



Deflection of light by a Coulomb charge in Born–Infeld electrodynamics

Jin Young Kim^a

Department of Physics, Kunsan National University, Kunsan 54150, South Korea

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Abstract We study the propagation of light under a strong electric field in Born–Infeld electrodynamics. The nonlinear effect can be described by the effective indices of refraction. Because the effective indices of refraction depend on the background electric field, the path of light can be bent when the background field is non-uniform. We compute the bending angle of light by a Born–Infeld-type Coulomb charge in the weak lensing limit using the trajectory equation based on geometric optics. We also compute the deflection angle of light by the Einstein–Born–Infeld black hole using the geodesic equation and confirm that the contribution of the electric charge to the total bending angle agree.

1 Introduction

In Maxwell’s electrodynamics the equations of motion for the electromagnetic field are linear and the speed of light is constant in vacuum. The path of light is not affected by the presence of electric or magnetic field. However, in quantum electrodynamics, nonlinear terms appear in the effective action from the vacuum polarization effect. Then the path of light can be bent when the light passes around a strong electric or magnetic field. In this sense, vacuum itself can be considered as a dielectric medium in nonlinear electrodynamics. The one-loop correction is described by the Euler–Heisenberg Lagrangian [1, 2]. In spite of several ground laboratory experiments to observe the nonlinearity in the presence of a strong magnetic field, the proof is still lacking [3–5].

When photons pass close to compact astronomical objects like a charged black hole or a magnetized neutron star, the path of light is affected by the electromagnetic effect as well as by the gravitational effect. The gravitational lensing by the Reissner–Nordstrom (RN) black hole is a typical example [6, 7]. Recently, inspired by the development in string theory, interests in the Born–Infeld (BI) theory were renewed.

The black hole solution in the Einstein–Born–Infeld (EBI) gravity, which is the nonlinear electromagnetic generalization of RN black hole, is one of the interesting topics. Diverse aspects of the propagation of light in Born–Infeld-type nonlinear electrodynamics were studied [8–13].

Most ground laboratory experiments and astronomical observations related to the nonlinearity of electromagnetism utilize the magnetic field. It seems natural to think utilizing the electric field to test such nonlinearity. One can consider the case when photons pass very close to heavy nuclei in ground laboratory experiment. In this case it is known that the relevant electric field is greater than 10^{21} V/m [14]. One can also consider the gravitational lensing of light by charged black holes. It seems not probable to observe charged black holes because the observed universe is charge neutral. Nonetheless we think it is interesting to think the propagation of light under the strong electric field by a Coulomb charge in Born–Infeld electrodynamics. It may also deepen our understanding when we study the nonlinearity using magnetic field.

In Born–Infeld-type nonlinear electrodynamics, the propagation of light in a uniform external electromagnetic field can be described by the effective index of refraction [11–13]. When the external field is non-uniform, for example the electric field by a spherically symmetric charge distribution, the effective index of refraction can change continuously. Then the path of light incoming from infinity can be bent by the gradient of the effective index of refraction generated by the background field. In this paper we compute the deflection angle of light by a Born–Infeld-type Coulomb charge.

The organization of this paper is as follows. In Sect. 2, we study the propagation of light under the background electric field in Born–Infeld electrodynamics. We calculate the effective indices of refraction for the uniform background electric field. In Sect. 3, we compute the bending angle of a light ray passing the strong electric field by a Born–Infeld-type Coulomb charge using the trajectory equation based on geo-

^a e-mail: jykim@kunsan.ac.kr (corresponding author)

metric optics. In Sect. 4, we compute the bending angle of the Einstein–Born–Infeld black hole in the weak field limit using the geodesic equation. We confirm that the two results agree in the appropriate limit. Finally in Sect. 5, we discuss our results.

2 Index of refraction in the background electric field

The generalized Born–Infeld action, which combines the classical Born–Infeld action [15,16] and the one-loop correction of quantum electrodynamics [1], can be described by the effective Lagrangian density [12]

$$\mathcal{L}_{eff} = \beta^2 \left(1 - \sqrt{1 + \frac{2S}{\beta^2} - \frac{P^2}{\beta^2\gamma^2}} \right). \tag{1}$$

Here S and P are two Lorentz-invariants, in the Lorentz–Heaviside system with $\hbar = c = \epsilon_0 = \mu_0 = 1$, given by

$$\begin{aligned} S &= \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2), \\ P &= \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}, \end{aligned} \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ is its dual tensor. The parameter β and γ are given by

$$\frac{1}{\beta^2} = \frac{1}{B^2} + 8a_{\text{QED}}, \quad \frac{1}{\gamma^2} = \frac{1}{B^2} + 2b_{\text{QED}}, \tag{3}$$

where B is the parameter characterizing the maximum value of field strength in classical Born–Infeld electrodynamics, a_{QED} and b_{QED} are the quantum electrodynamic corrections defined by, in terms the fine structure constant α and the electron mass m_e ,

$$a_{\text{QED}} = \frac{2}{45} \frac{\alpha^2}{m_e^4}, \quad b_{\text{QED}} = \frac{14}{45} \frac{\alpha^2}{m_e^4}. \tag{4}$$

When S/β^2 and $P^2/\beta^2\gamma^2$ are small, Eq. (1) can be expanded as

$$\mathcal{L}_{eff} \simeq -S + \frac{S^2}{2\beta^2} + \frac{P^2}{2\gamma^2}. \tag{5}$$

In the limit $B \rightarrow \infty$ this gives the Euler–Heisenberg Lagrangian.

From the Euler–Lagrange equation for vector field A_μ , we have

$$\partial_\mu \left[\frac{1}{\mathcal{R}} \left(F_{\mu\nu} - \frac{P}{\gamma^2} \tilde{F}_{\mu\nu} \right) \right] = 0, \tag{6}$$

where

$$\mathcal{R} = \sqrt{1 + \frac{2S}{\beta^2} - \frac{P^2}{\beta^2\gamma^2}}. \tag{7}$$

Also from the Bianchi identity for field strength tensors, we have

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{8}$$

Eqs. (6) and (8) can be considered as the generalized Maxwell equations. The electric displacement and magnetic field are given from Eq. (1)

$$\mathbf{D} = \frac{\partial \mathcal{L}_{eff}}{\partial \mathbf{E}} = \frac{1}{\mathcal{R}} \left(\mathbf{E} + \frac{1}{\gamma^2} \mathbf{B}(\mathbf{E} \cdot \mathbf{B}) \right), \tag{9}$$

$$\mathbf{H} = -\frac{\partial \mathcal{L}_{eff}}{\partial \mathbf{B}} = \frac{1}{\mathcal{R}} \left(\mathbf{B} - \frac{1}{\gamma^2} \mathbf{E}(\mathbf{E} \cdot \mathbf{B}) \right), \tag{10}$$

where $\mathbf{E}_i = F_{0i}$ and $\mathbf{B}_i = -(1/2)\epsilon_{ijk} F^{jk}$. From Eqs. (6) and (8), Maxwell equations can be written in terms of \mathbf{D} , \mathbf{B} , \mathbf{E} , and \mathbf{H} as

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0, & \frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, & \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0. \end{aligned} \tag{11}$$

From the definition $D_i = \epsilon_{ij} E_j$ and $B_i = \mu_{ij} H_j$, the electric permittivity tensor and the inverse magnetic permeability tensor were obtained by Kruglov [12] as

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{\mathcal{R}} \left(\delta_{ij} + \frac{1}{\gamma^2} B_i B_j \right), \\ (\mu^{-1})_{ij} &= \frac{1}{\mathcal{R}} \left(\delta_{ij} - \frac{1}{\gamma^2} E_i E_j \right). \end{aligned} \tag{12}$$

The above equations show that the vacuum is anisotropic in the generalized Born–Infeld electrodynamics. The eigenvalues and the inverse matrix which diagonalizes ϵ_{ij} are given by

$$\lambda_1(\epsilon) = \frac{1}{\mathcal{R}} \left(1 + \frac{\mathbf{B}^2}{\gamma^2} \right), \quad \lambda_2(\epsilon) = \frac{1}{\mathcal{R}} \text{ (degenerate)}, \tag{13}$$

$$(\epsilon^{-1})_{ij} = \mathcal{R} \left(\delta_{ij} - \frac{B_i B_j}{\gamma^2 + \mathbf{B}^2} \right). \tag{14}$$

The corresponding values for μ_{ij}^{-1} are

$$\lambda_1(\mu^{-1}) = \frac{1}{\mathcal{R}} \left(1 - \frac{\mathbf{E}^2}{\gamma^2} \right),$$

$$\lambda_2(\mu^{-1}) = \frac{1}{\mathcal{R}} \text{ (degenerate)}, \tag{15}$$

$$\mu_{ij} = \mathcal{R} \left(\delta_{ij} + \frac{E_i E_j}{\gamma^2 - \mathbf{E}^2} \right). \tag{16}$$

Now we consider the propagation of the plane electromagnetic wave (\mathbf{e} , \mathbf{b}) under the constant uniform background electric field

$$\mathbf{e} = \mathbf{e}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \mathbf{b} = \mathbf{b}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{17}$$

We take the direction of the background electric field as x -direction $\bar{\mathbf{E}} = (\bar{E}, 0, 0)$ and the direction of propagating wave as z -direction. We consider the case where the background electric field is much stronger than the photon's electric and magnetic fields. The electromagnetic fields are the superposition of the traveling wave and the background field. Substituting $\mathbf{E} = \mathbf{e} + \bar{\mathbf{E}}$ and $\mathbf{B} = \mathbf{b}$ in Eq. (1), up to the quadratic order in \mathbf{e} and \mathbf{b} , the effective Lagrangian is given by

$$\mathcal{L}_{eff}(\mathbf{e} + \bar{\mathbf{E}}, \mathbf{b}) = \beta^2 \left(1 - \sqrt{1 + \frac{\mathbf{b}^2 - (\mathbf{e} + \bar{\mathbf{E}})^2}{\beta^2} - \frac{(\mathbf{b} \cdot \bar{\mathbf{E}})^2}{\beta^2 \gamma^2}} \right). \tag{18}$$

The electric displacement and the magnetic field are obtained as

$$d_i = \frac{\partial \mathcal{L}}{\partial e_i} = \frac{1}{\kappa} \left(\delta_{ij} + \frac{\bar{E}_i \bar{E}_j}{\beta^2 \kappa^2} \right) e_j, \tag{19}$$

$$h_i = -\frac{\partial \mathcal{L}}{\partial b_i} = \frac{1}{\kappa} \left(\delta_{ij} - \frac{\bar{E}_i \bar{E}_j}{\gamma^2} \right) b_j, \tag{20}$$

where

$$\kappa = \sqrt{1 - \frac{\bar{\mathbf{E}}^2}{\beta^2}}. \tag{21}$$

The electric permittivity and the inverse magnetic permeability tensors are read as

$$\varepsilon_{ij} = \frac{1}{\kappa} \left(\delta_{ij} + \frac{\bar{E}_i \bar{E}_j}{\beta^2 \kappa^2} \right), \tag{22}$$

$$(\mu^{-1})_{ij} = \frac{1}{\kappa} \left(\delta_{ij} - \frac{\bar{E}_i \bar{E}_j}{\gamma^2} \right). \tag{23}$$

The above tensors for the constant uniform electric field can be obtained from those for constant uniform magnetic field [12] by replacing $\bar{\mathbf{B}}, \gamma^2, \beta^2 \kappa^2, \kappa = \sqrt{1 + \bar{\mathbf{B}}^2/\beta^2}$ with $\bar{\mathbf{E}}, \beta^2 \kappa^2, \gamma^2, \kappa = \sqrt{1 - \bar{\mathbf{E}}^2/\beta^2}$, respectively.

The Maxwell equations in (11) can be written in terms of \vec{k} and ω as

$$k_i d_i = k_i b_i = 0, \quad \mathbf{k} \times \mathbf{e} = \omega \mathbf{b}, \quad \mathbf{k} \times \mathbf{h} = -\omega \mathbf{d}, \tag{24}$$

and the wave equation for the electric field \mathbf{e} is obtained as

$$\left[\mathbf{k}^2 (\mu^{-1})_{ji} + \{k_a (\mu^{-1})_{al} k_l - \mathbf{k}^2 (\mu^{-1})_{aa}\} \delta_{ij} - k_l (\mu^{-1})_{jl} k_i + \omega^2 \varepsilon_{ij} \right] e_j = 0. \tag{25}$$

Substituting Eqs. (22) and (23) in Eq. (25) and using $\mathbf{k} \cdot \mathbf{e} = \mathbf{k} \cdot \bar{\mathbf{E}} = 0$, we obtain

$$\frac{\omega^2}{\kappa} \left[\left(1 - n^2 + n^2 \frac{\bar{\mathbf{E}}^2}{\gamma^2} \right) \delta_{ij} + \left(\frac{1}{\beta^2 \kappa^2} - \frac{n^2}{\gamma^2} \right) \bar{E}_i \bar{E}_j \right] e_j = 0, \tag{26}$$

where $n = k/\omega$ is the index of fraction. The condition for Eq. (26) to have nontrivial solutions is that the determinant of the following matrix is zero

$$\Lambda_{ij} = \left(1 - n^2 + n^2 \frac{\bar{\mathbf{E}}^2}{\gamma^2} \right) \delta_{ij} + \left(\frac{1}{\beta^2 \kappa^2} - \frac{n^2}{\gamma^2} \right) \bar{E}_i \bar{E}_j. \tag{27}$$

The eigenvalues of the matrix Λ_{ij} are obtained as

$$\lambda_1 = 1 - n^2 + n^2 \frac{\bar{\mathbf{E}}^2}{\gamma^2}, \quad \lambda_2 = 1 - n^2 + \frac{\bar{\mathbf{E}}^2}{\beta^2 \kappa^2}. \tag{28}$$

Using $\kappa = \sqrt{1 - \bar{\mathbf{E}}^2/\beta^2}$, the effective indices of refraction are obtained, from $\lambda_1 = 0$ and $\lambda_2 = 0$, as

$$n_{\perp} = \left(1 - \frac{\bar{\mathbf{E}}^2}{\gamma^2} \right)^{-\frac{1}{2}}, \quad n_{\parallel} = \left(1 - \frac{\bar{\mathbf{E}}^2}{\beta^2} \right)^{-\frac{1}{2}}. \tag{29}$$

Note that there is the vacuum birefringence effect for $\beta \neq \gamma$, coming from the QED corrections.

3 Deflection angle by geometric optics

When light passes a uniform constant background electric field, the propagation of light can be described by effective indices of refraction which depend on the background electric field. A non-uniform background electric field can make gradients for indices of refraction. If the light ray incoming from infinity passes a non-uniform electric field, the path of light can be bent by the gradients. The bending angle can be computed by calculus.

The simplest non-uniform and isotropic gradient can be made by a Coulomb charge or a spherically symmetric charge distribution. We consider the BI electrostatic case with $\mathbf{B} = \mathbf{H} = 0$ assuming a point charge Q is located at the origin

$$\nabla \cdot \mathbf{D} = Q \delta(\mathbf{r}). \tag{30}$$

The electric displacement is obtained as

$$\mathbf{D} = \frac{Q}{4\pi r^2} \hat{r}, \tag{31}$$

and, from Eq. (9), the electric field is obtained as

$$\mathbf{E} = \frac{Q}{4\pi \sqrt{r^4 + \rho_0^4}} \hat{r}, \tag{32}$$

where $\rho_0 = \sqrt{|Q|/4\pi\beta}$.

If we denote the unit vector in the direction of light by $\mathbf{u} = \mathbf{k}/k$ as in [17–20], the indices of refraction can be written as

$$n_{\perp} = \left(1 - \frac{(\bar{\mathbf{E}} \times \mathbf{u})^2}{\gamma^2}\right)^{-\frac{1}{2}}, \quad n_{\parallel} = \left(1 - \frac{(\bar{\mathbf{E}} \times \mathbf{u})^2}{\beta^2}\right)^{-\frac{1}{2}}. \tag{33}$$

The maximum field strength is $E_{\max} = \beta$ from Eq. (32). As mentioned before the lower bound of the classical BI parameter B in Eq. (3) is of the order $B > 10^{21}$ V/m while a_{QED} and b_{QED} are of the order E_c^{-2} , where E_c is the QED critical field strength defined by $E_c = m_e^2 c^3 / e \hbar = 1.32 \times 10^{18}$ V/m. In the limit $B \rightarrow \infty$, the bending of light comes purely from QED one-loop effect described by Euler–Heisenberg Lagrangian [21,21]. When the light ray is passing the region where the field strength E is comparable to the BI parameter B , one can compute the trajectory numerically. We consider the case when the light ray is passing the region where E is not as large as B . This corresponds to the weak field limit. The leading terms in Eq. (33) are

$$n_{\perp} = 1 + \frac{(\bar{\mathbf{E}} \times \mathbf{u})^2}{2\gamma^2}, \quad n_{\parallel} = 1 + \frac{(\bar{\mathbf{E}} \times \mathbf{u})^2}{2\beta^2}. \tag{34}$$

Because the electric field by the point charge Q is isotropic, we may assume that the light ray is confined to the equatorial plane $\theta = \pi/2$ and the trajectory of light $y(x)$ is on the xy plane. If we take the direction of incoming ray at $x = -\infty$ as the $+x$ direction, the slope $y' = dy/dx$ is the direction of the light ray (Fig. 1). Substituting $r = \sqrt{x^2 + y^2}$, $\mathbf{u} = (dx, dy)/\sqrt{dx^2 + dy^2} = (1, y')/\sqrt{1 + y'^2}$ and Eq. (32) into Eq. (34), the indices of refraction can be written as

$$n_{\perp} = 1 + \frac{1}{2} \frac{\beta^2}{\gamma^2} \frac{\rho_0^4}{r^4 + \rho_0^4} \frac{(y - xy')^2}{r^2(1 + y'^2)}, \tag{35}$$

$$n_{\parallel} = 1 + \frac{1}{2} \frac{\rho_0^4}{r^4 + \rho_0^4} \frac{(y - xy')^2}{r^2(1 + y'^2)}. \tag{36}$$

When the index of fraction is continually varying, the trajectory of light can be obtained from the following equation

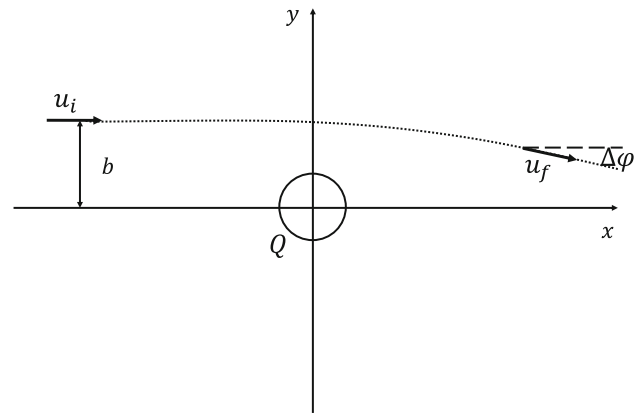


Fig. 1 Schematic of light bending in a Coulombic field

derived from Snell’s law [21,22]

$$\frac{d\mathbf{u}}{ds} = \frac{1}{n} (\mathbf{u} \times \nabla n) \times \mathbf{u}, \tag{37}$$

where s is the distance parameter $ds = |d\vec{r}|$ and $\mathbf{u} = d\mathbf{r}/ds$. This trajectory equation is equivalent to the eikonal equation of wave optics. For the light bending by the background magnetic dipole field, it has been confirmed that the result using this trajectory equation exactly agree with the result using the eikonal equation [22,23]. When the index of refraction is close to one, the trajectory equation in the leading order can be approximated as

$$\frac{d\mathbf{u}}{ds} = (\mathbf{u}_0 \times \nabla n) \times \mathbf{u}_0, \tag{38}$$

where \mathbf{u}_0 denotes the initial direction of the incoming ray. For the light ray coming from $x = -\infty$ and moving to $+x$ direction,

$$\mathbf{u}_0 = (1, 0, 0), \tag{39}$$

and the leading-order trajectory equation (38) can be written as

$$\frac{d^2x}{ds^2} = 0, \quad \frac{d^2y}{ds^2} = \frac{\partial n}{\partial y}. \tag{40}$$

From the first equation $ds = dx$ at the leading order and d^2y/ds^2 in the second equation can be replaced by d^2y/dx^2 .

Now we compute the deflection angle with the impact parameter b . The initial conditions can be written as

$$y(-\infty) = b, \quad y'(-\infty) = 0. \tag{41}$$

Also the y' in Eqs. (35) and (36) can be neglected in the leading order approximation. Then the trajectory equation to

compute the deflection angle is

$$y'' = \xi \rho_0^4 \left[\frac{y}{r^2(r^4 + \rho_0^4)} - \frac{y^3}{r^4(r^4 + \rho_0^4)} - \frac{2y^3}{(r^4 + \rho_0^4)^2} \right], \tag{42}$$

where $\xi = \beta^2/\gamma^2$ for the perpendicular mode n_\perp and $\xi = 1$ for the parallel mode n_\parallel . By integrating Eq. (42)

$$y'(\infty) - y'(-\infty) = \xi \rho_0^4 \int_{-\infty}^{\infty} \left[\frac{y}{r^2(r^4 + \rho_0^4)} - \frac{y^3}{r^4(r^4 + \rho_0^4)} - \frac{2y^3}{(r^4 + \rho_0^4)^2} \right] dx, \tag{43}$$

we can get the deflection angle by

$$\Delta\varphi = y'(\infty) - y'(-\infty). \tag{44}$$

The details of the integration are given in the appendix. Substituting the result in Eq. (84), we obtain the deflection angle as

$$\Delta\varphi = \xi \left[\frac{\pi}{2} - \frac{\pi}{\sqrt{2}} \frac{b}{\left(\sqrt{b^4 + \rho_0^4} + b^2\right)^{1/2}} - \frac{\pi}{2\sqrt{2}} \frac{b^3 \rho_0^4 \left(3\sqrt{b^4 + \rho_0^4} + 2b^2\right)}{\left(b^4 + \rho_0^4\right)^{3/2} \left(\sqrt{b^4 + \rho_0^4} + b^2\right)^{3/2}} \right]. \tag{45}$$

Let us consider the limiting case $B \rightarrow \infty$ to compare with the known result in the literature. Because $\rho_0 (= \sqrt{|Q|/4\pi\beta})$ is small when β is large, the leading term in Eq. (45) is

$$\Delta\varphi = -\xi \frac{9\pi}{16} \frac{\rho_0^4}{b^4} = -\xi \frac{9}{256\pi} \frac{Q^2}{\beta^2 b^4}. \tag{46}$$

The negative sign in Eq. (46) means that the bending is attractive toward the charge. From the definition of parameter β^2 in Eqs. (3) and (4), in the limit $B \rightarrow \infty$,

$$\frac{1}{\beta^2} \rightarrow \frac{16}{45} \frac{\alpha^2}{m_e^4}, \tag{47}$$

and Eq. (46) reduces to

$$\Delta\varphi = -\xi \frac{1}{80\pi} \frac{\alpha^2}{m^4} \frac{Q^2}{b^4}. \tag{48}$$

For parallel mode $\xi = 1$ and for perpendicular mode mode $\xi = (7/4)^2$. This result exactly agrees with the calculation from the Euler–Heisenberg Lagrangian [21].

4 Deflection angle by geodesic equation

In the previous section we calculated the bending angle by a Born–Infeld-type Coulomb charge from the trajectory equation based on geometric optics. We consider the general relativistic effect and compute the bending angle using the geodesic equation. We start from the black hole solution in Einstein–Born–Infeld gravity. The EBI action is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + L_{BI} \right), \tag{49}$$

where the BI Lagrangian density is given by Eq. (1), $L_{BI} = \mathcal{L}_{eff}$. Here we use the Gaussian system with $1/4\pi\epsilon_0 = \mu_0/4\pi = 1$, commonly adopted to study the general relativity coupled with electromagnetism.

For the static electrically-charged case, the solutions of the spherically symmetric black hole are given by [9,24]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{50}$$

$$\mathbf{D} = \frac{Q_G}{r^2} \hat{r}, \tag{51}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{2\beta_G^2}{r} \int_r^\infty \left(\sqrt{x^4 + \frac{Q_G^2}{\beta_G^2}} - x^2 \right) dx, \tag{52}$$

M is the ADM mass, Q_G is the electric charge, and β_G is the generalized Born–Infeld parameter in Gaussian unit system. The subscript G is explicitly added for later uses to compare with the calculation from geometric optics formalism.

The asymptotic form of the metric function in the large-distance limit $r \gg \sqrt{|Q_G|/\beta_G}$ can be written as

$$f(r) = 1 - \frac{2M}{r} + \frac{Q_G^2}{r^2} - \frac{Q_G^4}{20\beta_G^2 r^6} + \dots \tag{53}$$

It is obvious that the Reissner–Nordstrom metric is recovered in the limit $\beta_G \rightarrow \infty$.

Photons do not follow the geodesic of the gravitational field due to the coupling of nonlinear electrodynamics to gravity. They follow the effective null geodesic generated by the self-interaction of the nonlinear theory. The effective metric of light is given by [25,26]

$$ds_{eff}^2 = -\chi(r)f(r)dt^2 + \frac{\chi(r)}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{54}$$

where

$$\chi(r) = \left(1 + \frac{Q_G^2}{\beta_G^2 r^4}\right)^{-1}. \tag{55}$$

We will calculate the deflection angle using this effective metric.

We consider the motion of a photon in the static spherically symmetric metric of the form, following the notation and procedure in [6],

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{56}$$

The bending angle of light coming from infinity is given by

$$\Delta\varphi = 2|\varphi(r_0) - \varphi_\infty| - \pi, \tag{57}$$

where r_0 is the distance of closest approach and

$$\varphi(r_0) = \varphi_\infty + \int_{r_0}^\infty \left[\frac{A(r)}{C(r)}\right]^{1/2} \left[\frac{C(r)B(r_0)}{C(r_0)B(r)} - 1\right]^{-1/2} dr. \tag{58}$$

Substituting Eq.(54) in Eq.(58), we have

$$\begin{aligned} \varphi(r_0) - \varphi_\infty \\ = I(r_0) = \int_{r_0}^\infty \left[\frac{\chi(r)}{f(r)}\right]^{1/2} \left[\frac{r^2\chi(r_0)f(r_0)}{r_0^2\chi(r)f(r)} - 1\right]^{-1/2} \frac{dr}{r}. \end{aligned} \tag{59}$$

In general the bending angle of light by EBI black hole can be calculated from the above integral numerically [27]. Since we are interested in the bending by purely BI electromagnetic effect to compare with the result in the previous section, we first compute the deflection angle of EBI black hole in the weak deflection limit. The total bending angle is the sum of the contributions by mass term and generalized BI term. If we set $M = 0$ at the end, the bending angle is only from the generalized BI term. Then we can compare the bending angle with the result from geometric optics formalism. In this limit the Q_G^4 term in Eq. (53) is a subleading term compared with the Q_G^2 term. Thus, to the leading order, the metric function $f(r)$ in (59) is the same as the metric function of the RN black hole

$$f(r) = 1 - \frac{2M}{r} + \frac{Q_G^2}{r^2}. \tag{60}$$

Substituting Eqs. (60) and (55) in Eq. (59) and defining $x \equiv r_0/r$, up to the quadratic order in M and Q_G , we have

$$\begin{aligned} I(r_0) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[1 + \frac{M}{r_0} \left(1 + \frac{1}{1+x} \right) \right. \\ \left. + \frac{M^2}{r_0^2} \left(\frac{3x}{1+x} + \frac{3}{2(1+x)^2} + \frac{3}{2}x^2 \right) \right. \\ \left. - \frac{1}{2} \frac{Q_G^2}{r_0^2} (1+x^2) + \frac{1}{2} \frac{Q_G^2}{\beta_G^2 r_0^4} (1+x^2-x^4) \right]. \end{aligned} \tag{61}$$

The integral yields

$$\begin{aligned} I(r_0) = \frac{\pi}{2} + \frac{2M}{r_0} + \left(\frac{15\pi}{8} - 2 \right) \frac{M^2}{r_0^2} \\ - \frac{3\pi}{8} \frac{Q_G^2}{r_0^2} + \frac{9\pi}{32} \frac{Q_G^2}{\beta_G^2 r_0^4}. \end{aligned} \tag{62}$$

Inserting Eq. (62) in Eq. (57), we have

$$\begin{aligned} \Delta\varphi = \frac{4M}{r_0} + \left(\frac{15\pi}{4} - 4 \right) \frac{M^2}{r_0^2} \\ - \frac{3\pi}{4} \frac{Q_G^2}{r_0^2} + \frac{9\pi}{16} \frac{Q_G^2}{\beta_G^2 r_0^4}. \end{aligned} \tag{63}$$

Computation of higher order terms will be straightforward. Except for the last term, Eq. (63) is the deflection angle for the weak lensing by RN black hole [28]. When the bending is small we can replace r_0 with the impact parameter b . Also the sign of the last term is the same as the first term which is attractive. This means that the bending is toward the black hole. The third term comes from the general relativistic correction by $f(r) \neq 1$. The calculation in Sect. 3 is based on geometric optics formalism which corresponds to $f(r) = 1$. So this term does not appear in the previous section. The charge and electric field in the two unit systems are related as $q_G = q_{LH}/\sqrt{4\pi}$ and $E_G = \sqrt{4\pi} E_{LH}$. The last term in Eq. (63) exactly agrees with the result in Eq. (46) substituting $Q_G = Q/\sqrt{4\pi}$ and $\beta_G = \sqrt{4\pi}\beta$ so that $Q_G^2/\beta_G^2 = (1/16\pi^2)(Q^2/\beta^2)$.

5 Conclusion

We have studied the trajectory of a light ray under the background electric field of a Born–Infeld-type Coulomb charge. The nonlinear effect can be described by the effective indices of refraction. Assuming that the background electric field is much stronger than the photon’s electromagnetic field, we computed the bending angle in the weak lensing limit. We calculated the bending angle using the trajectory equation based on geometric optics. In the limit where the classical Born–Infeld parameter B is infinite, our computation yields the bending angle by Euler–Heisenberg action. Then we did the same calculation using the geodesic equation of

the Einstein–Born–Infeld black hole. We confirmed that the two results agree in the proper limit.

In general, the velocity of light depends on the effective nonlinear electromagnetic Lagrangian as well as the effective modification of the flat background by mass and charge. The relative sign of the third term and the fourth term in Eq. (63) is different. This means that the contribution to the total deflection angle by nontrivial metric function $f(r) \neq 1$ is repulsive while the contribution by nonlinear electrodynamics is attractive. So the total bending angle of EBI black hole is slightly larger than that of RN black hole. When one uses the magnetic field to test the nonlinearity of electrodynamics, one should also consider such general relativistic correction. It is interesting to study the bending of light by the magnetic dipole field in Born–Infeld electrodynamics.

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Appendix

In this appendix we compute the integral in Eq. (43)

$$I = \rho_0^4 \int_{-\infty}^{\infty} \left[\frac{y}{r^2(r^4 + \rho_0^4)} - \frac{y^3}{r^4(r^4 + \rho_0^4)} - \frac{2y^3}{(r^4 + \rho_0^4)^2} \right] dx. \tag{64}$$

In the leading order we can put $y = b$ and $r^2 = x^2 + b^2$ in the integrand

$$I = \rho_0^4 \int_{-\infty}^{\infty} \left[\frac{b}{(x^2 + b^2)\{(x^2 + b^2)^2 + \rho_0^4\}} - \frac{b^3}{(x^2 + b^2)^2\{(x^2 + b^2)^2 + \rho_0^4\}} - \frac{2b^3}{\{(x^2 + b^2)^2 + \rho_0^4\}^2} \right] dx. \tag{65}$$

By partial fraction of the first two terms, the integral can be written as

$$I = b \int_{-\infty}^{\infty} \left[\frac{1}{(x^2 + b^2)} - \frac{b^2}{(x^2 + b^2)^2} - \frac{x^2}{(x^2 + b^2)^2 + \rho_0^4} - \frac{2b^2\rho_0^4}{\{(x^2 + b^2)^2 + \rho_0^4\}^2} \right] dx. \tag{66}$$

The first and second integrals are easily computed as

$$I_1 = b \int_{-\infty}^{\infty} \frac{1}{x^2 + b^2} dx = b \left[\frac{1}{b} \tan^{-1} \left(\frac{x}{b} \right) \right]_{-\infty}^{\infty} = \pi, \tag{67}$$

$$I_2 = b^3 \int_0^{\infty} \frac{1}{(x^2 + b^2)^2} dx = b^3 \left[\frac{x}{2b^2(x^2 + b^2)} + \frac{1}{2b^3} \tan^{-1} \left(\frac{x}{b} \right) \right]_{-\infty}^{\infty} = \frac{\pi}{2}. \tag{68}$$

We do the third integral using the contour integral

$$I_3 = b \int_{-\infty}^{\infty} \frac{z^2}{(z^2 + b^2)^2 + \rho_0^4} dz. \tag{69}$$

There are four simple poles at

$$z = \pm(b^4 + \rho_0^4)^{1/4} e^{\pm i\alpha/2}, \tag{70}$$

where

$$\alpha = \tan^{-1} \left(-\frac{\rho_0^2}{b^2} \right), \quad (\pi/2 < \alpha < \pi). \tag{71}$$

If we take the contour a semicircle in the upper-half plane, the relevant poles are

$$z_1 = (b^4 + \rho_0^4)^{1/4} e^{i\alpha/2}, \quad z_2 = -(b^4 + \rho_0^4)^{1/4} e^{-i\alpha/2}, \tag{72}$$

and the corresponding residues are

$$a_{-1}(z_1) = \frac{e^{i\alpha/2}}{8i(b^4 + \rho_0^4)^{1/4} \sin(\alpha/2) \cos(\alpha/2)}, \tag{73}$$

$$a_{-1}(z_2) = \frac{e^{-i\alpha/2}}{8i(b^4 + \rho_0^4)^{1/4} \sin(\alpha/2) \cos(\alpha/2)}. \tag{74}$$

By the residue theorem, we compute the integral as

$$\int_{-\infty}^{\infty} \frac{z^2}{(z^2 + b^2)^2 + \rho_0^4} dz = 2\pi i [a_{-1}(z_1) + a_{-1}(z_2)] \\ = \frac{\pi}{2} \frac{1}{(b^4 + \rho_0^4)^{1/4} \sin(\alpha/2)}. \quad (75)$$

From Eq. (71),

$$\sin(\alpha/2) = \frac{1}{\sqrt{2}} \left(1 + \frac{b^2}{\sqrt{b^4 + \rho_0^4}} \right)^{1/2}, \quad (76)$$

and substituting $\sin(\alpha/2)$ into Eq. (75), we find

$$I_3 = \frac{\pi}{\sqrt{2}} \frac{b}{\left(\sqrt{b^4 + \rho_0^4} + b^2 \right)^{1/2}}. \quad (77)$$

The last integral can be written as

$$I_4 = 2b^3 \rho_0^4 \int_{-\infty}^{\infty} \frac{1}{[(z^2 + b^2)^2 + \rho_0^4]^2} dz \\ = 2b^3 \rho_0^4 \left[-\frac{\partial}{\partial c} \int_{-\infty}^{\infty} \frac{1}{(z^2 + b^2)^2 + c} dz \right]_{c=\rho_0^4}. \quad (78)$$

For the integrand $f(z) = 1/[(z^2 + b^2)^2 + \rho_0^4]$, the relevant poles are the same as the poles in I_3 and the corresponding residues are

$$a_{-1}(z_1) = \frac{e^{-i\alpha/2}}{8i(b^4 + \rho_0^4)^{3/4} \sin(\alpha/2) \cos(\alpha/2)}, \quad (79)$$

$$a_{-1}(z_2) = \frac{e^{i\alpha/2}}{8i(b^4 + \rho_0^4)^{3/4} \sin(\alpha/2) \cos(\alpha/2)}, \quad (80)$$

and, from the residue theorem, we have

$$\int_{-\infty}^{\infty} \frac{1}{(z^2 + b^2)^2 + \rho_0^4} dz \\ = \frac{\pi}{\sqrt{2}} \frac{1}{\sqrt{b^4 + \rho_0^4} \left(\sqrt{b^4 + \rho_0^4} + b^2 \right)^{1/2}}. \quad (81)$$

Using

$$-\frac{\partial}{\partial c} \left[\frac{\pi}{\sqrt{2}} \frac{1}{\sqrt{b^4 + c} \left(\sqrt{b^4 + c} + b^2 \right)^{1/2}} \right] \\ = \frac{\pi}{4\sqrt{2}} \frac{3\sqrt{b^4 + c} + 2b^2}{(b^4 + c)^{3/2} \left(\sqrt{b^4 + c} + b^2 \right)^{3/2}}, \quad (82)$$

we have

$$I_4 = \frac{\pi b^3 \rho_0^4}{2\sqrt{2}} \frac{3\sqrt{b^4 + \rho_0^4} + 2b^2}{(b^4 + \rho_0^4)^{3/2} \left(\sqrt{b^4 + \rho_0^4} + b^2 \right)^{3/2}}. \quad (83)$$

Substituting I_1 , I_2 , I_3 , and I_4 in Eq. (66), we finally obtain

$$I = \frac{\pi}{2} - \frac{\pi}{\sqrt{2}} \frac{b}{\left(\sqrt{b^4 + \rho_0^4} + b^2 \right)^{1/2}} \\ - \frac{\pi}{2\sqrt{2}} \frac{b^3 \rho_0^4 \left(3\sqrt{b^4 + \rho_0^4} + 2b^2 \right)}{(b^4 + \rho_0^4)^{3/2} \left(\sqrt{b^4 + \rho_0^4} + b^2 \right)^{3/2}}. \quad (84)$$

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