



# On the geometry of operator mixing in massless QCD-like theories

Marco Bohicchio<sup>a</sup>

Physics Department, INFN Roma1, Piazzale A. Moro 2, 00185 Rome, Italy

Received: 21 July 2021 / Accepted: 9 August 2021 / Published online: 20 August 2021  
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**Abstract** We revisit the operator mixing in massless QCD-like theories. In particular, we address the problem of determining under which conditions a renormalization scheme exists where the renormalized mixing matrix in the coordinate representation,  $Z(x, \mu)$ , is diagonalizable to all perturbative orders. As a key step, we provide a differential-geometric interpretation of renormalization that allows us to apply the Poincaré-Dulac theorem to the problem above: We interpret a change of renormalization scheme as a (formal) holomorphic gauge transformation,  $-\frac{\gamma(g)}{\beta(g)}$  as a (formal) meromorphic connection with a Fuchsian singularity at  $g = 0$ , and  $Z(x, \mu)$  as a Wilson line, with  $\gamma(g) = \gamma_0 g^2 + \dots$  the matrix of the anomalous dimensions and  $\beta(g) = -\beta_0 g^3 + \dots$  the beta function. As a consequence of the Poincaré-Dulac theorem, if the eigenvalues  $\lambda_1, \lambda_2, \dots$  of the matrix  $\frac{\gamma_0}{\beta_0}$ , in nonincreasing order  $\lambda_1 \geq \lambda_2 \geq \dots$ , satisfy the nonresonant condition  $\lambda_i - \lambda_j - 2k \neq 0$  for  $i \leq j$  and  $k$  a positive integer, then a renormalization scheme exists where  $-\frac{\gamma(g)}{\beta(g)} = \frac{\gamma_0}{\beta_0} \frac{1}{g}$  is one-loop exact to all perturbative orders. If in addition  $\frac{\gamma_0}{\beta_0}$  is diagonalizable,  $Z(x, \mu)$  is diagonalizable as well, and the mixing reduces essentially to the multiplicatively renormalizable case. We also classify the remaining cases of operator mixing by the Poincaré-Dulac theorem.

## 1 Introduction and physics motivations

In the present paper we revisit the operator mixing in asymptotically free gauge theories massless to all perturbative orders, such as QCD with massless quarks. We refer for short to such theories as massless QCD-like theories.

In fact, nonperturbatively, according to the renormalization group (RG), massless QCD-like theories develop a non-trivial dimensional scale that labels the RG trajectory – the

RG invariant –  $\Lambda_{RGI}$ :

$$\Lambda_{RGI} \sim \mu e^{-\frac{1}{2\beta_0 g^2}} g^{-\frac{\beta_1}{\beta_0}} c_0 \left( 1 + \sum_{n=1}^{\infty} c_n g^{2n} \right) \quad (1)$$

– the only free parameter [1, 2] in the nonperturbative S matrix of confining massless QCD-like theories [1, 2] – that any physical mass scale must be proportional to, with  $\beta_0$  and  $\beta_1$  the renormalization-scheme independent first-two coefficients of the beta function  $\beta(g)$ :

$$\frac{\partial g}{\partial \log \mu} = \beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \dots \quad (2)$$

and  $g = g(\mu)$  the renormalized coupling.

Hence, our main motivation is for the study of the ultraviolet (UV) asymptotics, implied by the RG, of 2-, 3- and  $n$ -point correlators of gauge-invariant operators for the general case of operator mixing, in relation to an eventual nonperturbative solution, specifically in the large- $N$  limit [3–6].

In this respect, the study of the UV asymptotics for correlators of multiplicatively renormalizable operators [7–9], apart from the intrinsic interest [10], sets powerful constraints [1, 2, 7–9, 11] on the nonperturbative solution of large- $N$  confining QCD-like theories.

Accordingly, the present paper is the first of a series, where we intend to study the structure of the UV asymptotics of gauge-invariant correlators implied by the RG in the most general case above, in order to extend the aforementioned nonperturbative results [1, 2, 7–9, 11] to operator mixing.

In particular, since operator mixing is ubiquitous in gauge theories, an important problem, which is hardly discussed in the literature, is to determine under which conditions it may be reduced, to all orders of perturbation theory, to the multiplicatively renormalizable case.

The aim of the present paper is to solve this problem, and also to classify the cases of operator mixing where the aforementioned reduction is not actually possible.

<sup>a</sup>e-mail: [marco.bohicchio@roma1.infn.it](mailto:marco.bohicchio@roma1.infn.it) (corresponding author)

## 2 Main results and plan of the paper

We can exemplify the structure of the UV asymptotics of 2-point correlators as follows. In massless QCD-like theories, we consider 2-point correlators in Euclidean space-time:

$$G_{ik}(x) = \langle O_i(x) O_k(0) \rangle \tag{3}$$

of renormalized local gauge-invariant operators  $O_i(x)$ :

$$O_i(x) = Z_{ik} O_{Bk}(x) \tag{4}$$

where  $O_{Bk}(x)$  are the bare operators that mix<sup>1</sup> under renormalization and  $Z$  is the bare mixing matrix.

The corresponding Callan–Symanzik equation [15–18] reads in matrix notation:

$$\left( x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2D \right) G + \gamma(g) G + G \gamma^T(g) = 0 \tag{5}$$

with  $\gamma^T$  the transpose of  $\gamma$ ,  $D$  the canonical dimension of the operators, and  $\gamma(g)$  the matrix of the anomalous dimensions:<sup>2</sup>

$$\gamma(g) = - \frac{\partial Z}{\partial \log \mu} Z^{-1} = \gamma_0 g^2 + \gamma_1 g^4 + \dots \tag{6}$$

The general solution has the form:

$$G(x) = Z(x, \mu) \mathcal{G}(x, g(\mu), \mu) Z^T(x, \mu) \tag{7}$$

with  $\mathcal{G}(x, g(\mu), \mu)$  satisfying:

$$\left( x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2D \right) \mathcal{G} = 0 \tag{8}$$

and:

$$Z(x, \mu) = P \exp \left( - \int_{g(x)}^{g(\mu)} \frac{\gamma(g)}{\beta(g)} dg \right) \tag{9}$$

where  $Z(x, \mu)$  is the renormalized mixing matrix in the coordinate representation,  $P$  denotes the path ordering of the exponential, and  $g(\mu)$ ,  $g(x)$  are short notations for the running couplings  $g(\frac{\mu}{\Lambda_{RGI}})$ ,  $g(x \Lambda_{RGI})$  at the corresponding scales, with UV asymptotics:

$$g^2(x \Lambda_{RGI}) \sim \frac{1}{\beta_0 \log(\frac{1}{x^2 \Lambda_{RGI}^2})} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log \log(\frac{1}{x^2 \Lambda_{RGI}^2})}{\log(\frac{1}{x^2 \Lambda_{RGI}^2})} \right). \tag{10}$$

<sup>1</sup> In fact [12–14], gauge-invariant operators also mix with BRST-exact operators and with operators that vanish by the equations of motion (EQM). But correlators of gauge-invariant operators with BRST-exact operators vanish, while correlators with EQM operators reduce to contact terms. Hence, for our purposes it suffices to take into account the mixing of gauge-invariant operators only.

<sup>2</sup> The sign of the coefficient matrices in Eq. (6),  $\gamma_0, \gamma_1, \dots$ , is the standard one, but opposite with respect to the convention employed in [1, 2, 7–9, 11].

We will discuss the UV asymptotics of  $\mathcal{G}(x, g(\mu), \mu)$  in a forthcoming paper,<sup>3</sup> while in the present paper we concentrate on the UV asymptotics<sup>4</sup> of  $Z(x, \mu)$ .

In the general case, because of the path-ordered exponential and the matrix nature of Eq. (9), it is difficult to work out the actual UV asymptotics of  $Z(x, \mu)$ . Of course, were  $\frac{\gamma(g)}{\beta(g)}$  diagonal, we would get immediately the corresponding UV asymptotics for  $Z(x, \mu)$ , as in the multiplicatively renormalizable case [17, 18].

Therefore, the main aim of the present paper is to find under which conditions a renormalization scheme exists where  $Z(x, \mu)$  is diagonalizable to all perturbative orders.

Another aim is to classify the cases of operator mixing where such a diagonalization is not possible.

We accomplish the aforementioned purposes in three steps:

In the first step (Sect. 3), we furnish an essential differential-geometric interpretation of renormalization: We interpret a change of renormalization scheme as a (formal) holomorphic gauge transformation,  $-\frac{\gamma(g)}{\beta(g)}$  as a (formal) meromorphic connection with a Fuchsian singularity at  $g = 0$ , and  $Z(x, \mu)$  as a Wilson line.

In the second step (Sect. 4), we employ the above interpretation to apply in the framework of operator mixing – for the first time, to the best of our knowledge – the theory of canonical forms, obtained by gauge transformations, for linear systems of differential equations with meromorphic singularities [19], and specifically (a formal version of) the Poincaré–Dulac theorem [20] for Fuchsian singularities, i.e., simple poles.

In the third step (Sect. 5), we provide a condensed proof of the Poincaré–Dulac theorem in the case (I) below, where  $Z(x, \mu)$  is diagonalizable to all orders of perturbation theory.

From the three steps above, our conclusions follow:

As a consequence of the Poincaré–Dulac theorem, if the eigenvalues  $\lambda_1, \lambda_2, \dots$  of the matrix  $\frac{\gamma_0}{\beta_0}$ , in nonincreasing order  $\lambda_1 \geq \lambda_2 \geq \dots$ , do not differ by a positive even integer (Sect. 4), i.e.:

$$\lambda_i - \lambda_j - 2k \neq 0 \tag{11}$$

for  $i \leq j$  and  $k$  a positive integer, then it exists a renormalization scheme where:

$$-\frac{\gamma(g)}{\beta(g)} = \frac{\gamma_0}{\beta_0} \frac{1}{g} \tag{12}$$

<sup>3</sup> M. Becchetti, M. Boichichio, *Canonical forms of operator mixing and UV asymptotics of OPE coefficients in massless QCD-like theories*, to appear in arXiv.

<sup>4</sup> In the present paper  $\gamma(g)$  and  $\beta(g)$  in Eq. (9) are actually only defined in perturbation theory by Eqs. (2) and (6). In this case, Eq. (9) only furnishes the UV asymptotics of  $Z(x, \mu)$ , thanks to the asymptotic freedom.

is one-loop exact to all orders of perturbation theory, with  $-\frac{\gamma(g)}{\beta(g)}$  defined in Eq. (16).

Moreover, according to the terminology of the Poincaré–Dulac theorem, our classification of operator mixing is as follows:

If a renormalization scheme exists where  $-\frac{\gamma(g)}{\beta(g)}$  can be set in the canonical form of Eq. (12), we refer to the mixing as nonresonant, that by Eq. (11) is the generic case. Otherwise, we refer to the mixing as resonant.

Besides,  $\frac{\gamma_0}{\beta_0}$  may be either diagonalizable<sup>5</sup> or nondiagonalizable.

Therefore, there are four cases of operator mixing:

- (I) Nonresonant diagonalizable  $\frac{\gamma_0}{\beta_0}$ .
- (II) Resonant diagonalizable  $\frac{\gamma_0}{\beta_0}$ .
- (III) Nonresonant nondiagonalizable  $\frac{\gamma_0}{\beta_0}$ .
- (IV) Resonant nondiagonalizable  $\frac{\gamma_0}{\beta_0}$ .

In the case (I),  $Z(x, \mu)$  is diagonalizable to all orders of perturbation theory, since the mixing is nonresonant and  $\frac{\gamma_0}{\beta_0}$  is diagonalizable.

The remaining cases, where  $Z(x, \mu)$  is not actually diagonalizable, will be analyzed in a forthcoming paper [21]. Specifically, we will work out in [21] the canonical form of  $-\frac{\gamma(g)}{\beta(g)}$  for resonant mixing – that is different from Eq. (12) –.

In the case (I), the UV asymptotics of  $Z(x, \mu)$  reduces essentially to the multiplicatively renormalizable case:

$$Z_i(x, \mu) = \exp\left(\int_{g(x)}^{g(\mu)} \frac{\gamma_{0i}}{\beta_0 g} dg\right) = \left(\frac{g(\mu)}{g(x)}\right)^{\frac{\gamma_{0i}}{\beta_0}} \tag{13}$$

in the diagonal basis, where  $Z_i(x, \mu)$  and  $\gamma_{0i}$  denote the eigenvalues of the corresponding matrices.

Of course,  $Z(x, \mu)$  in any other renormalization scheme can be reconstructed from the canonical diagonal form above – if it exists – by working out the other way around the appropriate change of basis according to Eq. (22).

Then, in the case (I) Eq. (7) reads:

$$G_{ik}(x) = Z_i(x, \mu) \mathcal{G}_{ik}(x, g(\mu), \mu) Z_k(x, \mu) \tag{14}$$

in the diagonal basis, where no sum on the indices  $i, k$  is understood. Eq. (14) furnishes the UV asymptotics of  $G_{ik}(x)$ , provided that the asymptotics of  $\mathcal{G}_{ik}(x, g(\mu), \mu)$  is known as well.

We believe that the aforementioned employment of the Poincaré–Dulac theorem makes the subject of operator mixing in the physics literature more transparent than in previous treatments [22].

<sup>5</sup> A sufficient condition for a matrix to be diagonalizable is that all its eigenvalues are different.

### 3 Differential geometry of renormalization

We point out that renormalization may be interpreted in a differential-geometric setting, where a (finite) change of renormalization scheme, i.e., a coupling-dependent change of the operator basis:

$$O'_i(x) = S_{ik}(g) O_k(x) \tag{15}$$

is interpreted as a matrix-valued (formal<sup>6</sup>) real-analytic invertible gauge transformation  $S(g)$ . Accordingly, the matrix  $A(g)$ :

$$\begin{aligned} A(g) = -\frac{\gamma(g)}{\beta(g)} &= \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_{2n} g^{2n} \right) \\ &= \frac{1}{g} \left( \frac{\gamma_0}{\beta_0} + \dots \right) \end{aligned} \tag{16}$$

that occurs in the system of ordinary differential equations defining  $Z(x, \mu)$  by Eqs. (6) and (2):

$$\left( \frac{\partial}{\partial g} + \frac{\gamma(g)}{\beta(g)} \right) Z = 0 \tag{17}$$

is interpreted as a (formal) real-analytic connection, with a simple pole at  $g = 0$ , that for the gauge transformation in Eq. (15) transforms as:

$$A'(g) = S(g) A(g) S^{-1}(g) + \frac{\partial S(g)}{\partial g} S^{-1}(g) \tag{18}$$

Moreover,

$$\mathcal{D} = \frac{\partial}{\partial g} - A(g) \tag{19}$$

is interpreted as the corresponding covariant derivative that defines the linear system:

$$\mathcal{D}X = \left( \frac{\partial}{\partial g} - A(g) \right) X = 0 \tag{20}$$

whose solution with a suitable initial condition is  $Z(x, \mu)$ .

As a consequence,  $Z(x, \mu)$  is interpreted as a Wilson line associated to the aforementioned connection:

$$Z(x, \mu) = P \exp\left(\int_{g(x)}^{g(\mu)} A(g) dg\right) \tag{21}$$

that transforms as:

$$Z'(x, \mu) = S(g(\mu)) Z(x, \mu) S^{-1}(g(x)) \tag{22}$$

for the gauge transformation  $S(g)$ .

Besides, by allowing the coupling to be complex valued, everything that we have mentioned applies in the (formal) holomorphic setting, instead of the real-analytic one.

<sup>6</sup> A formal series is not assumed to be convergent and, indeed, in the present paper we do not assume that the series in Eqs. (2) and (6) are convergent, since they arise from perturbation theory.

Hence, by summarizing, a change of renormalization scheme is interpreted as a (formal) holomorphic gauge transformation,  $-\frac{\gamma(g)}{\beta(g)}$  as a (formal) meromorphic connection with a Fuchsian singularity at  $g = 0$ , and  $Z(x, \mu)$  as a Wilson line.

#### 4 Canonical nonresonant form for $-\frac{\gamma(g)}{\beta(g)}$ by the Poincaré-Dulac theorem

According to the interpretation above, the easiest way to compute the UV asymptotics of  $Z(x, \mu)$  consists in setting the meromorphic connection in Eq. (16) in a canonical form by a suitable holomorphic gauge transformation.

Specifically, if the nonresonant condition in Eq. (11) is satisfied, a (formal) holomorphic gauge transformation exists that sets  $A(g)$  in Eq. (16) in the canonical nonresonant form – the Euler form [20] –:

$$A'(g) = \frac{\gamma_0}{\beta_0} \frac{1}{g} \tag{23}$$

according to the Poincaré–Dulac theorem.

In this respect, the only minor refinement that we need for applying the Poincaré–Dulac theorem to Eq. (16) is the observation that the inductive procedure in its proof [20] works as well by only restricting to the even powers of  $g$  in Eq. (27) that match the even powers of  $g$  in the brackets in the rhs of Eq. (16).

As a consequence, the nonresonant condition in Eq. (11) only involves positive even integers, as opposed to the general case (Sect. 5).

#### 5 A condensed proof of the Poincaré–Dulac theorem for nonresonant diagonalizable $A_0$

We provide a condensed proof of (the linear version of) the Poincaré–Dulac theorem [20] for nonresonant diagonalizable  $A_0$ , which includes the case (I) in the setting of operator mixing for a massless QCD-like theory.

The proof in the general case will be worked out in [21].

*Poincaré–Dulac theorem for nonresonant diagonalizable  $A_0$ :* The linear system in Eq. (20), where the meromorphic connection  $A(g)$ , with a Fuchsian singularity at  $g = 0$ , admits the (formal) expansion:

$$A(g) = \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_n g^n \right) \tag{24}$$

with  $A_0$  diagonalizable and eigenvalues  $\text{diag}(\lambda_1, \lambda_2, \dots) = \Lambda$ , in nonincreasing order  $\lambda_1 \geq \lambda_2 \geq \dots$ , satisfying the nonresonant condition:

$$\lambda_i - \lambda_j \neq k \tag{25}$$

for  $i \leq j$  and  $k$  a positive integer, may be set, by a (formal) holomorphic invertible gauge transformation, in the Euler normal form:<sup>7</sup>

$$A'(g) = \frac{1}{g} \Lambda \tag{26}$$

We only report the key aspects of the proof, leaving more details to [20].

*Proof:* The proof proceeds by induction on  $k = 1, 2, \dots$  by demonstrating that, once  $A_0$  and the first  $k - 1$  matrix coefficients,  $A_1, \dots, A_{k-1}$ , have been set in the Euler normal form above – i.e.,  $A_0$  diagonal and  $A_1, \dots, A_{k-1} = 0$  – a holomorphic gauge transformation exists that leaves them invariant and also sets the  $k$ th coefficient,  $A_k$ , to 0.

The 0 step of the induction consists just in setting  $A_0$  in diagonal form – with the eigenvalues in nonincreasing order as in the statement of the theorem – by a global (i.e., constant) gauge transformation.

At the  $k$ th step, we choose the holomorphic gauge transformation in the form:

$$S_k(g) = 1 + g^k H_k \tag{27}$$

with  $H_k$  a matrix to be found momentarily. Its inverse is:

$$S_k^{-1}(g) = (1 + g^k H_k)^{-1} = 1 - g^k H_k + \dots \tag{28}$$

where the dots represent terms of order higher than  $g^k$ .

The gauge action of  $S_k(g)$  on the connection  $A(g)$  furnishes:

$$\begin{aligned} A'(g) &= k g^{k-1} H_k (1 + g^k H_k)^{-1} \\ &\quad + (1 + g^k H_k) A(g) (1 + g^k H_k)^{-1} \\ &= k g^{k-1} H_k (1 + g^k H_k)^{-1} \\ &\quad + (1 + g^k H_k) \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_n g^n \right) (1 + g^k H_k)^{-1} \\ &= k g^{k-1} H_k (1 - \dots) \\ &\quad + (1 + g^k H_k) \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_n g^n \right) (1 - g^k H_k + \dots) \\ &= k g^{k-1} H_k + \frac{1}{g} \left( A_0 + \sum_{n=1}^k A_n g^n \right) \\ &\quad + g^{k-1} (H_k A_0 - A_0 H_k) + \dots \\ &= g^{k-1} (k H_k + H_k A_0 - A_0 H_k) \\ &\quad + A_{k-1}(g) + g^{k-1} A_k + \dots \end{aligned} \tag{29}$$

where we have skipped in the dots all the terms that contribute to an order higher than  $g^{k-1}$ , and we have set:

$$A_{k-1}(g) = \frac{1}{g} \left( A_0 + \sum_{n=1}^{k-1} A_n g^n \right) \tag{30}$$

<sup>7</sup> In the present paper, we refer to it as the canonical nonresonant diagonal form.

that is the part of  $A(g)$  that is not affected by the gauge transformation  $S_k(g)$ , and thus verifies the hypotheses of the induction – i.e., that  $A_1, \dots, A_{k-1}$  vanish –.

Therefore, by Eq. (29) the  $k$ th matrix coefficient,  $A_k$ , may be eliminated from the expansion of  $A'(g)$  to the order of  $g^{k-1}$  provided that an  $H_k$  exists such that:

$$A_k + (kH_k + H_k A_0 - A_0 H_k) = A_k + (k - ad_{A_0})H_k = 0 \quad (31)$$

with  $ad_{A_0} Y = [A_0, Y]$ . If the inverse of  $ad_{A_0} - k$  exists, the unique solution for  $H_k$  is:

$$H_k = (ad_{A_0} - k)^{-1} A_k \quad (32)$$

Hence, to prove the theorem, we should demonstrate that, under the hypotheses of the theorem,  $ad_{A_0} - k$  is invertible, i.e., its kernel is trivial.

Now  $ad_{A_0} - k$ , as a linear operator that acts on matrices, is diagonal, with eigenvalues  $\lambda_i - \lambda_j - k$  and the matrices  $E_{ij}$ , whose only nonvanishing entries are  $(E_{ij})_{ij}$ , as eigenvectors. The eigenvectors  $E_{ij}$ , normalized in such a way that  $(E_{ij})_{ij} = 1$ , form an orthonormal basis for the matrices.

Thus,  $E_{ij}$  belongs to the kernel of  $ad_{A_0} - k$  if and only if  $\lambda_i - \lambda_j - k = 0$ .

As a consequence, since  $\lambda_i - \lambda_j - k \neq 0$  for every  $i, j$  by the hypotheses of the theorem, the kernel of  $ad_{A_0} - k$  only contains the 0 matrix, and the proof is complete.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: There are no data associated to the present paper, since it is of theoretical nature.]

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Funded by SCOAP<sup>3</sup>.

## References

1. M. Bochicchio, The large- $N$  Yang–Mills S matrix is ultraviolet finite, but the large- $N$  QCD S matrix is only renormalizable. *Phys. Rev. D* **95**, 054010 (2017). <https://doi.org/10.1103/PhysRevD.95.054010>
2. M. Bochicchio, Renormalization in large- $N$  QCD is incompatible with open/closed string duality. *Phys. Lett. B* **783**, 341 (2018). <https://doi.org/10.1016/j.physletb.2018.06.072>
3. G. 't Hooft, A planar diagram theory for strong interactions. *Nucl. Phys. B* **72**, 461 (1974). [https://doi.org/10.1016/0550-3213\(74\)90154-0](https://doi.org/10.1016/0550-3213(74)90154-0)
4. G. Veneziano, Some aspects of a unified approach to gauge, dual and Gribov theories. *Nucl. Phys. B* **117**, 519 (1976). [https://doi.org/10.1016/0550-3213\(76\)90412-0](https://doi.org/10.1016/0550-3213(76)90412-0)
5. A.A. Migdal, Multicolor QCD as dual resonance theory. *Ann. Phys.* **109**, 365 (1977). [https://doi.org/10.1016/0003-4916\(77\)90181-6](https://doi.org/10.1016/0003-4916(77)90181-6)
6. E. Witten, Baryons in the  $\frac{1}{N}$  expansion. *Nucl. Phys. B* **160**, 57 (1979). [https://doi.org/10.1016/0550-3213\(79\)90232-3](https://doi.org/10.1016/0550-3213(79)90232-3)
7. M. Bochicchio, S.P. Muscinelli, Ultraviolet asymptotics of glueball propagators. *JHEP* **1308**, 064 (2013). [https://doi.org/10.1007/JHEP08\(2013\)064](https://doi.org/10.1007/JHEP08(2013)064)
8. M. Bochicchio, Glueball and meson propagators of any spin in large- $N$  QCD. *Nucl. Phys. B* **875**, 621 (2013). <https://doi.org/10.1016/j.nuclphysb.2013.07.023>
9. M. Bochicchio, An asymptotic solution of large- $N$  QCD, for the Glueball and meson spectrum and the collinear S-matrix. In: Proceedings, 16th International Conference on Hadron Spectroscopy (Hadron 2015). AIP Conf. Proc., vol. 1735, p. 030004 (2016). <https://doi.org/10.1063/1.4949387>
10. V. Prochazka, R. Zwicky, On finiteness of 2- and 3-point functions and the renormalisation group. *Phys. Rev. D* **95**, 065027 (2017). <https://doi.org/10.1103/PhysRevD.95.065027>
11. M. Becchetti, M. Bochicchio, OPE and a low-energy theorem in QCD-like theories. *JHEP* **03**, 088 (2019). [https://doi.org/10.1007/JHEP03\(2019\)088](https://doi.org/10.1007/JHEP03(2019)088)
12. S.D. Joglekar, B.W. Lee, General theory of renormalization of gauge invariant operators. *Ann. Phys.* **97**, 160 (1976). [https://doi.org/10.1016/0003-4916\(76\)90225-6](https://doi.org/10.1016/0003-4916(76)90225-6)
13. M. Henneaux, Remarks on the renormalization of gauge invariant operators in Yang–Mills theory. *Phys. Lett. B* **313**, 35 (1993). [Erratum *ibid.* **B 316**, 633 (1993)]. [https://doi.org/10.1016/0370-2693\(93\)91187-R](https://doi.org/10.1016/0370-2693(93)91187-R)
14. J.C. Collins, R.J. Scalise, Renormalization of composite operators in Yang–Mills theories using general covariant gauge. *Phys. Rev. D* **50**, 4117 (1994). <https://doi.org/10.1103/PhysRevD.50.4117>
15. C.G. Callan, Broken scale invariance in scalar field theory. *Phys. Rev. D* **2**, 15411547 (1970). <https://doi.org/10.1103/PhysRevD.2.1541>
16. K. Symanzik, Small distance behaviour in field theory and power counting. *Commun. Math. Phys.* **18**, 227 (1970). <https://doi.org/10.1007/BF01649434>
17. C. Itzykson, J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York City, NY, 1985)
18. M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, Boulder, CO, 1995)
19. D.G. Babbitt, V.S. Varadarajan, Formal reduction theory of meromorphic differential equations: a group theoretic view. *Pac. J. Math.* **109**, 1 (1983)
20. Y. Ilyashenko, S. Yakovenko, Lectures On Analytic Differential Equations, Graduate Studies in Mathematics, vol. 86 (2008)
21. M. Becchetti, M. Bochicchio, Operator mixing in massless QCD-like theories and Poincaré–Dulac theorem. [arXiv:2103.16220](https://arxiv.org/abs/2103.16220) [hep-th]
22. H. Sonoda, The rule of operator mixing. *Nucl. Phys. B* **366**, 629 (1991). [https://doi.org/10.1016/0550-3213\(91\)90032-S](https://doi.org/10.1016/0550-3213(91)90032-S)