Magnetized Kerr–Newman–Taub–NUT spacetimes

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Abstract We find a new class of exact solutions in the Einstein–Maxwell theory by employing the Ernst magnetization process to the Kerr–Newman–Taub–NUT spacetimes. We study the solutions and find that they are regular everywhere. We also find the quasilocal conserved quantities for the spacetimes, the corresponding Smarr formula and the first law of thermodynamics.

1 Introduction

Finding the exact solutions to the Einstein–Maxwell theory is always fascinating, as it opens a door to explore the new aspects of the gravitational physics. The exact solutions to the aforementioned theory contain the black hole solutions, such as the Kerr–Newman family, to a more general spacetime solutions of Plebanski–Demianski [1]. Different aspects of those solutions have been studied and reported, in which, some can be related to the real astrophysical phenomena, and others are still in vague. Among the latter, is the spacetime solutions with the NUT parameter, which is considered as the extension of the mass parameter. We note that the conserved quantities, such as the mass and angular momentum in a spacetime with a particular boundary, can be computed, which are related to the symmetry of the spacetime. However, the NUT parameter is not associated to any symmetry of the spacetime, and yet it also leads to some peculiar properties in the spacetime, such as conical singularity and the regular invariants such as squared Riemann tensor, at the origin of the coordinate system. Nevertheless, spacetimes with the NUT parameter has helped to shape our understanding of some gravitational and thermodynamical aspects of gravity theories [2–14]

The Kerr–Newman spacetime is a well known black hole solution in the Einstein–Maxwell theory. The solutions can

be extended to contain the NUT parameter, and usually referred to, as the Kerr–Newman–Taub–NUT spacetimes. Despite the conical singularities in the spacetimes, there are many research works to explore the different aspects of the Kerr–Newman–Taub–NUT black holes. We note the presence of the NUT parameter in the spacetime, leads to the loss of asymptotic flatness, if the corresponding null NUT counterpart has this asymptotic [1]. It can be shown that the Kerr–Newman–Taub–NUT spacetime is a special case of the Plebanski–Demianski spacetime which is considered as one of the most general solution in Einstein–Maxwell theory that can contain black holes [15].

In Einstein–Maxwell theory, it also exists an exact solution describing a universe filled by a homogeneous magnetic field known as the Melvin universe [16]. A black hole solution in this Melvin universe can be obtained by using the Ernst magnetization [17] applied to a known black hole spacetime in Einstein–Maxwell theory as a seed. In fact, performing Ernst magnetization to the Minkowski spacetime can give us the Melvin universe. In general, the magnetization can be done in two ways, namely at the level of perturbation as Wald introduced in [18], and as a strong field as Ernst proposed in [17]. In the Wald prescription, Maxwell field is introduced perturbatively by using the Killing vectors associated to the spacetime, while the presence of homogeneous magnetic field does not change the spacetime solution. We can infer that the magnetization by Wald does not change the asymptotic structure of the magnetized spacetime. The super-radiant instability in this weakly magnetized black hole had been investigated in [19], and this type of magnetization for Kerr–NUT–AdS spacetime had been performed in [20].

This Ernst magnetization itself can be viewed as a type of Harrison transformation [21] which maps an old solution to a new one in the theory. A number of aspects of the known magnetized black hole solutions had been reported in literature [22–38], and this shows the importance of such solution in shaping our knowledge on gravity. The most recent ones are the magnetization to Reissner–Nordstrom–Taub–

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NUT [39] and Kerr–Taub–NUT [40]. The work presented in this paper extends the previous works to magnetizing the Kerr–Newman–Taub–NUT (KNTN) spacetimes, which we refer to as the Melvin–Kerr–Newman–Taub–Nut (MKNTN) spacetimes. Though the idea is straightforward and the mechanism is well understood, but incorporating the functions in the solution are quite challenging.

In this paper, we perform the magnetization procedure to the Kerr–Newman–Taub–NUT spacetimes. We expect to get the magnetized KNTN spacetime solution, whose massless, null NUT, static, and neutral limit, is the Melvin magnetic universe [16]. Some aspects of the spacetime are discussed, such as the deformation of the horizon and the quasilocal conserved quantities associated to the solution.

The organization of this paper is as follows. In Sect. 2, after reviewing the Ernst magnetization process, we construct the MKNTN solutions by employing the Ernst magnetization to the KNTN metric as the seed solution. In Sect. 3, we study some properties of the MKNTN spacetimes. In Sect. 4, we obtain the quasilocal thermodynamical quantities for the MKNTN black holes, as well as the Smarr equation for the MKNTN black holes and verify the first law of thermodynamics. We consider the natural units $c = \hbar = k_B = G_4 = 1$.

2 Construction of the magnetized spacetimes

2.1 Ernst magnetization

Ernst magnetization is a transformation acting on a set of Ernst potentials which can be defined by using some functions appearing in the seed spacetime solution and the accompanying vector field in Einstein–Maxwell theory. The seed solution is typically expressed in the Lewis–Papapetrou–Weyl (LPW) form

$$ds^2 = -f^{-1}(\rho^2 dt^2 - e^{2\gamma} d\chi d\chi^*) + f(\omega dt - d\phi)^2, \quad (2.1)$$

where f , γ , and ω are function of χ . Here we have used the $-++$ signs convention for the spacetime, and $*$ notation representing the complex conjugation. Using the f function in the LPW line element above, accompanied by the vector $\mathbf{A} = A_\mu dx^\mu$, the gravitational Ernst potential,

$$\mathcal{E} = f + \Phi\Phi^* - i\Psi, \quad (2.2)$$

and the electromagnetic one

$$\Phi = A_\phi + i\tilde{A}_\phi, \quad (2.3)$$

can be constructed. The A_t component can be obtained after solving

$$\nabla A_t + \omega \nabla A_\phi + i \frac{\rho}{f} \nabla \tilde{A}_\phi = 0. \quad (2.4)$$

Note that the imaginary part of Φ is the vector field which constructs the dual field strength tensor

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \quad (2.5)$$

where $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$.

In equation above, the twist potential Ψ is given by the relation

$$\nabla \Psi = i \frac{f^2}{\rho} \nabla \omega + 2i \Phi^* \nabla \Phi. \quad (2.6)$$

Using the Ernst potentials, the following equations can be extracted from the equations of motion in Einstein–Maxwell theory,

$$(\mathcal{E} + \mathcal{E}^* + \Phi\Phi^*) \nabla^2 \mathcal{E} = 2(\nabla \mathcal{E} + 2\Phi^* \nabla \Phi) \cdot \nabla \mathcal{E}, \quad (2.7)$$

$$(\mathcal{E} + \mathcal{E}^* + \Phi\Phi^*) \nabla^2 \Phi = 2(\nabla \mathcal{E} + 2\Phi^* \nabla \Phi) \cdot \nabla \Phi. \quad (2.8)$$

The last equation is known as the Ernst equations, and is invariant under some transformation [41]. We note that all the incorporating functions in the metric (2.1) depend on ρ and z only, then the operator ∇ , in Eqs. (2.4), (2.6), (2.7) and (2.8) can be defined in the flat Euclidean space

$$d\chi d\chi^* = d\rho^2 + dz^2, \quad (2.9)$$

as $\nabla = \partial_\rho + i\partial_z$, where we have set the complex coordinate $d\chi = d\rho + idz$. Moreover, as we explain explicitly in Appendix A, we find the following differential equations for the function γ ,

$$\partial_z \gamma = \frac{1}{2f^2 \rho} \left\{ \rho^2 \partial_\rho f \partial_z f - f^4 \partial_\rho \omega \partial_z \omega + 2f \rho^2 (\partial_\rho \Phi \partial_z \Phi^* + \partial_\rho \Phi^* \partial_z \Phi) \right\}, \quad (2.10)$$

and

$$\partial_\rho \gamma = \frac{1}{4f^2 \rho} \left\{ \rho^2 ((\partial_\rho f)^2 - (\partial_z f)^2) + f^4 ((\partial_z \omega)^2 - (\partial_\rho \omega)^2) + 4f \rho^2 (\partial_\rho \Phi \partial_\rho \Phi^* - \partial_z \Phi^* \partial_z \Phi) \right\}. \quad (2.11)$$

According to Ernst, one can magnetized the seed solution described by the line element (2.1) and vector solution $\mathbf{A}$ above by transforming the corresponding Ernst potentials

$$\mathcal{E} \rightarrow \mathcal{E}' = \Lambda^{-1} \mathcal{E} \quad \text{and} \quad \Phi \rightarrow \Phi' = \Lambda^{-1} (\Phi - b\mathcal{E}), \quad (2.12)$$

where

$$\Lambda = 1 - 2b\Phi + b^2\mathcal{E}. \quad (2.13)$$

Here, the constant b is interpreted as the external magnetic field strength in the spacetime.¹ The transformation (2.12) leaves Eqs. (2.7) and (2.8) unchanged for the new potentials $\mathcal{E}'$ and Φ' . In other words, the new metric consisting the functions f' and ω' , together with the new vector potentials A'_t and A'_ϕ are also solutions to the Einstein–Maxwell field equations.

In particular, the transformed line element (2.1) resulting from the magnetization (2.12) has the components

$$f' = \text{Re}\{\mathcal{E}'\} - |\Phi'|^2 = |\Lambda|^{-2} f, \quad (2.14)$$

and

$$\nabla\omega' = |\Lambda|^2 \nabla\omega - \frac{\rho}{f} (\Lambda^* \nabla\Lambda - \Lambda \nabla\Lambda^*), \quad (2.15)$$

while the function γ remains unchanged. In Appendix A, we present an example, which shows the differential equations for the function γ , and the invariance of the function γ under the Ernst magnetization process.

We note that typical black hole solutions in the Einstein–Maxwell theory, are more compact where they are expressed in the Boyer–Lindquist type coordinates $\{t, r, x = \cos\theta, \phi\}$. Consequently, the LPW type metric (2.1) with stationary and axial Killing symmetries will have the metric function that depend on r and x , and the corresponding flat metric line element reads

$$d\chi d\chi^* = \frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x}, \quad (2.16)$$

where $\Delta_r = \Delta_r(r)$ and $\Delta_x = \Delta_x(x)$. Therefore, the corresponding operator ∇ will read $\nabla = \sqrt{\Delta_r}\partial_r + i\sqrt{\Delta_x}\partial_x$. Furthermore we can have $\rho^2 = \Delta_r\Delta_x$, then Eq. (2.4) gives us

$$\partial_r A_t = -\omega\partial_r A_\phi + \frac{\Delta_x}{f}\partial_x \tilde{A}_\phi, \quad (2.17)$$

and

$$\partial_x A_t = -\omega\partial_x A_\phi - \frac{\Delta_r}{f}\partial_r \tilde{A}_\phi. \quad (2.18)$$

The last two equations are useful later in obtaining the A_t component associated to the magnetized spacetime according to (2.12). To end some details on magnetization procedure, another equations which will be required to complete

the metric are

$$\partial_r \omega' = |\Lambda|^2 \partial_r \omega + \frac{\Delta_x}{f} \text{Im}\{\Lambda^* \partial_x \Lambda - \Lambda \partial_x \Lambda^*\}, \quad (2.19)$$

and

$$\partial_x \omega' = |\Lambda|^2 \partial_x \omega - \frac{\Delta_r}{f} \text{Im}\{\Lambda^* \partial_r \Lambda - \Lambda \partial_r \Lambda^*\}. \quad (2.20)$$

In the following section, we employ this magnetization scheme to the Taub–NUT spacetime.

2.2 The Melvin–Kerr–Newman–Taub–NUT spacetimes

To obtain the desired magnetized solution, we use the Ernst potentials that belong to Kerr–Newman–Taub–NUT system,

$$ds^2 = -\frac{\Delta_r}{\Sigma} (dt - (a\Delta_x - 2lx) d\phi)^2 + \Sigma \left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x} \right) + \frac{\Delta_x}{\Sigma} \left(a dt - (a^2 + l^2 + r^2) d\phi \right)^2, \quad (2.21)$$

and

$$A_\mu dx^\mu = \frac{qr}{\rho^2} (dt + (2lx - a\Delta_x) d\phi), \quad (2.22)$$

where $\Delta_r = r^2 - 2mr + a^2 + q^2 - l^2$, $\Delta_x = 1 - x^2$, and $\Sigma = r^2 + (ax + l)^2$. In the form of LPW line element (2.1), the above spacetime metric, associates to the functions

$$f = \Sigma^{-1} \left\{ [3x^2 + 1] l^4 - 4ax\Delta_x l^3 + [a^2 x^4 + (8mr - 8a^2 - 4q^2 - 6r^2)x^2 + 3a^2 + 2r^2] l^2 + 4ax\Delta_x [\Delta_r + l^2] l + \Delta_x [r^4 + r^2 a^2 (1 + x^2) + 2ra^2 m\Delta_x + a^2 (a^2 x^2 + q^2 x^2 - q^2)] \right\}, \quad (2.23)$$

$$\omega = [\Delta_x a (2l^2 + 2mr - q^2) + 2l\Delta_r x] \left\{ [3x^2 + 1] l^4 - 4ax\Delta_x l^3 + [a^2 x^4 + (8mr - 8a^2 - 4q^2 - 6r^2)x^2 + 3a^2 + 2r^2] l^2 + 4ax\Delta_x [\Delta_r + l^2] l + \Delta_x [r^4 + r^2 a^2 (1 + x^2) + 2ra^2 m\Delta_x + a^2 (a^2 x^2 + q^2 x^2 - q^2)] \right\}^{-1}, \quad (2.24)$$

$$e^{2\gamma} = \Delta_x r^4 + [2l^2 - a^2 x^4 - 4alx^3 - 6l^2 x^2 + 4alx + a^2] r^2 + 2m [ax^2 + 2lx - a]^2 r - a^2 [a^2 - l^2 + q^2] x^4 + [a^4 - 8a^2 l^2 + 2a^2 q^2 + 3l^4 - 4l^2 q^2] x^2 - 4al [a^2 - l^2 + q^2] x^3 + 4al [a^2 - l^2 + q^2] x$$

¹ For economical reason, we prefer to express the magnetic parameter as b instead of $B/2$ as appeared in [17]. The relation is $B = 2b$.

$$+3a^2l^2 - a^2q^2 + l^4, \quad (2.25)$$

and $\rho^2 = \Delta_x \Delta_r$.

From this seed solution, one can construct the corresponding Ernst potentials as follows

$$\Phi = \frac{qrx + iq(lx - a)}{l + ax + ir}, \quad (2.26)$$

and

$$\mathcal{E} = \frac{\mathcal{E}_R + i\mathcal{E}_I}{l + ax + ir}, \quad (2.27)$$

where

$$\begin{aligned} \mathcal{E}_R = & -a\Delta_r x^3 - 3l\Delta_r x^2 + a(a^2 - 5l^2 - 6mr + 2q^2 + r^2)x \\ & + l(3a^2 + l^2 - r^2), \end{aligned} \quad (2.28)$$

$$\begin{aligned} \mathcal{E}_I = & (2a^2m - a^2r + 2l^2m - 3l^2r + q^2r - r^3)x^2 \\ & - 2al(m - 2r)x + 2a^2m + a^2r + 3l^2r + r^3. \end{aligned} \quad (2.29)$$

The magnetized Ernst potentials can be obtained from the seed ones above, which yields to the magnetized metric with the new functions f' and ω' , while γ is unchanged.

In fact, the differential equations for the function $\gamma(r, x)$, are given by

$$\begin{aligned} & f^2(2\Delta_r x \partial_r \gamma - \Delta_x \partial_r \Delta_r \partial_x \gamma) \\ & + 2f\rho^2(\partial_r \Phi \partial_x \Phi^* + \partial_r \Phi^* \partial_x \Phi) \\ & - f^2(f^2 \partial_r \omega \partial_x \omega + x \partial_r \Delta_r) + \rho^2 \partial_r f \partial_x f = 0, \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} & 2\rho^2 f^2(\partial_r \Delta_r \partial_r \gamma + 2x \partial_x \gamma) - \rho^2(\Delta_r(\partial_r f)^2 - \Delta_x(\partial_x f)^2) \\ & + f^4(\Delta_r(\partial_r \omega)^2 - \Delta_x(\partial_x \omega)^2) \\ & - 4\rho^2 f(\Delta_r \partial_r \Phi \partial_r \Phi^* - \Delta_x \partial_x \Phi \partial_x \Phi^*) = 0. \end{aligned} \quad (2.31)$$

Similar equations hold for $\gamma'(r, x)$ with $f \rightarrow f'$, $\omega \rightarrow \omega'$ and $\Phi \rightarrow \Phi'$. We explicitly check that Eqs. (2.30) and (2.31) and their counterparts for γ' imply the metric function $\gamma'(r, x)$ is the same as $\gamma(r, x)$, which is given by (2.25).

To summarize the results, the Melvin–Kerr–Newman–NUT black hole is given by

$$\begin{aligned} ds^2 = & -f'^{-1}(r, x) \left\{ \rho^2(r, x) dt^2 - e^{2\gamma(r, x)} \left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x} \right) \right\} \\ & + f'(r, x) (\omega'(r, x) dt - d\phi)^2, \end{aligned} \quad (2.32)$$

together with the Maxwell's field

$$A' = A'_\phi(r, x) d\phi + A'_t(r, x) dt, \quad (2.33)$$

where the metric functions $f'(r, x)$ and $\omega'(r, x)$ are given by

$$f'(r, x) = \frac{\sum_{k=0}^4 c_k(r) x^k}{\sum_{j=0}^6 d_j(r) x^j}, \quad (2.34)$$

and

$$\omega'(r, x) = -\frac{\sum_{k=0}^6 \tilde{c}_k(r) x^k}{\sum_{j=0}^4 c_j(r) x^j}. \quad (2.35)$$

Moreover, the components of the Maxwell's field (2.33) are given by

$$A'_\phi(r, x) = \frac{\sum_{i=0}^6 a_i(r) x^i}{\sum_{i=0}^6 b_i(r) x^i}, \quad (2.36)$$

and

$$A'_t(r, x) = \frac{\sum_{i=0}^{11} e_i(r) x^i}{\sum_{i=0}^{10} f_i(r) x^i}. \quad (2.37)$$

The expressions for $a_i, b_i, \dots, f_i$, as functions of r , are given in Appendix B.

3 Some properties of the Melvin–Kerr–Newman–Taub–NUT spacetimes

The largest root of the metric function Δ_r describes the outer event horizon of the black hole (2.32), which is given by

$$r_H = m + \sqrt{m^2 + l^2 - a^2 - q^2}, \quad (3.1)$$

which implies $m^2 + l^2 \geq a^2 + q^2$, to have a real value for the outer event horizon. The inner event horizon is located at $r_- = m - \sqrt{m^2 + l^2 - a^2 - q^2}$. We note that the inner event horizon exists at a real positive (or zero) value $r_- \geq 0$, if $a^2 + q^2 \geq l^2$. Combining the former and the latter inequalities, we find the following range for the summation of squares of the rotational parameter and the electric charge of the spacetime (2.32).

$$l^2 \leq a^2 + q^2 \leq m^2 + l^2. \quad (3.2)$$

The trace of the energy-momentum tensor for the Maxwell's field (2.33) is identically zero. We have verified exactly that the metric (2.32) with the Maxwell's field (2.33) satisfy exactly all the Einstein–Maxwell field equations. The Ricci scalar of the spacetime is identically zero and the Ricci square invariant $\mathcal{R} = R^{\mu\nu} R_{\mu\nu}$ is regular everywhere including $r = 0$. The expression for the $\mathcal{R}$ is very long and so we

don't present it here. We also find the Kretschmann invariant $\mathcal{K} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{\mathcal{K}_1(r, x)}{\mathcal{K}_2(r, x)}$ where $\mathcal{K}_1(r, x)$ and $\mathcal{K}_2(r, x)$ are two functions with coefficients of the black hole parameters. Though the expression for $\mathcal{K}_1(r, x)$ is very complicated, however

$$\mathcal{K}_2(r, x) = 4f'^2(r, x)\rho^8(r, x)e^{12\gamma(r, x)}. \quad (3.3)$$

The location of curvature/coordinate singularities for the black hole (2.32) can be determined by the equation $\mathcal{K}_2(r, x) = 0$. In fact, beside $\rho(r, x) = 0$, we find the following equation for the location of singularities, which is expectedly independent of the magnetic field

$$\begin{aligned} & 3a^2l^2 - a^2q^2 + l^4 + (4a^3l - 4al^3 + 4alq^2)x \\ & + (a^4 - 8a^2l^2 + 2a^2q^2 + 3l^4 - 4l^2q^2)x^2 \\ & + (-4a^3l + 4al^3 - 4alq^2)x^3 \\ & + (-a^4 + a^2l^2 - a^2q^2)x^4 \\ & + 2m(ax^2 + 2lx - a)^2r \\ & + (-a^2x^4 - 4alx^3 - 6l^2x^2 + 4alx + a^2 + 2l^2)r^2 \\ & + (1 - x^2)r^4 = 0. \end{aligned} \quad (3.4)$$

The event horizon and $x = \pm 1$ are the roots of $\rho(r, x) = 0$, where $\mathcal{K}$ is regular and finite. Moreover, it seems $f'(r, x) = 0$ or Eq. (3.4) leads to other singular points, however, we verify that at those points, the Kretschmann invariant $\mathcal{K}$ remains completely finite.

We notice from Eq. (3.4) and Table 1, that the only magnetized spacetimes with the point singularity, at $r = 0$, are Melvin–Schwarzschild and Melvin–Reissner–Nordstrom spacetimes. All other magnetized spacetimes, i.e. Melvin, Melvin–Kerr, Melvin–NUT, Melvin–Kerr–Newmann, Melvin–Kerr–NUT, Melvin–Reissner–Nordstrom–NUT and MKNTN are completely regular at $r = x = 0$.

In the special case, where all the black hole parameters q , m , a , l approach zero, we find the metric (2.32) reduces to

$$ds_0^2 = \Theta(r, x) \left(-dt^2 + dr^2 + \frac{r^2}{\Delta_x} dx^2 \right) + \frac{r^2 \Delta_x}{\Theta(r, x)} d\phi^2, \quad (3.5)$$

where

$$\Theta(r, x) = (1 + b^2 r^2 \Delta_x)^2. \quad (3.6)$$

The metric (3.5) describes the spacetime of axisymmetric universe filled by parallel magnetic forcelines known as the Melvin magnetic universe [16]. Interestingly, these forcelines

do not contract and collapse under their own gravity, and the corresponding stability against radial perturbations had been investigated in [42]. In fact, the Maxwell's field (2.33) becomes

$$A = -\frac{br^2 \Delta_x}{\sqrt{\Theta(r, x)}} d\phi, \quad (3.7)$$

which generates the magnetic field

$$\vec{B} = \frac{2br^2 x}{\Theta(r, x)} \hat{r} + \frac{2br \Delta_x}{\Theta(r, x)} \hat{\theta}. \quad (3.8)$$

The Ricci scalar of the Melvin space-time (3.5) is identically zero, while the Kretschmann invariant is given by

$$\mathcal{K} = 64b^4 \frac{3b^4 r^4 \Delta_x^2 - 6b^2 r^2 \Delta_x + 5}{\Theta^4(r, x)}. \quad (3.9)$$

Moreover, we find the asymptotic behaviour of the black hole (2.32), where $r \rightarrow \infty$, by analyzing all the metric functions at large values of the radial coordinate. We find the asymptotic expressions for the metric functions, as

$$f' \sim \frac{\epsilon^2}{b^4 \Delta_x} + \mathcal{O}(\epsilon^3), \quad (3.10)$$

$$e^{2\gamma} \sim \frac{\Delta_x}{\epsilon^4} + \mathcal{O}(\epsilon^{-2}), \quad (3.11)$$

$$\rho^2 \sim \frac{\Delta_x}{\epsilon^2} + \mathcal{O}(\epsilon^{-1}), \quad (3.12)$$

$$\omega' \sim \frac{2b^4 l x (x^2 + 3)}{\epsilon^2} + \mathcal{O}(\epsilon^{-1}), \quad (3.13)$$

where $\epsilon \rightarrow 0$.

Using expressions (3.10)–(3.13), we find the asymptotic metric of the black hole (2.32), as

$$ds_{r \rightarrow \infty}^2 = ds_{0, r \rightarrow \infty}^2 + \frac{4lx(x^4 + 2x^2 + 3)}{\Delta_x^2} dt d\phi, \quad (3.14)$$

where $ds_{0, r \rightarrow \infty}^2$ is the asymptotic of the Melvin universe (3.5). We note that the presence of the NUT charge makes an off-diagonal term to the asymptotic metric of the black hole (2.32).

We consider now the set of 3-dimensional surfaces at a fixed value for the radial coordinate $r = \varrho$. The induced metric on the surface $r = \varrho$, is given by

$$\begin{aligned} ds_{r=\varrho}^2 = & -f'^{-1}(\varrho, x) \left(\rho^2(\varrho, x) dt^2 - e^{2\gamma(\varrho, x)} \frac{dx^2}{\Delta_x} \right) \\ & + f'(\varrho, x) (\omega'(\varrho, x) dt - d\phi)^2. \end{aligned} \quad (3.15)$$

The determinant of the metric (3.15) is

$$\det(g_{ij}) = -\frac{f'(\varrho, x)}{e^{2\gamma(\varrho, x)} (\varrho - r_H)(\varrho - r_-)}. \quad (3.16)$$

Table 1 Singularity at $r = 0 = x$

Spacetime	Is $r = 0 = x$ singularity?	M-Spacetime	Is $r = 0 = x$ singularity?
Minkowski	No	Melvin	No
Schwarzschild	Yes	M-Schwarzschild	Yes
Kerr	Yes	M-Kerr	No
NUT	No	M-NUT	No
Reissner–Nordstrom	Yes	M-Reissner–Nordstrom	Yes
Kerr–Newman	Yes	M-Kerr–Newman	No
Kerr–NUT	No	M-Kerr–NUT	No
Reissner–Nordstrom–NUT	No	M-Reissner–Nordstrom–NUT	No
Kerr–Newman–NUT	No	M-Kerr–Newman–NUT	No

From (3.16), we notice that the surface $r = \varrho$ describes a $2 + 1$ -dimensional space-time, if $\varrho > r_H$ or $\varrho < r_-$. On the other hand for $r_- < \varrho < r_H$, the surface $r = \varrho$ describes a 3-dimensional space. Of course for $\varrho = r_-$ or $\varrho = r_H$, the surface becomes a null surface.

We note that due to the presence of the NUT charge as well as the magnetic field, the horizon geometry is a distorted sphere. In fact the two-dimensional horizon is given by the line element

$$ds_H^2 = \frac{f'^{-1}(r_H, x)e^{2\gamma(r_H, x)}}{\Delta_x} dx^2 + f'(r_H, x) d\phi^2. \quad (3.17)$$

The two grand circles on the horizon, one at $x = 0$ and the other passing through $x = \pm 1$, have two different circumferences. The former circumference is given by

$$\begin{aligned} & \frac{C_{x=0}^2}{4\pi^2(3a^2l^2 + 2a^2mr_H - a^2q^2 + a^2r_H^2 + l^4 + 2l^2r_H^2 + r_H^4)} \\ &= 9a^4b^4l^2 + 4a^4b^4m^2 + 4a^4b^4mr_H \\ &+ 6a^2b^4l^4 + 12a^2b^4l^2mr_H + 4a^2b^4mr_H^3 + 2a^2b^4r_H^4 \\ &+ b^4l^6 + 7b^4l^4r_H^2 + 7b^4l^2r_H^4 + b^4r_H^6 + a^4b^4r_H^2 \\ &+ 8a^3b^3mq + 4a^3b^3qr_H + 12ab^3l^2qr_H + 4ab^3qr_H^3 \\ &+ 6a^2b^2l^2 + 4a^2b^2mr_H + 4a^2b^2q^2 + 2a^2b^2r_H^2 \\ &+ 4b^2l^2r^2 - H + 2b^2l^4 + 2b^2r_H^4 + 4abqr_H + l^2 + r_H^2. \end{aligned} \quad (3.18)$$

In Fig. 1, we plot the typical behaviour of the equatorial circumference versus the NUT charge l and the magnetic field b , where we set $a = 6$, $m = 7$ and $q = 1$.

The latter circumference involves an integral, which we can't find it explicitly as an exact form. In Fig. 2, we plot the result of numerical integration for the circumference, as a function of the magnetic field, where we set the other black hole parameters as $a = 6$, $l = 1$, $m = 7$ and $q = 1$.

The other interesting surfaces for the black hole (2.32) are the stationary limit surfaces. The stationary limit surfaces $r = r_s(x)$ are the roots of equation

$$\rho(r_s(x), x) \pm f'(r_s(x), x)\omega'(r_s(x), x) = 0. \quad (3.19)$$

Fig. 1 The equatorial circumference $C_{x=0}$ versus the NUT charge l and the magnetic field b , where we set the other black hole parameters $a = 6$, $m = 7$ and $q = 1$. The horizontal plane shows the equatorial circumference (which is equal to 87.4 in arbitrary unites), where the NUT charge and the magnetic field are zero

Fig. 2 The circumference of the great circle, passing through the north and south pole, versus the magnetic field b , where we set the other black hole parameters $a = 6$, $l = 1$, $m = 7$ and $q = 1$. The circumference with no magnetic field is equal to 71.9 in arbitrary units

Fig. 3 The scaled function $S(r, x)$ versus $r \geq r_H$ and $-1 \leq x \leq 1$, where we set the black hole parameters $a = 6$, $b = 2$, $l = 1$, $m = 7$ and $q = 1$. The stationary limit surfaces $r_s(x)$ are the intersection of the curve $S(r, x)$ with the horizontal plane at zero

For a generic black hole (2.32), the Eq. (3.19) turns out to be

$$S(r_s, x) \equiv \sum_{n=0}^{10} \alpha_n(a, b, l, m, q, x) r_s^n = 0, \quad (3.20)$$

which of course, is not solvable by radicals. Hence, we consider the black hole with the same parameters $a = 6$, $b = 2$, $l = 1$, $m = 7$ and $q = 1$, which we considered before in this section. The inner and outer event horizons are at $r_- = 3.4$ and $r_H = 10.6$, respectively.

In Fig. 3, we plot the function $S(r, x)$ versus $r \geq r_H$ and $-1 \leq x \leq 1$. The outer stationary limit surfaces $r = r_s(x)$ are the intersection of the curve with horizontal plane at 0, where $S(r_s, x) = 0$. The outer ergoregion for the black hole (2.32), is the region between $r = r_H$ and $r = r_s(x)$, where $S(r, x) > 0$.

In Fig. 4, we plot the function $S(r, x)$ versus $r \leq r_-$ and $-1 \leq x \leq 1$. The inner stationary limit surfaces $r_s(x)$ are the intersection of the curve $S(r, x)$ with the horizontal plane at 0, where $S(r_s, x) = 0$. The inner ergoregion for the black hole (2.32), is the region between $r = r_s(x)$ and $r = r_-$, where $S(r, x) > 0$. We also note that for $r_- < r < r_H$, the function $S(r, x)$ is positive everywhere, as shown in Fig. 5.

Now, we consider the rich structure of the electromagnetic fields, on and outside the event horizon. The electromagnetic field components are given by

$$E^i = F^{i\mu} u_\mu, \quad (3.21)$$

$$B^i = \tilde{F}^{i\mu} u_\mu \quad (3.22)$$

where $\tilde{F}$, the Hodge dual of two-form F , is given in (2.5), and u^μ is the 4-velocity of the observer. We find the exact forms for the electromagnetic fields, though their expressions are quite long, and so we do not present them here. In Fig. 6, we plot the typical behaviour of the components of the magnetic

Fig. 4 The scaled function $S(r, x)$ versus $0 \leq r \leq r_H$ and $-1 \leq x \leq 1$, where we set the black hole parameters $a = 6$, $b = 2$, $l = 1$, $m = 7$ and $q = 1$. The stationary limit surface $r_s(x)$ is the intersection of the function $S(r, x)$ with the horizontal plane at zero

Fig. 5 The scaled function $S(r, x)$ versus $r_- \leq r \leq r_H$ and $-1 \leq x \leq 1$, where we set the black hole parameters $a = 6$, $b = 2$, $l = 1$, $m = 7$ and $q = 1$

field, outside the event horizon, for a black hole with parameters $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$. Moreover in Fig. 7, we plot the typical behaviour of the components of the electric field, outside the event horizon, for the same black hole parameters.

In Figs. 8 and 9, we plot the behaviour of the electromagnetic fields on the event horizon. We notice the minimum and maximum of B_r and E_x appear quite away from the equatorial plane, however the maximum of B_x and the minimum of E_r occurs almost on the equatorial plane.

We also plot the polar electromagnetic fields outside the event horizon in Figs. 10 and 11.

We should notice that increasing behaviour in the polar electromagnetic fields doesn't extend to large values of the radial coordinate. In fact, we find the asymptotic behaviour of the electromagnetic fields for $r \rightarrow \infty$, as

Fig. 6 The r (left) and x (right) components of the magnetic field, versus r and x , for a black hole which the event horizon is located at $r_+ = 10.6$. We set the black hole parameters as $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$

Fig. 7 The r (left) and x (right) components of the electric field, versus r and x , for a black hole which the event horizon is located at $r_+ = 10.6$. We set the black hole parameters as $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$

Fig. 8 The r (left) and x (right) components of the magnetic field, on the event horizon, versus x , for a black hole which the event horizon is located at $r_+ = 10.6$. We set the black hole parameters as $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$

$$B_r = \mathcal{O}(r^{-2}), \quad B_x = \mathcal{O}(r^{-3}), \quad E_r = \mathcal{O}(r^{-8}), \quad E_x = \mathcal{O}(r^{-9}), \quad (3.23)$$

for an observer with the 4-velocity $u_\mu = (1, 0, 0, 0)$.

4 Thermodynamics of the Melvin–Kerr–Newman–Taub–NUT spacetimes

In this section, we discuss the thermodynamical quantities for the black hole (2.32) and then construct the mass of the black hole according to the Smarr relation.

We should emphasize that using the term “black hole” for the metric (2.32), with the NUT charge parameter l must

Fig. 9 The r (left) and x (right) components of the electric field, on the event horizon, versus x , for a black hole which the event horizon is located at $r_+ = 10.6$. We set the black hole parameters as $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$

Fig. 10 The r (left) and x (right) components of the polar magnetic field, outside the event horizon, versus r , for a black hole which the event horizon is located at $r_+ = 10.6$. We set the black hole parameters as $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$. In the left figure, the down curve is for $x = 1$ and the up curve is for $x = -1$. In the right figure, the down curve is for $x = -1$ and the up curve is for $x = 1$

Fig. 11 The r (left) and x (right) components of the polar electric field, outside the event horizon, versus r , for a black hole which the event horizon is located at $r_+ = 10.6$. We set the black hole parameters as $a = 6$, $q = 1$, $m = 7$, $l = 1$ and $b = 2$. In both figures, the down curve is for $x = 1$ and the up curve is for $x = -1$

be taken with some cautions, to avoid contradictions with the black hole uniqueness theorems. The event horizon is a global concept and, therefore, requires asymptotic flatness to be well defined. On the other hand, the NUT charge parameter l makes the spacetime (2.32) asymptotically locally flat, and so violates the global flatness condition. Hence, in general, a global event horizon does not exist. However, since the spacetime (2.32) possesses many physical quantities, similar to a black hole (such as event horizons, . . .), we often refer to the spacetime (2.32), as a black hole.²

The surface gravity for the black hole (2.32) is given by

$$\kappa = \sqrt{-\frac{1}{2} \nabla_\nu \xi_\mu \nabla^\nu \xi^\mu}, \quad (4.1)$$

where ξ^μ is the Killing vector $\xi^\mu = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$ and Ω_H is the angular velocity of the horizon, which is given by

$$\Omega_H = \left(\frac{\sum_{i=0}^4 o_i b^i}{\Xi} \right) \Big|_{r=r_+}, \quad (4.2)$$

where

$$\begin{aligned} o_0 &= 2am^2 + 2al^2 - aq^2 + 2am(r-m), \\ o_1 &= (-16l^2qm + 12q^3m - 16qm^3 + 8a^2mq - 8l^2q(r-m) \\ &\quad - 16m^2q(r-m) + 4q^3(r-m)), \\ o_2 &= -12am^2q^2 - 12(r-m)amq^2, \\ o_3 &= (-32a^2l^2mq + 16l^4qm - 80q^3m^3 + 20q^5m \\ &\quad + 64qm^5 - 64a^2m^3q + 24a^2mq^3 + 80l^2m^3q \\ &\quad - 40l^2mq^3 + (r-m)(64qm^4 - 8l^2q^3 - 48m^2q^3 \\ &\quad - 32qa^2m^2 + 48l^2m^2q + 4q^5)), \\ o_4 &= (-88a^3l^2m^2 + 16a^3l^2q^2 - 32a^3l^4 - 8al^6 - 4al^4q^2 \\ &\quad - 104am^2q^2l^2 - 80a^3m^4 + 96am^6 \\ &\quad + 28a^3m^2q^2 + 64am^2l^4 + 168am^4l^2 + 4al^2q^4 \\ &\quad - 120am^4q^2 + 30am^2q^4 \\ &\quad + (r-m)(-32al^2mq^2 - 32a^3m^3 + 96am^5 - 32a^3l^2m \\ &\quad + 16al^4m + 120al^2m^3 - 72am^3q^2 + 6amq^4)), \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \Xi &= 8(r-m)(8l^2m + 8m^3 - 4mq^2 \\ &\quad - 4a^2m^2 + 4l^4 + 12l^2m^2 - 4l^2q^2 + 8m^4 \\ &\quad - 8m^2q^2 + q^4). \end{aligned} \quad (4.4)$$

We note that for $b = 0$, we recover the angular velocity for the Kerr–Newman–NUT black hole [31]. In Figs. 12 and 13,

² We would like to thank anonymous referee for the above-mentioned comment.

we plot the angular velocity of the horizon versus different black hole parameters.

We find the surface gravity of the black hole

$$\kappa = \frac{1}{2r_+} \frac{r_+^2 + l^2 - a^2 - q^2}{r_+^2 + l^2 + a^2}, \quad (4.5)$$

is the same as the Kerr–Newman–NUT black hole, and so the Hawking temperature is

$$T_H = \frac{\kappa}{2\pi}. \quad (4.6)$$

The Coulomb potential on the horizon is given by

$$\Phi_H = -(A'_\mu \xi^\mu)|_{r=r_+} = \left(-A'_t - \frac{\Omega_H}{|\Lambda_0|^2} A'_\phi \right) \Big|_{r=r_+} - \Phi_H^{(0)}, \quad (4.7)$$

where we add a constant term $\Phi_H^{(0)}$ to the potential to make it regular at $r \rightarrow \infty$. The expression for the Φ_H is long and so we do not present it explicitly here. The area of the horizon is given by

$$\mathcal{A}_H = \mathcal{A}_0 |\Lambda_0|^2, \quad (4.8)$$

where $\mathcal{A}_0 = 4\pi(2l^2 + 2mr_+ - q^2)$ is the horizon area for the Kerr–Newman–NUT black hole. To find the electric charge of the black hole (2.32), we use the well-known equation [31]

$$Q = \frac{1}{4\pi} \int_\Sigma E_\perp, \quad (4.9)$$

where Σ is the two-dimensional hypersurface, parameterized with (θ, ϕ) in (2.32) and $E_\perp$ is the normal component of the electric field on Σ . We find

$$Q = \left(\frac{\sum_{i=0}^6 q_i b^i}{\sum_{i=0}^4 s_i b^i} \right) \Big|_{r=r_+}, \quad (4.10)$$

where

$$\begin{aligned} q_0 &= -a^4q - 2a^2qr^2 + l^4q - qr^4, \\ q_1 &= -4a^5m - 8a^3mr^2 + 4al^4m - 16al^4r \\ &\quad - 16al^2mr^2 + 8al^2q^2r - 4amr^4, \\ q_2 &= 4a^4l^2q - 5a^4q^3 - 40a^2l^4q - 52a^2l^2mqr \\ &\quad + 20a^2l^2q^3 - 16a^2l^2qr^2 - 10a^2q^3r^2 + 4l^6q \\ &\quad - 4l^4mqr + l^4q^3 + 16l^4qr^2 + 28l^2mqr^3 - 20l^2q^3r^2 \\ &\quad - 20l^2qr^4 - 5q^3r^4, \\ q_3 &= 32a^5l^2m - 40a^5mq^2 - 32a^3l^4m \\ &\quad + 64a^3l^4r - 64a^3l^2m^2r - 16a^3l^2mq^2 \\ &\quad + 64a^3l^2mr^2 - 96a^3l^2q^2r - 80a^3mq^2r^2 - 64al^6r \\ &\quad - 8al^4mq^2 - 160al^4mr^2 + 128al^4q^2r + 64al^4r^3 \end{aligned}$$

Fig. 12 The angular velocity of the horizon as function of a , b (left) and l , b (right), where we set the other black hole parameters to a set of fixed numbers

Fig. 13 The angular velocity of the horizon as function of q , b (left) and m , l (right), where we set the other black hole parameters to a set of fixed numbers

$$\begin{aligned}
 & -64al^2m^2r^3 + 144al^2mq^2r^2 + 32al^2mr^4 - 80al^2q^4r \\
 & -96al^2q^2r^3 - 40amq^2r^4, \\
 q_4 = & -80a^6m^2q + 16a^4l^4q - 116a^4l^2m^2q - 208a^4l^2mqr \\
 & -24a^4l^2q^3 + 48a^4l^2qr^2 - 160a^4m^2qr^2 \\
 & + 5a^4q^5 - 48a^2l^4m^2q - 128a^2l^4mqr \\
 & + 80a^2l^4q^3 - 96a^2l^4qr^2 - 8a^2l^2m^2qr^2 \\
 & + 40a^2l^2mq^3r - 352a^2l^2mqr^3 - 40a^2l^2q^5 \\
 & + 64a^2l^2q^3r^2 + 96a^2l^2qr^4 - 80a^2m^2qr^4 \\
 & + 10a^2q^5r^2 - 16l^8q - 12l^6m^2q - 48l^6mqr + 8l^6q^3 \\
 & + 80l^6qr^2 - 32l^4m^2qr^2 + 8l^4mq^3r + 192l^4mqr^3 \\
 & + 3l^4q^5 - 96l^4q^3r^2 - 112l^4qr^4 + 108l^2m^2qr^4 \\
 & - 120l^2mq^3r^3 - 144l^2mqr^5 + 40l^2q^5r^2 + 88l^2q^3r^4 \\
 & + 48l^2qr^6 + 5q^5r^4, \\
 q_5 = & -64a^7m^3 - 64a^5l^4m - 112a^5l^2m^3 - 192a^5l^2m^2r \\
 & - 32a^5l^2mq^2 + 64a^5l^2mr^2 - 128a^5m^3r^2 \\
 & + 12a^5mq^4 - 128a^3l^6m - 256a^3l^6r - 64a^3l^4m^3 \\
 & - 384a^3l^4m^2r + 224a^3l^4mq^2 - 384a^3l^4mr^2 \\
 & + 64a^3l^4q^2r - 224a^3l^2m^3r^2 + 256a^3l^2m^2q^2r \\
 & - 384a^3l^2m^2r^3 - 80a^3l^2mq^4 + 64a^3l^2mq^2r^2 \\
 & + 128a^3l^2mr^4 + 32a^3l^2q^4r - 64a^3m^3r^4 \\
 & + 24a^3mq^4r^2 - 64al^8m - 16al^6m^3 \\
 & - 128al^6m^2r + 64al^6mq^2
 \end{aligned}$$

$$\begin{aligned}
 & + 64al^6mr^2 + 64al^6q^2r - 64al^4m^3r^2 + 32al^4m^2q^2r \\
 & + 128al^4m^2r^3 + 4al^4mq^4 + 96al^4mq^2r^2 \\
 & - 64al^4mr^4 - 48al^4q^4r - 64al^4q^2r^3 \\
 & + 144al^2m^3r^4 - 128al^2m^2q^2r^3 - 192al^2m^2r^5 \\
 & + 32al^2mq^4r^2 + 96al^2mq^2r^4 + 64al^2mr^6 \\
 & + 8al^2q^6r + 32al^2q^4r^3 + 12amq^4r^4,
 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
 q_6 = & -64a^6l^2m^2q + 16a^6m^2q^3 - 64a^4l^6q \\
 & - 16a^4l^4m^2q - 320a^4l^4mqr + 48a^4l^4q^3 \\
 & - 64a^4l^4qr^2 + 64a^4l^2m^3qr + 4a^4l^2m^2q^3 \\
 & - 256a^4l^2m^2qr^2 + 144a^4l^2mq^3r \\
 & - 12a^4l^2q^5 + 16a^4l^2q^3r^2 + 32a^4m^2q^3r^2 \\
 & + a^4q^7 + 128a^2l^8q - 32a^2l^6m^2q + 448a^2l^6mqr \\
 & - 192a^2l^6q^3 - 128a^2l^6qr^2 - 16a^2l^4m^3qr \\
 & + 32a^2l^4m^2q^3 + 704a^2l^4m^2qr^2 - 512a^2l^4mq^3r \\
 & - 320a^2l^4mqr^3 + 88a^2l^4q^5 + 32a^2l^4q^3r^2 \\
 & + 320a^2l^2m^3qr^3 - 344a^2l^2m^2q^3r^2 - 192a^2l^2m^2qr^4 \\
 & + 140a^2l^2mq^5r + 96a^2l^2mq^3r^3 - 12a^2l^2q^7 \\
 & + 16a^2l^2q^5r^2 + 32a^2l^2q^3r^4 + 16a^2m^2q^3r^4 \\
 & + 2a^2q^7r^2 - 64l^{10}q - 16l^8m^2q - 256l^8mqr
 \end{aligned}$$

$$\begin{aligned}
& +144l^8q^3 + 192l^8qr^2 - 16l^6m^3qr + 12l^6m^2q^3 \\
& - 320l^6m^2qr^2 + 368l^6mq^3r + 512l^6mqr^3 \\
& - 108l^6q^5 - 272l^6q^3r^2 - 192l^6qr^4 - 144l^4m^3qr^3 \\
& + 240l^4m^2q^3r^2 + 336l^4m^2qr^4 - 132l^4mq^5r \\
& - 320l^4mq^3r^3 - 256l^4mqr^5 \\
& + 27l^4q^7 + 80l^4q^5r^2 + 112l^4q^3r^4 + 64l^4qr^6 \\
& + 36l^2m^2q^3r^4 - 36l^2mq^5r^3 - 48l^2mq^3r^5 + 12l^2q^7r^2 \\
& + 28l^2q^5r^4 + 16l^2q^3r^6 + q^7r^4.
\end{aligned} \quad (4.12)$$

Moreover, the functions s_i , $i = 0, \dots, 4$ are given by

$$\begin{aligned}
s_0 &= (a^2 + 2al + l^2 + r^2)(a^2 - 2al + l^2 + r^2), \\
s_1 &= 8(a^2 + 2al + l^2 + r^2)lqr, \\
s_2 &= (a^2 + 2al + l^2 + r^2) \\
&\quad (-8a^2l^2 + 6a^2q^2 + 12alq^2 + 8l^4 + 16l^2mr - 2l^2q^2 \\
&\quad - 8l^2r^2 + 6q^2r^2), \\
s_3 &= (a^2 + 2al + l^2 + r^2) \\
&\quad (16a^3mq + 24a^2lmq - 16a^2lqr \\
&\quad + 16al^2mq + 16amqr^2 + 8l^3mq \\
&\quad + 16l^3qr + 24lmqr^2 - 8lq^3r - 16lqr^3), \\
s_4 &= (a^2 + 2al + l^2 + r^2) \\
&\quad (16a^4m^2 + 16a^3lm^2 - 32a^3lmr + 16a^2l^4 \\
&\quad + 20a^2l^2m^2 + 16a^2l^2mr \\
&\quad - 8a^2l^2q^2 + 16a^2l^2r^2 + 16a^2m^2r^2 \\
&\quad + a^2q^4 + 32al^5 + 8al^3m^2 + 64al^3mr - 32al^3q^2 - 32al^3r^2 \\
&\quad + 48alm^2r^2 - 32almq^2r - 32almr^3 + 6alq^4 + 16l^6 \\
&\quad + 4m^2l^4 + 48l^4mr - 24l^4q^2 - 32l^4r^2 \\
&\quad + 36l^2m^2r^2 - 32l^2mq^2r - 48l^2mr^3 \\
&\quad + 9l^2q^4 + 24l^2q^2r^2 + 16l^2r^4 + q^4r^2).
\end{aligned} \quad (4.13)$$

In the limit of $l \rightarrow 0$, we recover exactly the results of [31] for the electric charge. In Figs. 14 and 15 we plot the electric charge (4.10) of the black hole, versus different black hole parameters.

We then find the angular momentum J of the black hole (2.32), according to [31]

$$J = \frac{1}{16\pi} \int_{\Sigma} *d\mathcal{P}, \quad (4.14)$$

where $\mathcal{P}^\mu = \frac{\partial}{\partial \phi}$ is the Killing vector in ϕ -direction. We find the integrand in (4.14) is given by

$$(*d\mathcal{P})_{x\phi} = -\frac{f'^2(\frac{\partial}{\partial r}w')}{2\rho} \sqrt{\frac{\Delta_r}{\Delta_x}}. \quad (4.15)$$

A straightforward and lengthy calculation shows that we get

$$J = \frac{\sum_{i=0}^{12} j_i(a, l, m, q)b^i}{\sum_{i=0}^8 k_i(a, l, m, q)b^i}, \quad (4.16)$$

where j_i and k_i depend on black hole parameters a, l, m and q . The expressions for j_i and k_i are very long to present, so we don't present them here. We verify that the expression (4.16) recovers exactly the results of [31] for Melvin–Kerr–Newman black holes, where the NUT charge l goes to zero. In Figs. 16 and 17 we plot the angular momentum (4.16) of the black hole, versus different black hole parameters.

Inspired by the dimensions of physical quantities for the black hole, as well as the presence of the NUT charge [43], we consider the total mass of the black hole (2.32) as

$$\mathcal{M} = \Phi_H Q + 2\Omega_H \mathcal{J} + \frac{\kappa}{4\pi} \mathcal{A}_H + 2\psi_N N_N + 2\psi_S N_S, \quad (4.17)$$

where N_N and ψ_N (N_S and ψ_S) are the thermodynamic charge and the potential, due to the presence of the NUT charge on north (south) pole, and Φ_H , Q , Ω_H , κ and $\mathcal{A}_H$ are given by Eqs. (4.7), (4.10), (4.2), (4.5) and (4.8), respectively. The total angular momentum $\mathcal{J}$ in Eq. (4.17) is

$$\mathcal{J} = J + J_N + J_S, \quad (4.18)$$

where J is given by (4.16), and J_N and J_S are contributions to the angular momentum from NUT charges on the north and south poles. The total mass for the black hole (2.32) is given by

$$\mathcal{M} = -\frac{1}{8\pi} \int_{r \rightarrow \infty} *d\mathcal{T}, \quad (4.19)$$

where $\mathcal{T}^\mu = \frac{\partial}{\partial t}$ is the time-like Killing vector. Evaluating the integral in Eq. (4.19), where $b = 0$, we find

$$\mathcal{M} = m. \quad (4.20)$$

Moreover, the thermodynamic potentials ψ_N and ψ_S are given by [43]

$$\psi_N = \frac{1}{4\pi} \sqrt{\frac{1}{2} \nabla_\mu \xi_{Nv} \nabla^\mu \xi_N^v}, \quad (4.21)$$

$$\psi_S = \frac{1}{4\pi} \sqrt{\frac{1}{2} \nabla_\mu \xi_{Sv} \nabla^\mu \xi_S^v}, \quad (4.22)$$

where ξ_N^μ is the Killing vector

$$\xi_N^\mu = \frac{\partial}{\partial t} + \Omega_N \frac{\partial}{\partial \phi}, \quad (4.23)$$

Fig. 14 The electric charge of the black hole as function of a , b (left) and l , b (right), where we set the other black hole parameters to a set of fixed numbers

Fig. 15 The electric charge of the black hole as function of q , b (left), and m , l (right), where we set the other black hole parameters to a set of fixed numbers

Fig. 16 The angular momentum of the black hole as function of a , b (left) and l , b (right), where we set the other black hole parameters to a set of fixed numbers

which generates the north pole Killing horizon, due to the presence of the NUT charge. The other Killing vector ξ_S^μ is

$$\xi_S^\mu = \frac{\partial}{\partial t} + \Omega_S \frac{\partial}{\partial \phi}, \quad (4.24)$$

which generates the south pole Killing horizon, due to the presence of the NUT charge. The functions Ω_N and Ω_S are given by

$$\Omega_N = -\frac{1}{2l} - 3\frac{q^2}{l}b^2 - \frac{8mq(2a-l)}{2l}b^3$$

$$\begin{aligned} \Omega_S = & -\frac{16a^2(l^2+m^2)+4l^2(al+m^2)-8alm^2+6alq^2+q^4}{2l}b^4, \\ & +\frac{1}{2l}+3\frac{q^2}{l}b^2+\frac{8mq(2a+l)}{2l}b^3 \\ & +\frac{16a^2(l^2+m^2)-4l^2(al-m^2)+8alm^2-6alq^2+q^4}{2l}b^4. \end{aligned} \quad (4.25)$$

Using the form of the Killing vectors on the north and south pole horizons, we find

$$\psi_N = \lim_{x \rightarrow 1, r \rightarrow \infty, \Omega \rightarrow \Omega_N} \Psi(r, x, \Omega), \quad (4.26)$$

Fig. 17 The angular momentum of the black hole as function of q , b (left) and m , l (right), where we set the other black hole parameters to a set of fixed numbers

and

$$\psi_S = \lim_{x \rightarrow -1, r \rightarrow \infty, \Omega \rightarrow \Omega_S} \Psi(r, x, \Omega), \quad (4.27)$$

where $\Psi(r, x, \Omega)$ is given by

$$\begin{aligned} & 64\pi^2 f'^2 e^{2\gamma} \rho^2 \Psi^2(r, x, \Omega) \\ &= -\Omega^2 \Delta_x \left(\frac{\partial}{\partial x} \omega' \right)^2 f'^6 - \Omega^2 \Delta_r f'^6 \left(\frac{\partial}{\partial r} \omega' \right)^2 \\ &+ 2\Omega \Delta_x \left(\frac{\partial}{\partial x} \omega' \right)^2 f'^6 \omega' + 2\Omega \Delta_r f'^6 \omega' \left(\frac{\partial}{\partial r} \omega' \right)^2 \\ &- \Delta_x \left(\frac{\partial}{\partial x} \omega' \right)^2 f'^6 \omega'^2 - \Delta_r f'^6 \omega'^2 \left(\frac{\partial}{\partial r} \omega' \right)^2 \\ &+ \Omega^2 \Delta_x \left(\frac{\partial}{\partial x} f' \right)^2 \rho^2 f'^2 + \Omega^2 \Delta_r \rho^2 \left(\frac{\partial}{\partial r} f' \right)^2 f'^2 \\ &- 2\Omega \Delta_x \left(\frac{\partial}{\partial x} f' \right)^2 \rho^2 f'^2 \omega' - 2\Omega \Delta_x \left(\frac{\partial}{\partial x} \omega' \right) \\ &\times \left(\frac{\partial}{\partial x} \rho^2 \right) f'^4 - 2\Omega \Delta_r \rho^2 \left(\frac{\partial}{\partial r} f' \right)^2 f'^2 \omega' \\ &- 2\Omega \Delta_r f'^4 \left(\frac{\partial}{\partial r} \omega' \right) \frac{\partial}{\partial r} \rho^2 + \Delta_x \left(\frac{\partial}{\partial x} f' \right)^2 \rho^2 f'^2 \omega'^2 \\ &+ \Delta_x \left(\frac{\partial}{\partial x} \omega' \right)^2 \rho^2 f'^4 \\ &+ \Delta_r \rho^2 \left(\frac{\partial}{\partial r} f' \right)^2 f'^2 \omega'^2 + \Delta_r \rho^2 f'^4 \left(\frac{\partial}{\partial r} \omega' \right)^2 \\ &+ 2\Delta_r f'^4 \omega' \left(\frac{\partial}{\partial r} \omega' \right) \frac{\partial}{\partial r} \rho^2 \\ &- \Delta_x \left(\frac{\partial}{\partial x} f' \right)^2 \rho^4 \\ &+ 2\Delta_x \left(\frac{\partial}{\partial x} f' \right) \left(\frac{\partial}{\partial x} \rho^2 \right) \rho^2 f' \\ &- \Delta_x \left(\frac{\partial}{\partial x} \rho^2 \right)^2 f'^2 - \Delta_r \rho^4 \left(\frac{\partial}{\partial r} f' \right)^2 \\ &+ 2\Delta_r \rho^2 \left(\frac{\partial}{\partial r} f' \right) f' \left(\frac{\partial}{\partial r} \rho^2 \right) \end{aligned}$$

$$\begin{aligned} & -\Delta_r f'^2 \left(\frac{\partial}{\partial r} \rho^2 \right)^2 + 2\Delta_x \left(\frac{\partial}{\partial x} \omega' \right) \\ & \times \left(\frac{\partial}{\partial x} \rho^2 \right) f'^4 \omega'. \end{aligned} \quad (4.28)$$

After substituting all the known functions f' , ω' , ρ , Δ_r and Δ_x in (4.28) and taking the limits in Eqs. (4.26) and (4.27), we find

$$\psi_N = \frac{1}{8\pi l}, \quad (4.29)$$

and

$$\psi_S = \frac{1}{8\pi l}. \quad (4.30)$$

As expected, we notice that neither the magnetic field nor the other parameters of the black hole (except the NUT charge) contribute to the thermodynamic potentials ψ_N and ψ_S , where $r \rightarrow \infty$. The independence of ψ_N and ψ_S from other parameters of the black hole and the magnetic field, is consistent with the notion of considering them as the “surface gravity” over the NUT tubes [43], similar to κ as the surface gravity (4.1) over the horizon r_H . The thermodynamic charges N_N and N_S are given by

$$N_N \Psi_N = \frac{1}{16\pi} \int_{T_N} *dT, \quad (4.31)$$

$$N_S \Psi_S = -\frac{1}{16\pi} \int_{T_S} *dT, \quad (4.32)$$

where the integrals are over the very narrow NUT tubes T_N and T_S , along the positive and negative z -axis, where $r_H \leq z < \infty$, $x \simeq 1$ and $-\infty < z \leq -r_H$, $x \simeq -1$, respectively. We find the integrand in Eqs. (4.31) and (4.32), is given by

$$(*dT)_{\phi r} = \frac{w' f'^3 \left(\frac{\partial}{\partial x} w' \right) - f' \left(\frac{\partial}{\partial x} \rho^2 \right) + \rho^2 \left(\frac{\partial}{\partial x} f' \right)}{2f'\rho} \sqrt{\frac{\Delta_x}{\Delta_r}}. \quad (4.33)$$

Evaluating the integrals in (4.31) and (4.32), we find very long expressions for the N_N and N_S in terms of the black hole parameters a, l, m and q and the magnetic field parameter b which are unfeasible to present here. We verify that in the special cases, the thermodynamic NUT potentials reduce exactly to the well-known results in [43].

As we notice from Eq. (3.14), the NUT charge contributes to an off-diagonal term in the asymptotic of MKNTN black holes (2.32). As a result, we find two contributions to the total angular momentum (4.18) from the north and south NUT tubes, which are given by

$$J_N = \frac{1}{16\pi} \int_{T_N} *d\mathcal{P}, \quad (4.34)$$

$$J_S = \frac{-1}{16\pi} \int_{T_S} *d\mathcal{P}, \quad (4.35)$$

respectively, where $\mathcal{P}^\mu = \frac{\partial}{\partial \phi}$ is the space-like Killing vector. We find the integrand in Eqs. (4.34) and (4.35), is given by

$$(*d\mathcal{P})_{\phi r} = -\frac{f'^2(\frac{\partial}{\partial x} w')}{2\rho} \sqrt{\frac{\Delta_x}{\Delta_r}}. \quad (4.36)$$

Evaluating the integrals in (4.34) and (4.35), we find very long expressions for the J_N and J_S in terms of the black hole parameters a, l, m and q and the magnetic field parameter b which are unfeasible to present here. We verify that in the special cases, the thermodynamic NUT potentials reduce exactly to the well-known results in [43].

To complete the calculation, we also mention that the area $\mathcal{S}_N$ and $\mathcal{S}_S$ of the north and south NUT tubes are given by

$$\mathcal{S}_N = n \int_{T_N} *d\xi_N, \quad (4.37)$$

$$\mathcal{S}_S = -n \int_{T_S} *d\xi_S, \quad (4.38)$$

respectively, where ξ_N and ξ_S are the Killing vectors (4.23) and (4.24), which generate the north and south pole tubes. The integrand in Eqs. (4.37) and (4.38), is given by

$$(*d\xi_Z)_{\phi r} = \sqrt{\frac{\Delta_x}{\Delta_r}} \frac{1}{2\rho f'} \left((w' - \Omega_Z) f'^3 \left(\frac{\partial}{\partial x} w' \right) - f' \left(\frac{\partial}{\partial x} \rho^2 \right) + \rho^2 \left(\frac{\partial}{\partial x} f' \right) \right), \quad (4.39)$$

where Z stands for N and S , respectively. After calculating the integrals in (4.37), (4.38), we find that $\mathcal{S}_N$ and $\mathcal{S}_S$, are independent of the magnetic field parameter b , and are given by

$$\mathcal{S}_N = \mathcal{S}_S = 4\pi l(r_\infty - r_H), \quad (4.40)$$

where $r_\infty - r_H$ is the length of the north (south) NUT tube along the positive (negative) z -axis.

Furnished by the results (4.17), (4.18), (4.29), (4.30), (4.31), (4.32), (4.34) and (4.35), we can verify that Eq. (4.17) is indeed the Smarr relation for the MKNTN black hole (2.32). Moreover, by construction of different thermodynamical quantities for the black hole (2.32), and the Smarr equation (4.17), we find the first law of thermodynamics, as given by

$$d\mathcal{M} = \Phi_H dQ + \Omega_H d\mathcal{J} + T_H dS + \psi_N dN_N + \psi_S dN_S. \quad (4.41)$$

In fact, a tedious but straightforward calculation shows that Eq. (4.17) implies that for $\mathcal{M} = \mathcal{M}(Q, \mathcal{J}, S, N_N, N_S)$, we have the first law of thermodynamics, as it is given by (4.41). We also verify that Eq. (4.41) reduces exactly to the well-known first law of thermodynamics for the Kerr-NUT and Melvin-Kerr-Newman black holes in [43] and [31].

5 Concluding remarks

In this work, we have constructed a new class of the exact solutions to the Einstein-Maxwell theory in four dimensions which describes the immersion of the Kerr-Newman-Taub-NUT spacetimes in an external magnetic field. The solutions are obtained by applying the Ernst magnetization procedure to the four dimensional Kerr-Newman-Taub-NUT spacetime as the seed. We discuss the properties of the MKNTN spacetimes and show that they are completely regular at $r = 0, x = 0$. Particularly, in addition to the extensive investigation on the space-time structure and the ergoregions for the MKNTN black hole, we also study the behaviour of the electromagnetic fields in the magnetized spacetime solutions. We find that the horizon has a non-trivial topology which leads to the eccentric horizon. The thermodynamical quasi-local conserved quantities of the spacetime are obtained, though they are generally quite complicated functions of the five independent parameters of MKNTN black holes, namely m, a, l, q and b . In this paper, we also establish the Smarr formula for the MKNTN black holes. Finally, we study the thermodynamics of the MKNTN spacetimes, and show the corresponding first law of thermodynamics.

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A The differential equations for $\gamma(\rho, z)$ in the LPW metric

Let us first consider the LPW metric (2.1) with $d\chi d\chi^*$ as given in (2.9), namely

$$ds^2 = -\frac{\rho^2}{f} dt^2 + \frac{e^{2\gamma}}{f} (d\rho^2 + dz^2) + f (\omega dt - d\phi)^2. \quad (\text{A.1})$$

If the system is electrovacuum, the accompanying vector field A_μ obeys Eq. (2.4), which then constructs the Ernst electromagnetic potential Φ , as defined in (2.3). Now let us write the Einstein equations for the electrovacuum system as

$$E_{\mu\nu} \equiv R_{\mu\nu} - \left(2F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) = 0. \quad (\text{A.2})$$

In fact, the Ernst equations (2.7) and (2.8) are obtained from the last equations. Particularly, from the $E_{\rho z}$ component of (A.2), we have

$$\partial_z \gamma = \frac{1}{2f^2 \rho} \left\{ \rho^2 \partial_\rho f \partial_z f - f^4 \partial_\rho \omega \partial_z \omega + 2f \rho^2 (\partial_\rho \Phi \partial_z \Phi^* + \partial_\rho \Phi^* \partial_z \Phi) \right\}, \quad (\text{A.3})$$

while from $E_{\rho\rho}$ and E_{zz} , we find

$$\begin{aligned} \partial_\rho \gamma = \frac{1}{4f^2 \rho} \left\{ \rho^2 \left((\partial_\rho f)^2 - (\partial_z f)^2 \right) \right. \\ \left. + f^4 \left((\partial_z \omega)^2 - (\partial_\rho \omega)^2 \right) \right. \\ \left. + 4f \rho^2 (\partial_\rho \Phi \partial_\rho \Phi^* - \partial_z \Phi^* \partial_z \Phi) \right\}. \end{aligned} \quad (\text{A.4})$$

Interestingly, despite the other metric functions f and ω transform under the Ernst magnetization (as reviewed in Sect. 2.1), the γ function remains invariant after the transformation. In other words, setting $\omega \rightarrow \omega'$, $f \rightarrow f'$, $\Phi \rightarrow \Phi'$ according to (2.12), (2.14), and (2.15) in (A.3) and (A.4), leaves the two differential equations (A.3) and (A.4), for γ unchanged. Indeed, the invariance of Eqs. (A.3) and (A.4) under the Ernst magnetization (2.12), (2.14), and (2.15) is not too obvious. Therefore, we provide a simple example which illustrate the invariance of the γ function in LPW metric (2.1)

under the Ernst transformation. We consider magnetizing Minkowski spacetime to get the magnetic Melvin spacetime. In the cylindrical type coordinate (t, ρ, z, ϕ) , Minkowski spacetime can be expressed as

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2. \quad (\text{A.5})$$

Obviously, the LPW metric functions for the Minkowski spacetime are $f = \rho^2$, $\gamma = \ln \rho$, and $\omega = 0$. Accordingly, the Ernst potentials for this seed solution are $\mathcal{E} = \rho^2$ and $\Phi = 0$, which give us $\Lambda = 1 + b^2 \rho^2$. The magnetized Ernst potentials then read

$$\mathcal{E}' = \Lambda^{-1} \rho^2 \quad \text{and} \quad \Phi' = -\Lambda^{-1} \rho^2 b, \quad (\text{A.6})$$

while the metric functions are

$$f' = \Lambda^{-2} \rho^2 \quad \text{and} \quad \omega' = 0. \quad (\text{A.7})$$

On the other hand, the associated vector solution obtained from Φ' has the components $A'_\phi = -\Lambda^{-1} b \rho^2$ and $\tilde{A}'_\phi = 0$, which lead to $A'_t = 0$, according to (2.4). Suppose we consider the function γ' in the Melvin spacetime, changes under the Ernst magnetization. To find γ' , we need to solve the remaining non-zero equations (A.2) for the Melvin spacetime with (A.6) and (A.7). They are given by

$$E'_{\rho\rho} = -\rho^{-2} \left\{ \rho^2 (\partial_z^2 \gamma' + \partial_\rho^2 \gamma') - \rho \partial_\rho \gamma' - 2 \right\}, \quad (\text{A.8})$$

$$E'_{\rho z} = -\rho^{-1} \partial_z \gamma', \quad (\text{A.9})$$

and

$$E'_{zz} = -\rho^{-1} \left\{ \rho (\partial_z^2 \gamma' + \partial_\rho^2 \gamma') + \partial_\rho \gamma' \right\}. \quad (\text{A.10})$$

We notice that the absence of external magnetic parameter b in the last three equations tells us that γ' obeys the same differential equations as γ . Hence, we can conclude γ' for the Melvin spacetime is the same as γ for the Minkowski spacetime. We also note that we get the same result $\gamma' = \gamma$, by plugging Eqs. (A.6) and (A.7) in (A.3) and (A.4). In regard to the Melvin–Kerr–Newman–Taub–NUT spacetime (2.32) with the Boyer–Lindquist type coordinate $(t, r, x = \cos \theta, \phi)$, we find the following differential equations for $\gamma(r, x)$ from the Einstein equations,

$$\begin{aligned} f^2 (2\Delta_r x \partial_r \gamma - \Delta_x \partial_r \Delta_r \partial_x \gamma) \\ + 2f \rho^2 (\partial_r \Phi \partial_x \Phi^* + \partial_r \Phi^* \partial_x \Phi) \\ - f^2 (f^2 \partial_r \omega \partial_x \omega + x \partial_r \Delta_r) + \rho^2 \partial_r f \partial_x f = 0, \end{aligned} \quad (\text{A.11})$$

and

$$\begin{aligned} -2\rho^2 f^2 (\partial_r \Delta_r \partial_r \gamma + 2x \partial_x \gamma) + \rho^2 (\Delta_r (\partial_r f)^2 - \Delta_x (\partial_x f)^2) \\ - f^4 (\Delta_r (\partial_r \omega)^2 - \Delta_x (\partial_x \omega)^2) \end{aligned}$$

$$+4\rho^2 f (\Delta_r \partial_r \Phi \partial_r \Phi^* - \Delta_x \partial_x \Phi \partial_x \Phi^*) = 0. \quad (\text{A.12})$$

We have explicitly checked that the seed solution $\{f, \omega, \Phi\}$ that belongs to Kerr–Newman–Taub–NUT system, and the magnetized version $\{f', \omega', \Phi'\}$ (as presented in Sect. 2.2), obey (A.11) and (A.12), with the same γ function, as given by (2.25).

B The magnetized metric functions for the Melvin–Kerr–Newman–Taub–NUT spacetimes

The coefficients that appear in the metric function f' for the Melvin–Kerr–Newman–NUT black hole, in Eq. (2.34), are given by

$$\begin{aligned} c_4 &= -a^2 \Delta_r, \\ c_3 &= -4la \Delta_r, \\ c_2 &= 3l^4 + (8rm - 8a^2 - 4q^2 - 6r^2)l^2 + a^4 \\ &\quad - 4a^2mr + 2a^2q^2 - r^4, \\ c_1 &= 4la \Delta_r, \\ c_0 &= 3a^2l^2 + 2a^2mr - a^2q^2 + a^2r^2 + l^4 + 2l^2r^2 + r^4, \\ d_6 &= a^2b^4\Delta_r^2, \\ d_5 &= 6ab^4l\Delta_r^2, \\ d_4 &= -b \left\{ 2a^6b^2 - 21a^4b^2l^2 - 4a^4b^2m^2 - 12a^4b^2mr \right. \\ &\quad + 6a^4b^2q^2 + 3a^4b^2r^2 + 28a^2b^2l^4 \\ &\quad - 8a^2b^2l^2m^2 + 84a^2b^2l^2mr - 32b^2a^2l^2q^2 \\ &\quad - 36a^2b^2l^2r^2 + 24a^2b^2m^2r^2 \\ &\quad - 24a^2b^2mq^2r - 12a^2b^2mr^3 + 4a^2b^2q^4 \\ &\quad + 8a^2b^2q^2r^2 - 9b^2l^6 - 4b^2l^4m^2 - 24b^2l^4mr \\ &\quad + 18b^2l^4q^2 + 9b^2l^4r^2 - 36b^2l^2m^2r^2 \\ &\quad + 32b^2l^2mq^2r + 40b^2l^2mr^3 - 9b^2l^2q^4 \\ &\quad - 12b^2l^2q^2r^2 - 15b^2l^2r^4 - b^2q^4r^2 + 2b^2q^2r^4 \\ &\quad - b^2r^6 - 4a^3bqr + 4abl^2qr \\ &\quad + 8abmq^2r - 4abq^3r - 4abqr^3 + 2a^4 - 2a^2l^2 \\ &\quad \left. - 4a^2mr + 2a^2q^2 + 2a^2r^2 \right\}, \\ d_3 &= -4b^2l \left\{ 3a^5b^2 - 10a^3b^2l^2 + 2a^3b^2m^2 - 20a^3b^2mr \right. \\ &\quad + 6a^3b^2q^2 + 6a^3b^2r^2 \\ &\quad + 7ab^2l^4 + 2ab^2l^2m^2 + 16ab^2l^2mr \\ &\quad - 10ab^2l^2q^2 - 2ab^2l^2r^2 + 18ab^2m^2r^2 \\ &\quad - 14ab^2mq^2r - 12ab^2mr^3 + 3ab^2q^4 \\ &\quad + 2ab^2q^2r^2 + 3ab^2r^4 + 2a^2bmqr - 4a^2bqr \\ &\quad + 2bl^2mq + 6bmqr^2 - 2bq^3r - 4bqr^3 + 2a^3 - 2al^2 \\ &\quad \left. - 4amr + 2aq^2 + 2ar^2 \right\}, \end{aligned}$$

$$\begin{aligned} d_2 &= a^6b^4 - 28a^4b^4l^2 + 8a^4b^4m^2 - 12a^4b^4mr + 4a^4b^4q^2 \\ &\quad + 37a^2b^4l^4 + 12a^2b^4l^2m^2 \\ &\quad + 84a^2b^4l^2mr - 38a^2b^4l^2q^2 - 18a^2b^4l^2r^2 \\ &\quad + 36a^2b^4m^2r^2 - 20a^2b^4mq^2r \\ &\quad - 12a^2b^4mr^3 + 4a^2b^4q^4 + 6a^2b^4q^2r^2 - 3a^2b^4r^4 \\ &\quad + 6b^4l^6 + 24b^4l^4mr - 6b^4l^4q^2 \\ &\quad - 30b^4l^4r^2 - 8b^4l^2mr^3 + 12b^4l^2q^2r^2 - 6b^4l^2r^4 \\ &\quad + 2b^4q^2r^4 - 2b^4r^6 + 8a^3b^3mq \\ &\quad - 8a^3b^3qr + 16ab^3l^2mq - 8ab^3l^2qr + 24ab^3mqr^2 \\ &\quad - 4ab^3q^3r - 8ab^3qr^3 + 2a^4b^2 \\ &\quad - 16a^2b^2l^2 - 8a^2b^2mr + 4a^2b^2q^2 \\ &\quad + 6b^2l^4 + 16b^2l^2mr - 2b^2l^2q^2 - 12b^2l^2r^2 \\ &\quad + 6b^2q^2r^2 - 2b^2r^4 - 4abqr + a^2, \\ d_1 &= 2l \left\{ 3a^5b^4 - 14a^3b^4l^2 - 4a^3b^4m^2 - 12a^3b^4mr \right. \\ &\quad + 6a^3b^4q^2 + 6a^3b^4r^2 - 5ab^4l^4 \\ &\quad - 12ab^4l^2mr + 2ab^4l^2q^2 + 18ab^4l^2r^2 \\ &\quad + 4ab^4mr^3 - 2ab^4q^2r^2 + 3ab^4r^4 - 8a^2b^3mq \\ &\quad - 8b^3l^2qr + 4b^2a^3 - 4ab^2l^2 - 8ab^2mr \\ &\quad \left. - 2ab^2q^2 + 4ab^2r^2 - 4bqr + a \right\}, \\ d_0 &= 9a^4b^4l^2 + 4a^4b^4m^2 + 4a^4b^4mr + a^4b^4r^2 \\ &\quad + 6a^2b^4l^4 + 12a^2b^4l^2mr + 4a^2b^4mr^3 \\ &\quad + 2a^2b^4r^4 + b^4l^6 + 7b^4l^4r^2 + 7b^4l^2r^4 \\ &\quad + b^4r^6 + 8a^3b^3mq + 4a^3b^3qr + 12ab^3l^2qr \\ &\quad + 4ab^3qr^3 + 6a^2b^2l^2 + 4a^2b^2mr \\ &\quad + 4a^2b^2q^2 + 2a^2b^2r^2 + 2b^2l^4 + 4b^2l^2r^2 \\ &\quad + 2b^2r^4 + 4abqr + l^2 + r^2. \end{aligned} \quad (\text{B.1})$$

The coefficients that appear in the metric function ω' for the Melvin–Kerr–Newman–NUT black hole, in Eq. (2.35), are given by

$$\begin{aligned} \tilde{c}_6 &= ab^4\Delta_r \left(2a^2l^2 - 4m^2a^2 + 6a^2mr \right. \\ &\quad - a^2q^2 - 2l^4 - 4l^2m^2 + 6l^2mr + 3l^2q^2 \\ &\quad \left. - 6l^2r^2 - 2mr^3 - q^4 + 3q^2r^2 \right), \\ \tilde{c}_5 &= -2b^3l\Delta_r \left(3a^4b - 6a^2bl^2 + 4a^2bq^2 - 6a^2br^2 \right. \\ &\quad + 3bl^4 + 4bl^2m^2 - 4bl^2mr - 4bl^2q^2 + 6bl^2r^2 \\ &\quad \left. + 4bmr^3 + bq^4 - 4bq^2r^2 - br^4 - 4qma + 4qar \right), \\ \tilde{c}_4 &= -b^2\Delta_r \left(32a^3b^2l^2 + 8a^3b^2m^2 - 6a^3b^2mr - 30ab^2l^4 \right. \\ &\quad - 20ab^2l^2m^2 + 18ab^2l^2mr + 25q^2l^2b^2a \\ &\quad - 42ab^2l^2r^2 - 14ab^2mr^3 - ab^2q^4 + 9r^2ab^2q^2 \\ &\quad \left. + 16a^2bmqr - 12ra^2bq - 16bl^2mq \right) \end{aligned}$$

$$\begin{aligned}
& +12rbl^2q - 4r^3bq + 6aq^2), \\
\tilde{c}_3 = & 4b^2l\Delta_r \left(a^4b^2 - 20a^2b^2l^2 - 8a^2b^2m^2 + 6a^2b^2mr \right. \\
& + 3a^2b^2q^2 - 6a^2b^2r^2 + b^2l^4 - 2b^2l^2mr \\
& - 2b^2l^2q^2 + 6b^2l^2r^2 + 2b^2mr^3 \\
& \left. - 2b^2q^2r^2 + b^2r^4 - 10abmq + 8abqr - 3q^2 \right), \\
\tilde{c}_2 = & 24a^5b^4l^2 + 8a^5b^4m^2 - 14ra^5b^4m - 92a^3b^4l^4 \\
& - 16a^3b^4l^2m^2 - 36ra^3b^4l^2m + 46a^3b^4l^2q^2 \\
& - 12r^2a^3b^4l^2 - 24ra^3b^4m^3 + 12a^3b^4m^2q^2 \\
& + 48a^3b^4m^2r^2 - 20a^3b^4mq^2r - 20r^3b^4a^3m \\
& + 6r^2a^3b^4q^2 + 28ab^4l^6 + 58rab^4l^4m - 46ab^4l^4q^2 \\
& - 24r^2ab^4l^2m^2 - 8rab^4l^2mq^2 \\
& + 108ab^4l^2mr^3 + 10ab^4l^2q^4 - 36r^2ab^4l^2q^2 \\
& - 36ab^4l^2r^4 + 24b^4am^2r^4 - 24r^3b^4amq^2 \\
& - 6ab^4mr^5 + 6r^2ab^4q^4 + 6b^4aq^2r^4 \\
& + 8a^4b^3mq - 16ra^4b^3q - 24a^2b^3l^2mq + 32ra^2b^3l^2q \\
& - 32ra^2b^3m^2q + 16a^2b^3mq^3 + 48a^2b^3mqr^2 \\
& - 20ra^2b^3q^3 - 16r^3b^3a^2q - 16rb^3l^4q \\
& - 24r^2b^3l^2mq + 12rb^3l^2q^3 + 16r^3b^3l^2q \\
& + 8b^3mqr^4 - 4r^3b^3q^3 - 12q^2l^2b^2a \\
& - 12rab^2mq^2 + 6ab^2q^4 - 4ra^2bq - 4rbl^2q \\
& - 4r^3bq + 2al^2 + 2amr - aq^2, \\
\tilde{c}_1 = & 2l\Delta_r \left(a^4b^4 + 18a^2b^4l^2 - 12a^2b^4mr - 2a^2b^4q^2 \right. \\
& \left. + 6a^2r^2b^4 + b^4l^4 - 6b^4l^2r^2 - 3b^4r^4 - 12ab^3qr - 1 \right), \\
\tilde{c}_0 = & 6a^5b^4l^2 + 4a^5b^4m^2 + 2ra^5b^4m + a^5b^4q^2 \\
& + 26a^3b^4l^4 + 4a^3b^4l^2m^2 - 7a^3b^4l^2q^2 \\
& + 6r^2a^3b^4l^2 \\
& - 12a^3b^4m^2r^2 + 6a^3b^4mq^2r - 4r^3b^4a^3m + a^3b^4q^4 \\
& + r^2a^3b^4q^2 + 8ab^4l^6 + 2rab^4l^4m - 12ab^4l^2mr^3 + 4r^2ab^4l^2q^2 \\
& - 6ab^4mr^5 + 8a^4b^3mq + 4ra^4b^3q + 8a^2b^3l^2mq \\
& - 8ra^2b^3l^2q - 8a^2b^3mqr^2 + 8ra^2b^3q^3 \\
& + 4rb^3l^4q - 4b^3qr^5 \\
& + 6a^3b^2q^2 + 6q^2l^2b^2a + 6r^2ab^2q^2 + 4ra^2bq \\
& + 4rbl^2q + 4r^3bq - 2al^2 - 2amr + aq^2. \tag{B.2}
\end{aligned}$$

The coefficients that appear in the Maxwell's field A'_ϕ for the Melvin–Kerr–Newman–NUT black hole, in Eq. (2.36), are given by

$$\begin{aligned}
a_0 = & -9a^4b^3l^2 - 4a^4b^3m^2 - 4a^4b^3mr - a^4b^3r^2 \\
& - 6a^2b^3l^4 - 12a^2b^3l^2mr - 4a^2b^3mr^3 - 2a^2b^3r^4 - b^3l^6 \\
& - 7b^3l^4r^2 - 7b^3l^2r^4 - b^3r^6 - 6a^3b^2mq - 3a^3b^2qr \\
& - 9ab^2l^2qr - 3ab^2qr^3 - 3a^2bl^2 - 2a^2bmr - 2a^2bq^2 \\
& - a^2br^2 - bl^4 - 2bl^2r^2 - br^4 - aqr, \\
a_1 = & -6a^5b^3l + 28a^3b^3l^3 + 8a^3b^3lm^2 + 24a^3b^3lmr
\end{aligned}$$

$$\begin{aligned}
& - 12a^3b^3lq^2 - 12a^3b^3lr^2 + 10ab^3l^5 + 24ab^3l^3mr \\
& - 4ab^3l^3q^2 - 36ab^3l^3r^2 - 8ab^3lmr^3 + 4ab^3lq^2r^2 \\
& - 6ab^3lr^4 + 12a^2b^2lmq + 12b^2l^3qr - 4a^3bl + 4abl^3 \\
& + 8ablmr + 2ablq^2 - 4ablr^2 + 2lqr, \\
a_2 = & -a^6b^3 + 28a^4b^3l^2 - 8a^4b^3m^2 + 12a^4b^3mr \\
& - 4a^4b^3q^2 - 37a^2b^3l^4 - 12a^2b^3l^2m^2 - 84a^2b^3l^2mr \\
& + 38a^2b^3l^2q^2 + 18a^2b^3l^2r^2 - 36a^2b^3m^2r^2 + 20a^2b^3mq^2r \\
& + 12a^2b^3mr^3 - 4a^2b^3q^4 - 6a^2b^3q^2r^2 + 3a^2b^3r^4 \\
& - 6b^3l^6 - 24b^3l^4mr + 6b^3l^4q^2 + 30b^3l^4r^2 + 8b^3l^2mr^3 \\
& - 12b^3l^2q^2r^2 + 6b^3l^2r^4 - 2b^3q^2r^4 + 2b^3r^6 \\
& - 6a^3b^2mq + 6a^3b^2qr - 12ab^2l^2mq + 6ab^2l^2qr \\
& - 18ab^2mqr^2 + 3ab^2q^3r + 6ab^2qr^3 - a^4b + 8a^2bl^2 \\
& + 4a^2bmr - 2a^2bq^2 - 3bl^4 - 8bl^2mr \\
& + bl^2q^2 + 6bl^2r^2 - 3bq^2r^2 + br^4 + aqr, \\
a_3 = & 12a^5b^3l - 40a^3b^3l^3 + 8a^3b^3lm^2 - 80a^3b^3lmr \\
& + 24a^3b^3lq^2 + 24a^3b^3lr^2 + 28ab^3l^5 + 8ab^3l^3m^2 \\
& + 64ab^3l^3mr - 40ab^3l^3q^2 - 8ab^3l^3r^2 + 72ab^3lm^2r^2 \\
& - 56ab^3lmq^2r - 48ab^3lmr^3 + 12ab^3lq^4 \\
& + 8ab^3lq^2r^2 + 12ab^3lr^4 + 6a^2b^2lmq \\
& - 12a^2b^2lqr + 6b^2l^3mq + 18b^2lmqr^2 - 6b^2lq^3r - 12b^2lqr^3 \\
& + 4a^3bl - 4abl^3 - 8ablmr + 4ablq^2 + 4ablr^2, \\
a_4 = & 2a^6b^3 - 21a^4b^3l^2 - 4a^4b^3m^2 - 12a^4b^3mr \\
& + 6a^4b^3q^2 + 3a^4b^3r^2 + 28a^2b^3l^4 - 8a^2b^3l^2m^2 \\
& + 84a^2b^3l^2mr - 32a^2b^3l^2q^2 - 36a^2b^3l^2r^2 + 24a^2b^3m^2r^2 \\
& - 24a^2b^3mq^2r - 12a^2b^3mr^3 + 4a^2b^3q^4 \\
& + 8a^2b^3q^2r^2 - 9b^3l^6 - 4b^3l^4m^2 - 24b^3l^4mr \\
& + 18b^3l^4q^2 + 9b^3l^4r^2 - 36b^3l^2m^2r^2 + 32b^3l^2mq^2r \\
& + 40b^3l^2mr^3 - 9b^3l^2q^4 - 12b^3l^2q^2r^2 \\
& - 15b^3l^2r^4 - b^3q^4r^2 + 2b^3q^2r^4 - b^3r^6 - 3a^3b^2qr + 3ab^2l^2qr \\
& + 6ab^2mqr^2 - 3ab^2q^3r - 3ab^2qr^3 + a^4b \\
& - a^2bl^2 - 2a^2bmr + a^2bq^2 + a^2br^2, \\
a_5 = & -6ab^3l\Delta_r^2, \\
a_6 = & -a^2b^3\Delta_r^2, \tag{B.3}
\end{aligned}$$

and

$$\begin{aligned}
b_0 = & (9a^4b^4l^2 + 4a^4b^4m^2 + 4a^4b^4mr + a^4b^4r^2 \\
& + 6a^2b^4l^4 + 12a^2b^4l^2mr + 4a^2b^4mr^3 + 2a^2b^4r^4 \\
& + b^4l^6 + 7b^4l^4r^2 + 7b^4l^2r^4 + b^4r^6 \\
& + 8a^3b^3mq + 4a^3b^3qr + 12ab^3l^2qr \\
& + 4ab^3qr^3 + 6a^2b^2l^2 + 4a^2b^2mr \\
& + 4a^2b^2q^2 + 2a^2b^2r^2 + 2b^2l^4 \\
& + 4b^2l^2r^2 + 2b^2r^4 + 4abqr + l^2 + r^2), \\
b_1 = & 6a^5b^4l - 28a^3b^4l^3 - 8a^3b^4lm^2 - 24a^3b^4lmr \\
& + 12a^3b^4lq^2 + 12a^3b^4lr^2 - 10ab^4l^5 - 24ab^4l^3mr \\
& + 4ab^4l^3q^2 + 36ab^4l^3r^2 + 8ab^4lmr^3 - 4ab^4lq^2r^2 \\
& + 6ab^4lr^4 - 16a^2b^3lmq - 16b^3l^3qr + 8a^3b^2l \\
& - 8ab^2l^3 - 16ab^2lmr - 4ab^2lq^2 + 8ab^2lr^2 - 8blqr + 2al, \\
b_2 = & a^6b^4 - 28a^4b^4l^2 + 8a^4b^4m^2 - 12a^4b^4mr \\
& + 4a^4b^4q^2 + 37a^2b^4l^4 + 12a^2b^4l^2m^2 \\
& + 84a^2b^4l^2mr - 38a^2b^4l^2q^2 - 18a^2b^4l^2r^2
\end{aligned}$$

$$\begin{aligned}
& +36a^2b^4m^2r^2 - 20a^2b^4mq^2r - 12a^2b^4mr^3 \\
& +4a^2b^4q^4 + 6a^2b^4q^2r^2 \\
& -3a^2b^4r^4 + 6b^4l^6 + 24b^4l^4mr - 6b^4l^4q^2 \\
& -30b^4l^4r^2 - 8b^4l^2mr^3 + 12b^4l^2q^2r^2 - 6b^4l^2r^4 + 2b^4q^2r^4 \\
& -2b^4r^6 + 8a^3b^3mq - 8a^3b^3qr + 16ab^3l^2mq - 8ab^3l^2qr \\
& +24ab^3mqr^2 - 4ab^3q^3r - 8ab^3qr^3 + 2a^4b^2 \\
& -16a^2b^2l^2 - 8a^2b^2mr + 4a^2b^2q^2 + 6b^2l^4 + 16b^2l^2mr \\
& -2b^2l^2q^2 - 12b^2l^2r^2 + 6b^2q^2r^2 - 2b^2r^4 \\
& -4abqr + a^2, \\
b_3 = & -12a^5b^4l + 40a^3b^4l^3 - 8a^3b^4lm^2 + 80a^3b^4lmr \\
& -24a^3b^4lq^2 - 24a^3b^4lr^2 - 28ab^4l^5 - 8ab^4l^3m^2 \\
& -64ab^4l^3mr + 40ab^4l^3q^2 + 8ab^4l^3r^2 \\
& -72ab^4lm^2r^2 + 56ab^4lmq^2r + 48ab^4lmr^3 - 12ab^4lq^4 \\
& -8ab^4lq^2r^2 - 12ab^4lr^4 - 8a^2b^3lmq + 16a^2b^3lqr \\
& -8b^3l^3mq - 24b^3lmqr^2 + 8b^3lq^3r + 16b^3lqr^3 \\
& -8a^3b^2l + 8ab^2l^3 + 16ab^2lmr - 8ab^2lq^2 \\
& -8ab^2lr^2, \\
b_4 = & -2a^6b^4 + 21a^4b^4l^2 + 4a^4b^4m^2 + 12a^4b^4mr \\
& -6a^4b^4q^2 - 3a^4b^4r^2 - 28a^2b^4l^4 + 8a^2b^4l^2m^2 \\
& -84a^2b^4l^2mr + 32a^2b^4l^2q^2 + 36a^2b^4l^2r^2 - 24a^2b^4m^2r^2 \\
& +24a^2b^4mq^2r + 12a^2b^4mr^3 - 4a^2b^4q^4 \\
& -8a^2b^4q^2r^2 + 9b^4l^6 + 4b^4l^4m^2 + 24b^4l^4mr - 18b^4l^4q^2 \\
& -9b^4l^4r^2 + 36b^4l^2m^2r^2 - 32b^4l^2mq^2r \\
& -40b^4l^2mr^3 + 9b^4l^2q^4 + 12b^4l^2q^2r^2 + 15b^4l^2r^4 \\
& +b^4q^4r^2 - 2b^4q^2r^4 + b^4r^6 + 4a^3b^3qr - 4ab^3l^2qr \\
& -8ab^3mqr^2 + 4ab^3q^3r + 4ab^3qr^3 - 2a^4b^2 + 2a^2b^2l^2 \\
& +4a^2b^2mr - 2a^2b^2q^2 - 2a^2b^2r^2, \\
b_5 = & 6ab^4l\Delta_r^2, \\
b_6 = & a^2b^4\Delta_r^2.
\end{aligned} \tag{B.4}$$

Finally, the coefficients that appear in the Maxwell's field A'_l for the Melvin–Kerr–Newman–NUT black hole, in Eq. (2.37), are given by

$$\begin{aligned}
e_0 = & -2a^6b^6mq - a^6b^6qr + 8a^4b^6l^2mq + 19a^4b^6l^2qr \\
& -8a^4b^6m^2qr - 2a^4b^6mq^3 - 4a^4b^6mqr^2 - a^4b^6q^3r \\
& -a^4b^6qr^3 - 2a^2b^6l^4mq + 23a^2b^6l^4qr \\
& -12a^2b^6l^2mqr^2 \\
& -3a^2b^6l^2q^3r - 8a^2b^6l^2qr^3 - 2a^2b^6mqr^4 \\
& -a^2b^6q^3r^3 + a^2b^6qr^5 - b^6l^6qr - 5b^6l^4qr^3 \\
& +5b^6l^2qr^5 + b^6qr^7 + 6a^5b^5l^2 \\
& +4a^5b^5m^2 + 2a^5b^5mr \\
& -2a^5b^5q^2 + 26a^3b^5l^4 + 4a^3b^5l^2m^2 \\
& +5a^3b^5l^2q^2 + 6a^3b^5l^2r^2 - 12a^3b^5m^2r^2 \\
& -18a^3b^5mq^2r - 4a^3b^5mr^3 \\
& -2a^3b^5q^4 - 5a^3b^5q^2r^2 + 8ab^5l^6 \\
& +2ab^5l^4mr - 3ab^5l^4q^2 - 12ab^5l^2mr^3 \\
& -14ab^5l^2q^2r^2 - 6ab^5mr^5 \\
& -3ab^5q^2r^4 + 4a^4b^4mq - a^4b^4qr + 4a^2b^4l^2mq \\
& +8a^2b^4l^2qr - 28a^2b^4mqr^2 - 11a^2b^4q^3r - 6a^2b^4qr^3 \\
& -b^4l^4qr - 18b^4l^2qr^3 - 5b^4qr^5 + 8a^3b^3l^2
\end{aligned}$$

$$\begin{aligned}
& +4a^3b^3m^2 + 4a^3b^3mr + 10ab^3l^4 + 4ab^3l^2mr - ab^3l^2q^2 \\
& +2ab^3l^2r^2 - 4ab^3mr^3 - 17ab^3q^2r^2 + 6a^2b^2mq \\
& +a^2b^2qr + b^2l^2qr - 5b^2qr^3 \\
& +2abl^2 + 2abmr + 2abq^2 + qr, \\
e_1 = & 16a^5b^6lmq + 18a^5b^6lqr - 24a^3b^6l^3mq - 20a^3b^6l^3qr \\
& +8a^3b^6lm^2qr + 4a^3b^6lmq^3 - 64a^3b^6lmqr^2 \\
& +24a^3b^6lq^3r + 20a^3b^6lqr^3 - 38ab^6l^5qr - 16ab^6l^3mqr^2 \\
& +12ab^6l^3q^3r + 28ab^6l^3qr^3 + 16ab^6lmqr^4 \\
& -8ab^6lq^3r^3 + 2ab^6lqr^5 + 2a^6b^5l + 34a^4b^5l^3 \\
& -28a^4b^5lmr + 22a^4b^5lq^2 + 14a^4b^5lr^2 - 34a^2b^5l^5 \\
& -48a^2b^5l^3mr + 4a^2b^5l^3q^2 + 12a^2b^5l^3r^2 \\
& +48a^2b^5lm^2r^2 + 8a^2b^5lmq^2r - 48a^2b^5lmr^3 + 2a^2b^5lq^4 \\
& -16a^2b^5lq^2r^2 + 6a^2b^5lr^4 - 2b^5l^7 - 4b^5l^5mr \\
& +2b^5l^5q^2 + 14b^5l^5r^2 + 24b^5l^3mr^3 \\
& -12b^5l^3q^2r^2 \\
& -6b^5l^3r^4 + 12b^5lmr^5 - 6b^5lq^2r^4 - 6b^5lr^6 + 12a^3b^4lqr \\
& -24ab^4l^3qr + 48ab^4lmqr^2 + 6ab^4lq^3r \\
& -36ab^4lqr^3 + 4a^4b^3l + 4a^2b^3l^3 - 32a^2b^3lmr \\
& +10a^2b^3lq^2 + 16a^2b^3lr^2 - 4b^3l^5 - 8b^3l^3mr + 4b^3l^3q^2 \\
& +8b^3l^3r^2 + 8b^3lmr^3 + 12b^3lq^2r^2 \\
& -4b^3lr^4 - 6ab^2lqr + 2a^2bl - 2bl^3 - 4blmr + 2blq^2 + 2blr^2, \\
e_2 = & 4a^6b^6mq + 7a^6b^6qr - 12a^4b^6l^2mq \\
& -91a^4b^6l^2qr - 8a^4b^6m^3q + 4a^4b^6m^2qr + 6a^4b^6mq^3 \\
& -10a^4b^6mqr^2 + 15a^4b^6q^3r + 11a^4b^6qr^3 \\
& +44a^2b^6l^4mq - 47a^2b^6l^4qr - 12a^2b^6l^2m^2qr - 8a^2b^6l^2mq^3 \\
& +144a^2b^6l^2mqr^2 - 44a^2b^6l^2q^3r - 2a^2b^6l^2qr^3 \\
& +28a^2b^6m^2qr^3 - 42a^2b^6mq^3r^2 - 16a^2b^6mqr^4 \\
& +9a^2b^6q^5r + 18a^2b^6q^3r^3 + a^2b^6qr^5 + 11b^6l^6qr \\
& +6b^6l^4mqr^2 - 9b^6l^4q^3r - b^6l^4qr^3 - 20b^6l^2mqr^4 \\
& +14b^6l^2q^3r^3 - 7b^6l^2qr^5 - 2b^6mq^6 + 3b^6q^3r^5 \\
& -3b^6qr^7 + 12a^5b^5l^2 - 18a^5b^5mr + 7a^5b^5q^2 \\
& -144a^3b^5l^4 - 24a^3b^5l^2m^2 - 36a^3b^5l^2mr + 6a^3b^5l^2q^2 \\
& -24a^3b^5l^2r^2 - 24a^3b^5m^3r - 12a^3b^5m^2q^2 \\
& +72a^3b^5m^2r^2 + 4a^3b^5mq^2r \\
& -12a^3b^5mr^3 + 10a^3b^5q^4 + 22a^3b^5q^2r^2 + 12ab^5l^6 + 54ab^5l^4mr \\
& -7ab^5l^4q^2 - 24ab^5l^2m^2r^2 - 20ab^5l^2mq^2r \\
& +132ab^5l^2mr^3 + ab^5l^2q^4 + 16ab^5l^2q^2r^2 - 36ab^5l^2r^4 \\
& +24ab^5m^2r^4 - 36ab^5mq^2r^3 + 6ab^5mr^5 - 3ab^5q^4r^2 \\
& +15ab^5q^2r^4 - 4a^4b^4mq - 20a^2b^4l^2mq \\
& -66a^2b^4l^2qr - 52a^2b^4m^2qr - 4a^2b^4mq^3 \\
& +80a^2b^4mqr^2 + 27a^2b^4q^3r + 16a^2b^4qr^3 \\
& -36b^4l^2mqr^2 + 6b^4l^2q^3r + 80b^4l^2qr^3 + 4b^4mqr^4 \\
& -14b^4q^3r^3 + 16b^4qr^5 - 6a^3b^3l^2 + 4a^3b^3m^2 \\
& -24a^3b^3mr + 3a^3b^3q^2 - 12ab^3l^4 + 32ab^3l^2mr - 10ab^3l^2q^2 \\
& -28ab^3l^2r^2 + 24ab^3m^2r^2 - 52ab^3mq^2r \\
& -8ab^3mr^3 + 4ab^3q^4 + 58ab^3q^2r^2 - 9a^2b^2qr + 3b^2l^2qr + 6b^2mqr^2 \\
& -9b^2q^3r + 9b^2qr^3 - 6abl^2 - 6abmr - 2qr, \\
e_3 = & 2a^5b^6lmq - 74a^5b^6lqr + 8a^3b^6l^3mq + 124a^3b^6l^3qr
\end{aligned}$$

$$\begin{aligned}
& +8a^3b^6lm^3q - 120a^3b^6lm^2qr + 26a^3b^6lmq^3 \\
& +300a^3b^6lmqr^2 - 104a^3b^6lq^3r - 84a^3b^6lqr^3 \\
& -34ab^6l^5mq + 70ab^6l^5qr - 32ab^6l^3m^2qr + 18ab^6l^3mq^3 \\
& -24ab^6l^3mqr^2 + 16ab^6l^3q^3r - 36ab^6l^3qr^3 \\
& -32ab^6lm^2qr^3 + 54ab^6lmq^3r^2 - 22ab^6lmqr^4 \\
& -18ab^6lq^5r + 8ab^6lq^3r^3 - 10ab^6lqr^5 - 152a^4b^5l^3 \\
& -32a^4b^5lm^2 + 72a^4b^5lmr - 16a^4b^5lq^2 \\
& -48a^4b^5lr^2 + 152a^2b^5l^5 + 32a^2b^5l^3m^2 + 224a^2b^5l^3mr \\
& -108a^2b^5l^3q^2 - 56a^2b^5l^3r^2 + 64a^2b^5lm^3r \\
& -8a^2b^5lm^2q^2 - 176a^2b^5lm^2r^2 - 112a^2b^5lmq^2r \\
& +176a^2b^5lmr^3 + 20a^2b^5lq^4 + 60a^2b^5lq^2r^2 \\
& -32a^2b^5lr^4 + 8b^5l^5mr - 4b^5l^5q^2 - 48b^5l^5r^2 + 16b^5l^3m^2r^2 \\
& -16b^5l^3mq^2r - 112b^5l^3mr^3 + 4b^5l^3q^4 \\
& +72b^5l^3q^2r^2 + 32b^5l^3r^4 - 16b^5lm^2r^4 + 48b^5lmq^2r^3 \\
& -24b^5lmr^5 - 20b^5lq^4r^2 - 4b^5lq^2r^4 + 16b^5lr^6 \\
& -20a^3b^4lmq - 32a^3b^4lqr + 20ab^4l^3mq + 56ab^4l^3qr \\
& +64ab^4lm^2qr - 8ab^4lmq^3 - 172ab^4lmqr^2 \\
& -32ab^4lq^3r + 112ab^4lqr^3 - 4a^4b^3l - 8a^2b^3l^3 \\
& -8a^2b^3lm^2 + 88a^2b^3lmr - 20a^2b^3lq^2 - 48a^2b^3lr^2 \\
& +4b^3l^5 - 8b^3l^3r^2 - 16b^3lm^2r^2 + 32b^3lmq^2r - 4b^3lq^4 \\
& -48b^3lq^2r^2 + 4b^3lr^4 - 6ab^2lmq + 18ab^2lqr \\
& -4a^2bl + 4bl^3 + 8blmr - 4blq^2 - 4blr^2,
\end{aligned}$$

$$\begin{aligned}
e_4 = & 4a^6b^6mq - 18a^6b^6qr - 68a^4b^6l^2mq + 206a^4b^6l^2qr \\
& +8a^4b^6m^3q - 16a^4b^6m^2qr + 14a^4b^6mq^3 \\
& +80a^4b^6mqr^2 - 46a^4b^6q^3r - 34a^4b^6qr^3 \\
& -28a^2b^6l^4mq - 50a^2b^6l^4qr - 8a^2b^6l^2m^3q \\
& +204a^2b^6l^2m^2qr - 60a^2b^6l^2mq^3 - 516a^2b^6l^2mqr^2 \\
& +192a^2b^6l^2q^3r + 104a^2b^6l^2qr^3 + 72a^2b^6m^3qr^2 \\
& -68a^2b^6m^2q^3r - 160a^2b^6m^2qr^3 + 14a^2b^6mq^5 \\
& +166a^2b^6mq^3r^2 + 84a^2b^6mqr^4 - 30a^2b^6q^5r \\
& -54a^2b^6q^3r^3 - 14a^2b^6qr^5 + 8b^6l^6mq - 18b^6l^6qr \\
& +12b^6l^4m^2qr - 8b^6l^4mq^3 - 8b^6l^4mqr^2 + 8b^6l^4q^3r \\
& +22b^6l^4qr^3 + 4b^6l^2m^2qr^3 - 20b^6l^2mq^3r^2 \\
& +56b^6l^2mqr^4 + 9b^6l^2q^5r - 32b^6l^2q^3r^3 - 6b^6l^2qr^5 \\
& -4b^6mq^3r^4 + 8b^6mqr^6 + 3b^6q^5r^3 - 8b^6q^3r^5 \\
& +2b^6qr^7 - 74a^5b^5l^2 - 20a^5b^5m^2 + 36a^5b^5mr \\
& -5a^5b^5q^2 + 272a^3b^5l^4 + 64a^3b^5l^2m^2 + 112a^3b^5l^2mr \\
& -114a^3b^5l^2q^2 + 40a^3b^5l^2r^2 + 64a^3b^5m^3r \\
& +4a^3b^5m^2q^2 - 128a^3b^5m^2r^2 + 28a^3b^5mq^2r \\
& +56a^3b^5mr^3 - a^3b^5q^4 - 32a^3b^5q^2r^2 - 78ab^5l^6 \\
& -20ab^5l^4m^2 - 156ab^5l^4mr + 105ab^5l^4q^2 \\
& -12ab^5l^4r^2 - 40ab^5l^2m^3r + 8ab^5l^2m^2q^2 \\
& +104ab^5l^2m^2r^2 + 192ab^5l^2mq^2r - 344ab^5l^2mr^3 \\
& -64ab^5l^2q^4 - 18ab^5l^2q^2r^2 + 114ab^5l^2r^4 \\
& +72ab^5m^2q^2r^2 - 76ab^5m^2r^4 - 62ab^5mq^4r + 56ab^5mq^2r^3 \\
& +20ab^5mr^5 + 10ab^5q^6 + 22ab^5q^4r^2 \\
& -27ab^5q^2r^4 - 12a^4b^4mq + 6a^4b^4qr + 44a^2b^4l^2mq \\
& +102a^2b^4l^2qr + 108a^2b^4m^2qr - 72a^2b^4mqr^2
\end{aligned}$$

$$\begin{aligned}
& -27a^2b^4q^3r - 12a^2b^4qr^3 - 8b^4l^4mq + 6b^4l^4qr \\
& -28b^4l^2m^2qr + 8b^4l^2mq^3 + 116b^4l^2mqr^2 \\
& -6b^4l^2q^3r - 132b^4l^2qr^3 + 12b^4mq^3r^2 - 12b^4mqr^4 \\
& -9b^4q^5r + 30b^4q^3r^3 - 18b^4qr^5 - 12a^3b^3l^2 \\
& -16a^3b^3m^2 + 32a^3b^3mr - 6a^3b^3q^2 - 6ab^3l^4 + 4ab^3l^2m^2 \\
& -88ab^3l^2mr + 23ab^3l^2q^2 + 58ab^3l^2r^2 \\
& -52ab^3m^2r^2 + 108ab^3mq^2r + 32ab^3mr^3 - 8ab^3q^4 \\
& -69ab^3q^2r^2 - 18a^2b^2mq + 15a^2b^2qr \\
& -9b^2l^2qr - 12b^2mqr^2 + 18b^2q^3r - 3b^2qr^3 + 6abl^2 \\
& +6abmr - 6abq^2 + qr,
\end{aligned}$$

$$\begin{aligned}
e_5 = & -72a^5b^6lmq + 116a^5b^6lqr + 112a^3b^6l^3mq \\
& -248a^3b^6l^3qr - 16a^3b^6lm^3q + 304a^3b^6lm^2qr \\
& -108a^3b^6lmq^3 - 528a^3b^6lmqr^2 + 184a^3b^6lq^3r \\
& +136a^3b^6lqr^3 + 32ab^6l^5mq + 12ab^6l^5qr \\
& -56ab^6l^3m^2qr + 24ab^6l^3mq^3 + 264ab^6l^3mqr^2 \\
& -132ab^6l^3q^3r - 56ab^6l^3qr^3 - 96ab^6lm^3qr^2 \\
& +96ab^6lm^2q^3r + 224ab^6lm^2qr^3 - 24ab^6lmq^5 \\
& -240ab^6lmq^3r^2 - 72ab^6lmqr^4 + 62ab^6lq^5r \\
& +40ab^6lq^3r^3 + 20ab^6lqr^5 - 12a^6b^5l \\
& +220a^4b^5l^3 + 64a^4b^5lm^2 - 48a^4b^5lmr - 60a^4b^5lq^2 \\
& +60a^4b^5lr^2 - 220a^2b^5l^5 - 72a^2b^5l^3m^2 \\
& -320a^2b^5l^3mr + 256a^2b^5l^3q^2 + 64a^2b^5l^3r^2 - 128a^2b^5lm^3r \\
& +16a^2b^5lm^2q^2 + 208a^2b^5lm^2r^2 + 264a^2b^5lmq^2r \\
& -240a^2b^5lmr^3 - 80a^2b^5lq^4 - 84a^2b^5lq^2r^2 \\
& +60a^2b^5lr^4 + 12b^5l^7 + 8b^5l^5m^2 - 24b^5l^5q^2 \\
& +60b^5l^5r^2 + 16b^5l^3m^3r - 8b^5l^3m^2q^2 - 56b^5l^3m^2r^2 \\
& -24b^5l^3mq^2r + 192b^5l^3mr^3 + 26b^5l^3q^4 - 96b^5l^3q^2r^2 \\
& -60b^5l^3r^4 - 48b^5lm^2q^2r^2 + 48b^5lm^2r^4 \\
& +52b^5lmq^4r - 72b^5lmq^2r^3 - 14b^5lq^6 + 22b^5lq^4r^2 \\
& +24b^5lq^2r^4 - 12b^5lr^6 + 44a^3b^4lmq + 24a^3b^4lqr \\
& -44ab^4l^3mq - 36ab^4l^3qr - 136ab^4lm^2qr + 20ab^4lmq^3 \\
& +212ab^4lmqr^2 \\
& +42ab^4lq^3r - 120ab^4lqr^3 \\
& -4a^4b^3l + 4a^2b^3l^3 + 16a^2b^3lm^2 - 80a^2b^3lmr \\
& +10a^2b^3lq^2 + 48a^2b^3lr^2 + 4b^3l^5 + 24b^3l^3mr \\
& -12b^3l^3q^2 - 8b^3l^3r^2 + 32b^3lm^2r^2 - 64b^3lmq^2r \\
& -24b^3lmr^3 + 8b^3lq^4 + 60b^3lq^2r^2 + 4b^3lr^4 \\
& +12ab^2lmq - 18ab^2lqr + 2a^2bl - 2bl^3 - 4blmr \\
& +2blq^2 + 2blr^2,
\end{aligned}$$

$$\begin{aligned}
e_6 = & -16a^6b^6mq + 22a^6b^6qr + 156a^4b^6l^2mq \\
& -230a^4b^6l^2qr + 8a^4b^6m^3q + 72a^4b^6m^2qr - 46a^4b^6mq^3 \\
& -140a^4b^6mqr^2 + 58a^4b^6q^3r + 46a^4b^6qr^3 - 88a^2b^6l^4mq \\
& +170a^2b^6l^4qr + 16a^2b^6l^2m^3q \\
& -420a^2b^6l^2m^2qr + 168a^2b^6l^2mq^3 + 716a^2b^6l^2mqr^2 \\
& -266a^2b^6l^2q^3r - 188a^2b^6l^2qr^3 \\
& -168a^2b^6m^3qr^2 + 156a^2b^6m^2q^3r + 280a^2b^6m^2qr^3
\end{aligned}$$

$$\begin{aligned}
& -32a^2b^6mq^5 - 234a^2b^6mq^3r^2 - 136a^2b^6mqr^4 \\
& + 38a^2b^6q^5r + 64a^2b^6q^3r^3 + 26a^2b^6qr^5 - 8b^6l^6mq \\
& - 2b^6l^6qr + 4b^6l^4mq^2 - 44b^6l^4mqr^2 \\
& + 30b^6l^4q^3r - 10b^6l^4qr^3 + 24b^6l^2m^3qr^2 - 28b^6l^2m^2q^3r \\
& - 52b^6l^2m^2qr^3 + 8b^6l^2mq^5 + 84b^6l^2mq^3r^2 \\
& - 32b^6l^2mqr^4 - 29b^6l^2q^5r + 12b^6l^2q^3r^3 + 10b^6l^2qr^5 \\
& - 2b^6mq^5r^2 + 12b^6mq^3r^4 - 12b^6mqr^6 + b^6q^7r \\
& - 7b^6q^5r^3 + 6b^6q^3r^5 + 2b^6qr^7 + 90a^5b^5l^2 \\
& + 20a^5b^5m^2 - 20a^5b^5mr - 7a^5b^5q^2 - 220a^3b^5l^4 \\
& - 72a^3b^5l^2m^2 - 120a^3b^5l^2mr + 202a^3b^5l^2q^2 \\
& - 36a^3b^5l^2r^2 - 48a^3b^5m^3r + 24a^3b^5m^2q^2 \\
& + 72a^3b^5m^2r^2 + 24a^3b^5mq^2r - 56a^3b^5mr^3 \\
& - 28a^3b^5q^4 + 14a^3b^5q^2r^2 + 90ab^5l^6 + 44ab^5l^4m^2 \\
& + 140ab^5l^4mr - 185ab^5l^4q^2 + 28ab^5l^4r^2 + 88ab^5l^2m^3r \\
& - 20ab^5l^2m^2q^2 - 152ab^5l^2m^2r^2 \\
& - 336ab^5l^2mq^2r + 360ab^5l^2mr^3 + 135ab^5l^2q^4 \\
& + 44ab^5l^2q^2r^2 - 126ab^5l^2r^4 - 156ab^5m^2q^2r^2 \\
& + 84ab^5m^2r^4 + 138ab^5mq^4r - 36ab^5mr^5 - 24ab^5q^6 \\
& - 39ab^5q^4r^2 + 21ab^5q^2r^4 + 20a^4b^4mq \\
& - 8a^4b^4qr - 44a^2b^4l^2mq - 38a^2b^4l^2qr - 60a^2b^4m^2qr \\
& + 12a^2b^4mq^3 + 16a^2b^4mqr^2 + 17a^2b^4q^3r \\
& + 16b^4l^4mq - 8b^4l^4qr + 56b^4l^2m^2qr \\
& - 16b^4l^2mq^3 - 124b^4l^2mqr^2 - 6b^4l^2q^3r + 96b^4l^2qr^3 \\
& - 24b^4mq^3r^2 + 12b^4mq^4r + 18b^4q^5r - 18b^4q^3r^3 \\
& + 8b^4qr^5 + 10a^3b^3l^2 + 4a^3b^3m^2 - 8a^3b^3mr \\
& + 3a^3b^3q^2 + 8ab^3l^4 - 8ab^3l^2m^2 + 64ab^3l^2mr - 12ab^3l^2q^2 \\
& - 40ab^3l^2r^2 + 32ab^3m^2r^2 - 60ab^3mq^2r \\
& - 24ab^3mr^3 + 4ab^3q^4 + 32ab^3q^2r^2 + 12a^2b^2mq - 7a^2b^2qr \\
& + 5b^2l^2qr + 6b^2mqr^2 - 9b^2q^3r - b^2qr^3 \\
& - 2abl^2 - 2abmr + 4abq^2,
\end{aligned}$$

$$\begin{aligned}
e_7 = & 76a^5b^6lmq - 84a^5b^6lqr - 156a^3b^6l^3mq + 208a^3b^6l^3qr \\
& + 8a^3b^6lm^3q - 288a^3b^6lm^2qr \\
& + 126a^3b^6lmq^3 + 424a^3b^6lmqr^2 - 156a^3b^6lq^3r \\
& - 104a^3b^6lqr^3 + 40ab^6l^5mq - 84ab^6l^5qr \\
& + 216ab^6l^3m^2qr - 106ab^6l^3mq^3 - 404ab^6l^3mqr^2 \\
& + 172ab^6l^3q^3r + 112ab^6l^3qr^3 + 200ab^6lm^3qr^2 \\
& - 200ab^6lm^2q^3r - 368ab^6lm^2qr^3 + 50ab^6lmq^5 \\
& + 330ab^6lmq^3r^2 + 156ab^6lmqr^4 - 72ab^6lq^5r \\
& - 76ab^6lq^3r^3 - 20ab^6lqr^5 + 16a^6b^5l - 120a^4b^5l^3 \\
& - 32a^4b^5lm^2 - 8a^4b^5lmr + 80a^4b^5lq^2 - 32a^4b^5lr^2 \\
& + 120a^2b^5l^5 + 48a^2b^5l^3m^2 + 160a^2b^5l^3mr \\
& - 204a^2b^5l^3q^2 - 8a^2b^5l^3r^2 + 64a^2b^5lm^3r - 8a^2b^5lm^2q^2 \\
& - 80a^2b^5lm^2r^2 - 224a^2b^5lmq^2r + 144a^2b^5lmr^3 \\
& + 92a^2b^5lq^4 + 52a^2b^5lq^2r^2 - 48a^2b^5lr^4 - 16b^5l^7 \\
& - 16b^5l^5m^2 - 8b^5l^5mr + 52b^5l^5q^2 - 32b^5l^5r^2 \\
& - 32b^5l^3m^3r + 16b^5l^3m^2q^2 + 64b^5l^3m^2r^2 + 96b^5l^3mq^2r \\
& - 144b^5l^3mr^3 - 64b^5l^3q^4 + 24b^5l^3q^2r^2 \\
& + 48b^5l^3r^4 + 96b^5lm^2q^2r^2 - 48b^5lm^2r^4 - 104b^5lmq^4r \\
& + 24b^5lmr^5 + 28b^5lq^6 + 16b^5lq^4r^2 - 12b^5lq^2r^4 \\
& - 28a^3b^4lmq + 28ab^4l^3mq + 80ab^4l^2mq
\end{aligned}$$

$$\begin{aligned}
& - 16ab^4lmq^3 - 100ab^4lmqr^2 - 12ab^4lq^3r + 48ab^4lqr^3 \\
& + 4a^4b^3l - 8a^2b^3lm^2 + 24a^2b^3lmr \\
& - 16a^2b^3lr^2 - 4b^3l^5 - 16b^3l^3mr + 8b^3l^3q^2 + 8b^3l^3r^2 \\
& - 16b^3lm^2r^2 + 32b^3lmq^2r \\
& + 16b^3lmr^3 - 4b^3lq^4 - 24b^3lq^2r^2 \\
& - 4b^3lr^4 - 6ab^2lmq + 6ab^2lqr,
\end{aligned}$$

$$\begin{aligned}
e_8 = & 14a^6b^6mq - 13a^6b^6qr - 100a^4b^6l^2mq + 111a^4b^6l^2qr \\
& - 8a^4b^6m^3q - 72a^4b^6m^2qr + 36a^4b^6mq^3 \\
& + 100a^4b^6mqr^2 - 33a^4b^6q^3r - 29a^4b^6qr^3 \\
& + 94a^2b^6l^4mq - 117a^2b^6l^4qr - 8a^2b^6l^2m^3q \\
& + 276a^2b^6l^2m^2qr - 124a^2b^6l^2mq^3 - 400a^2b^6l^2mqr^2 \\
& + 147a^2b^6l^2q^3r + 112a^2b^6l^2qr^3 + 120a^2b^6m^3qr^2 \\
& - 108a^2b^6m^2q^3r - 192a^2b^6m^2qr^3 + 22a^2b^6mq^5 \\
& + 138a^2b^6mq^3r^2 + 94a^2b^6mqr^4 - 22a^2b^6q^5r \\
& - 33a^2b^6q^3r^3 - 19a^2b^6qr^5 - 8b^6l^6mq \\
& + 19b^6l^6qr - 44b^6l^4m^2qr + 24b^6l^4mq^3 + 88b^6l^4mqr^2 \\
& - 48b^6l^4q^3r - 17b^6l^4qr^3 - 48b^6l^2m^3qr^2 + 56b^6l^2m^2q^3r \\
& + 92b^6l^2m^2qr^3 - 16b^6l^2mq^5 - 108b^6l^2mq^3r^2 \\
& - 24b^6l^2mqr^4 + 31b^6l^2q^5r + 16b^6l^2q^3r^3 + b^6l^2qr^5 \\
& + 4b^6mq^5r^2 - 12b^6mq^3r^4 + 8b^6mqr^6 - 2b^6q^7r \\
& + 5b^6q^5r^3 - 3b^6qr^7 - 36a^5b^5l^2 - 6a^5b^5mr + 11a^5b^5q^2 \\
& + 70a^3b^5l^4 + 28a^3b^5l^2m^2 + 48a^3b^5l^2mr \\
& - 111a^3b^5l^2q^2 + 18a^3b^5l^2r^2 - 12a^3b^5m^2q^2 \\
& + 12a^3b^5m^2r^2 - 58a^3b^5mq^2r + 12a^3b^5mr^3 + 29a^3b^5q^4 \\
& + 5a^3b^5q^2r^2 - 34ab^5l^6 - 28ab^5l^4m^2 - 38ab^5l^4mr \\
& + 98ab^5l^4q^2 - 20ab^5l^4r^2 - 56ab^5l^2m^3r \\
& + 16ab^5l^2m^2q^2 + 88ab^5l^2m^2r^2 + 176ab^5l^2mq^2r \\
& - 156ab^5l^2mr^3 - 82ab^5l^2q^4 - 28ab^5l^2q^2r^2 \\
& + 54ab^5l^2r^4 + 96ab^5m^2q^2r^2 - 36ab^5m^2r^4 \\
& - 90ab^5mq^4r - 24ab^5mq^2r^3 + 18ab^5mr^5 + 18ab^5q^6 \\
& + 24ab^5q^4r^2 - 6ab^5q^2r^4 - 8a^4b^4mq + 3a^4b^4qr \\
& + 16a^2b^4l^2mq - 6a^2b^4l^2qr + 4a^2b^4m^2qr - 8a^2b^4mq^3 \\
& + 4a^2b^4mqr^2 - 6a^2b^4q^3r + 2a^2b^4qr^3 - 8b^4l^4mq \\
& + 3b^4l^4qr - 28b^4l^2m^2qr + 8b^4l^2mq^3 + 44b^4l^2mqr^2 \\
& + 6b^4l^2q^3r - 26b^4l^2qr^3 + 12b^4mq^3r^2 - 4b^4mqr^4 \\
& - 9b^4q^5r + 2b^4q^3r^3 - b^4qr^5 + 4a^3b^3m^2 - 4a^3b^3mr \\
& + 4ab^3l^2m^2 - 12ab^3l^2mr + 8ab^3l^2r^2 - 4ab^3m^2r^2 \\
& + 4ab^3mq^2r + 4ab^3mr^3 - 4ab^3q^2r^2,
\end{aligned}$$

$$\begin{aligned}
e_9 = & -2b^4l\Delta_r \left(12a^3b^2mq - 13a^3b^2qr - 20ab^2l^2mq \right. \\
& + 21ab^2l^2qr - 28ab^2m^2qr + 14ab^2mq^3 \\
& + 34ab^2mqr^2 - 15ab^2q^3r - 5ab^2qr^3 + 3a^4b \\
& - 6a^2bl^2 + 10a^2bq^2 - 6a^2br^2 + 3bl^4 + 4bl^2m^2 \\
& - 4bl^2mr - 10bl^2q^2 + 6bl^2r^2 - 12bmqr^2 \\
& \left. + 4bmr^3 + 7bq^4 + 2bq^2r^2 - br^4 - 2amq + 2aqr \right),
\end{aligned}$$

$$e_{10} = -b^5\Delta_r \left(4a^4bmqr - 3a^4bqr - 12a^2bl^2mq + 12a^2bl^2qr \right)$$

$$\begin{aligned}
& -12a^2bm^2qr + 4a^2bmqr^3 \\
& +16a^2bmqr^2 - 4a^2bq^3r - 4a^2bqr^3 + 8bl^4mq \\
& -9bl^4qr + 12bl^2m^2qr - 8bl^2mq^3 - 16bl^2mqr^2 \\
& +10bl^2q^3r + 2bl^2qr^3 - bq^5r + 2bq^3r^3 \\
& -bqr^5 - 2a^3l^2 + 4a^3m^2 - 6a^3mr + 4a^3q^2 + 2al^4 \\
& +4al^2m^2 - 6al^2mr - 6al^2q^2 + 6al^2r^2 - 6amq^2r \\
& +2amr^3 + 4aq^4), \\
e_{11} = & +2ab^6lq(m-r)\Delta_r^2,
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
f_0 = & 9a^4b^4l^2 + 4a^4b^4m^2 + 4a^4b^4mr + a^4b^4r^2 \\
& +6a^2b^4l^4 + 12a^2b^4l^2mr \\
& +4a^2b^4mr^3 + 2a^2b^4r^4 + b^4l^6 \\
& +7b^4l^4r^2 + 7b^4l^2r^4 + b^4r^6 + 8a^3b^3mq + 4a^3b^3qr \\
& +12ab^3l^2qr + 4ab^3qr^3 + 6a^2b^2l^2 + 4a^2b^2mr \\
& +4a^2b^2q^2 + 2a^2b^2r^2 + 2b^2l^4 + 4b^2l^2r^2 \\
& +2b^2r^4 + 4abqr + l^2 + r^2, \\
f_1 = & 6a^5b^4l - 28a^3b^4l^3 - 8a^3b^4lm^2 - 24a^3b^4lmr + 12a^3b^4lq^2 \\
& +12a^3b^4lr^2 - 10ab^4l^5 - 24ab^4l^3mr \\
& +4ab^4l^3q^2 + 36ab^4l^3r^2 + 8ab^4lmr^3 - 4ab^4lq^2r^2 \\
& +6ab^4lr^4 - 16a^2b^3lmq - 16b^3l^3qr + 8a^3b^2l \\
& -8ab^2l^3 - 16ab^2lmr - 4ab^2lq^2 + 8ab^2lr^2 - 8blqr + 2al, \\
f_2 = & a^6b^4 - 46a^4b^4l^2 - 20a^4b^4mr + 4a^4b^4q^2 \\
& -2a^4b^4r^2 + 25a^2b^4l^4 + 12a^2b^4l^2m^2 + 60a^2b^4l^2mr \\
& -38a^2b^4l^2q^2 - 18a^2b^4l^2r^2 + 36a^2b^4m^2r^2 \\
& -20a^2b^4mq^2r - 20a^2b^4mr^3 + 4a^2b^4q^4 + 6a^2b^4q^2r^2 \\
& -7a^2b^4r^4 + 4b^4l^6 + 24b^4l^4mr - 6b^4l^4q^2 - 44b^4l^4r^2 \\
& -8b^4l^2mr^3 + 12b^4l^2q^2r^2 - 20b^4l^2r^4 + 2b^4q^2r^4 \\
& -4b^4r^6 - 8a^3b^3mq - 16a^3b^3qr + 16ab^3l^2mq \\
& -32ab^3l^2qr + 24ab^3mqr^2 - 4ab^3q^3r - 16ab^3qr^3 \\
& +2a^4b^2 - 28a^2b^2l^2 - 16a^2b^2mr - 4a^2b^2q^2 - 4a^2b^2r^2 \\
& +2b^2l^4 + 16b^2l^2mr - 2b^2l^2q^2 - 20b^2l^2r^2 \\
& +6b^2q^2r^2 - 6b^2r^4 - 12abqr + a^2 - 2l^2 - 2r^2, \\
f_3 = & -24a^5b^4l + 96a^3b^4l^3 + 8a^3b^4lm^2 \\
& +128a^3b^4lmr - 48a^3b^4lq^2 - 48a^3b^4lr^2 \\
& -8ab^4l^5 - 8ab^4l^3m^2 \\
& -16ab^4l^3mr + 32ab^4l^3q^2 - 64ab^4l^3r^2 - 72ab^4lm^2r^2 \\
& +56ab^4lmq^2r + 32ab^4lmr^3 - 12ab^4lq^4 \\
& -24ab^4lr^4 + 24a^2b^3lmq + 16a^2b^3lqr - 8b^3l^3mq \\
& +32b^3l^3qr - 24b^3lmqr^2 + 8b^3lq^3r + 16b^3lqr^3 \\
& -24a^3b^2l + 24ab^2l^3 + 48ab^2lmr \\
& -24ab^2lr^2 + 16blqr - 4al, \\
f_4 = & -4a^6b^4 + 86a^4b^4l^2 - 8a^4b^4m^2 + 40a^4b^4mr \\
& -14a^4b^4q^2 - 2a^4b^4r^2 - 96a^2b^4l^4 - 16a^2b^4l^2m^2 \\
& -240a^2b^4l^2mr + 108a^2b^4l^2q^2 + 72a^2b^4l^2r^2 \\
& -96a^2b^4m^2r^2 + 64a^2b^4mq^2r + 40a^2b^4mr^3 - 12a^2b^4q^4 \\
& -20a^2b^4q^2r^2 + 8a^2b^4r^4 - 2b^4l^6 + 4b^4l^4m^2 \\
& -24b^4l^4mr - 6b^4l^4q^2 + 58b^4l^4r^2 + 36b^4l^2m^2r^2
\end{aligned}$$

$$\begin{aligned}
& -32b^4l^2mq^2r - 24b^4l^2mr^3 + 9b^4l^2q^4 - 12b^4l^2q^2r^2 \\
& +34b^4l^2r^4 + b^4q^4r^2 - 6b^4q^2r^4 + 6b^4r^6 - 8a^3b^3mq \\
& +24a^3b^3qr - 32ab^3l^2mq + 24ab^3l^2qr - 56ab^3mqr^2 \\
& +12ab^3q^3r + 24ab^3qr^3 - 6a^4b^2 + 40a^2b^2l^2 \\
& +24a^2b^2mr - 6a^2b^2q^2 - 10b^2l^4 - 32b^2l^2mr \\
& +4b^2l^2q^2 + 28b^2l^2r^2 - 12b^2q^2r^2 + 6b^2r^4 + 12abqr \\
& -2a^2 + l^2 + r^2, \\
f_5 = & 36a^5b^4l - 120a^3b^4l^3 + 8a^3b^4lm^2 \\
& -208a^3b^4lmr + 72a^3b^4lq^2 + 72a^3b^4lr^2 \\
& +52ab^4l^5 + 16ab^4l^3m^2 \\
& +128ab^4l^3mr - 88ab^4l^3q^2 + 8ab^4l^3r^2 + 168ab^4lm^2r^2 \\
& -136ab^4lmq^2r - 112ab^4lmr^3 + 30ab^4lq^4 \\
& +24ab^4lq^2r^2 + 36ab^4lr^4 - 32a^2b^3lqr + 16b^3l^3mq \\
& -16b^3l^3qr + 48b^3lmqr^2 - 16b^3lq^3r - 32b^3lqr^3 \\
& +24a^3b^2l - 24ab^2l^3 - 48ab^2lmr + 12ab^2lq^2 + 24ab^2lr^2 \\
& -8blqr + 2al, \\
f_6 = & 6a^6b^4 - 72a^4b^4l^2 - 40a^4b^4mr + 18a^4b^4q^2 + 8a^4b^4r^2 \\
& +94a^2b^4l^4 - 4a^2b^4l^2m^2 + 256a^2b^4l^2mr \\
& -104a^2b^4l^2q^2 - 92a^2b^4l^2r^2 + 88a^2b^4m^2r^2 \\
& -72a^2b^4mq^2r - 40a^2b^4mr^3 + 13a^2b^4q^4 + 24a^2b^4q^2r^2 \\
& -2a^2b^4r^4 - 12b^4l^6 - 8b^4l^4m^2 - 24b^4l^4mr \\
& +30b^4l^4q^2 - 12b^4l^4r^2 - 72b^4l^2m^2r^2 + 64b^4l^2mq^2r \\
& +72b^4l^2mr^3 - 18b^4l^2q^4 - 12b^4l^2q^2r^2 \\
& -36b^4l^2r^4 - 2b^4q^4r^2 + 6b^4q^2r^4 - 4b^4r^6 + 8a^3b^3mq \\
& -16a^3b^3qr + 16ab^3l^2mq + 40ab^3mqr^2 - 12ab^3q^3r \\
& -16ab^3qr^3 + 6a^4b^2 - 20a^2b^2l^2 - 16a^2b^2mr \\
& +8a^2b^2q^2 + 4a^2b^2r^2 + 6b^2l^4 + 16b^2l^2mr \\
& -2b^2l^2q^2 - 12b^2l^2r^2 + 6b^2q^2r^2 - 2b^2r^4 - 4abqr + a^2, \\
f_7 = & -24a^5b^4l + 64a^3b^4l^3 - 8a^3b^4lm^2 + 128a^3b^4lmr \\
& -48a^3b^4lq^2 - 48a^3b^4lr^2 - 40ab^4l^5 - 8ab^4l^3m^2 \\
& -112ab^4l^3mr + 64ab^4l^3q^2 + 32ab^4l^3r^2 - 120ab^4lm^2r^2 \\
& +104ab^4lmq^2r + 96ab^4lmr^3 - 24ab^4lq^4 \\
& -32ab^4lq^2r^2 - 24ab^4lr^4 - 8a^2b^3lmq + 16a^2b^3lqr \\
& -8b^3l^3mq - 24b^3lmqr^2 + 8b^3lq^3r + 16b^3lqr^3 \\
& -8a^3b^2l + 8ab^2l^3 + 16ab^2lmr - 8ab^2lq^2 - 8ab^2lr^2, \\
f_8 = & -4a^6b^4 + 25a^4b^4l^2 + 4a^4b^4m^2 + 20a^4b^4mr \\
& -10a^4b^4q^2 - 7a^4b^4r^2 - 30a^2b^4l^4 + 8a^2b^4l^2m^2 \\
& -92a^2b^4l^2mr + 36a^2b^4l^2q^2 + 40a^2b^4l^2r^2 \\
& -32a^2b^4m^2r^2 + 32a^2b^4mq^2r + 20a^2b^4mr^3 - 6a^2b^4q^4 \\
& -12a^2b^4q^2r^2 - 2a^2b^4r^4 + 9b^4l^6 + 4b^4l^4m^2 \\
& +24b^4l^4mr - 18b^4l^4q^2 - 9b^4l^4r^2 + 36b^4l^2m^2r^2 \\
& -32b^4l^2mq^2r - 40b^4l^2mr^3 + 9b^4l^2q^4 + 12b^4l^2q^2r^2 \\
& +15b^4l^2r^4 + b^4q^4r^2 - 2b^4q^2r^4 + b^4r^6 + 4a^3b^3qr \\
& -4ab^3l^2qr - 8ab^3mqr^2 + 4ab^3q^3r + 4ab^3qr^3 - 2a^4b^2 \\
& +2a^2b^2l^2 + 4a^2b^2mr - 2a^2b^2q^2 - 2a^2b^2r^2, \\
f_9 = & 6ab^4l\Delta_r^2, \\
f_{10} = & a^2b^4\Delta_r^2.
\end{aligned} \tag{B.6}$$

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