Eur. Phys. J. C (2021) 81:476
https://doi.org/10.1140/epjc/s10052-021-09257-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Large N_f for multiple representations

Giacomo Cacciapaglia[a], **Shahram Vatani**[b]

Univ. Lyon, Université Claude Bernard Lyon 1, CNRS/IN2P3, UMR5822 IP2I, 69622 Villeurbanne, France

Received: 3 November 2020 / Accepted: 17 May 2021 / Published online: 29 May 2021

Abstract We present an extension of the large-N_f formalism that allows one to study cases with multiple fermion representations. The pole structure in the beta function is traced back to the intrinsic non-abelian nature of the gauge group, independently from the fermion representation. This result validates the conjectured existence of an interactive UV fixed point for non-abelian gauge theories with large fermion multiplicity. Finally, we apply our results to chiral gauge theories.

1 Introduction

The expansion in a large number, N_f, of fermionic matter fields has been used to study the dynamical properties of gauge theories. In particular, the gauge beta function, relevant for the renormalization group equation (RGE) of the gauge coupling and the mass anomalous dimension have been computed for both abelian [1,2] and non-abelian [3] theories. Some information about higher-order terms is also available [4–6]. An important property of this expansion is that the first order is scheme independent, as discussed in [7–9]. More recently, this technique has been reused to show that gauge theories in the large-N_f limit may feature a non-trivial, interacting Ultra-Violet (UV) fixed point [10] (see also [11]). This observation is very important in understanding the dynamics of gauge theories, as the presence of an UV fixed point would allow one to understand large-N_f gauge theories as fundamental, in the Wilsonian sense [12,13]. This effort falls in the larger quest for asymptotic freedom, first identified by Weinberg for quantum gravity [14] and later discovered by Sannino and Litim for perturbative gauge-Yukawa theories [15]. It is worth recalling that increasing N_f makes the theory lose the property of asymptotic freedom. This is where the large-N_f formalism comes to the rescue. A different limit, relevant for a complementary region in theory space, consists in taking large-N_c and large-N_f limits, while keeping the ratio $N_\mathrm{f}/N_\mathrm{c}$ fixed and relatively small. In this case, asymptotic freedom can be preserved in the UV, and an interactive fixed point may arise in the infra-red region [16,17].

The presence of a fixed point is linked to the fact that the first order in the large-N_f expansion has a negative pole at a given value of the gauge coupling, thus canceling the positive leading term when the gauge coupling grows near the singular value. This conclusion has been recently challenged in Ref. [18], where the resummation was re-organized thanks to non-perturbative information obtained from critical exponents. The result shows that a singularity in the leading large-N_f critical exponent does not necessarily imply a singularity in the beta function; however, without excluding the presence of an UV fixed point. Preliminary results from large-N_f lattice studies also remain inconclusive [19]. Thus, at the moment the presence of a physical UV fixed point cannot be excluded. This phenomenon has been applied in various contexts, from attempts to define a safe Standard Model [20–25], to grand unification [26,27], dark matter [28,29] and a variety of new physics scenarios [30–32]. In composite Higgs models with fermion partial compositeness [30], two *irreps* are needed to form spin-1/2 bound states that couple to the elementary quark and lepton fields. In general, the phenomenologically relevant application of theories with a UV fixed point consist in UV completions of the standard model, avoiding issues like vacuum stability and Landau poles.

The basic formulas for large-N_f resummation are known [33] for a simple gauge group $\mathcal{G}$, while an extension to semi-simple groups $\mathcal{G} = \times_\alpha \mathcal{G}_\alpha$ can be found in Ref. [34]. In all cases, the large-multiplicity fermions belong to a single irreducible representation (*irrep*) R_f of the gauge groups. In this note, we generalize the resummation to cases with a large multiplicity of fermions in multiple *irreps* of the gauge groups. We find that the pole structure is preserved, revealing that its presence is intrinsically linked to the non-abelian

[a] e-mail: g.cacciapaglia@ipnl.in2p3.fr
[b] e-mail: vatani@ipnl.in2p3.fr (corresponding author)

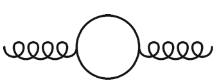
structure of the gauge group. This result is in agreement with the presence of the UV safety for non-abelian gauge groups and not for abelian ones. The latter was already in question due to the fact that the mass anomalous dimension diverges near the pole [10]. Our results are also relevant to understand the dynamical properties of some class of models, like gauge theories with a large number of chiral families, and composite Higgs models with top partial compositeness [30].

The paper is organized as follows: in Sect. 2 we review the main results useful for a large-N_f resummation, while presenting the results in a different form that can be applied to the case of multiple *irreps*. In Sect. 3 we present general formulas for the new case, before applying the results to physically interesting theories in Sect. 4. We offer our conclusion in Sect. 5.

2 Basic resummation results

In this section we will give a pedagogical introduction to the basic results for the large-N_f beta-function calculation, following Ref. [34]. We will, however, change some definitions, which will be useful to better understand the origin of the singularity and to extend the calculation to multiple *irrep* cases in the next section.

Let us consider a gauge–fermion theory with one species of Dirac fermions, Ψ, in the *irrep* R of a simple gauge group $\mathcal{G}$ (with gauge coupling constant g). To compute the β-function we need to calculate the radiative corrections to the 2-point function of the gauge boson propagator. For a large number of fermions, the leading contribution to the beta function comes from the one-loop fermion contribution:



$$= K \times \int \frac{\mathrm{d}^4 k}{4\pi^2} \frac{\mathrm{Tr}\left[\gamma^\mu \left(\not{k} - \not{p}\right) \gamma^\nu \not{k}\right]}{(p-k)^2 k^2}, \tag{1}$$

where we have defined an effective gauge coupling K, which takes into account the large fermion multiplicity by

$$K = \frac{g^2}{4\pi^2} N_\mathrm{f}\, T(R), \tag{2}$$

where $T(R)$ is the index of the fermion *irrep* R. K can be considered as the effective coupling controlling the perturbative expansion, thus the one-loop contribution can be counted at $\mathcal{O}(N_\mathrm{f}^0)$. This also allows one to define a chain of fermion bubbles, which are all contributing at $\mathcal{O}(N_\mathrm{f}^0)$, as shown in Fig. 1.

Fig. 1 Example of *bubble-chain* of length 5

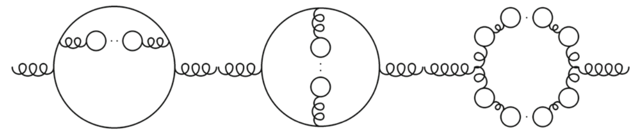


Fig. 2 Next-to-leading order diagrams. **C** is one representative of the diagrams containing gauge boson self-interactions

Two-loop corrections to the diagram in Eq. (1) correspond to attaching a gauge propagator to the fermion loop: as no additional N_f multiplicity is added, this diagram will effectively contribute to order K^2/N_f, thus providing next-to-leading order terms. Now, replacing the simple gauge propagator with a *bubble-chain*, will not increase the $1/N_\mathrm{f}$ order. In fact, the leading term is simply given by the resummation of the *bubble-chain* in the gauge propagator, as shown in the first two diagrams in Fig. 2. For a non-abelian theory, there are also contributions coming from gauge boson self-interactions: in this case, even the one-loop result is suppressed by $1/N_\mathrm{f}$ and should be considered at next-to-leading order. As before, the resummed results stem from dressing the gauge propagators with the *bubble-chain*, as exemplified in the third diagram in Fig. 2.

For each type of diagram $\mathbf{X}$, we can write the amplitude in the form $\delta^{ab} p^2 \Delta_{\mu\nu}(p)\, \Pi_{\mathbf{X}}^{(n)}(p)$, where $\Delta_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu / p^2$. Here n corresponds to the length of the *bubble-chain* in $\mathbf{A}$ and $\mathbf{B}$, while for $\mathbf{C}$ it is the total length of the two *bubble-chains*. A simple calculation gives

$$\Pi_{\mathbf{A}}^{(n)}(p) = N_\mathrm{f} g^4\, T(R)\, C(R)\, \frac{K^n}{(4\pi^2)^2} A_{\mathbf{A}}^{(n)}(p), \tag{3}$$

$$\Pi_{\mathbf{B}}^{(n)}(p) = N_\mathrm{f} g^4\, T(R)\, C(R) \left(1 - \frac{1}{2}\frac{C(G)}{C(R)}\right) \times \frac{K^n}{(4\pi^2)^2} A_{\mathbf{B}}^{(n)}(p), \tag{4}$$

$$\Pi_{\mathbf{C}}^{(n)}(p) = g^2\, C(G)\, \frac{K^n}{(4\pi^2)} A_{\mathbf{C}}^{(n)}(p), \tag{5}$$

where $C(R)$ the Casimir operator of the *irrep* R and $C(G)$ of the adjoint. The functions $A_{\mathbf{X}}^{(n)}$ are integrals of the loop momentum and contain the information needed to compute the leading order; however, we need to take into account all the n-long *bubble-chains* to extract the contribution to the β-function. The beauty of the large-N_f expansion stands in the fact that this resummation can be done, and the ϵ-dependence of the loop in dimensional regularization can be converted in a dependence on K of the resummed result (see Ref. [2] and the appendix of Ref. [34] for the proof).

To compute the evolution of the coupling we need to sum up all the diagrams: the total contribution thus reads

$$\Pi = \sum_n 2\Pi_{\mathbf{A}}^{(n)} + \Pi_{\mathbf{B}}^{(n)} + \Pi_{\mathbf{C}}^{(n)}$$

$$= \frac{K}{N_f T(R)} C(G) \sum_n K^n A_{\mathbf{C}}^{(n)}(p) + \frac{K^2}{N_f T(R)} C(R) \sum_n K^n \left[2\, A_{\mathbf{A}}^{(n)}(p) + \left(1 - \frac{1}{2}\frac{C(G)}{C(R)}\right) A_{\mathbf{B}}^{(n)}(p) \right], \tag{6}$$

where the factor of 2 comes from the two possible insertions of the *bubble-chain* in the diagram **A**. Upon closer inspection to the above formula, we can reorganize the sum as follows:

$$\Pi = \frac{K}{N_f T(R)} \sum_n \left\{ C(R)\, K^n \underbrace{\left[2A_{\mathbf{A}}^{(n-1)} + A_{\mathbf{B}}^{(n-1)}\right]}_{(*)} + C(G)\, K^{n-1} \underbrace{\left[A_{\mathbf{C}}^{(n-1)} - K A_{\mathbf{B}}^{(n-1)}/2\right]}_{(**)} \right\}. \tag{7}$$

The combination $(*)$ encodes the contribution to the beta function for an abelian gauge group (for which $C(G) = 0$ and $C(R) \to Q_f^2$) and was computed originally in Ref. [2], while the combination $(**)$ encodes the effect of the non-abelian dynamics. After resummation, we define two functions corresponding to the two combinations as follows:

$$\beta(K) = \frac{2K^2}{3} \left[1 + \frac{C(G)}{N_f T(R)} \left\{ \frac{-11}{4} + H(K) \right\} + \frac{C(R)}{N_f T(R)} F(K) \right], \tag{8}$$

where $F(K)$ stems from $(*)$ and $H(K)$ from $(**)$. Note that the $-11/4$ term isolates the one-loop contribution of gauge couplings, which is of order $1/N_f$, wile the 1 corresponds to the one-loop contribution of the fermions. Thus, in our definition, the functions $F(K)$ and $H(K)$ explicitly contain only the resummed higher-loop contribution. They are defined by [1]:

$$F(K) = \frac{3}{4} \int_0^K \mathrm{d}x\; \tilde{F}\left(0, \frac{2}{3}x\right), \quad H(K) = \frac{3}{4} \int_0^K \mathrm{d}x\; \tilde{F}\left(0, \frac{2}{3}x\right) \tilde{G}\left(0, \frac{1}{3}x\right); \tag{9}$$

with

$$\tilde{F}(0, \epsilon) = \frac{(1-\epsilon)\left(1 - \frac{\epsilon}{3}\right)\left(1 + \frac{\epsilon}{2}\right)\Gamma(4-\epsilon)}{3\Gamma^2\left(2 - \frac{\epsilon}{2}\right)\Gamma\left(3 - \frac{\epsilon}{2}\right)\Gamma\left(1 + \frac{\epsilon}{2}\right)}, \tag{10}$$

$$\tilde{G}(0, \epsilon) = \frac{20 - 43\epsilon + 32\epsilon^2 - 14\epsilon^3 + 4\epsilon^4}{4(2\epsilon - 1)(2\epsilon - 3)\left(1 - \epsilon^2\right)}. \tag{11}$$

The function $F(K)$ has a singularity at $K^* = 15/2$: this specifically arises from the singularity in the factor $\Gamma(4-\epsilon)$ in the loop integral $\tilde{F}(0, \epsilon)$. This term is relevant for abelian gauge groups. Instead, $H(K)$ has a singularity at $K^* = 3$, which stems from the $(1-\epsilon^2)$ factor in the denominator of $\tilde{G}(0, \epsilon)$. Thus, the presence of a pole at $K = 3$ for non-abelian gauge theories is to be traced back to the non-abelian nature of the gauge bosons. It has been observed in Ref. [34] that the mass anomalous dimension is finite in $K^* = 3$, while it diverges at $K^* = 15/2$, thus supporting the presence of an UV fixed point for the non-abelian gauge only. Furthermore, preliminary results for the $1/N_f^2$ contribution to the abelian β-function show that a discontinuity in $K^* = 3$ emerges. Both observations seem to support the idea that the non-abelian fixed point may be physical, while the abelian one is more arguable. In the remainder of this work, we will therefore focus on non-abelian gauge symmetries.

The resummation has been extended to semi-simple gauge groups in Ref. [34], and we will review the main results here. We consider a gauge group $\mathcal{G} = \times_\alpha \mathcal{G}_\alpha$, with gauge couplings g_α, and n_f fermions in the *irrep* $R_f = \times_\alpha R_\alpha$. Instead of defining a single effective N_f as in Ref. [34], we find it more convenient to define a fermion number N_α for each gauge group, as this will allow us to easily generalize the result to multiple *irreps*. We will consider each fermion number, defined as

$$N_\alpha = n_f \Pi_{\beta \neq \alpha} d(R_\beta), \tag{12}$$

to be of the same order for all gauge groups, i.e. $N_\alpha = \mathcal{O}(N_f)$. Similarly, we define effective gauge couplings as follows:

$$K_\alpha = \frac{g_\alpha^2}{4\pi^2} N_\alpha. \tag{13}$$

The only new ingredient in the case of semi-simple gauge groups is that the gauge couplings contribute to each other's β-function. As gauge bosons of different groups do not interact with each other, the leading order in N_f stems from diagrams of type **A** and **B**, where the gauge boson in the *bubble-chain* is different from the external ones, as shown in Fig. 3. The amplitudes can be written as

$$\Pi_{\mathbf{A'/B'}}^{(n)}(p) = g_\alpha^2 T(R_\alpha) \sum_{\beta \neq \alpha} g_\beta^2 C(R_\beta) d(R_\beta) \Pi_{\gamma \neq \alpha,\beta} d(R_\gamma)\, n_f \times \frac{K_\beta^n}{(4\pi^2)^2} A_{\mathbf{A/B}}^{(n)}(p), \tag{14}$$

where the integral functions are the same as before. Using $d(R_\beta)\Pi_{\gamma \neq \alpha,\beta} d(R_\gamma)\, n_f = N_\beta$, the total contribution can be written as

$$\Pi = \frac{K_\alpha}{N_\alpha} \sum_{\beta \neq \alpha} \frac{C(R_\beta)}{T(R_\beta)} K_\beta^{n+1} \left(2A_{\mathbf{A}}^{(n)}(p) + A_{\mathbf{B}}^{(n)}(p)\right), \tag{15}$$

[1] The functions F and G we define are related to the F_1 and H_1 functions of Ref. [34] by

$$F_1 = F, \qquad H_1 = F_1 + \frac{C(G)}{C(R)}\left(-\frac{11}{4} + H\right).$$

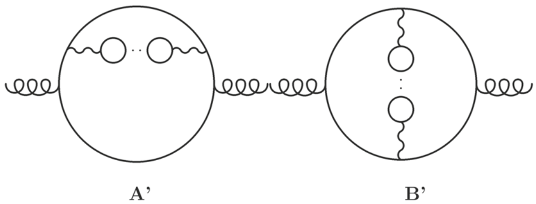


Fig. 3 Next-to-leading order diagrams for the mixed contributions, where the gauge bosons in the *bubble-chain* and on the external legs belong to different gauge groups

where we recognize the function also appearing in the abelian case.

Finally, the β-function for K_α can be written as

$$\beta(K_\alpha) = \frac{2K_\alpha^2}{3}\left[1 + \frac{C(G_\alpha)}{N_\alpha T(R_\alpha)}\left\{-\frac{11}{4} + H(K_\alpha)\right\} + \sum_\beta \frac{C(R_\beta)}{N_\alpha T(R_\beta)} F(K_\beta)\right]. \quad (16)$$

If we only consider non-abelian gauge groups, the second term, which includes the mixed contributions, will always remain finite and small as $K_\beta < 3$. Thus, the presence of an UV fixed point only comes from the first term, i.e. from the non-abelian nature of each gauge factor in the semi-simple group.

In order to offer a direct comparison with the results in the next section, we can also redefine the fermion multiplicities by absorbing the index of the *irrep*:

$$\tilde{N}_\alpha = N_\alpha T(R_\alpha), \quad K_\alpha = \frac{g_\alpha^2}{4\pi^2}\tilde{N}_\alpha, \quad (17)$$

with the β-function given by

$$\beta(K_\alpha) = \frac{2K_\alpha^2}{3}\left[1 + \frac{C(G_\alpha)}{\tilde{N}_\alpha}\left\{-\frac{11}{4} + H(K_\alpha)\right\} + \frac{1}{\tilde{N}_\alpha}\sum_\beta \frac{C(R_\beta)T(R_\alpha)}{T(R_\beta)} F(K_\beta)\right]. \quad (18)$$

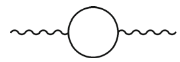
In the above form, the β-function can be easily extended to cases with a large multiplicity of multiple *irreps*, as we will show in the following section.

3 Extension to multiple *irreps*

We are now ready to extend the resummation formulae to cases with large numbers of multiple *irreps*. We will first start with a simple gauge group, and then generalize to semi-simple groups.

3.1 Simple gauge group

We start with the case of a simple gauge group $\mathcal{G}$ with n_i fermions Ψ_i in different *irreps* R_i. In such a case, the leading order contribution is given by the one-loop diagram below, where we sum over all the fermion species:



$$= \frac{g^2}{4\pi^2}\sum_i T(R_i)\, n_i \times \int \frac{\mathrm{d}^4 k}{4\pi^2}\frac{\mathrm{Tr}\left[\gamma^\mu\left(\not{k} - \not{p}\right)\gamma^\nu \not{k}\right]}{(p-k)^2 k^2}. \quad (19)$$

Following Eq. (1), we can define an effective gauge coupling as follows:

$$K = \frac{g^2}{4\pi^2}N, \qquad N = \sum_i n_i T(R_i), \quad (20)$$

where it is evident why we absorbed the index of the *irrep* in the definition of the fermion multiplicity. Interestingly, a large N can now also be due to the contribution of *irreps* with large index.

The next-to-leading order in $1/N$ is given by the same diagrams **A**, **B** and **C** in Fig. 2, yielding

$$\Pi_{\mathbf{A}}^{(n)}(p) = g^4 \sum_i T(R_i)\, C(R_i) n_i \,\frac{K^n}{(4\pi^2)^2} A_{\mathbf{A}}^{(n)}(p), \quad (21)$$

$$\Pi_{\mathbf{B}}^{(n)}(p) = g^4 \sum_i T(R_i)\, C(R_i)\left(1 - \frac{1}{2}\frac{C(G)}{C(R_i)}\right) \times n_i \,\frac{K^n}{(4\pi^2)^2} A_{\mathbf{B}}^{(n)}(p), \quad (22)$$

$$\Pi_{\mathbf{C}}^{(n)}(p) = g^2\, C(G)\,\frac{K^n}{(4\pi^2)} A_{\mathbf{C}}^{(n)}(p). \quad (23)$$

It is crucial that the loop factors do not depend on the *irrep* of the fermions. Furthermore, a sum on the fermion species can be introduced in the third contribution by inserting

$$\Pi_{\mathbf{C}}^{(n)}(p) \;= g^2 \sum_i \frac{n_i T(R_i)}{N} C(G)\,\frac{K^n}{(4\pi^2)} A_{\mathbf{C}}^{(n)}(p). \quad (24)$$

The total result can thus be written as

$$\begin{aligned}\sum_n 2\Pi_{\mathbf{A}}^{(n)} + \Pi_{\mathbf{B}}^{(n)} + \Pi_{\mathbf{C}}^{(n)} &= \frac{K}{N}\sum_i \left\{\sum_n \frac{n_i T(R_i) C(R_i)}{N} K^n \times \left[2A_{\mathbf{A}}^{(n-1)}(p) + A_{\mathbf{B}}^{(n-1)}(p)\right]\right. \\ &\quad \left. + \sum_n \frac{n_i T(R_i) C(G)}{N}\, K^{n-1} \times \left[A_{\mathbf{C}}^{(n-1)}(p) - K A_{\mathbf{B}}^{(n-1)}(p)/2\right]\right\}. \quad (25)\end{aligned}$$

We can now identify again the sums leading to the abelian $F(K)$ and non-abelian $H(K)$ functions, defined in Eq. (9).

The β-function can thus be expressed as

$$\beta(K) = \frac{2K^2}{3}\left[1 + \frac{C(G)}{N}\left\{\frac{-11}{4} + H(K)\right\}\right.$$
$$\left. + \frac{1}{N}\left(\sum_i \frac{T(R_i)\, C(R_i)\, n_i}{N}\right) F(K)\right]. \tag{26}$$

The coefficient in front of $F(K)$ evaluates to an $\mathcal{O}(1)$ number as it sums the degrees of freedom of the fermions weighted by the Casimir operators, thus the second term is genuinely an $\mathcal{O}(1/N)$ contribution. Nevertheless, we see that, for non-abelian gauge groups, the singularity driving the UV fixed point is the same as in the case of a single large-multiplicity *irrep*. Note how this result compares to Eq. (18).

This result proves that models with multiple *irreps* have the same dynamics as models with a single *irrep* in the large-N_f limit.

3.2 Semi-simple gauge group

We now consider the most general case of a semi-simple gauge group $\mathcal{G} = \times_\alpha \mathcal{G}_\alpha$ with n_i fermions Ψ_i in the *irrep* $R_i = \times_\alpha R_{i\alpha}$. Combining the definitions in the previous sections, we can define a fermion multiplicity for each gauge group as follows:

$$N_\alpha = \sum_i n_i \, \left(\Pi_{\beta\neq\alpha} d(R_{i\beta})\right)\, T(R_{i\alpha}), \qquad K_\alpha = \frac{g_\alpha^2}{4\pi^2} N_\alpha. \tag{27}$$

It is convenient to define effective fermion multiplicities that count the multiplicity of fermion specie Ψ_i relative to one or two gauge groups, respectively:

$$\tilde{n}_{i\alpha} = n_i \, \left(\Pi_{\beta\neq\alpha} d(R_{i\beta})\right), \quad \tilde{n}_{i\alpha\beta} = n_i \, \left(\Pi_{\gamma\neq\alpha,\beta} d(R_{i\gamma})\right), \tag{28}$$

so that $N_\alpha = \sum_i \tilde{n}_{i\alpha} T(R_\alpha)$.

The generalization of the β-function is now straightforward, starting from Eqs (18) and (26):

$$\beta(K_\alpha) = \frac{2K_\alpha^2}{3}\left[1 + \frac{C(G_\alpha)}{N_\alpha}\left\{-\frac{11}{4} + H(K)\right\}\right.$$
$$+\frac{1}{N_\alpha}\left\{\sum_i \frac{\tilde{n}_{i\alpha} T(R_{i\alpha}) C(R_{i\alpha})}{N_\alpha} F(K_\alpha)\right.$$
$$\left.\left. + \sum_{\beta\neq\alpha}\sum_i \frac{\tilde{n}_{i\alpha\beta} T(R_{i\alpha}) C(R_{i\beta})}{N_\beta} F(K_\beta)\right\}\right]. \tag{29}$$

The above result confirms that the UV dynamics of the theory is fully determined by the non-abelian structure, as the additional terms proportional to $F(K)$ remain finite for $K < 3$ in the case of non-abelian gauge groups. We also note that, as in the simple case, the β-function and thus its pole are scheme independent (see the proof in Appendix 1).

For completeness and reference, in Appendix 1 we provide the β-functions for Yukawa couplings from Ref. [34], adapted to our formalism and extended to the case of multiple *irreps*. As we will show in Appendix 1, they maintain the scheme independence property of the first order in the expansion.

4 Application to chiral gauge theories: generalized Georgi–Glashow and Bars–Yankielowicz models

Chiral gauge theories have received substantial attention in the literature, especially in the case of asymptotical freedom, because of the interesting low energy dynamics [35–38]. The simplest incarnations consist of two different species of fermions, whose multiplicities are chosen to cancel the gauge anomaly. Theories of this class can be constructed on a simple SU(N_c) gauge group, with one fermion in the symmetric or anti-symmetric *irrep* and a suitable number of conjugate fundamental to cancel the gauge anomaly. They go under the names of Bars–Yankielowicz (BY) [39] and generalized Georgi–Glashow (GG) [40] theories. Their low energy dynamics is still not fully understood in the asymptotically free case with a small number of chiral families [41,42]: indications of the possible allowed phases have been inferred by use of the minimization of degrees of freedom [37,38], or the recent idea of generalized anomalies [43,44]. In this work we are interested in the limit of a large number of fermions, where asymptotic freedom in the UV is lost, and the theory flows to a non-interacting point at low energies. Hence, we will consider a case with n_g chiral generations, with n_g much larger than N_c. Note that the results are also valid if some of these families are vector-like and heavy: for instance, a Georgi–Glashow model with $(n_\mathrm{g}-3)/2$ vector-like families may be a precursor of a UV-safe completion of $SU(5)$ grand unified theories.

The fermion content of the two template theories thus consists of

$$\mathrm{BY} \Rightarrow R_1 = \Box\Box \quad [n_\mathrm{g}], \quad R_2 = \bar{\Box} \quad [(N+4)n_\mathrm{g}]; \tag{30}$$

$$\mathrm{GG} \Rightarrow R_1 = \begin{smallmatrix}\Box\\ \Box\end{smallmatrix} \quad [n_\mathrm{g}], \quad R_2 = \bar{\Box} \quad [(N-4)n_\mathrm{g}]. \tag{31}$$

Following Eq. (20), we can define the following fermion multiplicity:

$$N_{\mathrm{BY/GG}} = n_\mathrm{g}\frac{N\pm 3}{2}, \tag{32}$$

which is large when n_g is large. Thus, the β-functions from Eq. (26) read

$$\beta(K)_{\mathrm{BY/GG}} = \frac{2K^2}{3}\left[1 + \frac{2N}{n_\mathrm{g}(N\pm 3)}\left\{-\frac{11}{4} + H(K)\right\}\right.$$

$$+\frac{1}{n_{\rm g}}\frac{3N^2\pm N-4}{2N(N\pm 3)}F(K)\Bigg], \tag{33}$$

where the large-$n_{\rm g}$ expansion is evident, and we recall that $N \geq 3$ for BY and $N \geq 5$ for GG. The UV dynamics of the theories, therefore, is determined by the $H(K)$ dependence, which produces an attractive fixed point for $K^* = 3$, corresponding to

$$\frac{g_*^2}{4\pi} = \frac{6\pi}{n_{\rm g}(N\pm 3)}. \tag{34}$$

We now investigate the UV properties of the BY and GG models with an additional Yukawa coupling. For this purpose, we focus of the simplest possibility: besides the two fermions Ψ_{R1} and Ψ_{R2}, we extend the models by a scalar ϕ in the anti-fundamental representation. This allows one to include a Yukawa interaction and a quartic coupling, as follows:

$$\mathcal{L} \supset -y\;\phi_a\;\Psi_{b\,R2}\;\Psi^{ab}{}_{R1} - \lambda\;\phi^{*a}\phi^{*b}\phi_a\phi_b + \text{h.c.}, \tag{35}$$

where a, b are gauge indices, and we omit the flavor index. The general formulas for the beta function for the new couplings can be found in Appendix 1. They can be simplified under the assumption that the gauge coupling quickly approaches its fixed point, so that we can replace $K = 3$. For the two models, we find

$$\beta\,(y)_{\rm BY/GG} = y^3\,\frac{3N+1}{32\pi^2} - y\,\frac{7N^2\pm 4N-1}{8n_{\rm g}(N\pm 3)N} \tag{36}$$

$$\beta\,(\lambda)_{\rm BY/GG} = \lambda^2\frac{2N+5}{16\pi^2} + \lambda\left(\frac{y^2N}{4\pi^2} - \frac{N^2-1}{n_{\rm g}N(N\pm 3)}\right) - \left(\frac{y^4N}{4\pi^2} + \frac{(N^2-1)(N^2-2)c_\lambda}{{n_{\rm g}}^2(N\pm 3)^2N}\right), \tag{37}$$

where $c_\lambda = 192$ is a numerical coefficient stemming from the gauge fixed point. The Yukawa beta function features a flow from an IR fixed point at

$$\frac{y_{\rm IR}^2}{4\pi^2} = \frac{7N^2\pm 4N-1}{N(N\pm 3)(3N+1)n_{\rm g}}, \tag{38}$$

to a free UV fixed point $y_{\rm UV} = 0$.

From the quartic coupling beta function, we can see that there always exist two real zeros for any value of y. Numerical values are shown in Fig. 4 for $n_{\rm g} = 10\,N$. These results can be generalized to any value of $n_{\rm g}$ knowing that the zeros scale behaves as $\lambda^* \sim n_{\rm g}^{-1}$ (as $y_{\rm IR}^2 \sim n_{\rm g}^{-1}$). Thus, for values below the positive zero, $\lambda < \lambda_+^*$, the couplings will flow in the UV towards the negative zero, $\lambda_{\rm UV} = \lambda_-^*$. The plot also shows that the running is not very sensitive to the value of the Yukawa coupling.

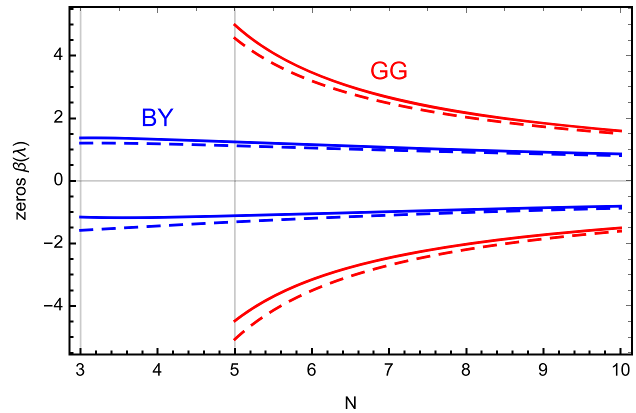


Fig. 4 Zeros of $\beta(\lambda)$ as a function of N for BY (blue) and GG (red) models. The solid lines correspond to $y = y_{\rm UV} = 0$, while the dashed ones correspond to $y = y_{\rm IR}$. The numerical values correspond to $n_{\rm g} = 10N$; however, other choices can be easily inferred knowing that the zeros scale behaves as $\lambda^* \sim n_{\rm g}^{-1}$

This analysis shows that chiral gauge theories as the BY and GG models, even when enriched with a scalar field, can feature a completely UV-safe behavior. The fact that the quartic coupling flows towards a negative value at high energy, though, may signal an instability in the scalar potential.

5 Conclusions

Large-$N_{\rm f}$ resummation has proven a useful tool to study the UV dynamics of gauge theories with large multiplicity of fermions. This has led to the conjecture of the emergence of an attractive interacting fixed point in the UV, which is due to the presence of a pole in the resummed leading order in the beta function. In this note, we extended the standard formalism to include cases with multiple fermion representations. We showed that the pole can be traced back to the intrinsic non-abelian nature of the gauge group, independently on the specific representations of the fermions. Thus, we support the conjecture for non-abelian gauge groups.

As a consequence, the pole and the UV fixed point are also found in theories with multiple representations. We apply these results to the simplest cases of chiral gauge theories, based on SU($N_{\rm c}$) gauge groups. As long as a large-enough number of chiral families are added, the theories develop an UV-safe dynamics.

Acknowledgements The authors acknowledge partial support from the France-China Particle Physics Lab (FCPPL) and the Labex-LIO (Lyon Institute of Origins) under grant ANR-10-LABX-66 (Agence Nationale pour la Recherche), and FRAMA (FR3127, Fédération de Recherche "André Marie Ampère").

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The results of this paper did not require any specific data.]

Funded by SCOAP[3].

Appendix A: Large-N_f β-functions for Yukawa and quartic couplings

For completeness, we will list here the resummed β-functions for Yukawa couplings [45] from Ref. [34], adapted to the notation we use in this paper, and to the case with multiple fermion *irreps*. We complete the model by adding a scalar field ϕ^A in the *irrep* $R_\phi = \times_\alpha R_{\phi\alpha}$, where A is a generic gauge index for the scalar *irrep*. The new Lagrangian for the interactions reads

$$\mathcal{L} \supset -y_{A\mathfrak{bc}}\, \phi^A\, \Psi_i^{\mathfrak{b}}\, \Psi_j^{\mathfrak{c}} - \lambda^{AB}_{CD}\, \phi^{*,A} \phi^{*,B} \phi_C \phi_D + \text{h.c.}, \tag{A1}$$

where $\mathfrak{b}$ and $\mathfrak{c}$ are gauge indices of the fermion *irreps*, and it is understood that the scalar gauge quantum numbers allow the Yukawa coupling to be gauge-invariant. First we note that the scalar will contribute to the gauge β-function with a $1/N_\mathrm{f}$ term,

$$\Delta\beta(K_\alpha)|_{\text{scalar}} \supset \frac{2K_\alpha^2}{3} \frac{T\left(R_{\phi\,\alpha}\right)}{4N_\alpha}, \tag{A2}$$

which corresponds to the one-loop result.

We recall that consistency of the expansion requires that the Yukawa and quartic couplings shall scale in a give way with N_f, as follows:

$$y^2 \sim \lambda \sim \frac{1}{N_\mathrm{f}}. \tag{A3}$$

This counting ensures that the non-gauge contribution to the beta function at one loop counts $1/N_\mathrm{f}$ in the expansion. The β-function for the Yukawa coupling reads

$$\begin{aligned}\beta\left(y_{A\mathfrak{bc}}\right) = \frac{1}{32\pi^2} \Big[&\left(y_D y^{*,D} y_A\right)_{\mathfrak{bc}} \\ &+ \left(y_A y^{*,D} y_D\right)_{\mathfrak{bc}} + 2\mathrm{Tr}\left[y_A y^{*,D}\right] y_{D\mathfrak{bc}} \Big] \\ &- y_{A\mathfrak{bc}} \sum_\alpha \frac{3K_\alpha}{4N_\alpha} H_0\left(\frac{2K_\alpha}{3}\right) \\ &\left[C\left(R_{i\,\alpha}\right) + C\left(R_{j\,\alpha}\right) + \frac{K_\alpha}{6} C\left(R_{\phi\,\alpha}\right)\right],\end{aligned}$$

where the traces and contractions are intended for the fermion gauge indices, and

$$H_0(x) = \frac{\left(1-\frac{x}{3}\right)\Gamma(4-x)}{3\Gamma^2\left(2-\frac{x}{2}\right)\Gamma\left(3-\frac{x}{2}\right)\Gamma\left(1+\frac{x}{2}\right)}. \tag{A4}$$

We recall that H_0 is a smooth function up to $K = 15/2$, where a pole is developed. Thus, for non-abelian semi-simple groups, the gauge contribution to the Yukawa running remains finite. If it dominates over the contribution of the pure Yukawa term, the coupling will flow towards a non-interactive fixed point (asymptotic free). If there is another Yukawa term, y', mixed contributions will play a role in the running as $\beta(y) \supset\sim yy'^2$. Thus we get the same behavior.

For the quartic coupling we obtain the following β-function:

$$\begin{aligned}&\beta\left(\lambda^{AB}_{\ \ CD}\right) \\ &= \frac{1}{16\pi^2}\left[2\lambda^{AE}_{\ \ CF}\lambda^{BF}_{\ \ DE} + 2\lambda^{AE}_{\ \ DF}\lambda^{BF}_{\ \ CE} + \lambda^{AB}_{\ \ EF}\lambda^{EF}_{\ \ CD}\right] \\ &+ \frac{1}{4\pi^2}\mathrm{Tr}\left[y_D y^{*,E}\right]\lambda^{AB}_{\ \ CE} \\ &- \frac{1}{4\pi^2}\mathrm{Tr}\left[y^A y_C^* y^B y_D^* + y^A y_D^* y^B y_C^*\right] \\ &- \lambda^{AB}_{\ \ CD}\sum_\alpha \frac{3}{N_\alpha} C\left(R_{\phi\,\alpha}\right) K_\alpha H_0\left(\frac{2K_\alpha}{3}\right) \\ &+ 48\pi^2 \sum_{\alpha<\beta} \frac{B^{AB}_{\alpha,\beta\ CD}}{N_\alpha N_\beta} \frac{K_\alpha K_\beta}{K_\alpha - K_\beta} \\ &\times\left[K_\alpha\left(1-\frac{K_\alpha}{6}\right)H_0\left(\frac{2K_\alpha}{3}\right) - K_\beta\left(1-\frac{K_\beta}{6}\right)H_0\left(\frac{2K_\beta}{3}\right)\right] \\ &+ 24\pi^2\sum_\alpha \frac{A^{AB}_{\alpha\ CD}}{N_\alpha^2}K_\alpha^2\left[\left(1-\frac{K_\alpha}{3}\right)H_0\left(\frac{2K_\alpha}{3}\right)\right. \\ &\left. + K_\alpha\left(1-\frac{K_\alpha}{6}\right)\frac{\partial}{\partial K_\alpha}H_0\left(\frac{2K_\alpha}{3}\right)\right],\end{aligned} \tag{A5}$$

where the tensors A and B are defined as

$$\begin{aligned}A^{AB}_{\alpha\ CD} &= \frac{1}{8}\left[\{T^a_{R_{\phi\,\alpha}}, T^b_{R_{\phi\,\alpha}}\}^A_{\ C}\{T^a_{R_{\phi\,\alpha}}, T^b_{R_{\phi\,\alpha}}\}^B_{\ D}\right. \\ &\left.+\{T^a_{R_{\phi\,\alpha}}, T^b_{R_{\phi\,\alpha}}\}^A_{\ D}\{T^a_{R_{\phi\,\alpha}}, T^b_{R_{\phi\,\alpha}}\}^B_{\ C}\right],\end{aligned} \tag{A6}$$

$$B^{AB}_{\alpha,\beta\ CD} = \frac{1}{2}\left[\left(T^a_{R_{\phi\,\alpha}} T^b_{R_{\phi\,\beta}}\right)^A_{\ C}\left(T^a_{R_{\phi\,\alpha}} T^b_{R_{\phi\,\beta}}\right)^B_{\ D}\right.$$

$$+\left(T^a_{R_{\phi\,\alpha}}T^b_{R_{\phi\,\beta}}\right)^A_{\ D}\left(T^a_{R_{\phi\,\alpha}}T^b_{R_{\phi\,\beta}}\right)^B_{\ C}\Bigg]. \tag{A7}$$

Appendix B: Scheme transformations

In this section we prove the scheme independence of all the β-functions above. For that purpose first we recall that a scheme transformation will map a gauge coupling $\alpha = \frac{g^2}{4\pi^2}$ to a new one $\alpha' = \frac{g'^2}{4\pi^2}$. This mapping must be invertible and as mentioned in [7,8] [9] can be parametrized as

$$\alpha = \alpha' \mathcal{F}(\alpha') \tag{B1}$$

$$\mathcal{F}(\alpha') = 1 + t_1\alpha' + t_2\alpha'^2 + \dots \tag{B2}$$

Thus if we perform a scheme transformation only for the gauge groupe $\mathcal{G}_\alpha$ we obtain

$$K_\alpha = K'_\alpha \mathcal{F}(K'_\alpha/N_\alpha) \tag{B3}$$

$$= K'_\alpha(1 + t_1 K'_\alpha/N_\alpha + t_2 K'^2_\alpha/N^2_\alpha + \dots). \tag{B4}$$

This feeds the gauge β-function in two ways. First the β-function of $\mathcal{G}_\alpha$ is already known to be scheme invariant as explained in [7–9]. We note that the contribution from the other gauge group is unchanged. Secondly, for the other β-functions, the contribution from K_a enters through functions that are already at order 1 in N. Changing the scheme requires one to expand those functions using (B4). But the expansion is in higher powers in N_α. Thus at first order all the β-functions here are scheme independent.

References

1. D. Espriu, A. Palanques-Mestre, P. Pascual, R. Tarrach, The γ function in the $1/N_f$ expansion. Z. Phys. C **13**, 153 (1982). https://doi.org/10.1007/BF01547679
2. A. Palanques-Mestre, P. Pascual, The $1/N_f$ expansion of the γ and beta functions in QED. Commun. Math. Phys. **95**, 277 (1984). https://doi.org/10.1007/BF01212398
3. J.A. Gracey, The QCD beta function at O(1/N(f)). Phys. Lett. B **373**, 178–184 (1996). https://doi.org/10.1016/0370-2693(96)00105-0. arXiv:hep-ph/9602214
4. M. Ciuchini, S.E. Derkachov, J.A. Gracey, A.N. Manashov, Quark mass anomalous dimension at O(1/N(f)**2) in QCD. Phys. Lett. B **458**, 117–126 (1999). https://doi.org/10.1016/S0370-2693(99)00573-0. arXiv:hep-ph/9903410
5. M. Ciuchini, S.E. Derkachov, J.A. Gracey, A.N. Manashov, Computation of quark mass anomalous dimension at O(1 / N**2(f)) in quantum chromodynamics. Nucl. Phys. B **579**, 56–100 (2000). https://doi.org/10.1016/S0550-3213(00)00209-1. arXiv:hep-ph/9912221
6. N.A. Dondi, G.V. Dunne, M. Reichert, F. Sannino, Towards the QED beta function and renormalons at $1/N_f^2$ and $1/N_f^3$. arXiv:2003.08397
7. T.A. Ryttov, R. Shrock, Scheme transformations in the vicinity of an infrared fixed point. Phys. Rev. D **86** (2012). https://doi.org/10.1103/physrevd.86.065032
8. T.A. Ryttov, R. Shrock, Analysis of scheme transformations in the vicinity of an infrared fixed point. Phys. Rev. D **86** (2012). https://doi.org/10.1103/physrevd.86.085005
9. R. Shrock, Study of possible ultraviolet zero of the beta function in gauge theories with many fermions. Phys. Rev. D **89** (2014). https://doi.org/10.1103/physrevd.89.045019
10. O. Antipin, F. Sannino, Conformal Window 2.0: the large N_f safe story. Phys. Rev. D **97**, 116007 (2018). https://doi.org/10.1103/PhysRevD.97.116007. arXiv:1709.02354
11. R. Shrock, Study of possible ultraviolet zero of the beta function in gauge theories with many fermions. Phys. Rev. D **89**, 045019 (2014). https://doi.org/10.1103/PhysRevD.89.045019. arXiv:1311.5268
12. K.G. Wilson, Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture. Phys. Rev. B **4**, 3174–3183 (1971). https://doi.org/10.1103/PhysRevB.4.3174
13. K.G. Wilson, Renormalization group and critical phenomena. 2. Phase space cell analysis of critical behavior. Phys. Rev. B **4**, 3184–3205 (1971). https://doi.org/10.1103/PhysRevB.4.3184
14. S. Weinberg, Ultraviolet divergences in quantum theories of gravitation. In: General Relativity: An Einstein Centenary Survey, pp. 790–831 (1980)
15. D.F. Litim, F. Sannino, Asymptotic safety guaranteed. JHEP **12**, 178 (2014). https://doi.org/10.1007/JHEP12(2014)178. arXiv:1406.2337
16. S. Girmohanta, T.A. Ryttov, R. Shrock, Large-nc and large-nf limits of su(nc) gauge theories with fermions in different representations. Phys. Rev. D **99** (2019). https://doi.org/10.1103/physrevd.99.116022
17. T.A. Ryttov, R. Shrock, Scheme-independent calculations of properties at a conformal infrared fixed point in gauge theories with multiple fermion representations. Phys. Rev. D **98** (2018). https://doi.org/10.1103/physrevd.98.096003
18. T. Alanne, S. Blasi, N.A. Dondi, Critical look at β-function singularities at large N. Phys. Rev. Lett. **123**, 131602 (2019). https://doi.org/10.1103/PhysRevLett.123.131602. arXiv:1905.08709
19. V. Leino, T. Rindlisbacher, K. Rummukainen, F. Sannino, K. Tuominen, Safety versus triviality on the lattice. arXiv:1908.04605
20. A.D. Bond, G. Hiller, K. Kowalska, D.F. Litim, Directions for model building from asymptotic safety. JHEP **08**, 004 (2017). https://doi.org/10.1007/JHEP08(2017)004. arXiv:1702.01727
21. S. Abel, F. Sannino, Framework for an asymptotically safe Standard Model via dynamical breaking. Phys. Rev. D **96**, 055021 (2017). https://doi.org/10.1103/PhysRevD.96.055021. arXiv:1707.06638
22. R. Mann, J. Meffe, F. Sannino, T. Steele, Z.-W. Wang, C. Zhang, Asymptotically safe standard model via vectorlike fermions. Phys. Rev. Lett. **119**, 261802 (2017). https://doi.org/10.1103/PhysRevLett.119.261802. arXiv:1707.02942
23. G.M. Pelaggi, A.D. Plascencia, A. Salvio, F. Sannino, J. Smirnov, A. Strumia, Asymptotically safe standard model extensions? Phys. Rev. D **97**, 095013 (2018). https://doi.org/10.1103/PhysRevD.97.095013. arXiv:1708.00437
24. S. Abel, E. Mølgaard, F. Sannino, Complete asymptotically safe embedding of the standard model. Phys. Rev. D **99**, 035030 (2019). https://doi.org/10.1103/PhysRevD.99.035030. arXiv:1812.04856
25. K. Kowalska, A. Bond, G. Hiller, D. Litim, Towards an asymptotically safe completion of the Standard Model. PoS **EPS–HEP2017**, 542 (2017). https://doi.org/10.22323/1.314.0542
26. E. Molinaro, F. Sannino, Z.W. Wang, Asymptotically safe Pati-Salam theory. Phys. Rev. D **98**, 115007 (2018). https://doi.org/10.1103/PhysRevD.98.115007. arXiv:1807.03669

27. Z.-W. Wang, AAl Balushi, R. Mann, H.-M. Jiang, Safe trinification. Phys. Rev. D **99**, 115017 (2019). https://doi.org/10.1103/PhysRevD.99.115017. arXiv:1812.11085
28. F. Sannino, I.M. Shoemaker, Asymptotically safe dark matter. Phys. Rev. D **92**, 043518 (2015). https://doi.org/10.1103/PhysRevD.92.043518. arXiv:1412.8034
29. C. Cai, H.-H. Zhang, Minimal asymptotically safe dark matter. Phys. Lett. B **798**, 134947 (2019). https://doi.org/10.1016/j.physletb.2019.134947. arXiv:1905.04227
30. G. Cacciapaglia, S. Vatani, T. Ma, Y. Wu, Towards a fundamental safe theory of composite Higgs and Dark Matter. arXiv:1812.04005
31. F. Sannino, J. Smirnov, Z.-W. Wang, Asymptotically safe clockwork mechanism. Phys. Rev. D **100**, 075009 (2019). https://doi.org/10.1103/PhysRevD.100.075009. arXiv:1902.05958
32. T.A. Ryttov, K. Tuominen, Safe glueballs and baryons. JHEP **04**, 173 (2019). https://doi.org/10.1007/JHEP04(2019)173. arXiv:1903.09089
33. B. Holdom, Large N flavor beta-functions: a recap. Phys. Lett. B **694**, 74–79 (2011). https://doi.org/10.1016/j.physletb.2010.09.037. arXiv:1006.2119
34. O. Antipin, N.A. Dondi, F. Sannino, A.E. Thomsen, Z.-W. Wang, Gauge–Yukawa theories: beta functions at large N_f. Phys. Rev. D **98**, 016003 (2018). https://doi.org/10.1103/PhysRevD.98.016003. arXiv:1803.09770
35. S. Raby, S. Dimopoulos, L. Susskind, Tumbling gauge theories. Nucl. Phys. B **169**, 373–383 (1980). https://doi.org/10.1016/0550-3213(80)90093-0
36. H. Georgi, L.J. Hall, M.B. Wise, Remarks on mass hierarchies from tumbling gauge theories. Phys. Lett. B **102**, 315 (1981). https://doi.org/10.1016/0370-2693(81)90625-0
37. T. Appelquist, Z.-Y. Duan, F. Sannino, Phases of chiral gauge theories. Phys. Rev. D **61**, 125009 (2000). https://doi.org/10.1103/PhysRevD.61.125009. arXiv:hep-ph/0001043
38. T. Appelquist, R. Shrock, Ultraviolet to infrared evolution of chiral gauge theories. Phys. Rev. D **88**, 105012 (2013). https://doi.org/10.1103/PhysRevD.88.105012. arXiv:1310.6076
39. I. Bars, S. Yankielowicz, Composite quarks and leptons as solutions of anomaly constraints. Phys. Lett. B **101**, 159–165 (1981). https://doi.org/10.1016/0370-2693(81)90664-X
40. T. Appelquist, A.G. Cohen, M. Schmaltz, R. Shrock, New constraints on chiral gauge theories. Phys. Lett. B **459**, 235–241 (1999). https://doi.org/10.1016/S0370-2693(99)00616-4. arXiv:hep-th/9904172
41. S. Bolognesi, K. Konishi, Dynamics and symmetries in chiral $SU(N)$ gauge theories. Phys. Rev. D **100**, 114008 (2019). https://doi.org/10.1103/PhysRevD.100.114008. arXiv:1906.01485
42. S. Bolognesi, K. Konishi, M. Shifman, Patterns of symmetry breaking in chiral QCD. Phys. Rev. D **97**, 094007 (2018). https://doi.org/10.1103/PhysRevD.97.094007. arXiv:1712.04814
43. S. Bolognesi, K. Konishi, A. Luzio, Dynamics from symmetries in chiral $SU(N)$ gauge theories. JHEP **09**, 001 (2020). https://doi.org/10.1007/JHEP09(2020)001. arXiv:2004.06639
44. S. Bolognesi, K. Konishi, A. Luzio, Probing the dynamics of chiral SU(N) gauge theories via generalized anomalies. arXiv:2101.02601
45. K. Kowalska, E.M. Sessolo, Gauge contribution to the 1/N$_f$ expansion of the Yukawa coupling beta function. JHEP **04**, 027 (2018). https://doi.org/10.1007/JHEP04(2018)027. arXiv:1712.06859