$O(25, 25)$ symmetry of bosonic string theory at order α'^2

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Abstract It has been recently observed that the imposition of the $O(1, 1)$ symmetry on the circle reduction of the classical effective action of string theory, can fix the effective action of the bosonic string theory at order α'^2 , up to an overall factor. In this paper, we use the cosmological reduction on the action and show that, up to one-dimensional field redefinitions and total derivative terms, it can be written in the $O(25, 25)$ -invariant form proposed by Hohm and Zwiebach.

1 Introduction

The string theory is a consistent theory of quantum gravity which includes the finite number of massless modes and the tower of infinite number of massive modes of the string excitations. At the low energies, the massive modes are integrated out to produce a higher-derivative effective action which includes only the massless fields. The effective action then has the genus-expansion and the stringy-expansion which is an expansion in terms of higher derivative couplings at each loop level. The classical effective action of the bosonic string theory on a spacetime manifold which has no boundary, has the following higher-derivative or α' -expansion:

$$S_{\text{eff}} = \sum_{m=0}^{\infty} \alpha'^m S_m = S_0 + \alpha' S_1 + \alpha'^2 S_2 + \alpha'^3 S_3 + \dots \quad (1)$$

The leading order action S_0 is the gauge invariant two-derivative couplings of the massless fields which includes, among other things, the Hilbert–Einstein term. This action and their appropriate higher derivative extensions may be found by the S-matrix method [1,2], by the sigma-mode method [3,4] or by exploring various symmetries in the string theory.

One of the symmetries in the perturbative string theory is T-duality [5,6] which appears when one compactifies theory

on a torus, e.g., the compactification of the full bosonic string theory on torus T^d is invariant under $O(d, d, Z)$ transformations. After integrating out the massive modes, however, the T-duality should appear as symmetry in the effective actions. It has been proved in [7] that the dimensional reduction of the classical effective actions of the bosonic string theory at each order of α' is invariant under $O(d, d, R)$ transformations.

In the most simple case, when one reduces the effective action on a circle, the invariance of the reduced action under the Z_2 -subgroup of the $O(1, 1)$ -group constrains greatly the couplings in the effective action. This constraint and the constraint that the couplings in the effective action must be invariant under the gauge transformations, fix the couplings at the leading order of α' , i.e.,

$$S_0 = -\frac{2}{\kappa^2} \int d^{26}x e^{-2\phi} \sqrt{-G} \left(R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} H^2 \right) \quad (2)$$

which is the standard effective action of the bosonic string theory. These constraints fix the couplings at order α' , α'^2 up to an overall factor [8,9], i.e.,

$$\begin{aligned} S_1 &= \frac{-2b_1}{\kappa^2} \int d^{26}x e^{-2\phi} \sqrt{-G} \\ &\quad \times \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{2} H_{\alpha}^{\delta\epsilon} H^{\alpha\beta\gamma} R_{\beta\gamma\delta\epsilon} \right. \\ &\quad \left. + \frac{1}{24} H_{\epsilon\delta\zeta} H^{\epsilon}{}_{\alpha}{}^{\beta} H^{\delta}{}_{\beta}{}^{\gamma} H^{\zeta}{}_{\gamma}{}^{\alpha} \right. \\ &\quad \left. - \frac{1}{8} H_{\alpha\beta}{}^{\delta} H^{\alpha\beta\gamma} H_{\gamma}{}^{\epsilon\zeta} H_{\delta\epsilon\zeta} \right) \\ S_2 &= \frac{-2b_2^2}{\kappa^2} \int d^{26}x e^{-2\phi} \sqrt{-G} \\ &\quad \times \left(-\frac{1}{12} H_{\alpha}^{\delta\epsilon} H^{\alpha\beta\gamma} H_{\beta\delta}{}^{\zeta} H_{\gamma}{}^{\iota\kappa} H_{\epsilon\iota}{}^{\mu} H_{\zeta\kappa\mu} \right. \\ &\quad \left. + \frac{1}{30} H_{\alpha\beta}{}^{\delta} H^{\alpha\beta\gamma} H_{\gamma}{}^{\epsilon\zeta} H_{\delta}{}^{\iota\kappa} H_{\epsilon\zeta}{}^{\mu} H_{\iota\kappa\mu} \right. \\ &\quad \left. + \frac{3}{10} H_{\alpha\beta}{}^{\delta} H^{\alpha\beta\gamma} H_{\gamma}{}^{\epsilon\zeta} H_{\delta\epsilon}{}^{\iota} H_{\zeta}{}^{\kappa\mu} H_{\iota\kappa\mu} \right) \end{aligned} \quad (3)$$

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$$\begin{aligned}
& + \frac{13}{20} H_{\alpha}^{\epsilon\zeta} H_{\beta}^{\iota\kappa} H_{\gamma\epsilon\zeta} H_{\delta\iota\kappa} R^{\alpha\beta\gamma\delta} \\
& + \frac{2}{5} H_{\alpha}^{\epsilon\zeta} H_{\beta\epsilon}^{\iota} H_{\gamma\zeta}^{\kappa} H_{\delta\iota\kappa} R^{\alpha\beta\gamma\delta} \\
& + \frac{18}{5} H_{\alpha\gamma}^{\epsilon} H_{\beta}^{\zeta\iota} H_{\delta\zeta}^{\kappa} H_{\epsilon\iota\kappa} R^{\alpha\beta\gamma\delta} \\
& - \frac{43}{5} H_{\alpha\gamma}^{\epsilon} H_{\beta}^{\zeta\iota} H_{\delta\epsilon}^{\kappa} H_{\zeta\iota\kappa} R^{\alpha\beta\gamma\delta} \\
& - \frac{16}{5} H_{\alpha\gamma}^{\epsilon} H_{\beta\delta}^{\zeta} H_{\epsilon}^{\iota\kappa} H_{\zeta\iota\kappa} R^{\alpha\beta\gamma\delta} \\
& - 2 H_{\beta\epsilon}^{\iota} H_{\delta\zeta\iota} R_{\alpha}^{\epsilon}{}_{\gamma}{}^{\zeta} R^{\alpha\beta\gamma\delta} \\
& - 2 H_{\beta\delta}^{\iota} H_{\epsilon\zeta\iota} R_{\alpha}^{\epsilon}{}_{\gamma}{}^{\zeta} R^{\alpha\beta\gamma\delta} \\
& - \frac{4}{3} R_{\alpha}^{\epsilon}{}_{\gamma}{}^{\zeta} R^{\alpha\beta\gamma\delta} R_{\beta\zeta\delta\epsilon} + \frac{4}{3} R_{\alpha\beta}^{\epsilon\zeta} R^{\alpha\beta\gamma\delta} R_{\gamma\epsilon\delta\zeta} \\
& + 3 H_{\beta}^{\zeta\iota} H_{\epsilon\zeta\iota} R^{\alpha\beta\gamma\delta} R_{\gamma}^{\epsilon}{}_{\alpha\delta} \\
& + 2 H_{\beta\epsilon}^{\iota} H_{\delta\zeta\iota} R^{\alpha\beta\gamma\delta} R_{\gamma}^{\epsilon}{}_{\alpha\zeta} + 2 H_{\alpha\beta\epsilon} H_{\delta\zeta\iota} R^{\alpha\beta\gamma\delta} R_{\gamma}^{\epsilon\zeta\iota} \\
& + \frac{13}{10} H_{\alpha}^{\gamma\delta} H_{\beta\gamma}^{\epsilon} H_{\delta}^{\zeta\iota} H_{\epsilon\zeta\iota} \nabla^{\beta} \nabla^{\alpha} \phi \\
& + \frac{13}{5} H_{\gamma}^{\epsilon\zeta} H_{\delta\epsilon\zeta} R_{\alpha}^{\gamma}{}_{\beta}{}^{\delta} \nabla^{\beta} \nabla^{\alpha} \phi \\
& - \frac{52}{5} H_{\beta\delta}^{\zeta} H_{\gamma\epsilon\zeta} R_{\alpha}^{\gamma\delta\epsilon} \nabla^{\beta} \nabla^{\alpha} \phi \\
& - \frac{26}{5} H_{\alpha\gamma\epsilon} H_{\beta\delta\zeta} R^{\gamma\delta\epsilon\zeta} \nabla^{\beta} \nabla^{\alpha} \phi \\
& + \frac{13}{5} \nabla^{\beta} \nabla^{\alpha} \phi \nabla_{\epsilon} H_{\beta\gamma\delta} \nabla^{\epsilon} H_{\alpha}^{\gamma\delta} \\
& + \frac{13}{10} H_{\beta\gamma}^{\epsilon} H^{\beta\gamma\delta} H_{\delta}^{\zeta\iota} \nabla^{\alpha} \phi \nabla_{\iota} H_{\alpha\epsilon\zeta} \\
& - \frac{13}{20} H_{\alpha}^{\beta\gamma} H_{\delta\epsilon}^{\iota} H^{\delta\epsilon\zeta} \nabla^{\alpha} \phi \nabla_{\iota} H_{\beta\gamma\zeta} \\
& + \frac{1}{20} H_{\alpha}^{\delta\epsilon} H^{\alpha\beta\gamma} \nabla_{\iota} H_{\delta\epsilon\zeta} \nabla^{\iota} H_{\beta\gamma}{}^{\zeta} \\
& + \frac{1}{5} H_{\alpha}^{\delta\epsilon} H^{\alpha\beta\gamma} \nabla_{\zeta} H_{\gamma\epsilon\iota} \nabla^{\iota} H_{\beta\delta}{}^{\zeta} \\
& - \frac{6}{5} H_{\alpha}^{\delta\epsilon} H^{\alpha\beta\gamma} \nabla_{\iota} H_{\gamma\epsilon\zeta} \nabla^{\iota} H_{\beta\delta}{}^{\zeta} \\
& - \frac{6}{5} H_{\alpha\beta}^{\delta} H^{\alpha\beta\gamma} \nabla_{\zeta} H_{\delta\epsilon\iota} \nabla^{\iota} H_{\gamma}^{\epsilon\zeta} \\
& + \frac{17}{10} H_{\alpha\beta}^{\delta} H^{\alpha\beta\gamma} \nabla_{\iota} H_{\delta\epsilon\zeta} \nabla^{\iota} H_{\gamma}^{\epsilon\zeta} \Big). \quad (4)
\end{aligned}$$

These constraints have been also used in [10,11] to find the NS-NS couplings in type II superstring theory at order α'^3 up to an overall factor. The couplings (3), for the overall factor $b_1 = 1/4$, are the standard effective action of the bosonic string theory at order α' which has been found in [12] by the S-matrix calculations.¹ The two Riemann cubed terms in (4) are the gravity couplings that have been found in [13] by the S-matrix method. Using field redefinitions, the above couplings can be rewritten in many other schemes.

¹ Note that there is a typo in the overall coefficient of $\mathbf{S}_2$ in [9], i.e., the factor b_1 in [9] should be b_1^2 .

The B -field couplings in (4) can be reproduced by the S-matrix method [14]. In this paper, we are going to check these couplings by studying the cosmological reduction of the couplings and show that they are invariant under $O(25, 25)$ transformations. Similar calculation has been done in [15,16] for the NS-NS couplings in the type II superstring theory at order α'^3 .

When one uses the cosmological reduction on the classical effective action, the resulting one-dimensional effective action should have $O(d, d, R)$ symmetry [7,17]. Using the most general one-dimensional field redefinitions, including the lapse function, and using integration by part, it has been shown in [18,19] that the cosmological reduction of the effective action (1) at order α' and higher, can be written in a scheme in which only the first time-derivative of the generalized metric $\mathcal{S}$ appears. Trace of odd number of the first-derivative of $\mathcal{S}$ is zero. It has been shown in [19] that the couplings which include $\text{tr}(\dot{\mathcal{S}}^2)$ can be removed by the lapse function transformation. Then the one-dimensional action can be written in a minimal scheme as the following expansion [18,19]:

$$\begin{aligned}
S_{\text{eff}}^c = S_0^c + \int dt e^{-\Phi} & (\alpha' c_{2,0} \text{tr}(\dot{\mathcal{S}}^4) + \alpha'^2 c_{3,0} \text{tr}(\dot{\mathcal{S}}^6) \\
& + \alpha'^3 [c_{4,0} \text{tr}(\dot{\mathcal{S}}^8) + c_{4,1} (\text{tr}(\dot{\mathcal{S}}^4))^2] \\
& + \alpha'^4 [c_{5,0} \text{tr}(\dot{\mathcal{S}}^{10}) + c_{5,1} \text{tr}(\dot{\mathcal{S}}^6) \text{tr}(\dot{\mathcal{S}}^4)] + \dots) \quad (5)
\end{aligned}$$

where the coefficients $c_{m,n}$ depends on the specific theory, e.g., $c_{2,0}, c_{3,0}$ are non-zero for the bosonic string theory whereas these numbers are zero for the superstring theory. The lapse function in above action is set $n = 1$.

The above minimal action has been found in [18,19] by imposing various equations of motion which is the same as using field redefinitions and removing total derivative terms. We are going to show that by imposing the most general field redefinitions and total derivative terms, the cosmological reduction of the actions (3) and (4) can be written in the above form. In this way, by keeping track the field redefinitions and total derivative terms, one can study the cosmological reduction of the boundary actions [20,21] as well, in which we are not interested in this paper.

Under the field redefinition $\psi \rightarrow \psi + \alpha' \delta \psi^{(1)} + \alpha'^2 \delta \psi^{(2)}$ where ψ collectively represents the one-dimensional functions, the cosmological reduction of the actions in (1) have the following expansions:

$$\begin{aligned}
\mathbf{S}_0^c(\psi + \alpha' \delta \psi^{(1)} + \alpha'^2 \delta \psi^{(2)}) & = \mathbf{S}_0^c(\psi) + \alpha' \delta_1 \mathbf{S}_0^c(\psi) \\
& + \frac{1}{2} \alpha'^2 \delta_1^2 \mathbf{S}_0^c(\psi) + \alpha'^2 \delta_2 \mathbf{S}_0^c(\psi) + \dots \\
\alpha' \mathbf{S}_1^c(\psi + \alpha' \delta \psi^{(1)} + \alpha'^2 \delta \psi^{(2)}) & = \alpha' \mathbf{S}_1^c(\psi) + \alpha'^2 \delta_1 \mathbf{S}_1^c(\psi) + \dots \\
\alpha'^2 \mathbf{S}_2^c(\psi + \alpha' \delta \psi^{(1)} + \alpha'^2 \delta \psi^{(2)}) & = \alpha'^2 \mathbf{S}_2^c(\psi) + \dots \quad (6)
\end{aligned}$$

where dots represent term at order α'^3 and higher in which we are not interested. In above equations, δ_i on the action means the action contains the first order perturbation $\delta\psi^{(i)}$ and δ_1^2 on the action means the action contains the second order perturbation $\delta\psi^{(1)}\delta\psi^{(1)}$. Then, up to some total derivative terms, the sum of terms at each order of α' , i.e.,

$$\begin{aligned} S_0^c &= S_0^c(\psi) \\ S_1^c &= S_1^c(\psi) + \delta_1 S_0^c(\psi) \\ S_2^c &= S_2^c(\psi) + \delta_1 S_1^c(\psi) + \frac{1}{2}\delta_1^2 S_0^c(\psi) + \delta_2 S_0^c(\psi) \end{aligned} \quad (7)$$

should be written in $O(25, 25)$ -invariant form as in (5). To find the above perturbations, one needs to know how the lapse function n appears in the cosmological action $S_0^c, S_1^c, \dots$ because the field redefinition should include the redefinition of all functions including the laps function. In general, the derivatives of the lapse function may appear in some complicated expressions in these actions. However, the $O(25, 25)$ -covariance forces this function to appear in the $O(25, 25)$ -invariant form of the actions by replacing the measure of integral as $dt \rightarrow dt/n^{(m-1)}$ where $2m$ is the number of time-derivative in the action, e.g., the laps function in S_1^c is as $dt \rightarrow dt/n^3$.

Since we know how the lapse function appears in the actions $S_0^c, S_1^c, S_2^c, \dots$, it is convenient to use the field redefinitions in these actions, i.e.,

$$\begin{aligned} S_0^c(\psi + \alpha'\delta\psi^{(1)} + \alpha'^2\delta\psi^{(2)}) \\ &= S_0^c(\psi) + \alpha'\delta_1 S_0^c(\psi) + \frac{1}{2}\alpha'^2\delta_1^2 S_0^c(\psi) + \alpha'^2\delta_2 S_0^c(\psi) + \dots \\ \alpha' S_1^c(\psi + \alpha'\delta_1\psi + \alpha'^2\delta_2\psi) &= \alpha' S_1^c(\psi) + \alpha'^2\delta_1 S_1^c(\psi) + \dots \\ \alpha'^2 S_2^c(\psi + \alpha'\delta_1\psi + \alpha'^2\delta_2\psi) &= \alpha'^2 S_2^c(\psi) + \dots \end{aligned} \quad (8)$$

where the perturbations can easily be calculated, once we know the $O(25, 25)$ -invariant form of the actions. Hence, one has to relate the perturbations $\delta S_0^c, \delta S_1^c, \dots$ in (7) to the perturbations $\delta S_0^c, \delta S_1^c, \dots$. It is obvious that $\delta S_0^c = \delta S_0^c$ because the leading order action is $O(25, 25)$ invariant up to a total derivative term. To find the relation between δS_1^c and δS_0^c , we expand the right-hand side of second equation in (7) and compare it with the right-hand side of the second equation in (8) to find the following relation:

$$\delta_1 S_1^c = \delta_1 S_1^c - \delta_1^2 S_0^c. \quad (9)$$

Inserting them into (7), one finds the following expressions should be invariant under the $O(25, 25)$ transformations:

$$\begin{aligned} S_0^c &= S_0^c(\psi) \\ S_1^c &= S_1^c(\psi) + \delta_1 S_0^c(\psi) \\ S_2^c &= S_2^c(\psi) + \delta_1 S_1^c(\psi) - \frac{1}{2}\delta_1^2 S_0^c(\psi) + \delta_2 S_0^c(\psi) \end{aligned} \quad (10)$$

which contains only the perturbation of the $O(25, 25)$ -invariant actions. In this paper, we are going to show that

when the cosmological reduction of the actions (2), (3) and (4) are inserted in the above expressions, they satisfy the $O(25, 25)$ symmetry and can be written in the minimal form of (5).

The outline of the paper is as follows: In Sect. 2, we review the observation that the cosmological reduction of the leading order action is invariant under the $O(25, 25)$ transformations. In Sect. 3, we show that when the cosmological reduction of the action (3) is inserted into the right-hand side of the second expression in (10), it can be written in the standard form of (5) after including the appropriate one-dimensional field redefinition $\delta_1 S_0^c(\psi)$ and total derivative terms. In Sect. 4, we show that when the cosmological reduction of the action (4) are inserted into the last expression in (10), the left-hand side can be written in the $O(25, 25)$ -invariant form of (5). This observation not only confirms the proposal (5), but also it confirms the effective action (4) that has been found in [9] by imposition of $O(1, 1)$ symmetry on the most general gauge invariant couplings.

2 Cosmological reduction at the leading order

In this section, we review the cosmological reduction of the leading order action [22–25]. When fields depend only on time, using the gauge symmetries it is possible to write the metric, B -field and dilaton as

$$\begin{aligned} G_{\mu\nu} &= \begin{pmatrix} -n^2(t) & 0 \\ 0 & G_{ij}(t) \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ij}(t) \end{pmatrix}, \\ 2\phi &= \Phi + \frac{1}{2} \log \det(G_{ij}) \end{aligned} \quad (11)$$

where the lapse function $n(t)$ can also be fixed to $n = 1$. The cosmological reduction of the action (2) then becomes

$$\begin{aligned} S_0^c &= -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{3}{4} \dot{G}_{ij} \dot{G}^{ij} \right. \\ &\quad \left. - G^{ij} \dot{G}_{ij} \dot{\Phi} - \dot{\Phi}^2 + G^{ij} \ddot{G}_{ij} \right] \end{aligned} \quad (12)$$

where $\dot{G}^{ij} \equiv G^{ik} G^{il} \dot{G}_{kl}$. Removing a total derivative term, one can write S_0^c as

$$S_0^c = -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} + \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} - \dot{\Phi}^2 \right]. \quad (13)$$

Using the generalized metric $\mathcal{S}$ which is defined as

$$\mathcal{S} \equiv \eta \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \quad (14)$$

where η is the metric of the $O(25, 25)$ group which in the non-diagonal form is

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (15)$$

one can write the action (13) as

$$S_0^c = -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[-\dot{\Phi}^2 - \frac{1}{8} \text{tr}(\dot{S}^2) \right] \quad (16)$$

which is invariant under the global $O(25, 25)$ transformations because the one-dimensional dilaton is invariant and the generalized metric transforms as

$$\mathcal{S} \rightarrow \Omega^T \mathcal{S} \Omega \quad (17)$$

where Ω belong to the $O(25, 25)$ group, i.e., $\Omega^T \eta \Omega = \eta$. The laps function can be inserted in the action (16) by replacing $dt \rightarrow dt/n$.

3 Cosmological reduction at order α'

In this section, we are going to show that the cosmological reduction of the couplings (3) can be written in $O(25, 25)$ -invariant after using appropriate one-dimensional field redefinitions and total derivative terms.

To find the cosmological reduction of these couplings we first find the cosmological reduction of the Riemann curvature and H . They are

$$\begin{aligned} R_{ijkl} &= -\frac{1}{4} \dot{G}_{il} \dot{G}_{jk} + \frac{1}{4} \dot{G}_{ik} \dot{G}_{jl}; \quad R_{i0jk} = 0; \\ R_{i0j0} &= \frac{1}{4} \dot{G}_{ik} \dot{G}_{j0} - \frac{1}{2} \ddot{G}_{ij} \\ H_{ijk} &= 0; \quad H_{ij0} = \dot{B}_{ij}. \end{aligned} \quad (18)$$

Using the above reductions, one finds the following cosmological reduction for the action (3):

$$\begin{aligned} S_1^c &= -\frac{2b_1}{\kappa^2} \int dt e^{-\Phi} \left[-\frac{3}{8} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_{kl} \right. \\ &\quad -\frac{1}{8} \dot{B}_{ij} \dot{B}^{ij} \dot{B}_{kl} \dot{B}^{kl} + \frac{1}{4} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_{jl} \\ &\quad -\frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_{kl} + \frac{1}{8} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_{kl} + \frac{1}{8} \dot{G}_{ij} \dot{G}^{ij} \dot{G}_{kl} \dot{G}^{kl} \\ &\quad \left. + \ddot{G}_{ij} \ddot{G}^{ij} + \dot{B}_i^k \dot{B}^{ij} \ddot{G}_{jk} - \dot{G}_i^k \dot{G}^{ij} \ddot{G}_{jk} \right] \end{aligned} \quad (19)$$

where we have used the gauge $n = 1$. This action is not invariant under $O(25, 25)$ transformations. Some of the non-invariant terms are total derivative terms which should be removed. Moreover, the action in terms of the variables G_{ij} , B_{ij} , Φ are not invariant. It should be invariant in terms of some other variables which involve higher derivatives of G_{ij} , B_{ij} , Φ .

To remove the total derivatives terms from (19), we add all total derivative terms at order α' with arbitrary coefficients to (19). We add the following total derivative terms:

$$-\frac{2}{\kappa^2} \int dt \frac{d}{dt} (e^{-\Phi} \mathcal{I}_1) \quad (20)$$

where $\mathcal{I}_1$ is all possible terms at three-derivative level with even parity which are constructed from $\dot{\Phi}$, $\dot{B}$, $\dot{G}$, $\ddot{\Phi}$, $\ddot{B}$, $\ddot{G}$, $\dots$. Using the package “xAct” [26], one finds there are 18 such terms, i.e.,

$$\mathcal{I}_1 = j_1 (\dot{\Phi})^3 + \dots \quad (21)$$

where the coefficients $J_1, \dots, J_{18}$ are 18 arbitrary parameters.

One can change the field variables in (11) as

$$\begin{aligned} G_{ij} &\rightarrow G_{ij} + \alpha' \delta G_{ij}^{(1)} \\ B_{ij} &\rightarrow B_{ij} + \alpha' \delta B_{ij}^{(1)} \\ \Phi &\rightarrow \Phi + \alpha' \delta \Phi^{(1)} \\ n &\rightarrow n + \alpha' \delta n^{(1)} \end{aligned} \quad (22)$$

where the matrices $\delta G_{ij}^{(1)}$, $\delta B_{ij}^{(1)}$ and $\delta \Phi^{(1)}$, $\delta n^{(1)}$ are all possible terms at 2-derivative level constructed from $\dot{\Phi}$, $\dot{B}$, $\dot{G}$, $\ddot{\Phi}$, $\ddot{B}$, $\ddot{G}$. The perturbations $\delta G_{ij}^{(1)}$, $\delta \Phi^{(1)}$, $\delta n^{(1)}$ contain even-parity terms and $\delta B_{ij}^{(1)}$ contains odd-parity terms, i.e.,

$$\begin{aligned} \delta n^{(1)} &= n_1 \dot{B}_i^j \dot{B}_j^i + \dots \\ \delta \Phi^{(1)} &= e_1 \dot{B}_i^j \dot{B}_j^i + \dots \\ \delta G_{ij}^{(1)} &= d_1 \dot{B}_i^k \dot{B}_{kj} + \dots \\ \delta B_{ij}^{(1)} &= f_1 \dot{G}_i^k \dot{B}_{kj} + \dots \end{aligned} \quad (23)$$

The coefficients $n_1, \dots, n_7$, $e_1, \dots, e_7$, $d_1, \dots, d_{12}$ and $f_1, \dots, f_4$ are arbitrary parameters. When the field variables in S_0^c are changed according to the above field redefinitions, they produce some couplings at order α' and higher. In this section we are interested in the resulting couplings at order α' , i.e.,

$$\begin{aligned} \delta_1 S_0^c &= -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[\delta n^{(1)} \left(-\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} + \dot{\Phi}^2 \right) \right. \\ &\quad + \delta \Phi^{(1)} \left(-\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} + \dot{\Phi}^2 \right) \\ &\quad - 2\dot{\Phi} \frac{d}{dt} \delta \Phi^{(1)} \\ &\quad + \delta G_{ij}^{(1)} \left(-\frac{1}{2} \dot{B}_k^j \dot{B}^{ki} - \frac{1}{2} \dot{G}_k^j \dot{G}^{ki} \right) \\ &\quad \left. + \frac{1}{2} \dot{G}^{ij} \frac{d}{dt} \delta G_{ij}^{(1)} + \frac{1}{2} \dot{B}^{ij} \frac{d}{dt} \delta B_{ij}^{(1)} \right] \end{aligned} \quad (24)$$

where we have used the fact that the lapse function appears in the action (13) by replacing $dt \rightarrow dt/n$. When replacing the perturbations (23) in the above equation, one finds, for some relations between the parameters in (23), the above equation produces total derivative terms. They are not correspond to the field redefinitions. One should remove such parameters in (23) to find the independent parameters which correspond to the pure field redefinitions.

Using the following field redefinitions:

$$\begin{aligned}\delta n^{(1)} &= b_1 \left(-\frac{1}{2} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{2} \dot{\Phi}^2 + 2\ddot{\Phi} \right) \\ \delta \Phi^{(1)} &= b_1 \left(\frac{1}{2} \dot{G}_{ij} \dot{G}^{ij} + \frac{1}{2} \dot{\Phi}^2 \right) \\ \delta G_{ij}^{(1)} &= b_1 \left(-\dot{B}_i^k \dot{B}_{jk} - 3\dot{G}_i^k \dot{G}_{jk} + 2\ddot{G}_{ij} \right) \\ \delta B_{ij}^{(1)} &= b_1 \left(\dot{B}_j^k \dot{G}_{ik} - \dot{B}_i^k \dot{G}_{jk} + 2\dot{B}_{ij} \dot{\Phi} \right)\end{aligned}\quad (25)$$

one finds the cosmological action (19) can be written as

$$\begin{aligned}S_1^c &= S_1^c + \delta_1 S_0^c = -\frac{2b_1}{\kappa^2} \int dt e^{-\Phi} \\ &\times \left[\frac{1}{8} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_{kl} - \frac{1}{4} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_{jl} \right. \\ &\left. + \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_{kl} + \frac{1}{8} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_{kl} \right]\end{aligned}\quad (26)$$

up to the following total derivative terms:

$$\begin{aligned}\mathcal{I}_1 &= b_1 \left(\dot{B}_i^k \dot{B}^{ij} \dot{G}_{jk} + \dot{G}_i^k \dot{G}^{ij} \dot{G}_{jk} - \frac{1}{2} \dot{B}_{ij} \dot{B}^{ij} \dot{\Phi} \right. \\ &\left. + \frac{1}{2} \dot{G}_{ij} \dot{G}^{ij} \dot{\Phi} - \dot{G}^{ij} \ddot{G}_{ij} \right).\end{aligned}\quad (27)$$

The action (26) has only the first time-derivative on matrices G_{ij} , B_{ij} and has no trace of one $\dot{G}$ and two $\dot{G}$ or $\dot{B}$.

Now using the definition of the generalized metric in (14), one finds

$$\begin{aligned}\text{tr}(\dot{S}^4) &= 2\dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_{kl} - 4\dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_{jl} + 8\dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_{kl} \\ &+ 2\dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_{kl}.\end{aligned}\quad (28)$$

Using the above $O(25, 25)$ -invariant expressions, one can write (26) as

$$S_1^c = -\frac{2b_1}{16\kappa^2} \int dt e^{-\Phi} \text{tr}(\dot{S}^4)\quad (29)$$

which is consistent with the cosmological action (5). The laps function can be inserted in the above action by replacing $dt \rightarrow dt/n^3$.

4 Cosmological reduction at order α'^2

In this section, we are going to show that up to one-dimensional field redefinitions and total derivative terms, the cosmological reduction of the couplings (4) can be written in $O(25, 25)$ -invariant form.

Since these couplings involve Riemann curvature, H , ∇H , $\nabla \phi$ and $\nabla \nabla \phi$, one needs the cosmological reduction of the Riemann curvature and H which are given in (18), and the reduction of ∇H , $\nabla \Phi$, $\nabla \nabla \Phi$, ∇G_{ij} , and $\nabla \nabla G_{ij}$

which are

$$\begin{aligned}\nabla_l H_{ijk} &= -\frac{1}{2} \dot{B}_{jk} \dot{G}_{il} + \frac{1}{2} \dot{B}_{ik} \dot{G}_{jl} \\ &- \frac{1}{2} \dot{B}_{ij} \dot{G}_{kl}; \quad \nabla_0 H_{ij0} = -\frac{1}{2} \dot{B}_j^k \dot{B}_{ik} - \frac{1}{2} \dot{B}_i^k \dot{G}_{jk} + \ddot{B}_{ij} \\ \nabla_0 H_{ijk} &= 0; \quad \nabla_k H_{ij0} = 0; \quad \nabla_0 \Phi \\ &= \dot{\Phi}; \quad \nabla_i \Phi = 0; \quad \nabla_0 \nabla_i \Phi = 0; \quad \nabla_0 \nabla_0 \Phi = \ddot{\Phi} \\ \nabla_i \nabla_j \Phi &= -\frac{1}{2} \dot{\Phi} \dot{G}_{ij}; \quad \nabla_0 G_{ij} \\ &= \dot{G}_{ij}; \quad \nabla_i G_{jk} = 0; \quad \nabla_0 \nabla_i G_{jk} = 0; \quad \nabla_0 \nabla_0 G_{ij} = \ddot{G}_{ij} \\ \nabla_i \nabla_j G_{kl} &= -\frac{1}{2} \dot{G}_{kl} \dot{G}_{ij}.\end{aligned}\quad (30)$$

Using the above reductions, one finds the following cosmological reduction for the action (4):

$$\begin{aligned}S_2^c &= -\frac{2b_1^2}{\kappa^2} \int dt e^{-\Phi} \left[-\frac{73}{60} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_k^m \dot{B}_l^n \dot{B}_{mn} \right. \\ &- \frac{1}{5} \dot{B}_i^k \dot{B}^{ij} \dot{B}_l^n \dot{B}^{lm} \dot{G}_{jm} \dot{G}_{kn} \\ &- \frac{39}{10} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}^{mn} \dot{G}_{km} \dot{G}_{ln} + \frac{3}{4} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_i^m \dot{G}_j^n \dot{G}_{km} \dot{G}_{ln} \\ &+ \frac{39}{5} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_k^m \dot{G}_l^n \dot{G}_{mn} - \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_j^m \dot{G}_l^n \dot{G}_{mn} \\ &+ \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_k^m \dot{G}_l^n \dot{G}_{mn} \\ &\left. - \frac{1}{6} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_k^m \dot{G}_l^n \dot{G}_{mn} + \dots \right]\end{aligned}\quad (31)$$

where dots represent the terms which have trace of $\dot{G}$, $\dot{G}\dot{G}$, $\dot{B}\dot{B}$ and $\dot{B}\dot{B}\dot{G}$, or have $\dot{\Phi}$, $\ddot{\Phi}$, $\ddot{G}$, $\ddot{B}$. All these terms are removable by field redefinitions and total derivative terms. In above action we have also choose the gauge $n = 1$.

To use the total derivatives terms and the field redefinition freedom to remove the dots in (31), we add all total derivative terms and all field redefinitions at order α'^2 with arbitrary coefficients to (31). We add the following total derivative terms:

$$-\frac{2}{\kappa^2} \int dt \frac{d}{dt} (e^{-\Phi} \mathcal{I}_2)\quad (32)$$

where $\mathcal{I}_2$ is all possible terms at five-derivative level with even parity which are constructed from $\dot{\Phi}$, $\dot{B}$, $\dot{G}$, $\ddot{\Phi}$, $\ddot{B}$, $\ddot{G}$, $\dots$. Using the package “xAct” [26], one finds there are 132 such terms, i.e.,

$$\mathcal{I}_2 = j_1 (\dot{\Phi})^5 + \dots\quad (33)$$

where the coefficients $J_1, \dots, J_{132}$ are 132 arbitrary parameters.

One can change the field variables in (11) as

$$\begin{aligned}G_{ij} &\rightarrow G_{ij} + \alpha' \delta G_{ij}^{(1)} + \alpha'^2 \delta G_{ij}^{(2)} \\ B_{ij} &\rightarrow B_{ij} + \alpha' \delta B_{ij}^{(1)} + \alpha'^2 \delta B_{ij}^{(2)} \\ \Phi &\rightarrow \Phi + \alpha' \delta \Phi^{(1)} + \alpha'^2 \delta \Phi^{(2)}\end{aligned}$$

$$n \rightarrow n + \alpha' \delta n^{(1)} + \alpha'^2 \delta n^{(2)} \quad (34)$$

where the first order perturbations $\delta G_{ij}^{(1)}$, $\delta B_{ij}^{(1)}$, $\delta \Phi^{(1)}$, $\delta n^{(1)}$ are given in (25) and the second order perturbations $\delta G_{ij}^{(2)}$, $\delta B_{ij}^{(2)}$, $\delta \Phi^{(2)}$, $\delta n^{(2)}$ are all possible terms at 4-derivative level constructed from $\dot{\Phi}$, $\dot{B}$, $\dot{G}$, $\ddot{\Phi}$, $\ddot{B}$, $\ddot{G}$, $\dots$. The perturbations $\delta G_{ij}^{(2)}$, $\delta \Phi^{(2)}$, $\delta n^{(2)}$ contain even-parity terms and $\delta B_{ij}^{(2)}$ contains odd-parity terms, i.e.,

$$\begin{aligned} \delta n^{(2)} &= n_1 \dot{B}_i^j \dot{B}_j^k \dot{B}_k^l \dot{B}_l^i + \dots \\ \delta \Phi^{(2)} &= e_1 \dot{B}_i^j \dot{B}_j^k \dot{B}_k^l \dot{B}_l^i + \dots \\ \delta G_{ij}^{(2)} &= d_1 \dot{B}_i^k \dot{B}_k^l \dot{B}_l^m \dot{B}_m j + \dots \\ \delta B_{ij}^{(2)} &= f_1 \dot{G}_i^k \dot{B}_k j \dot{B}_l^m \dot{B}_m^l + \dots \end{aligned} \quad (35)$$

The coefficients $n_1, \dots, n_{52}$, $e_1, \dots, e_{52}$, $d_1, \dots, d_{121}$ and $f_1, \dots, f_{59}$ are arbitrary parameters.

When the field variables in S_0^c are changed according to the above field redefinitions, they produce two sets of couplings at order α'^2 . One set is produced by the second order perturbations, i.e.,

$$\begin{aligned} \delta_2 S_0^c &= -\frac{2}{\kappa^2} \int dt e^{-\Phi} \\ &\times \left[\delta n^{(2)} \left(-\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} + \dot{\Phi}^2 \right) \right. \\ &+ \delta \Phi^{(2)} \left(-\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} + \dot{\Phi}^2 \right) \\ &- 2\dot{\Phi} \frac{d}{dt} \delta \Phi^{(2)} \\ &+ \delta G_{ij}^{(2)} \left(-\frac{1}{2} \dot{B}_k^j \dot{B}^{ki} - \frac{1}{2} \dot{G}_k^j \dot{G}^{ki} \right) \\ &\left. + \frac{1}{2} \dot{G}^{ij} \frac{d}{dt} \delta G_{ij}^{(2)} + \frac{1}{2} \dot{B}^{ij} \frac{d}{dt} \delta B_{ij}^{(2)} \right] \end{aligned} \quad (36)$$

which are similar to the first order perturbation (24), and the other set is reproduced by square of the first order perturbations, i.e.,

$$\begin{aligned} \frac{1}{2} \delta_1^2 S_0^c &= -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[\frac{1}{4} \frac{d}{dt} \delta B_{ij}^{(1)} \frac{d}{dt} \delta B^{(1)ij} \right. \\ &+ \frac{1}{4} \frac{d}{dt} \delta G_{ij}^{(1)} \frac{d}{dt} \delta G^{(1)ij} - \frac{d}{dt} \delta \Phi^{(1)} \frac{d}{dt} \delta \Phi^{(1)} \\ &- \dot{B}^{ij} \frac{d}{dt} \delta B_i^{(1)k} \delta G_{jk}^{(1)} - \dot{G}^{ij} \frac{d}{dt} \delta G_i^{(1)k} \delta G_{jk}^{(1)} \\ &+ \frac{1}{4} \dot{B}^{ij} \dot{B}^{kl} \delta G_{ik}^{(1)} \delta G_{jl}^{(1)} + \frac{1}{4} \dot{G}^{ij} \dot{G}^{kl} \delta G_{ik}^{(1)} \delta G_{jl}^{(1)} \\ &+ \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \delta G_j^{(1)l} \delta G_{kl}^{(1)} + \frac{1}{2} \dot{G}_i^k \dot{G}^{ij} \\ &\delta G_j^{(1)l} \delta G_{kl}^{(1)} - \frac{1}{2} \dot{B}^{ij} \frac{d}{dt} \delta B_{ij}^{(1)} \delta n^{(1)} \\ &\left. - \frac{1}{2} \dot{G}^{ij} \frac{d}{dt} \delta G_{ij}^{(1)} \delta n^{(1)} \right] \end{aligned}$$

$$\begin{aligned} &+ 2\dot{\Phi} \frac{d}{dt} \delta \Phi^{(1)} \delta n^{(1)} + \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \delta G_{jk}^{(1)} \delta n^{(1)} \\ &+ \frac{1}{2} \dot{G}_i^k \dot{G}^{ij} \delta G_{jk}^{(1)} \delta n^{(1)} \\ &+ \left(\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} + \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} - \dot{\Phi}^2 \right) \delta n^{(1)} \delta n^{(1)} \\ &+ \left(\frac{1}{8} \dot{B}_{ij} \dot{B}^{ij} + \frac{1}{8} \dot{G}_{ij} \dot{G}^{ij} \right. \\ &\left. - \frac{1}{2} \dot{\Phi}^2 \right) \delta \Phi^{(1)} \delta \Phi^{(1)} \\ &- \frac{1}{2} \dot{G}^{ij} \frac{d}{dt} \delta G_{ij}^{(1)} \delta \Phi^{(1)} + 2\dot{\Phi} \frac{d}{dt} \delta \Phi^{(1)} \delta \Phi^{(1)} \\ &+ \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \delta G_{jk}^{(1)} \delta \Phi^{(1)} + \frac{1}{2} \dot{G}_i^k \dot{G}^{ij} \delta G_{jk}^{(1)} \delta \Phi^{(1)} \\ &- \frac{1}{2} \dot{B}^{ij} \frac{d}{dt} \delta B_{ij}^{(1)} \delta \Phi^{(1)} + \left(\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} \right. \\ &\left. + \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} - \dot{\Phi}^2 \right) \delta n^{(1)} \delta \Phi^{(1)} \Big] \end{aligned} \quad (37)$$

where the first order perturbations are given in (25).

When the field variables in $O(25, 25)$ -invariant action (26) are changed according to the field redefinition (34), one also finds the following couplings at order α'^2 :

$$\begin{aligned} \delta_1 S_1^c &= -\frac{2b_1}{\kappa^2} \int dt e^{-\Phi} \left[-\dot{B}^{ij} \dot{G}_i^k \dot{G}_k^l \frac{d}{dt} \delta B_{jl}^{(1)} \right. \\ &+ \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \frac{d}{dt} \delta B_{kl}^{(1)} - \frac{1}{2} \dot{B}^{ij} \dot{G}_i^k \dot{G}_j^l \frac{d}{dt} \delta B_{kl}^{(1)} \\ &- \frac{1}{2} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \frac{d}{dt} \delta G_{jl}^{(1)} + \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \frac{d}{dt} \delta G_{kl}^{(1)} \\ &+ \frac{1}{2} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \frac{d}{dt} \delta G_{kl}^{(1)} \\ &- \frac{1}{2} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_i^m \dot{G}_{km} \delta G_{jl}^{(1)} - \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_l^m \delta G_{km}^{(1)} \\ &- \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_k^m \delta G_{lm}^{(1)} \\ &+ \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_j^m \delta G_{lm}^{(1)} - \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_k^m \delta G_{lm}^{(1)} \\ &- \frac{1}{2} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_k^m \delta G_{lm}^{(1)} \\ &- \frac{3}{8} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_{kl} \delta n^{(1)} + \frac{3}{4} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_{jl} \delta n^{(1)} \\ &- \frac{3}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_{kl} \delta n^{(1)} \\ &- \frac{3}{8} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_{kl} \delta n^{(1)} - \frac{1}{8} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_{kl} \delta \Phi^{(1)} \\ &+ \frac{1}{4} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_{jl} \delta \Phi^{(1)} \\ &- \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_{kl} \delta \Phi^{(1)} - \frac{1}{8} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_{kl} \delta \Phi^{(1)} \Big] \end{aligned} \quad (38)$$

where we have used the fact that the lapse function appears in the $O(25, 25)$ -invariant action (26) by replacing $dt \rightarrow$

dt/n^3 . The reason for this is that the action has two metric G^{00} which produce $1/n^4$, and the cosmological reduction of $e^{-2\phi}\sqrt{-G}$ is $ne^{-\Phi}$.

Inserting the first order perturbations (25) into (37), (38), and inserting the arbitrary second order perturbations (35) into (36), one finds the cosmological action (31) can be written as

$$\begin{aligned} S_2^c &= S_2^c + \delta_1 S_1^c - \frac{1}{2} \delta_1^2 S_0^c + \delta_2 S_0^c \\ &= -\frac{2b_1^2}{\kappa^2} \int dt e^{-\Phi} \left[\frac{1}{12} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_k^m \dot{B}_l^n \dot{B}_{mn} \right. \\ &\quad + \frac{1}{4} \dot{B}_i^k \dot{B}^{ij} \dot{B}_l^n \dot{B}^{lm} \dot{G}_{jm} \dot{G}_{kn} \\ &\quad - \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}^{mn} \dot{G}_{km} \dot{G}_{ln} + \frac{1}{4} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_i^m \dot{G}_j^n \dot{G}_{km} \dot{G}_{ln} \\ &\quad + \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{B}_j^l \dot{B}_k^m \dot{G}_l^n \dot{G}_{mn} - \frac{1}{2} \dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_j^m \dot{G}_l^n \dot{G}_{mn} \\ &\quad + \frac{1}{2} \dot{B}_i^k \dot{B}^{ij} \dot{G}_j^l \dot{G}_k^m \dot{G}_l^n \dot{G}_{mn} \\ &\quad \left. + \frac{1}{12} \dot{G}_i^k \dot{G}^{ij} \dot{G}_j^l \dot{G}_k^m \dot{G}_l^n \dot{G}_{mn} \right] \end{aligned} \quad (39)$$

for some specific values for the parameters $n_1, \dots, n_{52}$, $e_1, \dots, e_{52}$, $d_1, \dots, d_{121}$ and $f_1, \dots, f_{59}$ and up to some total derivative terms (33). We have chosen the parameters in the second order perturbations (35) and in the total derivative terms such that the terms in S_2^c which have trace of $\dot{G}$, $\dot{G}\dot{G}$, $\dot{B}\dot{B}$ and $\dot{B}\dot{B}\dot{G}$, or have $\dot{\Phi}$, $\ddot{\Phi}$, $\ddot{G}$, $\ddot{B}$ and their higher derivatives, are cancelled. Since we are not interested in studying the couplings at order α'^3 , we don't write here the explicit form of the second order field redefinitions and the total derivative terms.

Now using the definition of the generalized metric in (14), one finds the expression inside the bracket above is $-\text{tr}(\dot{S}^6)/24$. Hence the cosmological reduction of the couplings (4) can be written as

$$S_2^c = \frac{2b_1^2}{24\kappa^2} \int dt e^{-\Phi} \text{tr}(\dot{S}^6) \quad (40)$$

up to some field redefinitions and total derivative terms. It is consistent with the cosmological action (5). This confirms the effective action (4) that has been found in [9].

Note added: During the completion of this work, the preprint [27] appeared which has some overlaps with the results in this paper.

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