Sub-horizon modes and growth index in a linear scalar cosmological perturbations

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Abstract We test a four dimensional cosmological model embedded in a five dimensional bulk space by means of the dynamical Nash-Greene theorem. In a fluid approach, we apply a joint likelihood analysis to the data with the Markov Chain Monte Carlo (MCMC) method for cosmological parameter estimation. We use recent datasets as the “Gold 2018” growth-rate data, the Planck2018/ Λ CDM data on the cosmic microwave background (CMB) anisotropies, the Baryon acoustic oscillations (BAO) measurements, the Pantheon Supernovae type Ia and the data on the Hubble parameter $H(z)$ with redshift ranging from $0.01 < z < 2.3$. Performing the Information Criterion (IC) analysis, we find that the present model is in very good agreement with observations with a close statistical equivalence with w CDM cosmologies at $1-\sigma$ level with a slightly larger growth profiles. By modifications of CLASS(EFCLASS) code, we make a comparison between the models on their unlensed CMB TT temperature spectra. Moreover, the proposed model presents a low power spectrum by the reduction of the ISW effect at lower multipoles. We also find that the overall percentage relative difference of the growth index $\Delta\gamma(\%)$ is up to 1.4% as compared to w CDM pattern in sub-horizon scales.

1 Introduction

One of the fundamental question in contemporary observational cosmology is the understanding of matter distributions in the universe. The evolution of such distributions by the growth of large scale structure (LSS) impacts on cosmological models that must comply with such observational constraints. The current debate is on the fact that how such distribution may be affected by dark components in the context of

the dark matter and dark energy (DE). As commonly known, on the acceleration expansion problem, such dark components may have a time-independent (the cosmological constant) component with a dark energy equation of state (EOS) with negative fluid parameter $w = -1$ [1,2] but it is possible that it may evolve dynamically in time with mild deviations of the fluid parameter w . An interesting tool to evaluate and discriminate DE scalar fields and modified gravity models relies on the evolution of the linear growth of matter fluctuations and on the definition of the growth index γ which has been extensively studied by several cosmological models in literature (see [3–9] and refs. therein). This quantity is considered as a promising tool as a cosmological parameter. The reference value is assumed to be $\gamma = \frac{6}{11}$ for the Λ CDM model and deviations from that value sever as classifier for models with modifications of gravity. In this paper, we adopt the standard parameterization originally introduced by Peebles [10,11], inasmuch it is still the only ongoing mechanism that fits well the currently cosmological data.

An alternative indication for a new physics pinpoints that the universe may be embedded in a larger space (large extra dimensions). Most of these models have been Kaluza-Klein or/and string inspired, such as, for instance, the seminal works of the Arkani-Hamed, Dvali and Dimopolous (ADD) model [12], the Randall-sundrum model [13,14] and the Dvali-Gabadadze-Porrati model (DPG) [15]. In a different direction, apart from the brane-world archetypes, we use the dynamical embedding of geometries as an oriented mathematical background for a physical model [16–29]. In this paper, we propose an effective fluid approach for better confrontation of the present model to popular Λ CDM and w CDM models.

The Markov Chain Monte Carlo (MCMC) is used as a cosmological parameter estimator from a publicly available code [30–32]. The joint analysis is made from the latest data on

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Cosmic Microwave Background (CMB) Planck 2018 [2], the largest dataset of Pantheon SNIa [33] with redshift ranging from $0.01 < z < 2.3$, the Hubble parameter as a function of redshift $H(z)$ [34–39] and the “extended Gold 2018” compilation of growth-rate data using the data points of SDSS [40–42], 6dFGS [43], IRAS [44, 45], 2MASS [44, 46], 2dFGRS [47], GAMA [48], BOSS [49], WiggleZ [50], Vipers [51], FastSound [52], BOSS Q [53] and additional points from the 2018 SDSS-IV [54–56].

The paper is organized as follows: in the second section, we present the theoretical framework and its resulting cosmological model. In the third section, the perturbed equations are presented in a conformal Newtonian frame and also the growth equation in an effective fluid approach. The fourth section presents the outcomes and discussions from the comparison of the present model to Λ CDM and w CDM models in sub-horizon approximations. To analyse the response through the Integrated Sachs-Wolfe (ISW) effect, the full CMB spectrum is presented by using an adapted code from EFCLASS code [32] (a modification of the publicly available CLASS¹ code [57–59]) and the linear matter power spectrum $P(k)$ from the constrained cosmological parameters. As criteria for model selection, we use three information criteria such as the Akaike Information Criterion (AIC) [60], Modified Bayesian Information Criteria (MBIC) [61, 62] and Hannan–Quinn Criterion (HQC) [63]. In the final section, we conclude with our remarks and future prospects.

2 The induced four-dimensional cosmological model

The present model is inspired mostly on seminal studies in the 80’s inspired in the non-Abelian Kaluza–Klein theory [64–69]. In order to avoid any confusion with regular brane-world models [13–15, 70] because those models use different principles, e.g., the Israel–Lanczos condition [71] and $\mathbb{Z}_2$ -orbifold symmetry, which is inconsistent with the geometric deformation process in this paper, we do not identify the present model as brane-world one for the main following reason: the embedding of geometries is a dynamical character and the dynamics of the extrinsic curvature $k_{\mu\nu}$ is not replaced by an algebraic relation whatsoever and is regarded as a complement to the description of the whole gravitational field with normal and tangent components. In the dynamical embedding, the geometrical embedding process consists in the transference of the bulk dynamics to the embedded space once the integrability conditions of the embedding (Gauss, Codazzi and Ricci equations) are properly satisfied. This may sound a particularly interesting for elaborating a physical model, since we can study straightforwardly the dynamics of the gravity induced onto the four embedded space-time with-

out concerning the bulk dynamics for the aforementioned reasons. In the dynamical embedding, the bulk serves only as a curvature reference for the embedded space-time. On the other hand, for rigid embedding models, such DGP and variants, the anisotropic stress in the bulk poses the main challenge while calculating the growth of perturbations that lead to the appearance of ghosts in the theory. It is important to point out that there are several forms to embed a geometry. We adopt the dynamical embedding by means of smooth (differentiable) deformations originally applied to a flat D -dimensional Euclidean space embedding [72] and later on to Riemannian manifolds [73] by J. Nash. Years later, the Nash embedding theorem was extended to Pseudo-Riemannian manifolds by R. Greene [74] in the 70’s. Accordingly, we write the main result of the theorem such that

$$k_{\mu\nu} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial y}, \quad (1)$$

where $k_{\mu\nu}$ denotes the extrinsic curvature and y represents a coordinate on a direction orthogonal to the embedded geometry [73]. This process is generated by the appearance of the extrinsic curvature and is entirely consistent with general relativity (GR) adding to Einstein’s equations a new conserved quantity which can be therefore associated with an observable quantity and can be applied to general and arbitrary dimensions as originally proposed in [17–19].

2.1 Overview on dynamical embeddings

The gravitational action functional in the presence of confined matter field of a four-dimensional embedded space with thickness l can be generally embedded in a D -dimensional ambient space (bulk) has the form

$$S = -\frac{1}{2\kappa_D^2} \int \sqrt{|\mathcal{G}|} \mathcal{R} d^D x - \int \sqrt{|\mathcal{G}|} \mathcal{L}_m^* d^D x, \quad (2)$$

where κ_D^2 is the fundamental energy scale on the embedded space, $\mathcal{R}$ denotes the D -Ricci scalar of the bulk and $\mathcal{L}_m^*$ is the confined matter lagrangian. In this model, the matter energy momentum tensor occupies a finite hypervolume with constant radius l along the extra-dimensions. In this paper, we focus our attention on the induced four-dimensional model embedded into a five dimensional bulk. The variation of Einstein–Hilbert action in Eq. (2) with respect to the bulk metric $\mathcal{G}_{AB}$ leads to the Einstein equations for the bulk

$$\mathcal{R}_{AB} - \frac{1}{2} \mathcal{R} \mathcal{G}_{AB} = \alpha^* \mathcal{T}_{AB}, \quad (3)$$

where α^* is energy scale parameter and $\mathcal{T}_{AB}$ is the energy-momentum tensor for the bulk [18, 19, 22]. To generate an embedded space-time for a viable physical models is important to perturb the related background in accordance with the

¹ Available at https://github.com/lesgourg/class_public.

confinement hypothesis of gauge interactions that depends only on the four-dimensionality of the space-time [75–77]. It guarantees that the matter is localized in the embedded geometry and such geometry remains smooth (differentiable) after perturbations due to Nash-Greene perturbations. As an application, we focus our attention on the study of the dynamics of the induced four dimensional cosmological embedded in a five-dimensional bulk. Capital Latin indices run from 1 to 5. Small case Latin indices refer to the only one extra dimension considered. Moreover, let a Riemannian manifold V_4 be endowed with a non-perturbed metric $\bar{g}_{\mu\nu}$ being locally and isometrically embedded in a five-dimensional Riemannian geometry V_5 given by a differentiable and regular map $\mathcal{X} : V_4 \rightarrow V_5$ that satisfies the embedding equations

$$\mathcal{X}_{,\mu}^A \mathcal{X}_{,\nu}^B \mathcal{G}_{AB} = \bar{g}_{\mu\nu}, \quad (4)$$

$$\mathcal{X}_{,\mu}^A \bar{\eta}_a^B \mathcal{G}_{AB} = 0, \quad (5)$$

$$\bar{\eta}_a^A \bar{\eta}_b^B \mathcal{G}_{AB} = \bar{g}_{ab}, \quad (6)$$

where we denote by $\mathcal{G}_{AB}$ the metric components of V_5 in arbitrary coordinates, $\bar{\eta}$ denotes a non-perturbed unit vector field orthogonal to V_4 . This set of equations represents the isometry condition Eq. (4), orthogonality between the embedding coordinates $\mathcal{X}$ and $\bar{\eta}$ in Eq. (5), and also, the vector normalization $\bar{\eta}$ and $\bar{g}_{ab} = \epsilon_a \delta_{ab}$ with $\epsilon_a = \pm 1$ in which the signs represent the signatures of the extra-dimensions. Hence, the integration of the system of equations Eqs. (4), (5) and (6) assures the configuration of the embedding map $\mathcal{X}$.

The second fundamental form, or more commonly saying, the non-perturbed extrinsic curvature $\bar{k}_{\mu\nu}$ of V_4 is by definition the projection of the variation of $\bar{\eta}$ onto the tangent plane :

$$\bar{k}_{\mu\nu} = -\mathcal{X}_{,\mu}^A \bar{\eta}_{,\nu}^B \mathcal{G}_{AB} = \mathcal{X}_{,\mu\nu}^A \bar{\eta}^B \mathcal{G}_{AB}, \quad (7)$$

where the comma denotes the ordinary derivative.

The Nash-Greene geometrical deformation is better understood when we apply a Lie transport. Let a geometric object $\bar{\Omega}$ be in $\bar{V}_5$, its Lie transport along the flow for a small distance δy is given by $\Omega = \bar{\Omega} + \delta y \mathcal{L}_{\bar{\eta}} \bar{\Omega}$, where $\mathcal{L}_{\bar{\eta}}$ denotes the Lie derivative with respect to $\bar{\eta}$ [78]. In particular, the Lie transport of the Gaussian frame $\{\bar{X}_{\mu}^A, \bar{\eta}_a^A\}$, defined on $\bar{V}_n$ gives the perturbed frame

$$Z_{,\mu}^A = X_{,\mu}^A + \delta y \mathcal{L}_{\bar{\eta}} X_{,\mu}^A = X_{,\mu}^A + \delta y \eta_{,\mu}^A, \quad (8)$$

$$\eta^A = \bar{\eta}^A + \delta y [\bar{\eta}, \bar{\eta}]^A = \bar{\eta}^A. \quad (9)$$

However, from Eq. (7), we note that in general $\eta_{,\mu} \neq \bar{\eta}_{,\mu}$. The set of perturbed coordinates Z^A obtained by integrating the former equations does not necessarily describe another manifold. In order to be so, they need to satisfy embedding equations similar to Eqs. (4), (5) and (6) such as

$$Z_{,\mu}^A Z_{,\nu}^B \mathcal{G}_{AB} = g_{\mu\nu}, \quad Z_{,\mu}^A \eta^B \mathcal{G}_{AB} = 0, \quad \eta^A \eta^B \mathcal{G}_{AB} = 1. \quad (10)$$

Replacing Eqs. (8) and (9) in Eq. (10) and using the definition Eq. (7), we obtain the metric and extrinsic curvature of the new manifold

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - 2y \bar{k}_{\mu\nu} + y^2 \bar{g}^{\rho\sigma} \bar{k}_{\mu\rho} \bar{k}_{\nu\sigma}, \quad (11)$$

$$k_{\mu\nu} = \bar{k}_{\mu\nu} - 2y \bar{g}^{\rho\sigma} \bar{k}_{\mu\rho} \bar{k}_{\nu\sigma}. \quad (12)$$

Taking the derivative of Eq. (11) with respect to the coordinate y , we obtain Nash's deformation condition in Eq. (1). The orthogonal direction pointed by the y parameter leads to the extra-dimension in the ambient bulk space during the embedding process.

The integrability conditions for equations in Eq. (10) are given by the non-trivial components of the Riemann tensor of the embedding space², expressed in the new frame $\{Z_{,\mu}^A, \eta^B\}$ such as

$${}^5\mathcal{R}_{ABCD} Z_{,\alpha}^A Z_{,\beta}^B Z_{,\gamma}^C Z_{,\delta}^D = R_{\alpha\beta\gamma\delta} + (k_{\alpha\gamma} k_{\beta\delta} - k_{\alpha\delta} k_{\beta\gamma}), \quad (13)$$

$${}^5\mathcal{R}_{ABCD} Z_{,\alpha}^A Z_{,\beta}^B Z_{,\gamma}^C \eta^D = k_{\alpha[\beta;\gamma]}. \quad (14)$$

These are the Gauss–Codazzi equations (the third equation –the Ricci equation–does not appear in the case of just one extra dimension) [79]. The first of these equations (Gauss) shows that the Riemann curvature of the embedding space acts as a reference for the Riemann curvature of the embedded space-time. Both Riemann curvatures are ambiguous in the sense described by Riemann, but Gauss' equation in Eq. (13) shows that their difference is given by the extrinsic curvature. The second equation (Codazzi) complements this interpretation, stating that projection of the Riemann tensor of the embedding space along the normal direction is given by the tangent variation of the extrinsic curvature. Equations (11) and (12) describe the metric and extrinsic curvature of the deformed geometry V_4 . By varying y , they describe a continuous sequence of deformations in the embedding space. The existence of these deformations are given by the integrability conditions in Eqs. (13) and (14). The dynamics behind these deformations can be from the higher-dimensional Einstein's equations. As in Kaluza–Klein and in the brane-world theories, the bulk space V_5 has a metric geometry defined by

$${}^5\mathcal{R}_{AB} - \frac{1}{2} {}^5\mathcal{R} \mathcal{G}_{AB} = G_* T_{AB}^*, \quad (15)$$

where G_* is the new gravitational constant and T_{AB}^* denotes the components of the energy-momentum tensor of the known material sources. We can write in embedded vielbein $\{Z_{,\mu}^A, \eta^B\}$ for the metric of the bulk in the vicinity of V_4 as

$$\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}. \quad (16)$$

² To avoid confusion with the four dimensional Riemann tensor $R_{\alpha\beta\gamma\delta}$, the five-dimensional Riemann tensor is denoted by ${}^5\mathcal{R}_{ABCD}$. The extrinsic curvature terms in these equations follows from the five-dimensional Christoffel symbols together with the use of Eq. (1).

In this work, we adopt a five-dimensional bulk with constant curvature whose Riemann tensor is

$$\mathcal{R}_{ABCD} = K_* (\mathcal{G}_{AC}\mathcal{G}_{BD} - \mathcal{G}_{AD}\mathcal{G}_{BC}), \quad A, D = 1 \dots 5,$$

where $\mathcal{G}_{AB}$ denotes the bulk metric components in arbitrary coordinates. The constant curvature K_* is either zero (flat bulk) or positive (deSitter) or negative (anti-deSitter) constant curvature. In a pure geometrical sense, the prior definition of the bulk curvature is not really necessary for a dynamical embedding which depends on the smoothness of the embedding variables $\{Z^A_{,\mu}, \eta^B\}$, but for a physical model, it may be necessary such definition. The confinement condition implies that $K_* = \Lambda_*/6$, and Λ_* is the bulk cosmological constant.

From these dynamical equations in Eq. (15), we may derive the gravitational field in the embedded space-times, after the following observations:

1. Even though it does not exert any dynamical effect in our approach, the effective cosmological constant Λ was included in Eq. (15) for completeness once the bulk curvature serves as a reference for the embedded background. With this choice, the Nash perturbations begin in a curve space-time by the existence of the extrinsic curvature $k_{\mu\nu}$;
2. The confinement of the gauge fields to four dimensions is not a postulate. It is a consequence of the fact that only in four dimensions the three-form resulting from the derivative of the Yang-Mills curvature tensor is isomorphic to the one-form current. Consequently, all known observable sources of gravitation composing T_{AB} are confined to these deformable four-dimensional space-times, independently of the value of the extra coordinate y . Thus, such confinement can be implemented a very simple way by writing Einstein's equation in Eq. (15) in the Gaussian frame of every perturbed space-time with the energy-momentum tensor source $T_{\mu\nu}$ is such that

$$\begin{aligned} 8\pi GT_{\mu\nu} &= G_* Z^A_{,\mu} Z^B_{,\nu} T^*_{AB}, \quad Z^A_{,\mu} \eta^B T^*_{AB} = 0, \\ \eta^A \eta^B T^*_{AB} &= 0 \end{aligned} \quad (17)$$

3. Einstein's equations in Eq. (15) can be written as

$${}^5\mathcal{R}_{AB} = G_* (T^*_{AB} - \frac{1}{3} T^* \mathcal{G}_{AB}).$$

The tensor ${}^5\mathcal{R}_{AB}$ may be evaluated in the embedded space-times by contracting it with $Z^A_{,\mu} Z^B_{,\nu}$, $Z^A_{,\mu} \eta^B$ and $\eta^A \eta^B$ obtaining

$${}^5\mathcal{R}_{\mu\nu} = {}^5\mathcal{R}_{AB} Z^A_{,\mu} Z^B_{,\nu}.$$

Using Eqs. (1), (10) and the confinement conditions $\mathcal{G}_{55} = 1$ and $\mathcal{G}_{\mu 5} = 0$ and $\mathcal{G}_{\mu\nu} = g_{\mu\nu}$ with derivatives given by Eq. (1), Einstein's equations in Eq. (15) become

$${}^5\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{\partial k_{\mu\nu}}{\partial y} - 2k_{\mu\rho} k^\rho_{\nu} + h h_{\mu\nu}, \quad (18)$$

$${}^5\mathcal{R}_{\mu 5} = k^\rho_{\mu;\rho} + \frac{\partial \Gamma^\rho_{\mu 5}}{\partial y}. \quad (19)$$

where $h = \bar{g}^{\mu\nu} \bar{k}_{\mu\nu}$. For simplicity, in the previous calculation the cosmological constant was neglected. As a result, the Israel-Lanczos condition does not follow from Einstein's equations in Eq. (18) by themselves. To do so, it becomes necessary that the embedded geometry satisfy additional conditions such that the Ricci curvature of the embedding space coincide with the extrinsic curvature of the embedded space-time, that is ${}^5\mathcal{R}_{\mu\nu} = k_{\mu\nu}$, which is not generally true. One of these conditions is that the embedded space-time acts as a mirror boundary between two regions of the embedding space [71]. In this case we may evaluate the difference of ${}^5\mathcal{R}_{\mu\nu}$ from both sides of the space-times and the above mentioned boundary condition holds. However, in doing so the deformation given by Eq. (1) ceases to be. Therefore, to find the deformations caused by the extrinsic curvature, such special conditions in dynamical embedding are not applied and they are not needed in our present context.

2.2 The induced four-dimensional cosmological model

With the aforementioned remarks, using Eq. (1) to solve the integrability condition of the bulk and the embedded space given by Eqs. (13) and (14), in the Gaussian frame embedding vielbein $\{Z^A_{,\mu}, \eta^B\}$, one obtains the embedded four-dimensional equations (see Refs. [18, 19]) written as

$$R_{\mu\nu} - \frac{1}{2} R \bar{g}_{\mu\nu} + \Lambda \bar{g}_{\mu\nu} + \bar{Q}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}, \quad (20)$$

$$\bar{k}^\rho_{\mu;\rho} - h_{,\mu} = 0, \quad (21)$$

where Λ is the effective cosmological constant in four dimensions. The trace of Codazzi equations in Eq. (21), we denote the mean Gaussian curvature by $h^2 = h.h$ with $h = \bar{g}^{\mu\nu} \bar{k}_{\mu\nu}$. The semi-colon denotes the ordinary covariant derivative.

The resulting four-dimensional cosmological model is obtained standardly as we consider the Friedman-Lemaître-Robertson-Walker (FLRW) metric in coordinates (r, θ, ϕ, t) as

$$ds^2 = dt^2 - a^2 \left[dr^2 + f_\kappa^2(r) \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \quad (22)$$

where $a = a(t)$ is the scale expansion factor, $f(r)_\kappa = \sin r$, $r, \sinh r$ and κ corresponds to spatial curvature $(1, 0, -1)$,

respectively. For the present application, we consider a flat universe with $\kappa = 0$.

The $\bar{T}_{\mu\nu}$ tensor is the four-dimensional energy-momentum tensor of a perfect fluid, expressed in co-moving coordinates as

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})U_\mu U_\nu + \bar{p} \bar{g}_{\mu\nu}, \quad U_\mu = \delta_\mu^4, \quad (23)$$

where U_μ is the co-moving four-velocity. The related conservation equation is given by

$$\frac{d\bar{\rho}}{dt} + 3H(\bar{\rho} + \bar{p}) = 0, \quad (24)$$

where $\bar{\rho}$ and $\bar{p}$ denote the non-perturbed matter density and pressure, respectively, and the usual Hubble parameter is defined by $H \equiv H(t) = \dot{a}/a$. Moreover, the embedding between space-times leads to the appearance of the extrinsic terms that access the extra-dimension. The deformation tensor $\bar{Q}_{\mu\nu}$ is a geometrical term given by

$$\bar{Q}_{\mu\nu} = \bar{g}^{\rho\sigma} \bar{k}_{\mu\rho} \bar{k}_{\nu\sigma} - \bar{k}_{\mu\nu} h - \frac{1}{2} (K^2 - h^2) \bar{g}_{\mu\nu}, \quad (25)$$

where the term $K^2 = \bar{k}^{\mu\nu} \bar{k}_{\mu\nu}$ is the Gaussian curvature. It follows that $\bar{Q}_{\mu\nu}$ is conserved in the sense of Noether's theorem

$$\bar{Q}^{\mu\nu}{}_{;\nu} = 0. \quad (26)$$

To see what these observables may represent, consider again the one-parameter group of diffeomorphism defined by points in an embedded space-time, and the unit normal vector η . The orbit of each point is given by a curve $\alpha(y) = h_y(p)$, with tangent unit vector $\alpha' = \eta(p)$. The Frenet equation of each orbit tells that there is a transverse acceleration (that is, orthogonal to η and therefore tangent to the embedded space-time), given by a linear combination of the tangent basis $\{Z^A_{,\mu}\}$ as

$$\eta^A_{,\mu} = -g^{\rho\sigma} k_{\mu\rho} Z^A_{,\sigma}, \quad (27)$$

which is the definition of the extrinsic curvature (the minus sign is just conventional [79]). Thus, the extrinsic curvature represents an acceleration field in space-time, resulting from the flow of particles from one space-time to its deformation. Furthermore, since this acceleration always points to the concave side of the curve, for divergent flows we obtain a stress on the space-time geometry, which in principle can be detected by observing the motion of distant stars. This observable is determined from the conservation law associated with $Q_{\mu\nu}$. It is important to realize that such deformation does not appear in the purely intrinsic formulation of the geometry of space-time, because it originates from the variation of the normal vector η in Eq. (27). It does not appear also in models where

the extrinsic curvature is reduced to the confined matter as in [13, 14]. However, if this acceleration is associated with a conserved tensor, then it should have a dynamical equation.

Before proceeding further, we need to obtain the Friedmann equation for the evolution of the related background. Using spatially flat FLRW metric in Eq. (22), one obtains a solution of Eq. (21) such as

$$\bar{k}_{\mu\nu,\rho} - \Gamma_{\mu\rho}^\sigma \bar{k}_{\sigma\nu} = \bar{k}_{\mu\rho,\nu} - \Gamma_{\mu\nu}^\sigma \bar{k}_{\sigma\rho}, \quad (28)$$

which can be split into spatial and time parts. Hence, the main extrinsic quantities components are written in FLRW coordinates in a form

$$\bar{k}_{ij} = \frac{b}{a^2} \bar{g}_{ij}, \quad \bar{k}_{44} = -\frac{b}{a^2} \left(\frac{B}{H} - 1 \right), \quad (29)$$

$$\bar{K}^2 = \frac{b^2}{a^4} \left(\frac{B^2}{H^2} - 2 \frac{B}{H} + 4 \right), \quad (30)$$

$$h = \frac{b}{a^2} \left(\frac{B}{H} + 2 \right), \quad (31)$$

$$\bar{Q}_{ij} = -\frac{1}{3} \bar{Q}_{44} \left(2 \frac{B}{H} - 1 \right) \delta^{44} \bar{g}_{ij}, \quad \bar{Q}_{44} = -\frac{3b^2}{a^4}, \quad (32)$$

$$\bar{Q} = -(K^2 - h^2) = \frac{6b^2}{a^4} \frac{B}{H}. \quad (33)$$

Where the $a \equiv a(t)$ denotes the usual expansion parameter and $b(t) \equiv b = \bar{k}_{11}$ is the bending function. The dot symbol denotes the ordinary time derivative. Taking a similar notation for the Hubble function, we define the extrinsic parameter $B \equiv B(t) = \dot{b}/b$. As a result, one calculates the components of Eqs. (20) and (21), and obtains the background Hubble evolution $H(t)$

$$H^2(t) = \frac{8}{3} \pi G \rho + \frac{\Lambda}{3} + \frac{b^2}{a^4}, \quad (34)$$

where ρ is the total energy density and the bending function $b(t)$ remains undefined due to the Codazzi equations in Eq. (21) are homogeneous. It is important to note that the term B/H behaves as a cosmic accelerator. A first insight on such behavior is obtained if one calculates the minimum of the time component $\bar{k}_{44}$ by means of a Laplace transform, since such term does not appear in the spatial components $\bar{k}_{ij}$. Just keeping the linear terms, it is found that B/H has minimum with a de Sitter expansion plus an amplitude ξ_0 in a form $(B/H)(t) = e^{\sqrt{\Lambda}t + \xi_0}$. In the following section, we will check this idea by presenting an approach for dynamical equations for a spin-2 field.

2.3 The Einstein–Gupta equations

In each embedded space-time obtained by the smooth (differentiable) deformation, the metric and the extrinsic curvature

are independent variables as shown by Nash-Greene theorem (Eq. (1)). Therefore we have a total of 20 unknowns $g_{\mu\nu}$ and $k_{\mu\nu}$, against only 15 dynamical equations counting from Eq. (15). The remaining equations comes from the fact that $k_{\mu\nu}$ is an independent symmetric rank-2 tensor which corresponds also to a spin-2 field. A theorem due to S. Gupta tells that any such tensor necessarily satisfy an Einstein-like system of equations, having the Pauli-Fierz equation as its linear approximation³ [80,81]. Therefore, our equations for $k_{\mu\nu}$ must necessarily satisfy Gupta's equations.

In the following, we derive Gupta's equation for the extrinsic curvature in a space-time with metric $g_{\mu\nu}$, using an analogy with the derivation of Einstein's equations. We start by noting that $k_{\mu\nu}k^{\mu\nu} = K^2 \neq 4$, so that we normalize the extrinsic curvature, defining another temporary tensor

$$f_{\mu\nu} = \frac{2}{K} k_{\mu\nu}, \quad (35)$$

and define its inverse by $f^{\mu\rho} f_{\rho\nu} = \delta^\mu_\nu$. It follows that $f^{\mu\nu} = \frac{2}{K} k^{\mu\nu}$.

Denoting by $\parallel$ the covariant derivative with respect to a connection defined by $f_{\mu\nu}$, while keeping the usual semi-colon notation for the covariant derivative with respect to $g_{\mu\nu}$, the analogous to the "Levi-Civita" connection associated with $f_{\mu\nu}$ such that $f_{\mu\nu\parallel\rho} = 0$, is:

$$\Upsilon_{\mu\nu\sigma} = \frac{1}{2} (\partial_\mu f_{\sigma\nu} + \partial_\nu f_{\sigma\mu} - \partial_\sigma f_{\mu\nu}) \quad (36)$$

Defining

$$\Upsilon_{\mu\nu}{}^\lambda = f^{\lambda\sigma} \Upsilon_{\mu\nu\sigma}$$

The "Riemann tensor" for $f_{\mu\nu}$ has components

$$\mathcal{F}_{\nu\alpha\lambda\mu} = \partial_\alpha \Upsilon_{\mu\lambda\nu} - \partial_\lambda \Upsilon_{\mu\alpha\nu} + \Upsilon_{\alpha\sigma\mu} \Upsilon_{\lambda\nu}^\sigma - \Upsilon_{\lambda\sigma\mu} \Upsilon_{\alpha\nu}^\sigma$$

and the corresponding "f-Ricci tensor" and the "f-Ricci scalar" for $f_{\mu\nu}$ are, respectively by $\mathcal{F}_{\mu\nu} = f^{\alpha\lambda} \mathcal{F}_{\nu\alpha\lambda\mu}$ and $\mathcal{F} = f^{\mu\nu} \mathcal{F}_{\mu\nu}$. Finally, Gupta's equations for $f_{\mu\nu}$ can be obtained from the contracted Bianchi identity

$$\mathcal{F}_{\mu\nu} - \frac{1}{2} \mathcal{F} f_{\mu\nu} = \alpha_* \mathcal{T}_{\mu\nu}, \quad (37)$$

where $\mathcal{T}_{\mu\nu}$ stands for the source of this field such that $\mathcal{T}^{\mu\nu}{}_{\parallel\nu} = 0$ and α_* is a coupling constant. In spite of the resemblances, $k_{\mu\nu}$ is not a metric because it exists only after the Riemannian geometry has been defined for the metric $g_{\mu\nu}$. Moreover, the related cosmological constant is zero in this normalized extrinsic curvature space projection since

the cosmological constant problem is only associated to tangent (intrinsic) space-time if one relates the quantum vacuum energy to Λ , as originally formulated by Zeldovich [83].

If the universe is a submanifold embedded in a larger space, then the expansion of the universe must be towards an unobservable region of the embedding space. In this sense, the only option for the external source of Eq. (37) is the void characterized by $\mathcal{T}_{\mu\nu} = 0$. With such interpretation, Eq. (37) becomes simply a Ricci-flat-like equation

$$\mathcal{F}_{\mu\nu} = 0. \quad (38)$$

Such approach guarantees that even for a flat bulk, the Nash perturbations by means of the extrinsic curvature may appear in the embedded four-dimensional Minkowski M_4 .

Using the spatially flat FLRW metric in Eq. (22) and the definitions of Eq. (35) in such a way

$$f_{ij} = \frac{2}{K} g_{ij}, \quad i, j = 1..3, \quad f_{44} = -\frac{2}{K} \frac{1}{a} \frac{d}{dt} \left(\frac{b}{a} \right). \quad (39)$$

Thus, one derives the components of Eq. (36) and of the "f-curvature" $\mathcal{F}_{\mu\nu\rho\sigma}$, and finally, writes the Ricci-flat equations of Eq. (38) that result in

$$\frac{B}{H} = \beta_0 \pm \sqrt{4\eta_0^2 a^4 - 3}. \quad (40)$$

The β_0 parameter must obey the constraint $\beta_0 \geq 1$ in order to be consistent with the Null energy condition (NEC). This equation is readily integrated to obtain

$$b(t) = \tilde{\alpha}_0 a^{\beta_0} e^{\pm \gamma(a)}, \quad (41)$$

where $\gamma(a)$ is given by

$$\gamma(a) = \sqrt{4\eta_0^2 a^4 - 3} - \sqrt{3} \arctan \left(\frac{\sqrt{3}}{3} \sqrt{4\eta_0^2 a^4 - 3} \right), \quad (42)$$

and also can be written in terms of redshift by means of the usual relation $a = \frac{1}{1+z}$. Replacing Eq. (41) in Eq. (34), we obtain the Friedmann equation modified by the extrinsic curvature such as

$$H^2(a) = \frac{8}{3} \pi G \bar{\rho} + \frac{\Lambda}{3} + \tilde{\alpha}_0 a^{2\beta_0-4} e^{\gamma^\pm(a)}, \quad (43)$$

where $\tilde{\alpha}_0$ and β_0 denote integration constants. For the present application, we assume the total energy density $\rho = \bar{\rho}_{mat} + \rho_{rad}$. The quantities $\bar{\rho}_{mat}$ and ρ_{rad} represent the matter and radiation energy densities, respectively.

2.4 Effective fluid approach

In order to make easier the comparison of the present model to the current models in literature, an effective fluid approach is

³ One such spin-2 field was proposed by Salam et al, under the designation of strong gravity, acting as an intermediate field in the interaction between gravitation and hadrons [82].

presented mimicking a DE fluid by a geometric-based mechanism. In doing so, we can write Eq. (20) in a form

$$R_{\mu\nu} - \frac{1}{2}R\bar{g}_{\mu\nu} + \Lambda\bar{g}_{\mu\nu} = 8\pi G\bar{T}_{total}, \quad (44)$$

where $\bar{T}_{\mu\nu}^{total} = \bar{T}_{\mu\nu} + \frac{1}{8\pi G}\bar{T}_{\mu\nu}^{ext}$. The last previous term denotes the extrinsic contribution $\bar{T}_{\mu\nu}^{ext} \doteq \bar{Q}_{\mu\nu}$ and can be written as a perfect fluid as

$$\bar{T}_{\mu\nu}^{ext} = (\bar{\rho}_{ext} + \bar{p}_{ext})U_\mu U_\nu + \bar{p}_{ext}\bar{g}_{\mu\nu}, \quad U_\mu = \delta_\mu^4, \quad (45)$$

where U_μ is the co-moving four-velocity. The related conservation equation is given by

$$\frac{d\bar{\rho}_{ext}}{dt} + 3H(\bar{\rho}_{ext} + \bar{p}_{ext}) = 0, \quad (46)$$

where $\bar{\rho}_{ext}$ and $\bar{p}_{ext}$ denote the non-perturbed extrinsic density and extrinsic pressure, respectively. As discussed in the previous sections, the dynamical embedding guarantees the influence of the bulk over the embedded space by means of the Nash-Greene geometric perturbations. Hence, such influences are explicitly shown in the induced four-dimensional space-time by the appearance of the deformation tensor $Q_{\mu\nu}$ constructed by the intrinsic and extrinsic terms, i.e., $g_{\mu\nu}$ and $k_{\mu\nu}$ which is a conserved quantity in accordance with Eq. (26) and, then, $\bar{T}_{\mu\nu}^{total}$ is also conserved accordingly. Since we are not considering any exotic matter-energy source, the conservation of $\bar{T}^{ext}$ applies and differs from a DE fluid in standard rigid brane-world models which are not conserved to guarantee the necessity of the bulk-brane energy transfer. In a dynamical embedding, the deformation tensor $Q_{\mu\nu}$ plays such a role as the embedded space-time dynamically evolves in the bulk.

Based on previous cosmography tests [22,24–26,28], the parameter β_0 measures the magnitude of the deceleration parameter $q(z)$ and the values of parameter η_0 control the width of the transition phase redshift z_d from a decelerating to accelerating regime. In the aforementioned works, it was found that η_0 has a very small value $\eta_0 \rightarrow 0$ that leads to a transition redshift constraint $z_t < 1$ in compliance with observations and simplifies Eq. (41) [2]. Hence, the resulting model is defined in an effective fluid approach by Hubble evolution $H(z)$ simply written as

$$H(z) = H_0\sqrt{\Omega_m(z) + \Omega_{rad}(z) + \Omega_\Lambda + \Omega_{ext}(z)}, \quad (47)$$

where $H(z)$ is the Hubble parameter in terms of redshift z and H_0 is the current value of the Hubble constant. The matter density parameter is denoted by $\Omega_m(z) = \Omega_{m0}(1+z)^3$. The radiation density is given by $\Omega_{rad}(z) = \Omega_{rad0}(1+z)^4$ with $\Omega_{rad0} = \Omega_{m0}a_{eq}$ and Ω_Λ stands for the cosmological constant contribution. The term $\Omega_{ext}(z) = \Omega_{ext0}(1+z)^{4-2\beta_0}\alpha_0$

stands for the density parameter associated with the extrinsic curvature. The integration constants were all merged in the new integration constant α_0 . If $\Omega_{ext}(z)$ vanishes, one obtains the Λ CDM as a background limit. Concerning notation, the upper script “0” indicates the present value of any quantity. The equivalence number for the expansion factor a_{eq} given by

$$a_{eq} = \frac{1}{1+z_{eq}} = \frac{1}{(1+2.5 \times 10^4 \Omega_{m0} h^2 (T_{cmb}/2.7)^{-4})}, \quad (48)$$

where z_{eq} is the equivalence redshift. The CMB temperature is adopted for the value $T_{cmb} = 2.7255K$ and the Hubble factor $h = 0.67$. The complete form for Hubble parameter as in Eq. (47) at background level has been previously investigated in a series of studies [22,24–26,28]. Moreover, the current extrinsic contribution Ω_{ext}^0 is given by the normalization (flat) condition for redshift at $z = 0$ that results in

$$\Omega_{ext}^0 = (1 - \Omega_m^0 - \Omega_{rad}^0 - \Omega_\Lambda^0). \quad (49)$$

Thus, we write the related Friedman equation in a form

$$H^2 = \frac{8}{3}\pi G(\bar{\rho} + \bar{\rho}_\Lambda + \bar{p}_{ext}), \quad (50)$$

where $\bar{\rho}_{ext}(a) = \alpha_0 a^{2\beta_0-4}$ and we have maintained the Λ related terms just for completeness purposes. Once $\bar{\rho}_{ext}(a)$ is determined, the “extrinsic” pressure can be calculated straightforwardly using Eq. (46) to obtain

$$\bar{p}_{ext}(a) = -\frac{\alpha_0}{3}(2\beta_0 - 1)a^{2\beta_0-4}. \quad (51)$$

We point out that for values of $\beta_0 \geq 1$, NEC is satisfied. When $\alpha_0 = 0$, the effect of extrinsic curvature ceases to be for all extrinsic quantities $\bar{T}_{\mu\nu}^{ext}$, $\bar{\rho}_{ext}(a)$, $\bar{p}_{ext}(a)$, as expected and GR limit is obtained with $\Lambda = 0$. Thus, we define the effective Equation of State (EoS) for w_{ext} parameter by using the relation $w_{ext} = \frac{\bar{p}_{ext}}{\bar{\rho}_{ext}}$ to obtain

$$w_{ext} = -1 + \frac{1}{3}(4 - 2\beta_0). \quad (52)$$

When $\beta_0 = 2 - \frac{3}{2}(1+w)$, or equivalently, $w = -1 - \frac{1}{3}(2\beta_0 - 4)$, we obtain $w_{ext} = w$. Hence, we can write the dimensionless Hubble parameter $E(z)$ as

$$E^2(z) = \Omega_{m0}(1+z)^3 + \Omega_{rad0}(1+z)^4 + \Omega_{ext}^0(1+z)^{3(1+w)}. \quad (53)$$

Henceforth, the present model is denoted as β -model only to facilitate the referencing. Thus, we adopt the previous relation of β_0 parameter with fluid parameter w as shown in Eq. (52). As a result, we have completed the basis of the “fluid analogy” reinterpreting the β_0 parameter by the fluid parameter w which is the only extra model parameter to be considered from hereon to be constrained to data. As we are going to see in the next sections, at perturbation level, it marks the departure of the β -model from any analogy with both Λ CDM and w CDM due to the presence of the deformation tensor $\tilde{Q}_{\mu\nu}$ constructed by extrinsic terms that will imprint different perturbation equations.

3 Matter evolution equations in conformal Newtonian gauge

3.1 Perturbed equations in conformal Newtonian gauge

In longitudinal conformal Newtonian gauge in the embedded four dimensional space-time, the induced metric in Eq. (22) is given by

$$ds^2 = a^2[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)\delta_{ij}dx^i dx^j], \quad (54)$$

where $\Phi = \Phi(\mathbf{x}, \eta)$ and $\Psi = \Psi(\mathbf{x}, \eta)$ denote the Newtonian potential and the Newtonian curvature, respectively. The conformal time η is related with physical time as $dt = a(\eta)d\eta$.

The perturbed field equations of Eqs. (20) and (21) can be written as

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} - \delta T_{\mu\nu}^{ext}, \quad (55)$$

$$\delta k_{\mu\nu;\rho} = \delta k_{\mu\rho;\nu}. \quad (56)$$

where we omitted the Λ contribution. Concerning notation, from hereon the background quantities are represented by the tilde symbol. To obtain the explicit form for perturbed field equations in Eqs. (55) and (56), we need to determine both the perturbed metric $\delta g_{\mu\nu}$ and the perturbed extrinsic curvature $\delta k_{\mu\nu}$. Using the main result of the Nash-Greene theorem in Eq. (1), we revert such relation to

$$\delta g_{\mu\nu} = -2\tilde{k}_{\mu\nu}\delta y, \quad (57)$$

where δy denotes an infinitesimal displacement of the extra dimension coordinate y that accesses the bulk space. It is worth noting that such coordinate does not appear in the line element in Eq. (54) like those of rigid embedding models, since, by means of the Lie transport, it is applied to perturb the embedding coordinates in Eqs. (8) and (9). As a result, the dynamics (deformations) of the embedded four-dimensional space-time are fully correlated to the bulk and the related perturbed field equations ensure this characteristic once the

integrability conditions of the bulk and the embedded space were already satisfied (Eqs. (13) and (14)).

From Eqs. (11) and (12), we consider the linear perturbations of a new geometry $\tilde{g}_{\mu\nu}$ that is given by

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} - 2\delta y \tilde{k}_{\mu\nu}, \quad (58)$$

and the related perturbed extrinsic curvature

$$\tilde{k}_{\mu\nu} = \tilde{k}_{\mu\nu} - 2\delta y \tilde{g}^{\sigma\rho} \tilde{k}_{\mu\sigma} \tilde{k}_{\nu\rho}, \quad (59)$$

where we can identify $\delta k_{\mu\nu} = -2\delta y \tilde{g}^{\sigma\rho} \tilde{k}_{\mu\sigma} \tilde{k}_{\nu\rho}$. Using the Nash relation in Eq. (57), we obtain

$$\delta k_{\mu\nu} = \tilde{g}^{\sigma\rho} \tilde{k}_{\mu\sigma} \delta g_{\nu\rho}. \quad (60)$$

This is an important result since we can study the perturbations on the embedded four space-time with a given $\delta k_{\mu\nu}$ induced by the metric perturbations $\delta g_{\mu\nu}$ in Eq. (54). Hence, the induced perturbed equations in Eqs. (55) and (56) are found by using the standard theory of cosmological perturbations at a specific gauge [84].

The perturbation of the deformation tensor $T_{\mu\nu}^{ext}$ is made from its background form in Eq. (25) and the resulting $k_{\mu\nu}$ perturbations from Nash's fluctuations in Eq. (60). Thus, one obtains

$$\delta T_{\mu\nu}^{ext} = -\frac{3}{2}(K^2 - h^2)\delta g_{\mu\nu}. \quad (61)$$

It is worthy noting that due to the Nash fluctuations in Eq. (58), we notice that Codazzi equations and the Einstein-Gupta equations are invariant under perturbations and are confined to their background forms in Eqs. (56) and (38), respectively. Also, $\delta T_{\mu\nu}^{ext}$ is independently conserved, i.e., $\delta T_{\mu\nu;\nu}^{ext} = 0$. Then using the background relations in Eqs. (29), (30), (31), (32), and (33) in the Newtonian gauge, we can determine the components of $\delta T_{\mu\nu}^{ext}$ as

$$\delta T_{ij}^{ext} = \alpha_0 a^{-(1+3w)} \Psi \delta_{ij}, \quad (62)$$

$$\delta T_{i4}^{ext} = 0, \quad (63)$$

$$\delta T_{44}^{ext} = \alpha_0 a^{-(1+3w)} \Phi \delta_{44}, \quad (64)$$

where α_0 is an integration constant that carries an extrinsic curvature term from the bending function $b(t)$ as seen in Eq. (43).

For a perturbed fluid with pressure p and density ρ , one can write the perturbed components of the related stress-tensor

$$\delta T_{ij} = -\delta p \delta_{ij}, \quad (65)$$

$$\delta T_{i4} = \frac{1}{a}(\rho_0 + p_0)\delta u_{\parallel i}, \quad (66)$$

$$\delta T_{44} = \delta \rho, \quad (67)$$

where $\delta u_{\parallel i}$ denotes the tangent velocity potential and ρ_0 and p_0 denote the non-perturbed components of density and pressure, respectively. Moreover, for a pressureless matter and a null anisotropic matter stress, we obtain the closure condition $\Psi = \Phi$ and the following equation in the wave-number k -space of Fourier modes as

$$k^2 \Phi_k + 3\mathcal{H}(\Phi'_k + \Phi_k \mathcal{H}) = -4\pi G a^2 \delta \rho_k + 9\alpha_0 a^{1-3w} \Phi_k. \quad (68)$$

It is important to notice that when $\alpha_0 \rightarrow 0$ in Eq. (68), the standard GR correspondence is obtained. Since α_0 is not a large number, it can be set as 1 without loss of generality [85].

In the subhorizon approximation with $k^2 \gg \mathcal{H}^2$ or $k^2 \gg a^2 H^2$, after a Fourier transform, we perform the definition of the “contrast” matter density $\delta_m \equiv \frac{\delta \rho}{\rho_0}$. We use Eq. (68) and obtain a relation of Φ_k and δ_m given by

$$k^2 \Phi_k = -4\pi G_{eff} a^2 \rho_0 \delta_m, \quad (69)$$

where G_{eff} is the effective Newtonian constant and is given by

$$G_{eff}(a, k) = \frac{G}{1 - \frac{1}{k^2} a^{1-3w}}, \quad (70)$$

where G is the Newtonian gravitational constant. The corresponding equation of evolution of the contrast matter density $\delta_m(\eta)$ in conformal longitudinal Newtonian frame is written as

$$\delta''_m + \mathcal{H} \delta'_m - 4\pi G_{eff} a^2 \rho_0 \delta_m = 0, \quad (71)$$

where the prime symbols denote derivatives with respect to conformal time η . Thus, in terms of the expansion factor $a(t)$, we obtain the contrast matter density $\delta_m(a)$ as

$$\ddot{\delta}_m(a) + \left(\frac{3}{a} + \frac{\dot{H}(a)}{H(a)} \right) \dot{\delta}_m(a) - \frac{3\Omega_{m0} G_{eff}/G}{2(H^2(a)/H_0^2)} \delta_m(a) = 0, \quad (72)$$

where the dot symbols denote derivatives with respect to scale factor a . We point out that due to $G_{eff} \neq G$, Eq. (72) can only have numerical solutions and reinforces that the β -model obligatorily differs from Λ CDM and w CDM in the perturbation modes. Analytical solutions for growing modes of Eq. (72) can be obtained in specific cases as $G_{eff} = G$, i.e., when the extrinsic curvature vanishes. As it happens, one obtains an analogue result, for instance, as w CDM model, which the solution for the evolution of the contrast matter

density is given by [86–88]

$$\delta(a) = a \cdot {}_2F_1 \left(-\frac{1}{3w}, \frac{1}{2} - \frac{1}{2w}; 1 - \frac{5}{6w}; a^{-3w} (1 - \Omega_m^{-1}) \right), \quad (73)$$

where ${}_2F_1(a, b; c; z)$ is a hypergeometric function. An immediate solution is for Λ CDM, with $w = -1$, one obtains

$$\delta(a) = a \cdot {}_2F_1 \left(\frac{1}{3}, 1; \frac{11}{6}; a^3 (1 - \Omega_m^{-1}) \right). \quad (74)$$

3.2 The gauge choice

Our model is sensitive to the gauge choice since the modified perturbed equations, that change the GR standards, are led by the perturbed $\delta Q_{\mu\nu} = \delta T_{\mu\nu}^{ex}$ which is not gauge invariant but is a diagonal tensor, which means that it does not directly contribute to anisotropic shear viscosity in any gauge since in general $\delta Q_{\beta}^4 = \delta Q_4^{\beta} = 0$, or, accordingly, $\delta T_{\mu}^{ext}{}^4 = \delta T_4^{ext}{}^{\mu} = 0$, and relevant contributions/modifications, as compared with GR, rely on how the metric perturbations are accounted for in a specific gauge. For instance, adopting Ma and Bertschinger’s notation [89], in terms of the synchronous gauge, where the metric perturbations are included only in the spatial metric tensor, one has a flat FLRW metric

$$ds^2 = a^2(\tau) [d\tau^2 - (\delta_{ij} + h_{ij}) dx^i dx^j], \quad (75)$$

where τ is the conformal time and $|h_{ij}| \ll 1$. In the Fourier space k , the linearized modified Einstein perturbed equations are described by the functions $h = h(k, \tau)$ which is the trace of h_{ij} and $\eta = \eta(k, \tau)$, that the perturbed modified Einstein equations in the Fourier k -space wave modes are

$$\begin{aligned} k^2 \eta - \frac{1}{2} \mathcal{H} h' &= 4\pi G a^2 \delta T_4^4, \\ k^2 \eta' &= 4\pi G a^2 (\bar{\rho} + \bar{P}) \theta, \\ h'' + 2\mathcal{H} h' - 2k^2 \eta - 9\gamma_0 a^{(1-3w)} h &= -8\pi G a^2 \delta T_i^i, \\ h'' + 6\eta'' + 2\mathcal{H} (h' + 6\eta' - 2k^2 \eta) &= -24\pi G a^2 (\bar{\rho} + \bar{P}) \Theta, \end{aligned} \quad (76)$$

where the prime symbol denotes the ordinary derivative with respect to the conformal time τ , and the following quantities are defined as $(\bar{\rho} + \bar{P})\theta = ik^j \delta T_j^4$ and $(\bar{\rho} + \bar{P})\Theta = -(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij}) \Sigma_j^i$. The anisotropic shear tensor Σ_j^i is given by $\Sigma_j^i = T_j^i - \frac{1}{3} \delta_j^i T_k^k$. As a result, the only modification by extrinsic curvature lies at the trace space-space component. In these terms, the time-time, longitudinal time-space, and longitudinal traceless space-space parts of the modified Einstein equations remain unchanged. Even though the other

components remain the same and δ_m^{synch} growth equation is exact, and Eq. (76) cannot be neglected for the evolution of h and η because they cannot be obtained by differentiating the time-time, longitudinal time-space nor using energy-momentum conservation, which turns the equations in the synchronous gauge difficult to deal. Another aggravator is the fact that the synchronous gauge that may lead to nonphysical gauge modes and ambiguities due to a residual coordinate freedom [84, 89].

3.3 On the asymptotic and linear growth indices

In general, the most of modified gravity models shares a similar structure of the contrast matter density equation. Hence, Eq. (71) is rewritten in the physical time t using the following parametrization by means of $\mu(a, k)$ and $\nu(a, k)$ functions as

$$\ddot{\delta}_m + 2\nu H \dot{\delta}_m - 4\pi G \mu \rho_0 \delta_m = 0. \quad (77)$$

Concerning our present model, $\mu(a, k) = G_{eff}(a, k)/G$. To sum up, for the standard GR, $\mu = \nu = 1$ and for modified gravity model $\mu \neq 1$ and $\nu = 1$. Since such parametrization applies to our model, we use it to study both asymptotic and linear γ growth indices. The γ growth index is related to growth rate by the approximation [90]

$$f(a) = \frac{\ln \delta}{\ln a} \sim \Omega_m^\gamma(a). \quad (78)$$

and the definition

$$\Omega_m(a) = \frac{\Omega_m^0 a^{-3}}{E^2(a)}, \quad (79)$$

where dimensionless Hubble parameter $E(z)$ is given by Eq. (53). By differentiating Eq. (79) with respect to the expansion factor a and using Eq. (78) [5], one obtains

$$a \frac{df(a)}{da} + \left(2\nu + \frac{d \ln E(a)}{d \ln a} \right) f(a) + f(a)^2 = \frac{3}{2} \mu(a) \Omega_m(a). \quad (80)$$

Moreover, the asymptotic growth index is given by

$$\gamma_\infty = \frac{3(M_0 + M_1) - 2(H_1 + N_1)}{2 + 2X_1 + 3M_0}, \quad (81)$$

where the related quantities are defined as

$$M_0 = \mu|_{\omega=0}, \quad M_1 = \frac{d\mu}{d\omega}|_{\omega=0}, \quad N_1 = \frac{d\nu}{d\omega}|_{\omega=0}, \quad (82)$$

and also

$$H_1 = -\frac{X_1}{2} = \frac{d \left(\frac{d \ln E}{d \ln a} \right)}{d\omega} \Big|_{\omega=0}. \quad (83)$$

The relation $\omega = \ln \Omega_m(a)$ applies at $z \gg 1$ ($a \rightarrow 0$ and $\Omega_m(a) \rightarrow 1$, then $\omega \rightarrow 0$) [3, 4]. Moreover, the linear growth parameter $\gamma(a)$ is obtained by a Taylor expansion at the point $a = 1$ and results in

$$\gamma(a) = \gamma_0 + \gamma_1(1 - a), \quad (84)$$

with $\gamma_0 = \gamma(1)$ and γ_1 is given by

$$\gamma_1 = \frac{\Omega_{m0}^{\gamma_0} - 3\gamma_0 + 2\nu_0 - 2 \left(\gamma_0 - \frac{1}{2} \right) \frac{d \ln E}{d \ln a} \Big|_{a=1} - \frac{3}{2} \mu_0 \Omega_{m0}^{1-\gamma_0}}{\ln \Omega_{m0}}, \quad (85)$$

where $\mu_0 = \mu(1)$ and $\nu_0 = \nu(1)$. In addition, the asymptotic growth is related to the linear growth by

$$\gamma_\infty = \gamma_0 + \gamma_1. \quad (86)$$

With the aid of Eq. (53), one finds the quantity

$$\frac{d \ln E(a)}{d \ln a} = -\frac{3}{2} - \frac{3}{2} w(1 - \Omega_m(a)). \quad (87)$$

Moreover, the parameter $\mu = G_{eff}/G$ is straightforwardly given by Eq. (70) and $\nu = 1$. The function $\mu(a, k)$ for the β -model is given by

$$\mu(a, k) = \frac{k^2}{k^2 - \left(\frac{\Omega_m^0}{\Omega_m(a)} \frac{(1 - \Omega_m(a))}{(1 - \Omega_m^0)} \right)^{-\frac{1-3w}{3w}}}. \quad (88)$$

Independently of the k -dependence in the former expression of $\mu(a, k)$, in the limit $\mu|_{\omega=0}$ with $\Omega_m(a) = e^\omega$, it leads to $\mu = 1$. As pointed out in [5], G_{eff} has a k -dependence (so is μ), one needs to solve numerically Eq. (80) to obtain values of γ_0 and then γ_1 in Eq. (85).

As a result, one obtains the following set of parameters for the present model

$$\{M_0, M_1, N_1, X_1\} = \left(1, 0, \frac{3w}{2}, -3w \right), \quad (89)$$

and the related asymptotic growth [3–9]

$$\gamma_\infty = \frac{3(w - 1)}{6w - 5}. \quad (90)$$

We also point out that when $w = -1$, Eq. (90) gives $\gamma_\infty = \frac{6}{11}$ that reproduces as expected the result for the Λ CDM model.

4 Observational constraints: analysis and results

4.1 Statistical considerations on the data

In the following, we point out the main equations and statistical tools for the observational data on Growth fluctuations, CMB, BAO and the Pantheon SNIa. A complete set and description of these formulas can be found in detail in Refs. [30–32,91].

4.1.1 Matter growth fluctuations

The σ_8 parameter measures the growth of r.m.s fluctuations on the scale of $8h^{-1}\text{Mpc}$. This is performed by the measure of the quantity

$$f\sigma_8(a) \equiv f(a) \cdot \sigma_8(a), \quad (91)$$

where $f(a) = \frac{\ln \delta}{\ln a}$ is the growth rate and the growth factor $\delta(a)$ is given by Eq. (72). To compatibilize the data dependence from the fiducial cosmology and another cosmological survey, it is necessary to rescale the growth-rate data by the ratio $r(z)$ of the Hubble parameter $H(z)$ and the angular distance $d_A(z)$ by using the relation

$$r(z) = \frac{H(z)d_A(z)}{H_f(z)D_fA(z)}, \quad (92)$$

where the subscript “ f ” corresponds a quantity of fiducial cosmology. Accordingly, the angular distance $d_A(z)$ is defined as

$$d_A(z) = \frac{c}{(1+z)} \int_z^0 \frac{1}{H(z')} dz'. \quad (93)$$

Moreover, the regulation of the χ^2 statistics is made from the definition of

$$\chi^2(\Omega_{m0}, w, \sigma_8) = V^i C_{ij}^{-1} V_j, \quad (94)$$

where $V^i \equiv f\sigma_{8,i} - r(z_i)f\sigma_8(z_i, \Omega_{m0}, w, \sigma_8)$ denotes a set of vectors that goes up to i th-datapoints at redshift z_i for each $i = 1 \dots N$ number of datapoints. The set of $f\sigma_{8,i}$ datapoints come from theoretical predictions [30]. The set of C_{ij}^{-1} defines the inverse covariance matrix. To disentangle the correlations of datapoints from WiggleZ dark energy survey the covariant matrix C_{ij} [50] is given by

$$C_{ij}^{wigglez} = 10^{-3} \begin{pmatrix} 6.400 & 2.570 & 0.000 \\ 2.570 & 3.969 & 2.540 \\ 0.000 & 2.540 & 5.184 \end{pmatrix} \quad (95)$$

and the resulting total matrix C_{ij}^{tot}

$$C_{ij}^{tot} = 10^{-3} \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots \\ 0 & C_{ij}^{wigglez} & 0 & \dots \\ 0 & 0 & \dots & \sigma_N^2 \end{pmatrix} \quad (96)$$

where the set of σ^2 's denote the N -variances.

4.1.2 CMB Planck2018 data

For the CMB data, we use the Planck2018 released [2] with χ^2 -statistics given by

$$\chi_{CMB}^2 = X_{Planck2018}^T C_{CMB}^{-1} X_{Planck2018}, \quad (97)$$

where the covariant matrix for the parameters for $R, l_A, \Omega_{b0}h^2$ is

$$X_{Planck2018} = \begin{pmatrix} R - 1.7502 \\ l_A - 301.471 \\ \omega_b - 0.02236 \end{pmatrix} \quad (98)$$

where $\omega_b = \Omega_{b0}h^2$. The two shift parameters R and l_A are defined as the scale distance and acoustic scale, respectively, as

$$R = \frac{\sqrt{\Omega_{m0}}}{c} d_A(z_{CMB})(1 + z_{CMB}), \quad (99)$$

$$l_A = \frac{\pi d_A(z_{CMB})(1 + z_{CMB})}{r_s(z_{CMB})}, \quad (100)$$

where the angular distance d_A is given by Eq. (93) and the related redshift at recombination z_{cmb} is given by

$$z_{CMB} = 1048[1 + 0.00124(\Omega_b h^2)^{-0.738}][1 + g_1(\Omega_{m0}h^2)^{g_2}], \quad (101)$$

and the parameters (g_1, g_2) are defined accordingly as

$$g_1 = \frac{0.0783(\Omega_b h^2)^{-0.238}}{1 + 39.5(\Omega_b h^2)^{0.763}}, \quad g_2 = \frac{0.560}{1 + 21.1(\Omega_b h^2)^{1.81}}. \quad (102)$$

The comoving sound horizon $r_s(z)$ is given by

$$r_s(z) = c \int_z^\infty \frac{c_s(z')}{H(z')} dz', \quad (103)$$

and the related sound speed c_s

$$c_s(z) = \frac{1}{\sqrt{3(1 + \bar{R}_b/(1+z))}}, \quad (104)$$

with $\bar{R}_b = 31500\Omega_{b0}h^2(T_{CMB}/2.7K)^{-4}$. Moreover, the inverse of the covariant matrix C_{CMB}^{-1} for the parameters for $(l_a, R, \Omega_{b0}h^2)$ is given by $C_{CMB}^{-1} = \sigma_i\sigma_j C$, with $\sigma_i = (0.0046, 0.090, 0.00015)$. Thus, for the normalized covariance matrix one has

$$C = \begin{pmatrix} 1.00 & 0.46 & -0.66 \\ 0.46 & 1.00 & -0.37 \\ -0.66 & -0.33 & 1.00 \end{pmatrix} \quad (105)$$

4.1.3 BAO joint data

The BAO dataset used in this paper relies on the set of probes of SDSS [40–42], 6dFGS [43], IRAS [44,45], 2MASS [44,46], 2dFGRS [47], GAMA [48], BOSS[49], WiggleZ [50], Vipers [51], FastSound [52], BOSS Q [53] and additional points from the 2018 SDSS-IV [54–56].

The χ^2 statistics for WiggleZ is defined as

$$\chi_{WiggleZ}^2 = (\bar{A}_{obs} - \bar{A}_{th})C_{WiggleZ}^{-1}(\bar{A}_{obs} - \bar{A}_{th})^T, \quad (106)$$

where $\bar{A}_{obs} = (0.447, 0.442, 0.424)$ sets the observational values for data vectors at $z = (0.44, 0.60, 0.73)$ compared with the theoretical predictions $\bar{A}_{th} = (z, p_i)$. The latter is defined as

$$\bar{A}_{th} = d_V(z) \frac{\sqrt{\Omega_{m0}H_0^2}}{cz}, \quad (107)$$

with the related dilation scale $d_V(z)$ as

$$d_V(z) = \frac{1}{H_0} \left[(1+z)^2 d_A(z)^2 \frac{cz}{E(z)} \right]^{1/3}. \quad (108)$$

Moreover, the inverse of the covariant matrix $C_{WiggleZ}^{-1}$ is given by

$$C_{WiggleZ}^{-1} = \begin{pmatrix} 1040.3 & -807.5 & 336.8 \\ -807.5 & 3720.3 & -1551.9 \\ 336.8 & -1551.9 & 2914.9 \end{pmatrix}. \quad (109)$$

Likewise, the χ^2 statistic of SDSS data is defined as

$$\chi_{SDSS}^2 = (\bar{d}_{obs} - \bar{d}_{th})C_{SDSS}^{-1}(\bar{d}_{obs} - \bar{d}_{th})^T, \quad (110)$$

where the observable distance $d_{obs} = (0.1905, 0.1097)$ that is calculated at $z = 0.2$ and $z = 0.35$, and theoretical distance d_{th} that is given by

$$\bar{d}_{th} = \frac{r_s(z_d)}{d_V(z)}, \quad (111)$$

where the related comoving sound horizon $r_s(z)$ and sound speed $c_s(z)$ were given previously by Eqs. (103) and (104), respectively. The drag redshift z_{drag} is defined as

$$z_{drag} = \frac{1291(\Omega_m h^2)^{0.251}}{1 + 0.659(\Omega_m h^2)^{0.828}} [1 + b_1(\Omega_m h^2)^{b_2}], \quad (112)$$

where $b_1 = 0.313(\Omega_m h^2)^{-0.419} [1 + 0.607(\Omega_m h^2)^{0.674}]$ and $b_2 = 0.238(\Omega_m h^2)^{0.223}$.

The inverse of the covariant matrix C_{SDSS}^{-1} is given by

$$C_{SDSS}^{-1} = \begin{pmatrix} 30124 & -17227 \\ -17227 & 86977 \end{pmatrix}. \quad (113)$$

Finally, we combine all the BAO data as

$$\begin{aligned} \chi_{BAO}^2 &= \chi_{WiggleZ}^2 + \chi_{SDSS}^2 + \chi_{6dF}^2 + \chi_{SDSS-MGS}^2 \\ &\quad + \chi_{BOSS-LOWZ}^2 + \chi_{BOSS-Ly\alpha}^2. \end{aligned} \quad (114)$$

4.1.4 The Pantheon supernova type Ia data

An important quantity to constrain SNIa data relies on the theoretical distance modulus $\mu_{th}(z)$ defined as

$$\mu_{th}(z) = 5 \log_{10}(d_L(z)) + \mu_0, \quad (115)$$

where $\mu_0 = 42.38 - 5 \log_{10} h$ with $h = 0.672$. The luminosity distance d_L related to Hubble expansion rate is given by

$$d_L(z|s, \mu_0) = (1+z) \int_0^z \frac{du}{E(u|s)}, \quad (116)$$

where s denotes the free parameters of a model. We use the prior for the density parameter of visible baryonic matter $\Omega_{b0} = 2.236/100 h^2$. The χ^2 statistics are used in a form

$$\chi_{SNIa}^2(s|\mu_0) = \sum_{i=1}^n \frac{[\mu_{th,i}(s, \mu_0|z_i) - \mu_{obs,i}(z_i)]^2}{\sigma_{\mu i}^2}, \quad (117)$$

where $n = 1048$ is the number of events of the Pantheon SNIa data [33], the distance modulus obtained from observations is denoted by $\mu_{obs,i}(z_i)$, and $\sigma_{\mu i}$ is the total uncertainty of the observational data.

4.2 Results and discussion

The methodology used in this paper relies on the Markov Chain Monte Carlo (MCMC) sample technique adapted from a publicly available code of a modified Metropolis–Hastings algorithm [31,32] used as a parameter estimator. We perform our analysis using the joint likelihood of kinematic probes on the CMB Planck 2018 data [2], the Pantheon

Table 1 A summary of best-fit values of the main cosmological parameters from MCMC chains. The χ^2_{bf} denotes the χ^2 best-fit values from MCMC chains of each individual model

Model	Ω_{m0}	$100 \Omega_{b0} h^2$	h	σ_8	Model parameters	χ^2_{bf}
Λ CDM	0.317 ± 0.006	2.234 ± 0.015	0.674 ± 0.004	0.758 ± 0.028	$w = -1$	1087.82
w CDM	0.312 ± 0.008	2.233 ± 0.015	0.679 ± 0.008	0.760 ± 0.028	$w = -1.023 \pm 0.033$	1087.40
β -model	0.316 ± 0.006	2.238 ± 0.015	0.674 ± 0.004	0.756 ± 0.028	$w = -1.005 \pm 0.009$	1088.16

Table 2 A summary of mean values of background parameters calculated by using MCMC chains

Model	Ω_{m0}	$100 \Omega_{b0} h^2$	h	σ_8	Model parameters
Λ CDM	0.316 ± 0.006	2.235 ± 0.015	0.674 ± 0.004	0.761 ± 0.028	$w = -1$
w CDM	0.312 ± 0.007	2.234 ± 0.015	0.680 ± 0.008	0.759 ± 0.028	$w = -0.993 \pm 0.027$
β -model	0.320 ± 0.006	2.238 ± 0.016	0.669 ± 0.005	0.761 ± 0.028	$w = -1.023 \pm 0.009$

Table 3 A summary of the obtained values of AIC, MBIC and HQC for the studied models, and the Jeffreys scale with the strength evidence against a model

Model	AIC	ΔAIC	Evidence	$MBIC$	$\Delta MBIC$	HQC	ΔHQC	Evidence
Λ CDM	1095.86	0	Null	1108.56	0	1103.41	0	Null
w CDM	1097.45	1.60	Weak	1113.32	4.76	1106.89	3.48	Positive
β -model	1098.21	2.36	(barely) weak	1114.08	5.52	1107.65	4.24	Positive

SN Ia [33] with redshift ranging from $0.01 < z < 2.3$, the Hubble parameter $H(z)$ as a function of redshift [34–39] and the “extended Gold 2018” growth-rate data compilation of SDSS [40–42], 6dFGS [43], IRAS [44,45], 2MASS [44,46], 2dFGRS [47], GAMA [48], BOSS [49], WiggleZ [50], Vipers [51], FastSound [52], BOSS Q [53] and additional points from the 2018 SDSS-IV [54–56].

To apply our χ^2 -statistics, we extract the data points from the Pantheon SNIa, CMB, BAO, Hubble parameter and growth, with the amount of 1048, 3, 9, 36 and 25 data points, respectively, with a total of 1121 data points. In order to keep the analysis on the sub-horizon linear scale, we set the minimum value of expansion parameter as $a_{min} = 0.001$, $k = 300$ $H_0 \sim 0.1 \text{ hMpc}^{-1}$ and the parameter vectors $\{\Omega_{m0}, 100\Omega_b h^2, h, w, \sigma_8\}$ for both β -model and w CDM. For the Λ CDM, one has $\{\Omega_{m0}, \Omega_{b0} h^2, -1, h, \sigma_8\}$. The priors for the β -model were set as $\{(0.001, 1), (0.001, 0.08), (0.4, 1), (-1.07, -0.999), (0.1, 1.8)\}$, respectively. To the CMB temperature, we adopt a reference value $T_{cmb} = 2.7255 \text{ K}$. Moreover, the joint analysis was also implemented by the product of the particular likelihoods $\mathcal{L}$ for each data set

$$\mathcal{L}_{tot} = \mathcal{L}_{Pantheon} \cdot \mathcal{L}_{BAO} \cdot \mathcal{L}_{CMB} \cdot \mathcal{L}_{H(z)} \cdot \mathcal{L}_{growth} \quad (118)$$

and the sum of individual χ^2 to get the total χ^2

$$\chi^2_{tot} = \chi^2_{Pantheon} + \chi^2_{BAO} + \chi^2_{CMB} + \chi^2_{H(z)} + \chi^2_{growth} \quad (119)$$

The list of full values for the $H(z)$ data can be found in Tables (1) and (2) of Ref.[32]. In Tables 1 and 2, we present the best-fit and mean values for the studied models (Table 3).

The statistical relevance of the models as compared to observations is made with model selection criteria using maximum likelihood estimation, the so-called IC classifiers used to estimate the strength of tension between the data fitting and particular models. To sum up, they are used to infer how much information is lost when the *true* (unknown) model is replaced by a particular one. In general, the model with higher IC tends to aggravate the tension which commonly happens when more parameters are allowed. To apply this methodology, we adopt the errors from the data being as Gaussian. The first IC we use is the Akaike criterion (AIC) [60] that evaluates the fit-to-data for small samples sizes [92,93] by the formula

$$AIC = \chi^2_{bf} + 2k \frac{2k(k+1)}{N-k-1}, \quad (120)$$

where χ^2_{bf} is the best fit χ^2 of the model, k represents the number of the uncorrelated (free) parameters and N is the number of the data point in the related dataset. The difference $|\Delta AIC| = AIC_{model(2)} - AIC_{model(1)}$ represents the Jeffreys’ scale [94] that measures the intensity of tension between two competing models. Higher values for $|\Delta AIC|$ denotes more tension between models (more statistically uncorrelated). For $|\Delta AIC| \leq 2$ the models are statistically consistent with a considerable level of empirical support.

Table 4 A summary of the obtained values of growth index for the studied models in the $k \sim 0.1 h\text{Mpc}^{-1}$ scale. The ΛCDM model is adopted as a reference

Model	γ_∞	γ_0	γ_1
ΛCDM	0.5454 ± 0.0000	0.7591 ± 0.0012	-0.2136 ± 0.0012
$w\text{CDM}$	0.5456 ± 0.0007	0.7588 ± 0.0050	-0.2132 ± 0.0050
$\beta\text{-model}$	0.5449 ± 0.0002	0.7566 ± 0.0028	-0.2117 ± 0.0033

For $4 < \Delta AIC < 7$ indicates a positive tension against the model with a higher value of AIC. For $|\Delta AIC| \geq 10$ defines a strong evidence against the model with a higher AIC. In Table 4, it is shown the results of the classification of the AIC analyses showing a statistical equivalence between the ΛCDM and $w\text{CDM}$ models and a nearly equivalence of these models with the β -model that has ΔAIC roughly close to 2. Another common selection criteria relies on the Modified Bayesian Information Criterion (MBIC) that is given by in order to correct BIC [61] change-point processes [62], given by

$$MBIC = \chi_{bf}^2 + k(\ln[N] - \ln[2\pi]) . \quad (121)$$

From the Jeffreys' scale, higher values for $|\Delta MBIC|$ denotes more tension between competing models and more statistically uncorrelated they are. For $|\Delta MBIC| \leq 2$, it is not worth more than a bare mention and the models are statistically consistent with a considerable level of empirical support. For $2 < \Delta MBIC \leq 6$ indicates a positive tension against the model with a higher value of BIC. For $6 < \Delta MBIC \leq 10$, it defines a strong evidence against the model with a higher MBIC value and a very strong evidence against the model with a higher BIC value happens when $\Delta MBIC > 10$ and a strong evidence with $\Delta MBIC$ against the models with at least one extra parameter such as the $w\text{CDM}$ and the proposed β -model.

To complement our analysis, we also use Hannan–Quinn information Criterion (HQC) [63] as an alternative to AIC and MBIC based the law of the iterated logarithm. It penalizes a model with an $(\ln \ln N)$ factor by the relation

$$HQC = \chi_{bf}^2 + 2k \ln[\ln[N]] . \quad (122)$$

In terms of the Jeffreys' scale for HQC, we use the same scale like that of MBIC. All these criteria focus on the penalty of free parameters which is a sort of Occam's razor realization to avoid model complexity with overfitting and considering only the maximum likelihood analysis it may induce false positives (or negatives). This must be seen with caution since it may be critical for cosmological models once the probes of cosmic data present a significant level of tension between them. This situation turns a non trivial task to decide between the model simplicity over complexity [95–

99]. Particularly, MBIC penalizes additional free parameters much heavily than AIC and HQC. The results are presented in Table 4 that show that the beta-model is statistically closer to $w\text{CDM}$ model than ΛCDM . In Fig. 1, we present the evolution of the growth density $\delta_m(a)$ in different k -modes that reinforce the differences of the models due to their different growth equations and the influence of the $G_{eff}(a, k)$ function, in the case of the β -model. In the left top panel, at the scale $k \sim 0.1 h\text{Mpc}^{-1}$, the β -model shows a growth evolution larger than ΛCDM and $w\text{CDM}$. For larger scales ($k \sim 0.01 h\text{Mpc}^{-1}$ and $k \sim 0.001 h\text{Mpc}^{-1}$), the top right panel shows a close equivalence to the previous case but also shows a mild density growth for β -model larger than the previous $k \sim 0.1 h\text{Mpc}^{-1}$ scale. Such small difference is better visualized in the bottom panel in which β -model behaviour in different scales (solid lines) does not overpass a percentage difference of 1%.

In Table 4, it is presented the results for the growth indices ($\gamma_\infty, \gamma_0, \gamma_1$) for the studied models at the $k \sim 0.1 h\text{Mpc}^{-1}$ scale from the MCMC chains. The evolution of the $\gamma(z)$ parameter is shown in Fig. 2. The left top and bottom panels show the evolution of $\gamma(z)$ and the percentage difference $\Delta\gamma(\%)$ between the models at the $k \sim 0.1 h\text{Mpc}^{-1}$ scale, respectively. In the right top and bottom panels are shown the profiles of the former quantities $\gamma(z)$ and $\Delta\gamma(\%)$ at the $k \sim 0.01 h\text{Mpc}^{-1}$ and $k \sim 0.001 h\text{Mpc}^{-1}$ scales, respectively, and present a larger different profile as compared to the $k \sim 0.1 h\text{Mpc}^{-1}$ scale. Since $\gamma(z)$ is somewhat related to matter distribution, the results in Fig. 2 are in agreement with the density profiles in Fig. 1. In addition, at the $k \sim 0.01 h\text{Mpc}^{-1}$ and $k \sim 0.001 h\text{Mpc}^{-1}$ scales, we have the values for $\gamma_0 = \{(0.7565 \pm 0.0028); (-0.2116 \pm 0.0028)\}$ and $\gamma_1 = \{(0.7483 \pm 0.0029); (-0.2034 \pm 0.0029)\}$, respectively. As expected, we obtain $\gamma_{\Lambda\text{CDM}} = 6/11$ and the values of the growth index deviate from $\gamma_{\Lambda\text{CDM}}$. In all cases, the asymptotic values of γ_∞ are closer to $\gamma_{\Lambda\text{CDM}}$. For the β -model γ_∞ is just $\sim 10^{-4}\%$ smaller than γ_∞ for ΛCDM which may be caused by the perturbations lowering γ_∞ [100]. For the $w\text{CDM}$ model, γ_∞ is just higher $\sim 10^{-4}\%$ smaller than γ_∞ for ΛCDM .

In Fig. 3, we have the analysis on the Hubble function $H(z)$ in units of $\text{km s}^{-1} \text{Mpc}^{-1}$ in terms of redshift ($z \sim 2$) using the values presented in Table 2. In the left panel, it shown a compatibility between all models and the curves

Fig. 1 Density growth profiles in different scales for the considered models. The values of the parameters were extracted from Table 2. Top left panel: Density growth in $k \sim 0.1 \text{ h Mpc}^{-1}$ scale. Top right panel: Density growth in $k \sim 0.01 \text{ h Mpc}^{-1}$ and $k \sim 0.001 \text{ h Mpc}^{-1}$ scales.

The black and blue solid lines denote the β -model profile. Bottom panel: Evolution of $k^{1.5} \delta_m(a)$, with k in units of h Mpc^{-1} . Solid lines denote β -model profile in different scales

are nearly overlapped for the Hubble background evolution where we used data points from [33, 36] and some “clustering” measurements of $H(z)$ [101]. In the right panel, we have the percentage difference $\%(H_i - H_j)/H_i$ between w CDM and Λ CDM (dashed blue line) that goes up to 1.6% at early times. The percentage difference between w CDM and the β -model (solid red line) goes roughly up to 2.15% at early times. When β -model is compared with Λ CDM (solid black line), the percentage difference goes roughly up to 1.35% at early times. In all cases, the damping pattern occurs leading to a convergence of the models at late times. The observed spikes are possibly produced by the BAO influence on the $H(z)$ measurements. In the case of the percentage difference between w CDM and the β -model we have a spike around $z \sim 0.7$ when matches the transition range for DE profiles and voids [102]. For the case of β -model and Λ CDM, the spike occurs around $z \sim 1.1$ showing the underlying influence of SDSS and Vipers data at the range $z = [0.6 - 1.1]$ [103].

We also calculate the BBN speed-up factor for the β -model as compared with w CDM and Λ CDM. The BBN speed-up factor is a consequence of the modification of the value of the gravitational constant during BBN epoch

$z_{\text{BBN}} \sim 10^9$ [104, 105]. It serves as a reference for constraints in which the bound relies on the BBN epoch $|\frac{\Delta H^2}{H_{\Lambda\text{CDM}}^2}| < 10\%$

bound is fulfilled with $|\frac{\Delta H_{\beta}^2}{H_{\Lambda\text{CDM}}^2}| \sim 0.64\%$. If we assume $|\frac{\Delta H_{\beta}^2}{H_{w\text{CDM}}^2}| < 10\%$, we obtain $|\frac{\Delta H^2}{H_{w\text{CDM}}^2}| < 1.27\%$.

Finally, we apply an important initial test for the β -model that must provide a correct power spectra of matter growth and CMB signatures in comparison with w CDM and Λ CDM. In Fig. 4, we present the resulting unlensed spectrum of temperature anisotropy C_l^{TT} . We used the modified version of CLASS code (EFCLASS [32]) from the mean values of Table 2. The left panel presents lower multipole values for the β -model (orange curve) due to a reduction of ISW effect as compared with Λ CDM and w CDM models that is in agreement with the observed large scale (low- l) CMB spectrum. For high multiples no differences were observed and the curves are overlapped. In the right panel, it is presented the linear matter power spectrum $P(k)$ with and overall overlapping between the curves of the studied models. It is interesting to note that in the β -model curve (orange line) presents a slight but higher amplitude starting from $k \sim 0.01 \text{ h Mpc}^{-1}$ scale leading to a higher growth matter profile in concordance

Fig. 2 Growth index evolution in different scales (top panels). In the bottom panels, the percentage difference is presented. Λ CDM is considered as a reference model. The values of the parameters were extracted from Tables 2 and 4

Fig. 3 In the left panel, Hubble function in units of $\text{km s}^{-1} \text{Mpc}^{-1}$ is presented with its evolution in terms of redshift. In the right panel, it is shown the percentage difference between the studied models in loga-

rithm scale. The values of the parameters were extracted from Table 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4 The left panel shows the unlensed CMB TT power spectrum for the models using CLASS (EFCLASS). In the right panel, the matter power spectrum P_k is presented. The values of the parameters were extracted from Table 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with the results presented in Table 2 from MCMC chains and hence the resulting matter density profiles in Fig. 1.

5 Remarks

In this paper, we discussed the dark energy problem with a proposal of a geometric model. Differently from the standard Brane-world models where the embedded is rigid, we opted to explore the dynamical Nash-Greene embedding theorem to construct an induced four dimensional cosmological model embedded in five dimensions. Our first results are related to the obtainment of the perturbed equations. Using the linear Nash-Greene fluctuations, we showed that only the gravitational tensor equation propagates cosmological perturbations, but the trace of Codazzi equation in Eq. (21) and Gupta equations in Eq. (38) remain confined to their background form in the embedded space-time. We showed that once the perturbed $\delta Q_{\mu\nu}$ is a diagonal tensor, it does not directly contribute to anisotropic shear viscosity in any gauge and its influence on cosmological perturbations will depend on how the metric perturbations are accounted for a specific gauge. In the sub-horizon regime of the longitudinal conformal Newtonian gauge, we used the corresponding modified Friedman equation proposing an effective fluid approach in which $\delta Q_{\mu\nu}$ plays the role of an “extrinsic” energy tensor $T_{\mu\nu}^{ext}$. We confronted the present model to the Λ CDM and w CDM through a joint analysis on recent pack of datasets on the Pantheon SNIa, CMB, BAO, cosmic growth and the Hubble $H(z)$ evolution. By means of the IC model selection analyses, we found that the β -model statistically approaches w CDM rather than Λ CDM in all selection criteria (AIC, MBIC and HQC), since the former models pre-sent one extra-parameter as compared to the latter. Thus, they are naturally more penalized in selection criteria, such as MBIC and HQC that intend to avoid model complexity, resulting in a positive evidence against both β -model and w CDM as compared to Λ CDM. On the other hand, a weak evidence against the models are obtained using the AIC criteria, as shown in Table 4.

In this direction, we used a Markov Chain Monte Carlo analysis (MCMC) from a modified Metropolis-Hastings algorithm [31,32] to determine the cosmic parameters. We

studied matter density profiles $\delta_m(z)$ for the models indicating a higher growth density for the β -model in the scale $k \sim 0.1$. We also verified that the pattern for larger growth are suchlike for the other scales $k \sim 0.01$ and $k \sim 0.001$. We computed the growth index $\gamma(z)$, reproducing the related value $\gamma_0 = 6/11$ for Λ CDM and the expected deviations of $\gamma(z)$ from the models in comparison. As a result, we showed that the asymptotic values of growth index γ_∞ present a mild lower value as compared to that one from Λ CDM due to, in terms of fluid approach, the DE perturbations. Finally, we analysed the evolution of $H(z)$, the full spectrum of unlensed temperature anisotropy C_l^{TT} and the linear matter power spectrum $P(k)$. The unlensed temperature anisotropy C_l^{TT} for β -model shows a diminishing of the ISW effect in low multipoles and the resulting $P(k)$ shows a slight but higher signs of the amplitude as compared to Λ CDM and w CDM patterns. The obtained results of such analyses corroborate the patterns observed in the growth profiles of $\delta_m(z)$ and $\gamma(z)$. As prospects, we intend to make a larger analysis taking into account possible effects of anisotropies of DE fluid on C_l^{TT} and $P(k)$ profiles. This process is in due course and will be reported elsewhere.

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