Dynamical analysis and statefinder of Barrow holographic dark energy

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Abstract Based on the holographic principle and the Barrow entropy, Barrow holographic dark energy had been proposed. In order to analyze the stability and the evolution of Barrow holographic dark energy, we, in this paper, apply the dynamical analysis and statefinder methods to Barrow holographic dark energy with different IR cutoff and interacting terms. In the case of using Hubble horizon as IR cutoff with the interacting term $Q = \frac{\lambda}{H} \rho_m \rho_D$, we find this model is stable and can be used to describe the whole evolution of the universe when the energy transfers from the pressureless matter to the Barrow holographic dark energy. When the dynamical analysis method is applied to this stable model, an attractor corresponding to an accelerated expansion epoch exists and this attractor can behave as the cosmological constant. Furthermore, the coincidence problem can be solved in this case. Then, after using the statefinder analysis method to this model, we find this model can be discriminated from the standard Λ CDM model. Finally, we have discussed the turning point of Hubble diagram in Barrow holographic dark energy and find the turning point does not exist in this model.

1 Introduction

Observations [1–6] indicate that the universe is currently undergoing a phase of accelerated expansion. Nevertheless, Einstein's General Relativity with only radiation and matter cannot result in an accelerated expansion of the universe. In order to explain this observational result, a mysterious matter called dark energy was introduced to explain this accelerated phase. Since dark energy has not been detected directly, there is no reason to assume dark energy resembles known forms of matter or energy, and it becomes one of the biggest

mysteries in modern cosmology. The simplest dark energy model is the cosmological constant (Λ CDM) in which the constant vacuum energy density can drive the accelerated expansion. Although it is well in agreement with the current observations [7], it suffers from the fine-tuning problem and the coincidence problem.

An interesting approach to describe the origin of dark energy is considering the holographic principle [8,9] in the cosmological framework. After the holographic principle was introduced in cosmology, the holographic dark energy (HDE) arose [10]. However, the original HDE is not successful since it cannot explain the current accelerated expansion of the universe [11,12]. To remedy this situation, a new HDE was proposed [12]. Then, the HDE model becomes a physically viable dark energy candidate and has been widely studied theoretically [13–30] and observationally [31–38]. In HDE, various observational data strongly constrains the free parameter c of HDE being less than 1 [31,33,37,39–47], indicating HDE would lead to a phantom universe with big rip [48]. To solve the big rip, an interaction between dark energy and dark matter within HDE model was introduced [48,49]. In HDE, this interaction was also used to alleviate the cosmic coincidence problem [16,50,51] and resolved the classical instability [51] which stems from a negative sound speed [52]. And the interaction terms within HDE have been tested and constrained by various astronomical observations [21,32,50,53–55] including Planck 2015 [45].

Horizon entropy is the most important cornerstone of HDE models, any change of it will lead to different HDE models. Inspired by the holographic principle and using different horizon entropy, some new HDE models were proposed recently, such as, Tsallis holographic dark energy (THDE) [56], Renyi holographic dark energy [57], Ricci–Gauss–Bonnet holographic dark energy [58] and Sharma–Mittal holographic dark energy [59]. Another cornerstone of

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HDE models is IR cutoff, different IR cutoffs can also result in new HDE models [10, 12, 60]. These new proposed HDE models were studied in various scenario [60–75].

Based on the generic holographic principle and using the Barrow entropy [76] instead of the Bekenstein–Hawking entropy, Barrow holographic dark energy (BHDE) was proposed by considering the future event horizon as IR cutoff [77, 78]. When the IR cutoff was chosen as the Hubble horizon, a new BHDE was proposed [79]. Then, the thermodynamics of this new BHDE with an interaction term was studied [80]. However, this new BHDE is unstable since its squared sound speed is negative [79]. It is noted that an interaction term between dark energy and dark matter can be used to solve the classical instability in HDE [51] and THDE [63], and this interaction term can also alleviate the cosmic coincidence problem [16, 50, 51]. So, in BHDE models, it is unclear whether the classical instability and the cosmic coincidence problem can be solved when an interaction term is taken into account.

When a dark energy model is proposed, it should fit the conventional standard cosmology as well as explain the current accelerated expansion. For a viable cosmological model, it should describe the whole evolution history of the universe. Namely, the universe stems from a radiation dominated epoch, and then enters into a matter dominated epoch to enable the formation of large-scale structures, and eventually evolves into a dark energy dominated epoch. To analyze the evolution of the universe, the dynamical analysis method is introduced, which is an excellent method to investigate the qualitative behavior of a given cosmological model and can avoid the difficulty in solving non-linear cosmological equations. In this method, the dynamics of the universe can be described by analyzing the behavior of critical points of the dynamical system of a model, and the critical points always indicate the main evolution epoch of the universe. The stable point is used to describe a late time epoch dominated by the dark energy, the saddle point can denote the matter dominated epoch, and the unstable point corresponds to the early radiation dominated epoch. This method was used with great success in analyzing the evolution of the universe within HDE models [63, 68, 81–85]. These positive results in HDE lead to some interesting questions in BHDE and motivate us to study BHDE: Whether a stable BHDE model can describe the whole evolution of the universe. How to discriminate the BHDE model from the standard Λ CDM model if the BHDE model describes the cosmic evolution successfully? The main task of this paper is to answer these questions.

Furthermore, a turning point of Hubble diagram in HDE was studied in Ref. [86] which shows that HDE model may be at odds with the cosmological principle. Here, the free parameter region $0 < c < 1$, which was constrained by the current data, was considered. For $0.5 < c < 1$, this turning point exists in the future, while it occurs in the past

and is observable for $c < 0.5$. In BHDE with the future event horizon as IR cutoff, the combination $H(z) + SNIa$ place constraints on the free parameter $C = 3.421^{+1.753}_{-1.611}$ and the parameter of Barrow entropy $\Delta = 0.094^{+0.094}_{-0.101}$ [78]. It is interesting to discuss whether this turning point exists in BHDE with the future event horizon as IR cutoff when the constraints $C = 3.421^{+1.753}_{-1.611}$ and $\Delta = 0.094^{+0.094}_{-0.101}$ are considered.

Regarding that BHDE may offer an interesting framework to study phenomenology beyond to HDE, the main goal of this paper is to discuss six BHDE models: we first consider BHDE with different IR cutoffs, and then, two kinds of interaction terms are used. The analysis is performed in four respects: Firstly, by analyzing the squared sound speed, we study the stability of BHDE models. Secondly, we use the dynamical analysis method to analyze the phase space behavior of the stable BHDE models and discuss the cosmic coincidence problem. Thirdly, in order to discriminate the stable BHDE model from the standard Λ CDM model, we perform the statefinder diagnostic by depicting the evolution trajectories of statefinder parameters. Finally, we plot the evolution curves of $H(z)$ and study the turning point of Hubble diagram in BHDE models.

The paper is organized as follows. In Sect. 2, we investigate the evolution and stability of the universe in BHDE with different IR cutoff and interacting terms. In Sect. 3, we use the dynamical analysis method to analyze the phase space behavior of the stable BHDE model. In Sect. 4, the statefinder diagnostic pairs are used to discriminate the stable BHDE model from the standard Λ CDM model. In Sect. 5, we discuss the turning point of Hubble diagram in BHDE. Finally, our main conclusions are presented in Sect. 6.

2 The Universe evolution

Using the Barrow entropy [76], the energy density of BHDE is given as [77]

$$\rho_D = CL^{\Delta-2}. \quad (1)$$

Here, C is a parameter with the dimension $[L]^{-2-\Delta}$ and Δ satisfies the relation $0 \leq \Delta \leq 1$. For the case $\Delta = 0$, ρ_D provides a smooth spacetime structure, i.e. the standard holographic dark energy, and $\Delta = 1$ corresponding to the most intricate structure.

We consider a homogeneous and isotropic flat Friedmann–Robertson–Walker universe

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\Omega^2), \quad (2)$$

where $a(t)$ is the scale factor with t being cosmic time, and setting $\kappa^2 = 8\pi G$. The Friedmann equation is given as

$$H^2 = \frac{\kappa^2}{3}(\rho_r + \rho_m + \rho_D), \quad (3)$$

where ρ_r , ρ_m and ρ_D denote the energy density of radiation, pressureless matter and BHDE, respectively. The radiation, pressureless matter and BHDE conservation equations take the form

$$\dot{\rho}_r + 4H\rho_r = 0, \quad (4)$$

$$\dot{\rho}_m + 3H\rho_m = Q, \quad (5)$$

$$\dot{\rho}_D + 3H(1 + w_D)\rho_D = -Q, \quad (6)$$

in which $w_D = \frac{p_D}{\rho_D}$ denotes the equation of state parameter of BHDE and Q represents the energy exchange between pressureless matter and BHDE. For $Q > 0$, energy transfers from BHDE to pressureless matter, and energy transfers from pressureless matter to BHDE for $Q < 0$. In this paper, we consider two interacting cases $Q = H(\alpha\rho_m + \beta\rho_D)$ [63, 83, 87] and $Q = \frac{\lambda}{H}\rho_m\rho_D$ [87, 88]. Here, α , β and λ are coupling constants.

After introducing the following dimensionless density parameters

$$\Omega_r = \frac{\kappa^2\rho_r}{3H^2}, \quad \Omega_m = \frac{\kappa^2\rho_m}{3H^2}, \quad \Omega_D = \frac{\kappa^2\rho_D}{3H^2}, \quad \sigma = \frac{\kappa^2 Q}{3H^3}, \quad (7)$$

we can express the Friedmann equation (3) as

$$\Omega_r + \Omega_m + \Omega_D = 1. \quad (8)$$

Differentiating the Friedmann equation (3) and combining Eqs. (4), (5), (6), (7) and (8), we obtain

$$\frac{\dot{H}}{H^2} = \frac{1}{2}[\Omega_m + (1 - 3w_D)\Omega_D] - 2. \quad (9)$$

And the deceleration parameter q can be calculated as

$$q = -1 - \frac{\dot{H}}{H^2} = 1 - \frac{1}{2}[\Omega_m + (1 - 3w_D)\Omega_D]. \quad (10)$$

Defining $\Omega' = d\Omega/d(\ln a)$ and differentiating Eq. (7), we get

$$\Omega'_m = [(3w_D - 1)\Omega_D - \Omega_m + 1]\Omega_m + \sigma, \quad (11)$$

$$\Omega'_D = [(3w_D - 1)(\Omega_D - 1) - \Omega_m]\Omega_D - \sigma. \quad (12)$$

Here, Eqs. (5), (6), (7), (8) and (9) are used.

To discuss the stability of BHDE model, we need to analyze its squared sound speed

$$v_s^2 = \frac{d\rho_D}{d\rho_D} = \frac{\dot{\rho}_D}{\dot{\rho}_D} = \frac{\rho_D}{\dot{\rho}_D}\dot{\omega}_D + \omega_D. \quad (13)$$

For $v_s^2 > 0$, this model can be stable against perturbations. Otherwise, it is unstable.

To find a stable and suitable model for BHDE, we will analyze six models in the following:

- (i) Model I: Non-interacting BHDE with future event horizon as IR cutoff (BHDEF);
- (ii) Model II: Interacting BHDE with future event horizon as IR cutoff and $Q = H(\alpha\rho_m + \beta\rho_D)$ (IBHDEFA);
- (iii) Model III: Interacting BHDE with future event horizon as IR cutoff and $Q = \frac{\lambda}{H}\rho_m\rho_D$ (IBHDEFL);
- (iv) Model IV: Non-interacting BHDE with Hubble horizon as IR cutoff (BHDEH);
- (v) Model V: Interacting BHDE with Hubble horizon as IR cutoff and $Q = H(\alpha\rho_m + \beta\rho_D)$ (IBHDEHA);
- (vi) Model VI: Interacting BHDE with Hubble horizon as IR cutoff and $Q = \frac{\lambda}{H}\rho_m\rho_D$ (IBHDEHL).

2.1 BHDE with future event horizon

Considering the future event horizon as IR cutoff, the energy density of BHDE takes the form [77]

$$\rho_D = C R_h^{\Delta-2}, \quad (14)$$

with

$$R_h = a \int_t^\infty \frac{dt}{a} = a \int_a^\infty \frac{da}{Ha^2}, \quad (15)$$

which satisfies the relation $\dot{R}_h = HR_h - 1$. For the case $\Delta = 0$ with $C = 3c^2$, it reduces to the standard holographic dark energy.

In the following, we assume the present scale factor $a_0 = 1$ so that the redshift z satisfies the relation $z = \frac{1}{a} - 1$. Since the combination $H(z) + SNIa$ constrain $C = 3.421^{+1.753}_{-1.611}$ and $\Delta = 0.094^{+0.094}_{-0.101}$ [78], we consider $C = 3.5$ and $\Delta = 0.1$ within BHDE with future event horizon as IR cutoff.

2.1.1 Non-interacting

When there is no energy exchange between the pressureless matter and BHDE, we obtain $Q = 0$ and $\sigma = 0$. By differentiating Eq. (14) and using Eq. (6), one can obtain the equation of state parameter of BHDE

$$w_D = \frac{\Delta - 2}{3}F - \frac{\Delta + 1}{3}, \quad (16)$$

with $F = \frac{1}{HR_h}$ and F' satisfies

$$F' = -\left[\frac{1}{2}(\Omega_m + (1 - 3w_D)\Omega_D) - F - 1\right]F. \quad (17)$$

Using Eqs. (14) and (16), the squared sound speed can be written as

$$v_s^2 = \frac{F'}{3(1-F)} + \frac{\Delta-2}{3}F - \frac{\Delta+1}{3}. \quad (18)$$

Substituting Eq. (17) into the above relation, $v_s^2 > 0$ leads to

$$0 \leq \Omega_D < \frac{4 - \Omega_m}{4}, \frac{\Omega_m + (\Delta + 2)\Omega_D - 4\Delta}{2[(\Delta - 2)\Omega_D + 2(3 - \Delta)]} + \frac{1}{2}\sqrt{\xi} < F < 1,$$

or

$$\frac{4 - \Omega_m}{4} < \Omega_D \leq 1, 1 < F < \frac{\Omega_m + (\Delta + 2)\Omega_D - 4\Delta}{2[(\Delta - 2)\Omega_D + 2(3 - \Delta)]} + \frac{1}{2}\sqrt{\xi},$$

in which

$$\xi = \frac{\Omega_m^2 + [2(\Delta + 2)\Omega_D - 8\Delta]\Omega_m}{[(\Delta - 2)\Omega_D + 2(3 - \Delta)]^2} + \frac{(2 + \Delta)^2\Omega_D^2 - 8(3\Delta + 2)\Omega_D + 16(3 + 2\Delta)}{[(\Delta - 2)\Omega_D + 2(3 - \Delta)]^2}.$$

These results mean the condition $v_s^2 > 0$ cannot be satisfied during the whole evolution of the universe. So, this model is unstable.

Throughout this paper, we choose $\Omega_r^0 = 0.0001$, $\Omega_m^0 = 0.3152$, $\Omega_D^0 = 0.6847$ and $H_0 = 67.36 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [7] as the initial conditions. For the cases of $\Delta = 0.1$, $C = 3.5$, $\Delta = 0.2$, $C = 3.5$ and $\Delta = 0.2$, $C = 4.0$ with the initial conditions, by solving Eqs. (11), (12) and (17) numerically, we get the evolution curves of Ω_D , Ω_m , w_D and q which are plotted in Figs. 1 and 2. And the black dotted line in Figs. 1 and 2 represents the situation of $\Delta = 0$ which corresponds to the standard entropy.

From Fig. 1, we can see $\Omega_D \rightarrow 1$ and $\Omega_m \rightarrow 0$ at the late time, and the universe can be dominated by BHDE at the late time evolution. The left panel of Fig. 2 shows that the BHDE behave as the quintessence and w_D can cross the phantom line. The right panel of Fig. 2 indicates a suitable range for the transition redshift is obtainable. But this model is unstable since $v_s^2 > 0$ cannot be satisfied, and some evolution curves of v_s^2 are depicted in Fig. 3.

2.1.2 Interacting with $Q = H(\alpha\rho_m + \beta\rho_D)$

When the energy exchange $Q = H(\alpha\rho_m + \beta\rho_D)$ between the pressureless matter and BHDE exists, the equation of state parameter w_D has the form

$$w_D = \frac{\Delta-2}{3}F - \frac{\alpha\Omega_m + \beta\Omega_D}{3\Omega_D} - \frac{\Delta+1}{3}. \quad (19)$$

And the squared sound speed v_s^2 is given as

$$v_s^2 = \frac{(\Delta-2)\Omega_D^2 F' - \alpha\Omega_m'\Omega_D + \alpha\Omega_m\Omega_D'}{3(\Delta-2)(1-F)\Omega_D^2} + \frac{\Delta-2}{3}F - \frac{\alpha\Omega_m + \beta\Omega_D}{3\Omega_D} - \frac{\Delta+1}{3}. \quad (20)$$

Here, F' is given in Eq. (17). Since the expression of v_s^2 is too complicated, we cannot get the analytical solution for $v_s^2 > 0$ and then resort to numerical solution.

Using the initial conditions with $\Delta = 0.1$, $C = 3.5$ and solving Eqs. (11), (12) and (17), we obtain the evolution curves for Ω_D , Ω_m , w_D , q and v_s^2 which are plotted in Figs. 4, 5 and 6. The specific case of $\Delta = 0$ is plotted by the black dotted line, and the values of α , β and C are 0.1, 0.1 and 0.8, respectively.

It can be seen from Fig. 4 where Ω_D is larger than 1 and Ω_m is less than 0 for a negative β . And for the positive β , the universe can be dominated by BHDE at the late time. The evolution curves of w_D and q is shown in Fig. 5. The left panel of Fig. 5 shows that BHDE can behave as quintessence and cross the phantom line for a negative α , and the right one shows that the late time acceleration can be achieved and the transition redshift is obtainable. These results mean that $\alpha < 0$ and $\beta > 0$ can be used to describe the evolution of the universe. However, the result from Fig. 6 means this model is unstable since the evolution curve of v_s^2 is not always positive during the whole evolution epoch.

2.1.3 Interacting with $Q = \frac{\lambda}{H}\rho_m\rho_D$

When the energy exchange $Q = \frac{\lambda}{H}\rho_m\rho_D$ between the pressureless matter and BHDE is adopted, w_D and v_s^2 are written as

$$w_D = \frac{\Delta-2}{3}F - \lambda\Omega_m - \frac{\Delta+1}{3}. \quad (21)$$

and

$$v_s^2 = \frac{F'}{3(1-F)} + \frac{\Delta-2}{3}F - \frac{\Delta+1}{3} - \frac{\lambda\Omega_m'}{(\Delta-2)(1-F)} - \lambda\Omega_m. \quad (22)$$

Here, F' is given by Eq. (17).

Then, solving Eqs. (11), (12) and (17) with $\Delta = 0.1$ and $C = 3.5$, we get the evolution curves of Ω_D , Ω_m , w_D , q and v_s^2 which are plotted in Figs. 7, 8 and 9. The black dotted line in Figs. 7, 8 and 9 denotes the specific case of $\Delta = 0$ with $\lambda = 0.1$ and $C = 0.8$. From these figures, one can see that this model can produce suitable results if proper values are chosen, but it is unstable.

Fig. 1 Evolution curves of Ω_D and Ω_m versus redshift parameter $\ln(1+z)$ for BHDEF

Fig. 2 Evolution curves of w_D and q versus redshift parameter $1+z$ for BHDEF

Fig. 3 Evolution curves of v_s^2 versus redshift parameter for BHDEF

Fig. 4 Evolution curves of Ω_D and Ω_m versus redshift parameter $\ln(1+z)$ for IBHDEFA

Fig. 5 Evolution curves of w_D and q versus redshift parameter $1+z$ for IBHDEFA

Fig. 6 Evolution curves of v_s^2 versus redshift parameter for IBHDEFA. The left panel was plotted for $\Delta = 0.1$, $C = 3.5$, while the right one was shown for $\Delta = 0.2$, $C = 4.0$

Fig. 7 Evolution curves of Ω_D and Ω_m versus redshift parameter $\ln(1+z)$ for IBHDEFL

Fig. 8 Evolution curves of w_D and q versus redshift parameter $\ln(1+z)$ for IBHDEFL

2.2 BHDE with Hubble horizon

and

Using the Hubble horizon as IR cutoff, the energy density of BHDE leads to

$$\rho_D = CH^{2-\Delta}. \quad (23)$$

For $\Delta = 2(\delta - 1)$, the energy density of BHDE has the same form as that of Tsallis holographic dark energy (THDE) in mathematics [56]. Different from THDE, the parameter Δ in BHDE is limited to $0 \leq \Delta \leq 1$.

2.2.1 Non-interacting

For the case $Q = 0$, differentiating Eq. (23) and using Eq. (6), the equation of state parameter w_D and the squared sound speed v_s^2 lead to

$$w_D = \frac{(\Delta - 2)(\Omega_m + \Omega_D - 1) - 3\Delta}{3(\Delta - 2)\Omega_D + 6}. \quad (24)$$

$$v_s^2 = \frac{4(\Omega_m + \Omega_D - 1)}{3(\Omega_m + 4\Omega_D - 4)} + \frac{2\Delta(\Omega_m + 4\Omega_D - 4)}{3[(\Delta - 2)\Omega_D + 2]^2}. \quad (25)$$

Using the initial conditions and solving Eqs. (11) and (12), we obtain the evolution curves of Ω_D , Ω_m , w_D , q and v_s^2 which have been plotted in Figs. 10, 11 and 12. In Figs. 10, 11 and 12, the case $\Delta = 0$ is depicted by the black dotted line. It can be seen from above figures that the current acceleration can be realized and the whole evolution of universe can be described by this model. The universe can evolve into the era depicted by the standard Λ CDM model since w_D approach to -1 at the late time, and a suitable range for the transition redshift is obtainable for the large value of Δ . For BHDE with Hubble horizon as IR cutoff, the observable Hubble data favor a large value of Δ which can be seen in the right panel of Fig. 27. Thus, for BHDE with Hubble horizon as IR cutoff, we set $\Delta = 0.8$. The right panel of Fig. 12 which is plotted for $\Delta = 0.8$ shows that this model can describe the evolution of universe. However, this model is also unstable

Fig. 9 Evolution curves of v_s^2 versus redshift parameter for IBHDEFL. The left panel was plotted for $\Delta = 0.1$, $C = 3.5$, while the right one was shown for $\Delta = 0.2$, $C = 4.0$

Fig. 10 Evolution curves of Ω_D and Ω_m versus redshift parameter $\ln(1+z)$ for BHDEH

Fig. 11 Evolution curves of w_D and q versus redshift parameter $1+z$ for BHDEH

Fig. 12 Evolution curves of v_s^2 and density parameters versus redshift parameter for BHDEH. The right panel was plotted for $\Delta = 0.8$

Fig. 13 Evolution curves of Ω_D and Ω_m versus redshift parameter $\ln(1+z)$ for IBHDEHA

Fig. 14 Evolution curves of w_D and q versus redshift parameter $1+z$ for IBHDEHA

Fig. 15 Evolution curves of v_s^2 and density parameters versus redshift parameter for IBHDEHA. The right panel was plotted for $\alpha = -0.7, \beta = 0.2$

Fig. 16 Evolution curves of Ω_D and Ω_m versus redshift parameter $\ln(1+z)$ for IBHDEHL

Fig. 17 Evolution curves of w_D and q versus redshift parameter $1+z$ for IBHDEHL

Fig. 18 Evolution curves of v_s^2 and density parameters versus redshift parameter for IBHDEHL. The right panel was plotted for $\lambda = -0.5$

since the evolution curves of v_s^2 in the left panel of Fig. 12 means $v_s^2 < 0$.

2.2.2 Interacting with $Q = H(\alpha\rho_m + \beta\rho_D)$

For the situation $Q = H(\alpha\rho_m + \beta\rho_D)$, we can write w_D and v_s^2 as follows

$$w_D = \frac{[(\Delta - 2)(\Omega_m + \Omega_D - 1) - 3\Delta]\Omega_D - 2(\alpha\Omega_m + \beta\Omega_D)}{[3(\Delta - 2)\Omega_D + 6]\Omega_D}. \quad (26)$$

and

$$v_s^2 = \frac{4(\Omega_m + \Omega_D - 1) + \frac{2\alpha[(\alpha-3)\Omega_m + \beta\Omega_D]}{(\Delta-2)\Omega_D}}{3[(\alpha+1)\Omega_m + (\beta+4)\Omega_D - 4]} + \frac{2\Delta(\alpha+1)\Omega_m + 4(2\Delta + \beta)(\Omega_D - 1)}{3[(\Delta-2)\Omega_D + 2]^2} \quad (27)$$

In Figs. 13, 14 and 15, the evolution curves of Ω_D , Ω_m , w_D , q and v_s^2 have been plotted with the initial conditions and $\Delta = 0.8$. Here, we have plotted the case $\Delta = 0$ with $\alpha = 0.7$ and $\beta = 0.2$ by the black dotted line. From Fig. 13, we can see that Ω_D is larger than 1 and the value of Ω_m becomes negative at the late time era for the negative β . The left panel of Fig. 15 shows that this model can be stable for the negative α . So, a negative α with a positive β can realize the late time acceleration and ensure this model is stable. Unfortunately, the result from the right panel of Fig. 15 shows that Ω_r dominates the evolution of the universe at the high redshift region $z > 2 \times 10^{11}$ in which a approaches to 0.

2.2.3 Interacting with $Q = \frac{\lambda}{H}\rho_m\rho_D$

For the case $Q = \frac{\lambda}{H}\rho_m\rho_D$, after some tediously calculations, w_D and v_s^2 lead to

$$w_D = \frac{[(\Delta - 2)(\Omega_m + \Omega_D - 1) - 3\Delta]\Omega_D - 6\lambda\Omega_m\Omega_D}{[3(\Delta - 2)\Omega_D + 6]\Omega_D}, \quad (28)$$

and

$$v_s^2 = \frac{(\Delta - 2)\Omega_D + (\Delta - 2 - 6\lambda)\Omega_m + 2 - 4\Delta}{3[(\Delta - 2)\Omega_D + 2]} + \frac{[(\Delta - 2 - 6\lambda)\Omega_m - 4\Delta]\Omega'_D}{3[(\Delta - 2)\Omega_D + 2](3\lambda\Omega_m\Omega_D + \Omega_m + 4\Omega_D - 4)} - \frac{(\Delta - 2 - 6\lambda)\Omega'_m}{3(\Delta - 2)(3\lambda\Omega_m\Omega_D + \Omega_m + 4\Omega_D - 4)}. \quad (29)$$

The evolution curves of Ω_D , Ω_m , w_D , q and v_s^2 are plotted in Figs. 16, 17 and 18 with the initial conditions and $\Delta = 0.8$. The black dotted line in Figs. 16, 17 and 18 represents the case $\Delta = 0$ with $\lambda = 0.5$. From these figures, we can see that

the current acceleration and the whole evolution of universe can be realized in this model, i.e. the universe evolves from the radiation dominated era to the pressureless matter dominated era, and then enters into the BHDE dominated era. Since both w_D and q approach to -1 , BHDE behaves as the cosmological constant at the late time evolution, and the universe will eventually evolve into an epoch described by the standard Λ CDM model. In addition, the left panel of Fig. 18 shows that this model is stable for some negative coupling constant λ , while it is unstable for a positive coupling constant. These results means when the energy transfers from the pressureless matter to BHDE, this model is stable. Thus, this model can be stable under some specific conditions.

3 Dynamical analysis of these stable models with Hubble horizon as IR cutoff

In previous section, we have discussed the cosmic evolution and stability of BHDE models, and it is found that only IBHDEHA and IBHDEHL model can be stable against perturbations under some specific conditions. In the following, we will analyze the dynamical behaviour of these two stable models.

In order to investigate the complete asymptotic behaviour of IBHDEHA and IBHDEHL models, we will use the dynamical system techniques to study the dynamical behaviour of these models. To achieve this goal, following Refs. [63, 87, 89–91], we can obtain the critical points by solving the autonomous system equations

$$\Omega'_m = \Omega'_D = 0. \quad (30)$$

By linearizing the autonomous system equations, we obtain the corresponding first order differential equations. Then, the stability of critical points can be determined by the eigenvalues of the coefficient matrix of the first order differential equations. For a critical point, if all eigenvalues are negative, it is a stable point which represents an attractor. If all eigenvalues are positive, the corresponding point is unstable. If the eigenvalues have different signs, it denotes a saddle point.

3.1 Interacting with $Q = H(\alpha\rho_m + \beta\rho_D)$

In the case for taking the Hubble horizon as IR cutoff, the autonomous system consists of Eqs. (11) and (12). Then, solving $\Omega'_m = \Omega'_D = 0$, we obtain three critical points given in Table 1. The value of Ω_r can be obtained by Eq. (8). For $q < 0$, the corresponding critical point represents an acceleration phase, otherwise it is a deceleration one.

Thus, point A_1 denotes the radiation dominated deceleration epoch, point A_2 represents the pressureless matter dominated deceleration epoch, and point A_3 is an accelera-

Table 1 Critical points and the stability conditions of IBHDEHA

Label	Critical points (Ω_m, Ω_D)	w_D	q	Eigenvalues	Conditions	Points
A_1	$(0, 0)$	$\frac{1-\alpha-\beta-2\Delta}{3}$	1	$(1 + \alpha, 2\Delta)$	$0 < \Delta < 1, -1 < \alpha \leq 1$	Unstable point
A_2	$(1, 0)$	—	$\frac{1-\alpha}{2}$	$(-1 - \alpha, \frac{3-\alpha}{2}\Delta)$	$0 < \Delta < 1, -1 < \alpha \leq 1$	Saddle point
A_3	$(\frac{\beta}{3-\alpha+\beta}, \frac{3-\alpha}{3-\alpha+\beta})$	$\frac{3-\alpha+\beta}{3-\alpha}$	-1	$(-4, -\frac{(3-\alpha)(3-\alpha+\beta)\Delta}{(3-\alpha)\Delta+2\beta})$	$-1 \leq \alpha \leq 1, -1 \leq \beta \leq 0, \frac{2\alpha}{\alpha-3} < \Delta < 1$ or $-1 \leq \alpha \leq 1, 0 < \beta \leq 0, 0 < \Delta < 1$	Stable point

Fig. 19 Phase space trajectories for IBHDEHA. The red points represent the critical points in Table 1 and the black point denotes the current value. Here, we consider $\alpha = -0.7$ and $\beta = 0.2$. The left panel is plotted for the behavior of attractor A_3 with $\Delta = 0.8$, the right panel is plotted for the evolution trajectories with different value of $0 < \Delta < 1$

Fig. 20 Phase space trajectories for IBHDEHA with $\alpha = -0.7$ and $\beta = 0$. The left panel is plotted for the behavior of attractor A_3 with $\Delta = 0.8$, the right panel is plotted for the evolution trajectories with different value of $0 < \Delta < 1$

Fig. 21 Evolution curves of Ω_r for IBHDEHA with $-1 < \beta < 1$. The left panel is plotted for $\Delta = 0.8$ and $\alpha = -0.7$, the right panel is for $\Delta = 0.9$ and $\alpha = -0.6$

tion epoch determined by the value of α and β . For the case $\beta = 0$, point A_3 becomes $(0, 1)$ and $w_D = -1$ is obtained. Thus, A_3 can behave as the cosmological constant Λ dominated acceleration era under the condition $\beta = 0$.

To analyze the stability of these critical points, we linearize this autonomous system and then obtain the eigenvalues of the corresponding critical points. These results are also shown in Table 1. Under the constraints conditions $0 < \Delta < 1$ and $-1 < \alpha \leq 1$, point A_1 is unstable and P_2 is a saddle point, the stability of A_3 is determined by α and β . The stability of these points indicates that the universe stems from the radiation dominated era (A_1) and then evolves into the pressureless matter dominated era (A_2), and eventually

enters into the BHDE dominated late time acceleration epoch (A_3).

Since A_3 is a stable point with $\Omega_m = \frac{\beta}{3-\alpha+\beta}$ and $\Omega_D = \frac{3-\alpha}{3-\alpha+\beta}$, the ratio of energy density $r = \frac{\rho_m}{\rho_D} = \frac{\Omega_m}{\Omega_D}$ can approach a constant, which means the coincidence problem can be solved. An example of this case is shown in Fig. 19. The behavior of attractor A_3 is shown in the left panel of Fig. 19, and the trajectories are shown in the right one.

For the case $\beta = 0$, the universe evolves from the radiation dominated era (A_1) into the pressureless matter dominated era (A_2), and eventually enters the BHDE dominated late time acceleration epoch (A_3) which can behave as the

Table 2 Critical points and the stability conditions of IBHDEHL

Label	Critical points (Ω_m, Ω_D)	w_D	q	Eigenvalues	Conditions	Points
P_1	(0, 0)	$\frac{1}{3}(1 - 2\Delta)$	1	(1, 2Δ)	$0 < \Delta < 1$	Unstable point
P_2	(1, 0)	$-\frac{1}{2}\Delta - \lambda$	$\frac{1}{2}$	$(-1, \frac{3}{2}\Delta)$	$0 < \Delta < 1$	Saddle point
P_3	(0, 1)	-1	-1	$(-4, -3(1 - \lambda))$	$0 < \Delta < 1, \lambda < 1$	Stable point
P_4	$(1 - \frac{1}{\lambda}, \frac{1}{\lambda})$	$-\lambda$	-1	$(-4, -\frac{3\Delta(1-\lambda)}{2(1-\lambda)-\Delta})$	$0 < \Delta < 1, \lambda > 1$	Stable point

Fig. 22 Phase space trajectories for IBHDEHL with $\lambda = -0.5$. The left panel is plotted for the behavior of attractor P_3 with $\Delta = 0.8$, the right panel is plotted for the evolution trajectories with different value of $0 < \Delta < 1$

cosmological constant Λ . The phase space and evolution trajectories of this case have been plotted in Fig. 20.

Although the IBHDEHA model is stable under some specific conditions and can describe the complete evolution epoch of the universe, Ω_r cannot dominate the evolution of the universe in the suitable redshift regions. Some examples are plotted in Fig. 21 and the right panel of Fig. 15. So, this model cannot describe the whole evolution history of the universe.

3.2 Interacting with $Q = \frac{\lambda}{H} \rho_m \rho_D$

Solving $\Omega'_m = \Omega'_D = 0$, we obtain four critical points shown in Table 2. From this table, it can be seen that point P_1 denotes a deceleration epoch dominated by radiation, point P_2 represents a deceleration epoch dominated by the pressureless matter, and point P_3 denotes the BHDE dominated acceleration era. Point P_4 is fully determined by the value of λ . It is easily seen that P_3 can behave as the cosmological constant Λ dominated acceleration era since $w_D = -1$ and $q = -1$.

By linearizing this autonomous system, we obtain the eigenvalues of the corresponding critical points. These results are also shown in Table 2. Under the constraint condition $0 < \Delta < 1$, point P_1 is unstable and P_2 is a saddle point, the stability of P_3 and P_4 are determined by λ . For $\lambda < 1$, P_3 is a stable point, while a stable point P_4 requires $\lambda > 1$. Although P_3 and P_4 both behave as an attractor, they cannot be stable simultaneously.

Thus, for $\lambda < 1$, these results show that the universe evolves from the radiation dominated era (P_1) into the pressureless matter dominated era (P_2), and eventually enters the BHDE dominated late time acceleration epoch (P_3) which can behave as the cosmological constant Λ . Some examples of this situations are plotted in Fig. 22. The left panel describes the behavior of attractor P_3 . And the right panel depicts the phase space trajectories on the $\Omega_m - \Omega_D$ plane in which all trajectories stem from P_1 , approach to P_2 , and eventually converge into P_3 . In this situation, the whole history of the universe can be described by this model.

For $1 < \lambda < 2$, the stability of these points indicates that the universe stems from the radiation dominated era (P_1) and then evolves close to the pressureless matter dominated era (P_2), and eventually enters into the BHDE dominated late time acceleration epoch (P_4) in which the coincidence problem can be solved. An example of this case is shown in Fig. 23. The behavior of attractor P_4 is shown in the left panel of Fig. 23, and the trajectories is shown in the right one. In this case, the coincidence problem is solved.

For $\lambda \geq 2$, the universe will finally evolve into an accelerated epoch dominated by the pressureless matter rather than the BHDE. Although the coincidence problem can be solved in this case, the acceleration in this case is caused by the pressureless matter.

Fig. 23 Phase space trajectories for IBHDEHL with $\lambda = 1.2$. The left panel is plotted for the behavior of attractor P_4 with $\Delta = 0.8$, the right panel is plotted for the evolution trajectories with different value of $0 < \Delta < 1$

Fig. 24 Illustrative examples for the statefinder diagnostic $\{r, s\}$ of IBHDEHL. Present values of $\{r, s\}$ are marked by black dots, while the present values of the Λ CDM model is a fixed point $\{r, s\} = \{1, 0\}$ marked by the red dot. Here, we take $\Delta = 0.8$

4 Statefinder analysis

In previous section, we have discussed the dynamical evolution of the universe in IBHDEHA and IBHDEHL models. We find only IBHDEHL model can explain the current accelerated expansion and describe the whole evolution of the universe. And an attractor behaving as the cosmological constant Λ exists in this model. Then, how to discriminate this model from the standard Λ CDM model becomes another problem. To solve this problem, we use the statefinder diagnostic pairs $\{r, s\}$, which was introduced by Sahni et al. [92], to analyze this model.

The statefinder parameters r and s are defined as [92]

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-\frac{1}{2})}. \quad (31)$$

The statefinder parameter is a geometrical diagnostic since it only depends on a . By differentiating Eq. (9), we can express the statefinder parameters r and s by Ω'_m and Ω'_D . Then, solving Eqs. (11) and (12) numerically, we can obtain the evolution curves of the universe in the parameter space $r-s$.

In Fig. 24, we have plotted an example for the evolution curves of the statefinder diagnostic pairs $\{r, s\}$. In the left panel, we take some positive λ , and use the negative one in the right panel. The black dotted line in Fig. 24 represents **HDE model**. From Fig. 24, one can see that the statefinder pairs

$\{r, s\}$ start from the left side of the Λ CDM fixed point $(0, 1)$, i.e. $s < 0$ and $r > 1$ which is the characteristic of Chaplygin gas, and then enter into the region $s > 0$, $r < 1$ where the trajectories of quintessence and phantom stay in these regions. After passing through these regions, they finally tend to the Λ CDM fixed point $(0, 1)$. It is noticeable that the behaviour of the statefinder pairs $\{r, s\}$ in these model is similar to the quintom dark energy [93] and interacting vacuum energy model [94]. In addition, for $\lambda = -0.5$, the statefinder pairs start from the Chaplygin gas region and then approach to the Λ CDM fixed point. For $\lambda = -0.3$, the present value of $\{r, s\}$ is close to the Λ CDM fixed point, and it is not for the other cases. These trajectories in $r-s$ plane demonstrate the contrasting behavior of this model and the standard Λ CDM model strikingly.

For the sake of complementarity, we have plotted the evolution trajectories for another statefinder diagram $\{r, q\}$ in Fig. 25. From Fig. 25, one can see that the standard Λ model starts from the standard cold dark matter fixed point $(0.5, 1)$, while the evolution curves for IBHDEHL have a smaller deviation from this fixed point. And both models can evolve into the de Sitter expansion fixed point $(-1, 1)$ in the future.

In addition, under the condition $\Omega_r = 0$ and $\Omega_D = 0.73$, the evolution curves of HDE in Figs. 24 and 25 become that in [95].

Fig. 25 Illustrative examples for the statefinder diagnostic $\{r, q\}$ of IBHDEHL. Present values of $\{r, q\}$ are marked by black dots, the de Sitter expansion fixed point marked by the purple dot and the standard cold dark matter fixed point marked by the red dot. Here, we take $\Delta = 0.8$

Table 3 Examples for best fitting values of c in HDE model

Data	c	References
$H(z) + SNIa + CMB + BAO$	$0.88^{+0.21}_{-0.15}$	[32]
Planck + WP + BAO + HST + lensing	0.495 ± 0.039	[37]
$H(z) + SN + CMB + BAO$	$0.7331^{+0.0354}_{-0.0421}$	[45]
$H(z) + SNIa + CMB + BAO + BBN$	$0.785^{+0.042}_{-0.056}$	[46]
Planck + BAO + R19	0.51 ± 0.02	[47]
CMB + BAO + SNE	0.621 ± 0.026	[86]

5 Turning point in Hubble diagram

In HDE, it was found that there exists a turning point in the Hubble diagram $H(z)$, which leads to HDE model conflicting with the cosmological paradigm [86]. This turning point is determined by the constant parameter c in HDE model, and this point occurs in the future for $0.5 \leq c < 1$, while it becomes observable for $c \leq 0.5$. This result was obtained in [86] and it was also shown in the left panel of Fig. 26. When the cosmological observational data are used to constrain the free parameter c in HDE model, the combination of different data provides different best fitting values of c , and some examples are shown in Table 3 where most results indicate that the turning point can be postponed to the future.

Then, in order to discuss the turning point in BHDE models, we have plotted the evolution curves of Hubble parameter H in Figs. 26 and 27. The error bar in Figs. 26 and 27 represent the observational Hubble parameter data [46, 96]. In the right panel of Fig. 26, the evolution curve of H for BHDEF model was plotted with $\Delta = 0.1$. The results show that the turning point moves to lower redshift with the increasing of C , and vanishes when $C \geq 4.45$. The evolution curve of H for BHDEF model with $C = 3.5$ was plotted in the left panel of Fig. 27 which shows that this turning point moves to lower redshift regions with the decreasing of Δ . For $\Delta \leq 0.04$, the turning point disappears. Recently, the combination $H(z) + SNIa$ place constraints on $C = 3.421^{+1.753}_{-1.611}$ and $\Delta = 0.094^{+0.094}_{-0.101}$ [78]. Thus, under the constraints $C = 3.421^{+1.753}_{-1.611}$ and $\Delta = 0.094^{+0.094}_{-0.101}$, the turning point of Hubble diagram in BHDEF is nonexistent.

The evolution curves of H for BHDEH with different Δ are shown in the right panel of Fig. 27. This figure shows that the turning point occurs in the future, and it vanishes with the increasing of Δ . For $\Delta > 0.7$, the turning point can be avoided. And for $\Delta \sim 0.1$, the turning point occurs near $z = -1$ which means $a \rightarrow \infty$. When $\Delta \leq 0.02$, this point disappears. Thus, in the region $\Delta > 0.7$ or $\Delta \leq 0.02$, the turning point of Hubble diagram does not exist in BHDEH.

Recently, considering BHDE with the apparent horizon as IR cutoff (BHDEA), the Big Bang Nucleosynthesis (BBN) place a constraint on the parameter of Barrow entropy $\Delta \leq 1.4 \times 10^{-4}$ in order not to spoil the BBN epoch [97]. Under this constraint, BHDEA model can return to HDE model and the turning point in Hubble diagram cannot be avoided. In BHDEF model, the combination $H(z) + SNIa$ give looser constraints $\Delta = 0.094^{+0.094}_{-0.101}$ and $C = 3.421^{+1.753}_{-1.611}$, which provide us an approach to solve the turning point in Hubble diagram. In BHDEA and BHDEF models, the constraints indicate that a small Δ is favored by the current observations. However, in BHDEH model, the right panel of Fig. 27 indicates the observable Hubble data favor $\Delta > 0.7$, which is different from the values of Δ in BHDEA and BHDEF models. To obtain a best fitting value of the parameters in BHDEA, BHDEF and BHDEH models, one require to combine more observational data to constrain the parameters. It is interesting to compare BHDEA, BHDEF and BHDEH models and combine more current observations to constrain the parameters in these models, and this is what we shall do in the future.

Fig. 26 Evolution curves of H . The left panel was plotted for HDE, while the right one was shown for BHDEF with $\Delta = 0.1$

Fig. 27 Evolution curves of H . The left panel was plotted for BHDEF with $C = 3.5$, while the right one was shown for BHDEH

6 Conclusion

Based on the holographic principle and the Barrow entropy, which results from the modification of the black hole surface due to quantum-gravitational effects, a new HDE model named BHDE has been proposed. In this paper, by considering the future event horizon and the Hubble horizon as IR cutoff, we analyze the evolution and stability of BHDE models with different interaction terms. We find all of these BHDE models can explain the current accelerated expansion, but only IBHDEHA and IBHDEHL can be stable under some specific conditions in which the coupling constant requires to be negative. A negative coupling constant may result in $Q < 0$, which indicates that the pressureless matter is giving energy to BHDE. When we study the evolution of the universe in IBHDEHA and IBHDEHL models, we find only IBHDEHL model can describe the whole evolution of the universe, i.e. the universe stems from the radiation dominated epoch, passes through the pressureless matter dominated epoch, and eventually approaches the dark matter dominated acceleration epoch. Since both w_D and q can approach to -1 at the late time evolution of the universe, the BHDE in IBHDEHL model can behave as the cosmological constant.

Then, we apply the dynamical analysis techniques to IBHDEHA and IBHDEHL models which are stable against perturbations. Although IBHDEHA model can describe the entire evolution epoch of the universe, Ω_r cannot dominate the evolution of the universe in the corresponding redshift regions. Thus, only IBHDEHL model can describe the whole evolution of the universe. In this model, the results show that

there exist three different critical points for $\lambda < 1$, namely the unstable point represents the radiation dominated epoch, the saddle point denotes the pressureless matter dominated epoch, and the stable point corresponds to the BHDE dominated epoch. For the stable point, since both w_D and q equal to -1 , this point can behave as the cosmological constant. The attractor behavior and evolution curves of the stable point show that IBHDEHL model can describe the thermal history of the universe. Thus, IBHDEHL model is a viable cosmological model. In addition, for $1 < \lambda < 2$, the stable point can alleviate the coincidence problem since the ratio of the energy densities can approach to a constant

In order to discriminate IBHDEHL model from the standard Λ CDM model, we apply the statefinder analysis method to IBHDEHL model. The statefinder diagrams show that the evolution curves of IBHDEHL model will eventually approach to the Λ CDM fixed point in $r-s$ phase plane and the de Sitter expansion fixed point in $r-q$ phase plane. Furthermore, the statefinder diagrams indicate that the interaction between the pressureless matter and BHDE has a significant effect on the evolution of the universe, and the IBHDEHL model can be distinguished from the standard Λ CDM model by this method.

Finally, we discuss the turning point of Hubble diagram in BHDE models. For BHDEF model, the turning point is nonexistent under the current observational results. For BHDEH model, this point vanishes for $\Delta > 0.7$, and this point can also be avoided for $\Delta \sim 0.1$ in which the turning point occurs in the region $z \sim -1$ which means $a \rightarrow \infty$. When $\Delta \leq 0.02$, this point disappears. These results indi-

cate that the turning point in Hubble diagram does not exist in BHDE models under the acceptable conditions.

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