Wave packet approach to quantum correlations in neutrino oscillations

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Abstract Quantum correlations provide a fertile testing ground for investigating fundamental aspects of quantum physics in various systems, especially in the case of relativistic (elementary) particle systems as neutrinos. In a recent paper, Ming et al. (Eur Phys J C 80:275, 2020), in connection with results of Daya-Bay and MINOS experiments, have studied the quantumness in neutrino oscillations in the framework of plane-wave approximation. We extend their treatment by adopting the wave packet approach that accounts for effects due to localization and decoherence. This leads to a better agreement with experimental results, in particular for the case of MINOS experiment.

1 Introduction

The study of quantum correlations [1] is a very active research area in view of applications such as quantum communication and computation, and quantum cryptography. They have been studied in a variety of physical contexts, such as quantum optics and condensed matter systems but, more recently, attention has also been directed towards subatomic physics. A particular focus has been concentrated on relativistic systems of neutrinos and mesons [2–14], which are interesting candidates for applications of quantum information beyond photons; investigations in this direction can also provide a possible “feedback” effect allowing better understanding of fundamental physical properties of such particles.

The phenomenon of neutrino oscillations offers a rare example of quantum correlations on macroscopic scale. Neutrino oscillations have been investigated both from a theoretical perspective, and in relation to the available data from sev-

eral experiments, confirming the intrinsic quantum nature of this phenomenon [15]. One of the most important and useful aspects concerning quantum correlations in neutrinos is that they can be expressed in terms of the oscillation probabilities, which are directly obtainable from experiments.

In a recent article [16], Ming et al. have investigated quantum correlations in neutrino oscillations by referring to Daya Bay [17–19], and MINOS experiments [20,21]. They found interesting results by verifying the violation of some specific bounds by quantum markers such as the nonlocal advantage of quantum coherence (NAQC), the steering and the Bell nonlocality. In both the experimental situations they obtained an oscillatory behavior of the markers in function of the ratio L/E between length and energy. It is to be remarked that the Bell nonlocality marker, although oscillating, remains above its bound value, thus implying that the system remains always in a quantum regime. However, the NAQC marker lies alternately above and below its bound value. The authors then conclude that the NAQC criterion is more restrictive, and indicates in some regions a stronger level of coherence with respect to the Bell nonlocality.

The results in [16] have been obtained in the framework of the plane-wave approach which, as well known, does not account for the effects due to localization and decoherence. In Ref. [22], it has been addressed the question under what conditions the plane-wave approach is sufficient to describe adequately the phenomenon of neutrino oscillation, and when, instead, one must resort to the in principle more realistic description provided by the wave-packet approach for neutrino oscillations, as introduced in Refs. [23,24]. The answer is not simple or trivial; in addition, it is even more complicated by the difficulties of precisely determining the values of the associated physical parameters.

In this paper, we extend the study of quantum correlations associated to neutrino oscillations of Ming et al. [16], by adopting the wave packet approach. For simplicity, we

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will limit ourselves to consider only two of the quantum markers considered in Ref. [16], namely NAQC and Bell localization. We find that these quantities keep the same formal expression in terms of oscillation probabilities which now, however, cannot only be expressed in terms of L/E , but depend separately on these quantities. We find that, in the case of Daya Bay experiment, corrections provided by the wave-packet approach with respect the plane-wave one are practically irrelevant, in agreement also to the analysis carried out in [19]. At variance, when MINOS experiment is considered, the corrections are noticeable, and lead to a better description of experimental data.

Moreover, as a byproduct of our analysis an interesting feature of NAQC emerges: at large distances, when oscillations are washed away, the violation of NAQC bound depends solely on the mixing angle, leading to a violation in the case of MINOS at variance with Daya Bay case.

The plan of the paper is as follow: in Sect. 2 we recall the notions and the meaning of NAQC and Bell nonlocality. In Sect. 3 we review the study of Ming et al. carried out in the plane-wave approximation. In Sect. 4, we generalize the study of Sect. 3 within the wave-packet approach to neutrino oscillations and compare our results with those obtained in Ref. [16]. Section 5 is devoted to conclusions and outlook. Two appendices containing some technical issues are also provided.

2 NAQC and Bell nonlocality

In this section we assume a bipartite scenario that can be applied to the case of two-flavor neutrino oscillations, and we briefly review some definitions and properties of NAQC and Bell nonlocality, following Refs. [25–27]. In recent years, interest has been focused on a quantitative characterization of coherence, that expresses the level of quantumness content in a given system. As well discussed in the paper of Ming et al., quantum correlations and effectiveness in detecting coherence and quantumness, are arguments which hide subtle aspects (particularly for mixed states). Coherence arises if in a system made by two subsystems A and B (for example, two qubits), A (B) cannot be considered to have a “own individuality” that is separated by B (A). For example, in the most simple, and well known, case of an entangled bipartite system in a pure state, the information one can obtain from A and B separately is lower than that obtained by the whole system $A \cup B$. Nonlocality can be declined in many, not equivalent, ways. Wiseman et al. found a hierarchy among different nonlocality criteria, i. e. entanglement, steering, and Bell nonlocality, showing the strict inclusion relation: Entanglement $\supset$ Steering $\supset$ Bell nonlocality. The above requirements thus become more and more restrictive in proceeding from left to right. The criterium underlying all nonlocality check

is the verification of the impossibility for whatever classical theory to describe the system of interest and, being the set of correlations admitting hidden variables models a convex set, it is completely described by some set of linear inequalities.

Bell nonlocality: Bell’s theorem, and the connected Bell nonlocality, was originally obtained in the hypothesis of perfect anti-correlation, by exploiting EPR argument [28], and assuming pre-determined values [29]. Later, it has been extended by relaxing the hypothesis of perfect correlation and allowing small deviations and “continuous dependence” on the correlation itself [30]. This last formulation is expressed by the Clauser–Horne–Shimony–Holt (CHSH) inequality.

We consider the subsystems A and B . On each subsystem we carry out experiments. The experiments on A will be indicated with a, a' , and those on B with b, b' . The quantities a, a', b, b' take values in $\{\pm 1\}$. Indicating with $E(a, b)$ etc. the expectation value of the product of the outcomes of the experiment, it is possible to write the CHSH inequality as:

$$B(\rho_{AB}) = |B_{CHSH}| \leq 2, \quad (1)$$

where $B_{CHSH} = E(a, b) + E(a, b') + E(a', b) - E(a', b')$ is the Bell operator.

The maximum violation of CHSH inequality can be written as $B_{max}(\rho_{AB}) = 2\sqrt{M(\rho_{AB})}$, where:

$$M(\rho_{AB}) = \max(u_i + u_j) \leq 1, \quad i \neq j. \quad (2)$$

that is, the maximum of the sum of the eigenvalues u_i ($i = 1, 2, 3$) of the matrix $T^\dagger T$, where $T_{m,n} = \text{Tr}[\rho_{AB}(\sigma_m \otimes \sigma_n)]$ are the elements of a correlation matrix T and $\sigma_{m,n}$, $m, n = 1, 2, 3$, are the Pauli matrices. ρ_{AB} is the density matrix associated with the state of interest. If the inequality is violated, i. e. if the quantity in the left member lies above the value 1, the system cannot be described by any classical theory, and it exhibits Bell nonlocality and the connected level of coherence and quantumness.

Nonlocal advantage of quantum coherence: More recently, Mondal et al. [26] introduced the concept of non-local advantage of quantum coherence (NAQC). Again, from the requirement that the system of interest cannot be described by any classical theory it is obtained a suitable inequality, whose violation detects and measures the coherence and quantumness content. In this case, however, a more direct and clear connection to the concept of quantum coherence is established; in fact, the central quantity is given by the distance (expressed by a suitable l_1 -norm) of a given state by the set of the incoherent states, which are characterized by the absence of off-diagonal elements in their density matrix. At least from the point of view of the general setting, it is worth to go a little bit inside the procedure adopted in [26], referring however to the appendix for a more precise treatment. At first, one considers a single system (for example, a qubit), and defines the l_1 -norm as:

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{i,j}| \quad (3)$$

where ρ is the density matrix of the given state, and the sum is extended to the absolute values of the off diagonal elements. It is evident that, if $\rho_{i,j} = 0 \forall i \neq j$ the state is incoherent while, if this condition is not fulfilled, the state exhibits a certain level of coherence.

If the qubit is prepared in either spin up or spin down state along z-direction, then the qubit is incoherent when we calculate the coherence in z-basis ($C_{l_1}^z = 0$) and it is fully coherent in x- and y-basis ($C_{l_1}^{x(y)} = 1$). One then asks what is the upper bound of $C_{l_1} = C_{l_1}^x + C_{l_1}^y + C_{l_1}^z$. This limit for a general state of a single system is provided by:

$$\sum_{i=x,y,z} C_{l_1}^i(\rho) \leq C_{max}, \quad (4)$$

where one can verify that the upper bound C_{max} is state-independent, and given by $\sqrt{6}$. The equality sign holds for a pure state $\rho_{max} = \frac{1}{2} \left[\frac{1}{\sqrt{3}} (\sigma_x + \sigma_y + \sigma_z) + \mathbf{1} \right]$.

Once this first result for a single *system* has been established, one moves to consider a bipartite system, made of two *subsystems* which are managed by two (space-like separated) participants: Alice (who has available the first subsystem) and Bob (who has available the second one). What is under examination is the ability of Alice of influencing measurements of Bob on the second subsystem without resorting to predetermined classical correlations (this ability is defined as “steering”). To this aim, this “steerability of local coherence” is presented by a game between Alice and Bob. Alice performs local measurements on the first subsystem, and communicates the results to Bob by classical channels. Bob does not trust Alice, and thus must check that she didn’t resort to predetermined classical correlations. Therefore, he verifies if the *average* coherence of conditional state of the second subsystem overcomes the limit of coherence of a single system, Eq. (4), previously established. If this happens, to the second subsystem cannot be attributed a its own identity, and it cannot be considered as a separated entity with respect the first one. In this case, one says that a non local advantage of quantum coherence can be obtained by the conditional state of the second subsystem. More precisely, let us suppose that Alice performs a measurement Π_i^b on the eigenbasis of σ_i on A and obtains the outcome $b = \{0, 1\}$ with probability $p_{\Pi_i^b} = \text{Tr}[(\Pi_i^b \otimes \mathbf{1})\rho_{AB}]$. The measured state for the two-qubit state can be obtained as $\rho_{AB|\Pi_i^b} = (\Pi_i^b \otimes \mathbf{1})\rho_{AB}(\Pi_i^b \otimes \mathbf{1})/p_{\Pi_i^b}$ and the conditional state for qubit B is $\rho_{B|\Pi_i^b} = \text{Tr}_A(\rho_{AB|\Pi_i^b})$. Then Alice tells Bob her measurement choice and Bob has to measure the coherence of qubit B at random in the eigenbases of the other two Pauli matrices σ_j and σ_k .

If Eq. (4) is violated then we cannot have a single-system description of the coherence of subsystem B. The criterion for achieving a NAQC of qubit B can be written as:

$$N_{l_1}(\rho_{AB}) = \frac{1}{2} \sum_{i,j,b} p(\rho_{B|\Pi_i^b}) C_{l_1}^{\sigma_i}(\rho_{B|\Pi_j^b}) > \sqrt{6}. \quad (5)$$

via all possible probabilistic averaging methods. Here $\rho_{B|\Pi_i^b}$ is the conditional state for B after a measurement on the eigenbasis of σ_i on A, $p(\rho_{B|\Pi_i^b})$ is its probability and $C_{l_1}^{\sigma_i}(\rho_{B|\Pi_j^b})$ is the l_1 -norm of coherence of the conditional state for B in the basis of eigenvalues of Pauli observables σ_j and σ_k .

3 Quantum correlations in neutrino oscillations: plane waves

Following Ref. [16] we now study quantum correlations in the plane wave approach in the case of two-flavor oscillations.

The time evolution of the state for two-flavor neutrino oscillations gives us:

$$|v_\alpha(t)\rangle = a_{\alpha\alpha}(t)|v_\alpha\rangle + a_{\alpha\beta}(t)|v_\beta\rangle, \quad (6)$$

with $\alpha, \beta = e, \mu$. From this equation is simple to see that the survival probability to find a neutrino of flavor α after a time t is given by $P_{\alpha\alpha}(t) = |a_{\alpha\alpha}(t)|^2$, while the transition probability is given by $P_{\alpha\beta}(t) = |a_{\alpha\beta}(t)|^2$. First we see that it is possible to rewrite Eqs. (5) and (2) for the NAQC and the Bell nonlocality in terms of neutrino oscillation probabilities (see Appendix A) as:

$$N_{l_1}(\rho_{AB}) = 2 + 2\sqrt{P_{\alpha\alpha}(t)(1 - P_{\alpha\alpha}(t))} > \sqrt{6}, \quad (7)$$

and

$$M(\rho_{AB}) = 1 + 4P_{\alpha\alpha}(t)(1 - P_{\alpha\alpha}(t)) \leq 1. \quad (8)$$

The survival probability is given by:

$$P_{\alpha\alpha}(L) = 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4c\hbar E} \right) \quad (9)$$

where θ is the mixing angle, Δm^2 is the mass-squared difference, E is the neutrino energy and $L = ct$ is the distance between the production and the detection points after a time t .

In Fig. 1 we show the violations of the NAQC and Bell-CHSH inequalities, using the data from the Daya Bay Reactor Neutrino [17–19] and MINOS [20,21] experiments, as reported in Ref. [16].

Fig. 1 NAQC and Bell-CHSH inequalities as a function of the distance. **a** The plot is made using the data from Daya Bay experiment: $\sin^2 2\theta_{13} = 0.084^{+0.005}_{-0.005}$ and $\Delta m_{ee}^2 = 2.42^{+0.10}_{-0.11} \times 10^{-3} \text{ eV}^2$. The value of the energy is $E = 2 \text{ MeV}$. **b** The plot is made using the data from MINOS experiment: $\sin^2 2\theta_{23} = 0.95^{+0.035}_{-0.036}$ and $\Delta m_{32}^2 =$

$2.32^{+0.12}_{-0.08} \times 10^{-3} \text{ eV}^2$. The value of the energy is $E = 0.5 \text{ GeV}$. The L -axis is in logarithmic scale. The magenta and cyan dot-dashed horizontal lines are the bounds of the NAQC and Bell-CHSH inequalities, respectively

Fig. 2 On the left panel is shown the survival transition for an electronic neutrino in the wave packet approach. The plot is done with the following values of parameters: $E = 2 \text{ MeV}$, $\xi = 0$, $\sin^2 2\theta_{13} = 0.084 \pm 0.005$ and $\Delta m_{ee}^2 = 2.42^{+0.10}_{-0.11} \times 10^{-3} \text{ eV}^2$ and

$\sigma_x = 3.3 \times 10^{-6} \text{ m}$. The L -axis is in logarithmic scale. On the right panel is shown the NAQC inequalities for this survival probability. The dot-dashed horizontal line is the bound of the NAQC inequality

Fig. 3 NAQC and Bell-CHSH inequalities as a function of the distance. The plot is made using the data from Daya Bay experiment: $\sin^2 2\theta_{13} = 0.084 \pm 0.005$ and $\Delta m_{ee}^2 = 2.42_{-0.11}^{+0.10} \times 10^{-3} \text{ eV}^2$ and $\sigma_x = 1.25 \times 10^{-6} m$. The value of the energy is $E = 4 \text{ MeV}$. The darker

magenta and the lighter blue dashed horizontal lines are the bounds of the NAQC and Bell-CHSH inequalities, respectively. The solid and dot-dashed lines represent the plot for the wave packet approach and plane waves approximation, respectively

Fig. 4 NAQC inequality as a function of the distance for three different wave packet widths σ_x : $\sigma_x = 5 \times 10^{-6} m$ (green line), $\sigma_x = 2.5 \times 10^{-6} m$ (blue line) and $\sigma_x = 1.7 \times 10^{-6} m$ (orange line). The value of the energy is $E = 2 \text{ MeV}$. The dot-dashed horizontal line is the bound of the NAQC inequality

Note that, while in Ref. [16], the inequalities are plotted as a function of L/E , here we express them as a function of distance L alone. This will be useful for making the comparison with the wave packet treatment of next section.

In Fig. 1 a violation of these inequalities signals quantum coherence. On the left panels we see how we can reach a non local advantage of quantum coherence for certain regular range of distances, strongly dependent on the oscillation probability of the neutrino. On the other hand, on the right panels we observe that the Bell nonlocality is present for all values of the distance L . As highlighted in Ref. [16], this means that NAQC is a stronger quantum correlation than Bell nonlocality.

4 Quantum correlations in neutrino oscillations: wave packets

In this section, we use the wave packet approach to neutrino oscillations to extend the result of the previous section.

In this approach, Eq. (6) becomes:

$$|v_\alpha(x, t)\rangle = \sum_j U_{\alpha j}^* \psi_j(x, t) |v_j\rangle, \quad (10)$$

where $U_{\alpha j}$ denotes the elements of the PMNS mixing matrix. $\psi_j(x, t)$ is the wave function of the mass eigenstate $|v_j\rangle$ with mass m_j :

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}} (2\pi\sigma_p^{P^2})^{-\frac{1}{4}} \times \int dp \exp\left\{-\frac{(p-p_j)^2}{4\sigma_p^{P^2}}\right\} e^{ipx - iE_j(p)t}, \quad (11)$$

where we assume a Gaussian distribution for the momentum of the massive neutrino v_j . From Eq. (11) it is possible to obtain the neutrino oscillation probability in the wave packet approach (see Appendix B).

4.1 Electron neutrino oscillations

In order to compare the plane waves and the wave packet approaches to neutrino oscillation, we start to consider an electronic neutrino at the initial time $t = 0$. In Fig. 2, we plot the electronic neutrino survival probability, given by Eq. (38), together with the NAQC inequality as functions of the distance, in the wave packet approach.

Fig. 5 NAQC and Bell-CHSH inequalities as a function of the distance. The plot is made using the data from MINOS experiment: $\sin^2 2\theta_{23} = 0.95^{+0.035}_{-0.036}$ and $\Delta m_{32}^2 = 2.32^{+0.12}_{-0.08} \times 10^{-3} \text{eV}^2$. The value of the energy is $E = 0.5 \text{GeV}$ and $\sigma_x = 7 \times 10^{-9} \text{m}$. The L -axis is in

logarithmic scale. The darker magenta and the lighter blue dashed horizontal lines are the bounds of the NAQC and Bell-CHSH inequalities, respectively. The solid and dot-dashed lines represent the plot for the wave packet approach and plane wave approximation, respectively

In Fig. 3 we compare the plots of the NAQC and the Bell-CHSH inequalities obtained with the approximation of plane waves and those obtained with the wave packet approach. On the right panel of the figure we observe a violation of the Bell inequality for each value of the distance L . Nevertheless, from a certain distance onwards the violation decreases until it reaches a constant value for large L . Certainly the most interesting behavior is observed on the left panel of the figure. We can see how we can still reach a non local advantage of quantum coherence, but only up to a certain distance. Indeed at great distances we go down the value $\sqrt{6}$ due to the spatial separation of the wave packets. The effects of interference are destroyed by the decoherence due to localization.

Another interesting behavior that emerges from the wave packet treatment is that the amount of coherence depends by the wave packet width σ_x . In Fig. 4 is shown as it increases by σ_x . This behavior is due to the overlapping of the mass eigenstates that increases by σ_x and more coherence is expected [31].

4.2 Muon neutrino oscillations

Now, we consider the case of MINOS experiment, which deals with a muon neutrino at the initial time. In this case, the length and energy scales involved are very different from the case of Daya-Bay experiment. In Fig. 5, using the same parameter values as in Ref. [16], we compare the plots of the NAQC and the Bell-CHSH inequalities obtained with the approximation of plane waves and those obtained with the wave packet approach.

It is evident from Fig. 5 that exists a considerable difference between the two approaches. On the left panel, we see that we reach a non local advantage of quantum coherence for any value above some distance, which does not occur for

the case by plane wave approach. From Fig. 5 of Ref. [16], where also the values computed by exploiting the experimental data (dotted points) are shown, it appears that the present approach based on wave packets describe experimental data better than the plane wave curve: one can in fact observe that experimental points reveal attenuation and saturation on the maximum value at large distance.

For the case of Bell nonlocality, both approaches give curves above the bound, but again the description by the wave packet curve looks better due to the attenuation of the oscillations on the distance scale involved.

In definitive, our results show how in the case in which long spatial extensions and high energies are involved, the wave packet approach turns out to be fundamental for a more realistic description of neutrino oscillations.

It is interesting to comment on the different long-distance behavior of NAQC in the two cases treated above: from Fig. 3 we see that in the Daya Bay case the NAQC lies below the bound in such long distance regime, while for MINOS, Fig. 5, the bound is always violated. This behavior may appear surprising, but can be understood as follows. Let us consider the wave packet oscillation probability in the simplest two-flavor case:

$$P_{\alpha \rightarrow \alpha} = 1 - \frac{1}{2} \sin^2(2\theta) \left\{ 1 - \cos\left(\frac{\Delta m L}{2E}\right) \exp\left[-\left(\frac{L}{L^{coh}}\right)^2\right] - 2\pi^2(1 - \xi)^2 \left(\frac{\sigma_x}{L^{osc}}\right)^2 \right\} \quad (12)$$

For simplicity, we neglect the corrections due to the production and detection processes, setting $\xi = 0$. In the limit of large distance, the exponential term goes to zero and this allows us to approximate the oscillation probability with:

$$P_{\alpha \rightarrow \alpha} = 1 - \frac{1}{2} \sin^2(2\theta) \quad (13)$$

By considering Eq. (7) it is possible, using Eq. (13), to obtain the value of the mixing angle for which the value of the NAQC is exactly $\sqrt{6}$. By solving this simple equation:

$$2 + 2\sqrt{\left(1 - \frac{1}{2} \sin^2(2\theta)\right)\left(\frac{1}{2} \sin^2(2\theta)\right)} = \sqrt{6} \quad (14)$$

or

$$\sin(2\theta)\sqrt{2 - \sin^2(2\theta)} = \sqrt{6} - 2 \quad (15)$$

we find $\sin^2(2\theta) = 0.106$, which corresponds to an angle of about 10° . From these considerations it is easy to understand why the NAQC behaves differently at large distances in the two experiments considered. In fact, in the Daya Bay experiment, the value of the $\sin^2(2\theta)$ is less than that found here and this justifies why at large distances the NAQC lies below the bound. On the other hand, in the MINOS experiment, the value of $\sin^2(2\theta)$ is considerably higher than the value corresponding to bound $\sqrt{6}$ and leads to the high value of NAQC.

5 Conclusions

In this paper we have extended a recent study by Ming et al. [16] on quantum coherence in neutrino oscillations, by adopting a wave-packet approach, in contrast with their treatment based on plane waves. In particular, we have considered two quantum markers there studied, namely the nonlocal advantage of quantum coherence (NAQC) and Bell nonlocality, which in the wave-packet approach exhibit a non-trivial dependence on distance and energy.

We find that, in the case of Daya Bay experiment, the wave packet treatment does not add significant corrections to the result by Ming et al. This is in agreement with the analysis of Ref. [19], where it was shown that plane waves are sufficient to describe rather accurately such short-baseline, low energy, neutrino oscillation experiment. On the other hand, in the MINOS experiment, due to the long baseline and high energies involved, we found a remarkable correction and a better description of experimental data of our treatment with respect to the analysis of Ref. [16]. In particular, we find a NAQC marker constantly above the bound for large distances and, both in the NAQC and Bell nonlocality cases, an attenuation within the length scale of the experiment. This is due to a longer spatial extension and a greater energy of the MINOS with respect to the Daya Bay experiment.

The different long-distance behavior of NAQC is due to the different values of the mixing angles for the two experiments here considered. In fact, we find that the value of the mixing angle for which NAQC reaches the bound $\sqrt{6}$ is $\sin^2(2\theta) = 0.106$: the NAQC for Daya Bay lies below such

bound while for MINOS stays above. It however remains to understand the physical significance of such a phenomenon, namely the violation of this bound, in connection with possible quantum protocols that can be realized or not in the two cases.

We would like to remark one important aspect concerning the neutrino wave packet dispersion σ_x , whose value is not a priori known, as also discussed in Ref. [19]. There a wide range of values for such parameter was indicated, which allows us to agree reasonably well with the experimental values for the quantum markers reported in Ref. [16].

We also remark the subtle behaviour of the correlations above discussed. Indeed, it may appear counter-intuitive that, even after oscillations have been washed away, i.e. at distances much larger than the coherence length, still markers of coherence such as the Bell nonlocality and NAQC may detect high levels of coherence, especially in the case of MINOS experiment. This is due to the difference between static correlations (associated to mixing only) and dynamical correlations (associated to flavor oscillations). These concepts, for the particular case of entanglement, have been discussed in Refs. [4, 5].

We plan to extend our study to the case of three-flavor neutrino oscillations, which could be interesting from a theoretical point of view, due to the presence of the CP-violating phase. Furthermore, a similar approach can be exploited for studying correlations of other particles, as mesons, also taking into account other quantum markers, beyond those here exploited.

Finally, we plan to consider the extension of present work in the framework of the quantum field theory approach to neutrino mixing and oscillations [32–34]. In particular, in Ref. [35], neutrino oscillations have been studied by means of wave packets and relativistic flavor currents, which give a complete characterization of the space-time features of this phenomenon and which should then account for the quantum correlations considered in this paper.

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Appendix A: NAQC and Bell nonlocality criterions in terms of neutrino oscillation probabilities

We consider the state:

$$|\nu_\alpha(t)\rangle = a_{\alpha\alpha}(t)|\nu_\alpha\rangle + a_{\alpha\beta}(t)|\nu_\beta\rangle, \quad (16)$$

with $\alpha, \beta = e, \mu$.

The corresponding density matrix is given by:

$$\rho_{AB}^\alpha(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a_{\alpha\beta}(t)|^2 & a_{\alpha\beta}(t)a_{\alpha\alpha}^*(t) & 0 \\ 0 & a_{\alpha\alpha}(t)a_{\alpha\beta}^*(t) & |a_{\alpha\alpha}(t)|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (17)$$

in the orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

A.1: NAQC

We first see how we can write the NAQC criterion in terms of neutrino oscillation probability. We want to perform Pauli measurement σ_x on qubit A . For this aim, the post measurement states for the initial electron flavor state ρ_{AB}^α are expressed as:

$$\rho_{AB|\sigma_{x_k}}^\alpha(t) = [(|x_k\rangle\langle x_k| \otimes \mathbf{1})\rho_{AB}^\alpha(t)(|x_k\rangle\langle x_k| \otimes \mathbf{1})]/p_{\sigma_{x_k}}, \quad (18)$$

where:

$$p_{\sigma_{x_k}} = \text{Tr}[(|x_k\rangle\langle x_k| \otimes \mathbf{1})\rho_{AB}^\alpha(t)(|x_k\rangle\langle x_k| \otimes \mathbf{1})] \quad (19)$$

Here $|x_k\rangle$ ($k = 1, 2$) are the eigenstates of Pauli observables σ_x .

The conditional state for particle B can be expressed as:

$$\rho_{B|\sigma_{x_k}} = \text{Tr}_A(\rho_{AB|\sigma_{x_k}}^\alpha(t)). \quad (20)$$

The l_1 -norm coherence for the conditional state for B in the basis of eigenvector of Pauli observables σ_y and σ_z can be obtained as:

$$C_{l_1}^{\sigma_{x_k}}(\rho_{B|\sigma_{x_k}}) = |\langle y_1|\rho_{B|\sigma_{x_k}}|y_2\rangle| + |\langle y_2|\rho_{B|\sigma_{x_k}}|y_1\rangle| + |\langle z_1|\rho_{B|\sigma_{x_k}}|z_2\rangle| + |\langle z_2|\rho_{B|\sigma_{x_k}}|z_1\rangle|. \quad (21)$$

We show the explicit calculation for $C_{l_1}^{\sigma_{x_1}}(\rho_{B|\sigma_{x_1}})$.

We remember that:

$$|x_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (22)$$

Then, we have:

$$(|x_1\rangle\langle x_1| \otimes \mathbf{1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad (23)$$

$$\rho_{AB|\sigma_{x_1}}^\alpha(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{\alpha\beta}(t)a_{\alpha\alpha}^*(t) & |a_{\alpha\beta}(t)|^2 & a_{\alpha\beta}(t)a_{\alpha\alpha}^*(t) & |a_{\alpha\beta}(t)|^2 \\ |a_{\alpha\alpha}(t)|^2 & a_{\alpha\alpha}(t)a_{\alpha\beta}^*(t) & |a_{\alpha\alpha}(t)|^2 & a_{\alpha\alpha}(t)a_{\alpha\beta}^*(t) \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (24)$$

where:

$$p_{\sigma_{x_1}} = \frac{1}{2} [|a_{\alpha\beta}(t)|^2 + |a_{\alpha\alpha}(t)|^2] = \frac{1}{2}. \quad (25)$$

From Eq. (24) follows:

$$\rho_{B|\sigma_{x_1}} = \begin{pmatrix} |a_{\alpha\alpha}(t)|^2 & a_{\alpha\alpha}(t)a_{\alpha\beta}^*(t) \\ a_{\alpha\beta}(t)a_{\alpha\alpha}^*(t) & |a_{\alpha\beta}(t)|^2 \end{pmatrix} \quad (26)$$

Remembering that:

$$|y_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, |y_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad (27)$$

$$|z_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |z_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

it is simple to see that:

$$C_{l_1}^{\sigma_{x_1}}(\rho_{B|\sigma_{x_k}}) = 1 + 4\sqrt{P_{\alpha\alpha}(t)(1 - P_{\alpha\alpha}(t))}. \quad (28)$$

where $P_{\alpha\alpha}(t) = |a_{\alpha\alpha}(t)|^2$.

In the same way we calculate $C_{l_1}^{\sigma_{x_2}}(\rho_{B|\sigma_{x_2}})$.

Likewise, the same procedure can be applied on performing Pauli measurement σ_y or σ_z .

After all calculation, we find that:

$$N_{l_1}(\rho_{AB}^\alpha(t)) = 2 + 2\sqrt{(1 - P_{\alpha\alpha}(t))P_{\alpha\alpha}(t)}, \quad (29)$$

A.2: Bell nonlocality

Now we see how to rewrite the Bell nonlocality criterion in terms of neutrino oscillation probability.

We calculate the correlation matrix T whose elements are $T_{m,n} = \text{Tr}[\rho(\sigma_m \otimes \sigma_n)]$, where σ_i , $i = 1, 2, 3$ are the Pauli matrices.

$$T = \begin{pmatrix} a_{\alpha\beta}a_{\alpha\alpha}^* + a_{\alpha\alpha}a_{\alpha\beta}^* & -ia_{\alpha\beta}a_{\alpha\alpha}^* + ia_{\alpha\alpha}a_{\alpha\beta}^* & 0 \\ ia_{\alpha\beta}a_{\alpha\alpha}^* - ia_{\alpha\alpha}a_{\alpha\beta}^* & a_{\alpha\beta}a_{\alpha\alpha}^* + a_{\alpha\alpha}a_{\alpha\beta}^* & 0 \\ 0 & 0 & -|a_{\alpha\alpha}|^2 - |a_{\alpha\beta}|^2 \end{pmatrix} \quad (30)$$

It is simple to calculate:

$$T^\dagger = \begin{pmatrix} a_{\alpha\beta}a_{\alpha\alpha}^* + a_{\alpha\alpha}a_{\alpha\beta}^* & ia_{\alpha\beta}a_{\alpha\alpha}^* - ia_{\alpha\alpha}a_{\alpha\beta}^* & 0 \\ -ia_{\alpha\beta}a_{\alpha\alpha}^* + ia_{\alpha\alpha}a_{\alpha\beta}^* & a_{\alpha\beta}a_{\alpha\alpha}^* + a_{\alpha\alpha}a_{\alpha\beta}^* & 0 \\ 0 & 0 & -|a_{\alpha\alpha}|^2 - |a_{\alpha\beta}|^2 \end{pmatrix} \quad (31)$$

From the matrix product calculation $T^\dagger T$ we found that the eigenvalues of this matrix are:

$$u_1 = (-|a_{\alpha\alpha}|^2 - |a_{\alpha\beta}|^2)^2 = (-P_{\alpha\alpha} - P_{\alpha\beta})^2 = (-1)^2 = 1,$$

$$u_2 = u_3 = 4a_{\alpha\alpha}a_{\alpha\beta}a_{\alpha\alpha}^*a_{\alpha\beta}^* = 4P_{\alpha\alpha}(1 - P_{\alpha\alpha}),$$

where we have used $P_{\alpha\alpha} + P_{\alpha\beta} = 1$.

Appendix B: Wave packet description of neutrino oscillations

In this appendix we briefly review the wave packet approach to neutrino oscillations [23, 24].

Let us consider a neutrino with definite flavor α ($\alpha = e, \mu, \tau$), that propagates along x axis. We can write:

$$|v_\alpha(x, t)\rangle = \sum_j U_{\alpha j}^* \psi_j(x, t) |v_j\rangle, \quad (32)$$

where $U_{\alpha j}$ denotes the elements of the PMNS mixing matrix and $\psi_j(x, t)$ is the wave function of the mass eigenstate $|v_j\rangle$ with mass m_j . If we assume a Gaussian distribution for the momentum of the massive neutrino v_j :

$$\psi_j(p) = (2\pi\sigma_p^2)^{-\frac{1}{4}} \exp\left\{-\frac{(p - p_j)^2}{4\sigma_p^2}\right\} \quad (33)$$

where p_j is the average momentum and σ_p^2 is the momentum uncertainty determined by the production process, the wave function is:

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}} \int dp \psi_j(p) e^{ipx - iE_j(p)t}, \quad (34)$$

where the energy is $E_j(p) = \sqrt{p^2 + m_j^2}$. Now we assume that the Gaussian momentum distribution, Eq. (33), is strongly peaked around p_j , that is, we assume the condition $\sigma_p^2 \ll E_j^2(p_j)/m_j$. This allows us to approximate the energy with:

$$E_j(p) \simeq E_j + v_j(p - p_j), \quad (35)$$

where $E_j = \sqrt{p_j^2 + m_j^2}$ is the average energy and $v_j = \frac{\partial E_j(p)}{\partial p} \Big|_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet of the massive neutrino v_j .

Using these approximations we can perform an integration on p of Eq. (34), obtaining:

$$\psi_j(x, t) = (2\pi\sigma_x^2)^{-\frac{1}{4}} \exp\left[-iE_j t + ip_j x - \frac{(x - v_j t)^2}{4\sigma_x^2}\right] \quad (36)$$

where $\sigma_x^2 = \frac{1}{2\sigma_p^2}$ is the spatial width of the wave packet.

At this point, by substituting Eq. (36) in Eq. (32) it is possible to obtain the density matrix operator by $\rho_\alpha(x, t) = |v_\alpha(x, t)\rangle\langle v_\alpha(x, t)|$ which describes the neutrino oscillations in space and time. Although in laboratory experiments it is

possible to measure neutrino oscillations in time through the measurement of both the production and detection processes, due to the long time exposure in time of the detectors it is convenient to consider an average in time of the density matrix operator. In this way $\rho_\alpha(x)$ is the relevant density matrix operator and it can be obtained by a gaussian time integration

In the case of ultra-relativistic neutrinos, it is useful to consider the following approximations: $E_j \simeq E + \xi_P \frac{m_j^2}{2E}$, where E is the neutrino energy in the limit of zero mass and ξ_P is a dimensionless quantity that depends on the characteristics of the production process, $p_j \simeq E - (1 - \xi_P) \frac{m_j^2}{2E}$ and $v_j \simeq 1 - \frac{m_j^2}{2E^2}$. Considering these approximations, $\rho_\alpha(x)$ becomes:

$$\rho_\alpha(x) = \sum_{j,k} U_{\alpha j}^* U_{\alpha k} \exp\left[-i \frac{\Delta m_{jk}^2 x}{2E} - \left(\frac{\Delta m_{jk}^2 x}{4\sqrt{2}E^2\sigma_x^P}\right)^2 - \left(\xi_P \frac{\Delta m_{jk}^2}{4\sqrt{2}E\sigma_p^P}\right)^2\right] |v_j\rangle\langle v_k|, \quad (37)$$

where $\Delta m_{jk}^2 = m_j^2 - m_k^2$.

Taking into account that the detection process, described by the operator $\mathcal{O}_\beta(x - L)$, takes place at a distance L from the origin of the coordinates, the transition probability is given by:

$$P_{\nu_\alpha \rightarrow \nu_\beta}(L) = \text{Tr}(\rho_\alpha(x) \mathcal{O}_\beta(x - L))$$

$$= \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta j}^* U_{\beta k} \exp\left[-2\pi i \frac{L}{L_{jk}^{osc}} - \left(\frac{L}{L_{jk}^{coh}}\right)^2 - 2\pi^2 (1 - \xi)^2 \left(\frac{\sigma_x}{L_{jk}^{osc}}\right)^2\right], \quad (38)$$

where L_{jk}^{osc} is the oscillation length and L_{jk}^{coh} the coherence length, defined by:

$$L_{jk}^{osc} = \frac{4\pi E}{\Delta m_{jk}^2}, \quad L_{jk}^{coh} = \frac{4\sqrt{2}E^2}{|\Delta m_{jk}^2|} \sigma_x, \quad (39)$$

with $\sigma_x^2 = \sigma_x^{P^2} + \sigma_x^{D^2}$ and $\xi^2 \sigma_x^2 = \xi_P^2 \sigma_x^{P^2} + \xi_D^2 \sigma_x^{D^2}$, where σ^D is the uncertainty of the detection process and ξ_D depends from the characteristics of the detection process.

We note that the wave packet description confirms the standard value of the oscillation length. The coherence length is the distance beyond which the interference of the massive neutrinos v_j and v_k is suppressed. This because the separation of their wave packets when they arrive at the detector is so large that they cannot be absorbed coherently. The last term in the exponential of Eq. (39) implies that the interference of the neutrinos is observable only if the localization of the production and detection processes is smaller than the oscillation length.

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