr}\rho = 1$. After taking the trace of the density operator over the closed superstring states and applying $\text{Tr}\rho = 1$, we obtain the normalization factor equal to the partition function $\mathcal{N}_B = Z_{\text{NS-NS}}(2\epsilon)$.

In the paper [5] there are two regularization approaches, which are corresponding to the boundary state and Ishibashi states. Each approach possesses its own normalization factor. As it has been shown in [5] the regularization of the Ishibashi states can correctly recover the spatial topological entanglement entropy for Chern-Simons theories while the first approach of regularization cannot recover it. However, unlike the topological theories, e.g. the Chern-Simons theories, our action does not represent a topological theory. That is, we don’t have a topological sector, and hence, there is no any topological entanglement entropy. Thus, we don’t normalize the Ishibashi states individually. Therefore, for the regularization we applied only the first approach.

An interpretation of the numerator of ρ is that a closed superstring propagates for the time $t = \epsilon$, then it is absorbed by a D-brane. It is immediately emitted by an identical D-brane and again propagates for the duration $t = \epsilon$. However, the interpretation of the partition function in the denominator of (3.1), i.e. $Z_{\text{NS-NS}}(2\epsilon)$, is that a closed superstring is emitted by a D-brane, then it propagates for the time $t = 2\epsilon$ and then it is absorbed by an identical D-brane.

The partition function can be conveniently extracted from the amplitude (2.19) as in the following

$$\begin{aligned} Z_{\text{NS-NS}}(2\epsilon) &= {}_{\text{NS-NS}}\langle B| e^{-2\epsilon H} |B\rangle_{\text{NS-NS}} \\ &= \frac{T_p^2 V_{p+1}}{2(2\pi)^{9-p}} \frac{1}{\det(U/8\pi)} \prod_{m=1}^{\infty} \frac{|\det M_{(m-1/2)}|^2}{|\det M_{(m)}|^2} \\ &\quad \times \left(\sqrt{\frac{1}{4\epsilon}} \right)^{9-p} \frac{1}{q} \left(\prod_{m=1}^{\infty} \left[\left(\frac{1+q^{2m-1}}{1-q^{2m}} \right)^{7-p} \right. \right. \end{aligned}$$

$$\begin{aligned} &\left. \times \frac{\det(\mathbf{1} + Q_{(m-1/2)}^\dagger Q_{(m-1/2)} q^{2m-1})}{\det(\mathbf{1} - Q_{(m)}^\dagger Q_{(m)} q^{2m})} \right] \\ &- \prod_{m=1}^{\infty} \left[\left(\frac{1-q^{2m-1}}{1-q^{2m}} \right)^{7-p} \right. \\ &\left. \times \frac{\det(\mathbf{1} - Q_{(m-1/2)}^\dagger Q_{(m-1/2)} q^{2m-1})}{\det(\mathbf{1} - Q_{(m)}^\dagger Q_{(m)} q^{2m})} \right] \Bigg). \tag{3.2} \end{aligned}$$

Since we have identical branes in the same position, the indices 1 and 2 and also the y -dependence have been omitted. Similar to the stringy literature, in which for simplification various numeric values are chosen for the slope α' [8, 9, 58–61], we have selected the choice $\alpha' = 2$.

3.2 The associated LREE to the setup

The first step for computing the LREE of our setup is the calculation of the Rényi entropy. Accordingly, we need to find $\text{Tr}\rho_L^n$, where the reduced density operator ρ_L is derived via the trace over the right-moving oscillators. We utilize the replica trick, which for the real “ n ” gives

$$\text{Tr}\rho_L^n \sim \frac{Z_{\text{NS-NS}}(2n\epsilon)}{Z_{\text{NS-NS}}^n(2\epsilon)} \equiv \frac{Z_{n\text{NS-NS}}(L)}{Z_{\text{NS-NS}}^n}. \tag{3.3}$$

The quantity $Z_{n\text{NS-NS}}$ is called the “replicated partition function”.

Since there are various approaches to sum over the spin structure (η) and momentum, there are different ways to acquire the replicated partition function and replicated normalization constant [9]. Explicitly, if we first sum over η and then we do the replication, the spin structure of each copy will be disconnected from the other copies. This case is called the *uncorrelated* spin structure. Another possibility is that: at first replicate each spin structure separately and then compute sum over them. This case is called the *correlated* spin structure. In the same way, the uncorrelated and correlated momentum are constructed by integrating over the momenta before and after the replication, respectively. Besides, if the normalization constant $K_p^{1/2}$ (see Eq. (3.5)) is raised to the power n , through the replication process, we call it *replicated* normalization constant. Otherwise, it will be called the *unreplicated* normalization constant.

In fact, all of the above possibilities can be studied. However, here we choose only one of them which is invariant under the open-closed string duality. This reliable case possesses the unreplicated normalization constant, the correlated momentum and the correlated spin structure. For an NS-NS brane we have

$$\int_0^\infty dl {}_{\text{NS-NS}}\langle B, \eta| e^{-lH_c} |B, \eta\rangle_{\text{NS-NS}}$$

$$\begin{aligned}
 &= \mathcal{N}^2 \int_0^\infty dl \left(\frac{1}{l}\right)^{\frac{9-p}{2}} \frac{f_3^8(q)}{f_1^8(q)} \\
 &= \mathcal{N}^2 \frac{32(2\pi)^{p+1}}{V_{p+1}} \int_0^\infty \frac{dt}{2t} \text{Tr}_{\text{NS}} \left[e^{-tH_o} \right], \\
 &\int_0^\infty dl_{\text{NS-NS}} \langle B, \eta | e^{-lH_c} | B, -\eta \rangle_{\text{NS-NS}} \\
 &= \mathcal{N}^2 \int_0^\infty dl \left(\frac{1}{l}\right)^{\frac{9-p}{2}} \frac{f_4^8(q)}{f_1^8(q)} \\
 &= \mathcal{N}^2 \frac{32(2\pi)^{p+1}}{V_{p+1}} \int_0^\infty \frac{dt}{2t} \text{Tr}_{\text{R}} \left[e^{-tH_o} \right],
 \end{aligned}$$

where the integral variables l and t exhibit the length of the cylinder in closed string channel and the circumference of the cylinder in the open string channel, respectively. For a non-BPS brane the normalization constant is

$$\mathcal{N}_{\text{non-BPS}}^2 = \frac{V_{p+1}}{64(2\pi)^{p+1}}.$$

The replicated partition function with the correlated momentum gives rise to the factor $(1/nl)^{(9-p)/2}$, while Z_n with uncorrelated momentum leads to the factor $(1/l)^{n(9-p)/2}$, which is not invariant under the modular transformation. Besides, the correlated spin structure leads to a factor 2, while the uncorrelated spin structure introduces the factor 2^{2n-1} . In addition, the unreplicated normalization \mathcal{N}^2 is chosen instead of the replicated normalization \mathcal{N}^{2n} . These imply that to satisfy the open-closed duality we have to apply the replicated partition function with the correlated momentum, unreplicated normalization constant, and the correlated spin structure.

As ϵ tends to zero the quantity $q = e^{-4\pi\epsilon}$ does not vanish. Therefore, we apply the transformation $4\epsilon \rightarrow 1/4\epsilon$ to go to the open string channel. Here, we work with the quantity $\tilde{q} = \exp(-\frac{\pi}{4\epsilon})$ which in the limit $\epsilon \rightarrow 0$ tends to zero. Thus, we can expand Eq. (3.3) for small \tilde{q} as in the following

$$\begin{aligned}
 \frac{Z_{n \text{ NS-NS}}}{Z_{\text{NS-NS}}^n} &\approx 2^{1-n} K_p^{1-n} \left((2\sqrt{\epsilon})^{1-n} \sqrt{n} \right)^{9-p} \\
 &\times \exp \left[\frac{\pi}{4\epsilon} \left(\frac{1}{n} - n \right) \right] \prod_{m=1}^\infty 2^{1-n} C_{(m-1/2)}^{1-n} \\
 &\times \left\{ \tilde{q}^{\frac{2m-1}{n} - n(2m-1)} + C_{(m)} \tilde{q}^{\frac{4m-1}{n} - n(2m-1)} \right. \\
 &- n C_{(m)} \tilde{q}^{\frac{2m-1}{n} - n(2m-1) + 2m} \\
 &- n C_{(m)}^2 \times \tilde{q}^{\frac{4m-1}{n} - n(2m-1) + 2m} \\
 &\left. + \frac{n(n+1)}{2} \times C_{(m)}^2 \tilde{q}^{\frac{2m-1}{n} - n(2m-1) + 4m} + \mathcal{O}(\tilde{q}^{6m}) \right\}, \quad (3.4)
 \end{aligned}$$

where K_p , $C_{(m)}$ and $C_{(m-1/2)}$ are defined by

$$K_p = \frac{T_p^2 V_{p+1}}{2(2\pi)^{9-p}} \frac{1}{\det(U/8\pi)} \prod_{m=1}^\infty \frac{|\det M_{(m-1/2)}|^2}{|\det M_{(m)}|^2},$$

$$C_{(t)} = \text{Tr} \left(Q_{(t)}^\dagger Q_{(t)} \right) + 7 - p. \quad (3.5)$$

The index “ t ” is a positive integer “ m ” or a positive half-integer “ $m - 1/2$ ”.

Now for obtaining the LREE we should take the limit $n \rightarrow 1$ of the Rényi entropy, which yields

$$\begin{aligned}
 S_{\text{LREE}}^{(p)} &\approx \frac{1}{2} \ln 2 + \ln K_p + \frac{9-p}{2} (2 \ln 2 + \ln \epsilon - 1) + \frac{\pi}{3\epsilon} \\
 &+ \sum_{m=1}^\infty \left\{ \ln C_{(m-1/2)} + C_{(m)} \left(1 - \frac{m\pi}{2\epsilon} \right) e^{-m\pi/2\epsilon} \right. \\
 &\left. - \frac{1}{2} C_{(m)}^2 \left(1 - \frac{m\pi}{\epsilon} \right) e^{-m\pi/\epsilon} + \mathcal{O}(\exp(-3m\pi/2\epsilon)) \right\}. \quad (3.6)
 \end{aligned}$$

The first term comes from the sum over the spin structure and a contribution from the oscillators. The second term shows the boundary entropy of the brane, the third term originates from the zero-modes, and the rest terms are regarding to the contributions of the oscillators and conformal ghosts. The parameters of the setup have been appeared in K_p , $C_{(m)}$ and $C_{(m-1/2)}$. Besides, the mode dependence of the LREE is a consequence of the presence of the tachyonic field.

3.3 The LREE and the thermodynamic entropy

To investigate the thermal properties of our system we can associate a temperature to it. This temperature is proportional to the inverse of the correlation length, i.e. $\beta = 2\epsilon$. Applying the definition of the thermodynamic entropy and using the partition function (3.2), in the high temperature limit of the system $\epsilon \rightarrow 0$, we find

$$\begin{aligned}
 S_{\text{thermal}} &= \beta^2 \frac{\partial}{\partial \beta} \left(-\frac{1}{\beta} \ln Z_{\text{NS-NS}} \right) \\
 &\approx \frac{1}{2} \ln 2 + \ln K_p + \frac{9-p}{2} (\ln 2\beta - 1) + \frac{2\pi}{3\beta} \\
 &+ \sum_{m=1}^\infty \left\{ \ln C_{(m-1/2)} + C_{(m)} \left(1 - \frac{m\pi}{\beta} \right) e^{-m\pi/\beta} \right. \\
 &- \frac{1}{2} C_{(m)}^2 \left(1 - \frac{2m\pi}{\beta} \right) e^{-2m\pi/\beta} \\
 &\left. + \mathcal{O}(\exp(-3m\pi/\beta)) \right\}. \quad (3.7)
 \end{aligned}$$

We observe that the thermodynamic entropy of the system exactly is equivalent to its LREE. This similarity between the thermal and entanglement entropies also has been obtained in the literature, e.g., see Refs. [47–51].

Since the constants $\{C_{(m)} | m \in \mathbb{N}\}$ depend on the mode numbers calculation of the summation of the series in Eq. (3.7) is very complicated. Therefore, we don't have an explicit form of the entropy function $S_{\text{thermal}}(T)$, in which

$\beta = 1/T$. Hence, the phase transition of the corresponding system is not clear.

4 Condensing the tachyon

4.1 Evolution of the LREE under the tachyon condensation

Presence of an open string tachyon on a D-brane drastically makes it unstable. Through the tachyon condensation process the D-brane collapses, i.e., it loses some of its directions. Ultimately, one receives the closed string vacuum or at most an intermediate stable D-brane [38, 39]. Under the tachyon condensation at least one of the elements of the tachyon matrix $U_{\alpha\beta}$ tends to infinity. For instance, if we apply $U_{pp} \rightarrow \infty$ the condensation occurs in the x^p -direction.

Before imposing the condensation on the tachyon we compute the LREE in the large value of the tachyon matrix, that is $U \gg 2(\Delta - \mathcal{F})$. This tachyon matrix accompanied by the conditions (2.8) yield

$$\begin{aligned} \tilde{S}_{\text{LREE}}^{(p)} \approx & \ln 2 + \ln K_p + \frac{9-p}{2} (2 \ln 2 + \ln \epsilon - 1) + \frac{\pi}{3\epsilon} \\ & + \sum_{m=1}^{\infty} \left\{ \ln H_{(m-1/2)} + H_{(m)} \left(1 - \frac{m\pi}{2\epsilon} \right) e^{-m\pi/2\epsilon} \right. \\ & - \frac{1}{2} H_{(m)}^2 \left(1 - \frac{m\pi}{\epsilon} \right) e^{-m\pi/\epsilon} \\ & \left. + \mathcal{O}(\exp(-3m\pi/2\epsilon)) \right\}, \end{aligned} \tag{4.1}$$

up to the order $\mathcal{O}(U^{-3})$, where we defined

$$H_{(t)} = 8 - 512 t^2 \text{Tr} \left(\omega^2 U^{-2} \right), \tag{4.2}$$

The index “ t ” is a positive integer “ m ” or a positive half-integer “ $m - 1/2$ ”.

Now, suppose that the tachyon is condensed only in the x^p -direction of the brane. In this case, one finds

$$\lim_{U_{pp} \rightarrow \infty} \ln K_p = \ln K_{p-1} + \ln \left(\frac{\pi L_p}{\bar{U}_{pp}} \right), \tag{4.3}$$

in which the infinite value of U_{pp} was called \bar{U}_{pp} , and L_p is the infinite length of the brane in the x^p -direction. For acquiring this result, the trusty relation $T_p = T_{p-1}/(2\pi\sqrt{\alpha'})$ and the regularization schemes $\prod_{n=1}^{\infty} n \rightarrow \sqrt{2\pi}$ and $\prod_{n=1}^{\infty} (2n-1) \rightarrow \sqrt{2}$ have been exerted. The third phrase of Eq. (4.1) can be rephrased as

$$\begin{aligned} \frac{9-p}{2} (2 \ln 2 + \ln \epsilon - 1) = & \frac{9-(p-1)}{2} (2 \ln 2 + \ln \epsilon - 1) \\ & - \frac{1}{2} (2 \ln 2 + \ln \epsilon - 1). \end{aligned} \tag{4.4}$$

By taking the limit $U_{pp} \rightarrow \infty$, the factor $\text{Tr}(\omega U^{-2})$ reduces to $\text{Tr}(\omega U^{-2})'$, where the prime indicates a $p \times p$ matrix. Accordingly, under the tachyon condensation experience the LREE finds the form

$$\lim_{U_{pp} \rightarrow \infty} \tilde{S}_{\text{LREE}}^{(p)} = \tilde{S}_{\text{LREE}}^{(p-1)} + \lambda, \tag{4.5}$$

$$\lambda \equiv \ln \left(\frac{\pi L_p}{2\bar{U}_{pp}} \right) - \frac{1}{2} (\ln \epsilon - 1). \tag{4.6}$$

In fact, when the tachyon condensation acts on one direction of an unstable Dp -brane, it collapses to a $D(p-1)$ -brane [39]. Here, the associated LREE with the $D(p-1)$ -brane is exactly given by $\tilde{S}_{\text{LREE}}^{(p-1)}$. The infinite parameters L_p and \bar{U}_{pp} can be accurately adjusted such that their ratio to be a finite value.

The extra contribution to the entropy, i.e. λ , can be interpreted as the entropy of the released closed superstrings via the collapse of the Dp -brane. In comparison with the bosonic case [37], the extra entropy λ has reduced by $-\ln(2\bar{U}_{pp})$, which can be interpreted as reduction of superstring radiation during the collapse of the brane. For example, consider the case that the total entropies of the bosonic and superstring systems, after tachyon condensation, are equal. Then, the inequality $\lambda_{\text{bosonic}} > \lambda_{\text{superstring}}$ induces the following inequality

$$\left(\tilde{S}_{\text{LREE}}^{(p-1)} \right)_{\text{superstring}} > \left(\tilde{S}_{\text{LREE}}^{(p-1)} \right)_{\text{bosonic string}}.$$

Thus, one may deduce that under the tachyon condensation the resultant $D(p-1)$ -brane in the superstring theory is more stable than that in the bosonic string theory.

4.2 The second law of thermodynamics for the LREE

The thermal and entanglement entropies have some close connections [47–51]. For instance, in Refs. [47–49] it has been demonstrated that the entanglement entropy obeys relations which are similar to the laws of thermodynamics. In Sec. (3.3) we proved that the LREE and thermal entropy of our setup possess an identical feature. This similarity stimulated us to check the second law of thermodynamics for the LREE under the tachyon condensation process.

Now we compare the LREE of our initial state, which is the Dp -brane, with that of the final state, i.e. the resultant $D(p-1)$ -brane and the released closed superstrings. Thus, we have

$$S_{\text{initial}} = \tilde{S}_{\text{LREE}}^{(p)},$$

$$S_{\text{final}} = \lim_{U_{pp} \rightarrow \infty} \tilde{S}_{\text{LREE}}^{(p)} = \tilde{S}_{\text{LREE}}^{(p-1)} + \lambda. \tag{4.7}$$

The second law of thermodynamics implies that, the entropy should be increased during the process. Therefore, we should check the inequality $S_{\text{final}} - S_{\text{initial}} > 0$,

$$\begin{aligned} & \tilde{S}_{\text{LREE}}^{(p-1)} + \lambda - \tilde{S}_{\text{LREE}}^{(p)} \\ &= \ln \left(\frac{\pi}{2\tilde{U}_{pp}} \right) - \ln \left(\frac{\det U'}{\det U} \right) \\ & \quad - \sum_{m=1}^{\infty} \left\{ 2 \ln \left(\frac{\det M'_{(m)} \det M_{(m-1/2)}}{\det M_{(m)} \det M'_{(m-1/2)}} \right) \right. \\ & \quad + \ln \left(\frac{H_{(m-1/2)}}{H'_{(m-1/2)}} \right) \\ & \quad + \left(H_{(m)} - H'_{(m)} \right) \left(1 - \frac{m\pi}{2\epsilon} \right) e^{-m\pi/2\epsilon} \\ & \quad \left. - \frac{1}{2} \left(H_{(m)}^2 - H'^2_{(m)} \right) \left(1 - \frac{m\pi}{\epsilon} \right) e^{-m\pi/\epsilon} \right\}. \end{aligned} \tag{4.8}$$

The primes represent the $p \times p$ matrices and $H_{(m)}$ was defined by Eq. (4.2). There are many parameters, i.e. the various matrix elements, which control the value of this difference. The minimal condition for positivity of (4.8) is given by

$$\begin{aligned} & \frac{\det U}{\det U'} \prod_{m=1}^{\infty} \left[\left(\frac{\det M_{(m)} \det M'_{(m-1/2)}}{\det M'_{(m)} \det M_{(m-1/2)}} \right)^2 \frac{H_{(m-1/2)}}{H'_{(m-1/2)}} \right] \\ & > \frac{2\tilde{U}_{pp}}{\pi}, \end{aligned} \tag{4.9}$$

up to the leading order. According to the following formula

$$\cos \theta = \prod_{m=1}^{\infty} \left[1 - \frac{\theta^2}{(m-1/2)^2 \pi^2} \right],$$

we can write

$$\begin{aligned} \prod_{m=1}^{\infty} \frac{H_{(m-1/2)}}{H'_{(m-1/2)}} &= \frac{\cos \phi}{\cos \phi'} \prod_{m=1}^{\infty} \frac{\text{Tr}(\omega^2 U^{-2})}{\text{Tr}(\omega^2 U^{-2})'} \\ &= \frac{\cos \phi}{\cos \phi'} \left(\frac{\text{Tr}(\omega^2 U^{-2})}{\text{Tr}(\omega^2 U^{-2})'} \right)^N, \end{aligned} \tag{4.10}$$

where $N = \sum_{m=1}^{\infty} 1$, and the angle ϕ has the definition

$$\phi = \frac{\pi}{8\sqrt{\text{Tr}(\omega^2 U^{-2})}}.$$

Now we impose an additional condition

$$R \equiv \frac{\text{Tr}(\omega^2 U^{-2})}{\text{Tr}(\omega^2 U^{-2})'} > 1. \tag{4.11}$$

This inequality inspires that the second factor in the RHS of Eq. (4.10) is infinite. In fact, the infinities in the LHS and RHS of (4.9) completely are independent. However,

the value of the quantity R depends on all matrix elements of the matrices U and ω . By adjusting the parameters $\{U_{\alpha\beta}, \omega_{\alpha\beta} | \alpha, \beta = 0, 1, \dots, p\}$ we can receive a large value for R such that the infinity in the LHS of (4.9) to be dominant to \tilde{U}_{pp} , and the ratio R^N / \tilde{U}_{pp} to be fixed. Finally, these conditions reliably confirm the preservation of the second law of thermodynamics for the LREE of the setup.

5 Conclusions

In the context of the type IIA/IIB superstring theories we investigated the left-right entanglement entropy of a non-BPS unstable Dp -brane. The brane has tangential dynamics. Besides, they have been dressed by the $U(1)$ gauge potential, the anti-symmetric tensor field and the open string tachyon field. For achieving this, the boundary state formalism in the NS–NS sector was employed and the interaction amplitude between two identical dynamical Dp -branes with the fore-going fields was introduced.

The parameters of the dynamics and background fields were entered into the LREE, and hence, they generalized the form of the LREE. Therefore, the value of the LREE can be accurately controlled by adjusting these parameters. Because of the presence of the tachyon field, the closed string mode numbers drastically appeared in the LREE through the infinite product and the series. However, as we chose only the NS–NS sector, both the integer and half-integer modes were entered.

Effect of the tachyon condensation on the LREE was also studied. The LREE of the initial Dp -brane was decomposed to the LREE of a new unstable dressed-dynamical $D(p-1)$ -brane and an extra contribution which belongs to the emitted closed superstrings through the brane collapse. In comparison with the bosonic case [37], the extra entropy has been reduced, which indicates a smaller amount of string radiation. This reveals that after the tachyon condensation the resultant D-brane in the superstring theory is more stable than its counterpart in the bosonic string theory.

Furthermore, we defined a temperature for our system to derive the thermodynamic entropy via the partition function. We found that the thermal entropy of the configuration exactly is equivalent to its LREE. Similar equivalence relations have been demonstrated in Refs. [9,36,37]. The common properties of the thermodynamic entropy and LREE motivated us to check the second law of thermodynamics for the LREE under the tachyon condensation process. In fact, preservation of the second law of thermodynamics for the LREE imposes two prominent conditions among the parameters of the setup.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This paper is a theoretical work. No experimental data were used.]

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