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The magnetic moment of $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state

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Abstract In this paper, we tentatively assign the $P_c(4312)$ to be a $\bar{D}\Sigma_c$ molecular state with quantum number $J^P = \frac{1}{2}^-$, and calculate its magnetic moment using the QCD sum rule method in external weak electromagnetic field. Starting with the two-point correlation function in external electromagnetic field and expanding it in power of the electromagnetic interaction Hamiltonian, we extract the magnetic moment from the linear response to the external electromagnetic field. The numerical value of the magnetic moment of $P_c(4312)$ is $\mu_{P_c} = 1.75^{+0.15}_{-0.11}$.

1 Introduction

In Ref. [1], LHCb collaboration reported the discoveries of two pentaquark states $P_c(4380)$ and $P_c(4450)$ in the $J/\psi p$ invariant mass spectrum of the process $\Lambda_b \to J/\psi p K$. In 2019, they confirmed the $P_c(4450)$ state consisting of two narrow overlapping peaks $P_c(4440)$ and $P_c(4457)$, and observed a new narrow pentaquark state $P_c(4312)$ [2]. Following these experimental discoveries, there have been many theoretical studies concerning these pentaquark states through various models/methods, such as the meson-baryon molecular scenario [3–30], the compact five quark states [31–47], kinematical triangle singularity [48] and so on.

In Ref. [30], we assumed the $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state with quantum number $\frac{1}{2}^-$, and studied the decay of $P_c(4312)$ to $J/\psi p$ and to $\eta_c p$ with the QCD sum rule method. The QCD sum rule method [49,50] is a nonperturbative analytic formalism firmly entrenched in QCD with minimal modeling and has been successfully applied in almost every aspect of strong interaction physics. In Ref. [51–53], the QCD sum rule method was extended to calculate the magnetic moments of the nucleon and hyperon in the external field method. In this method, a static electromagnetic field is introduced which couples to the quarks and polarizes the QCD vacuum, and the magnetic moments of hadrons can be extracted from the linear response to this field. Later, a more systematic studies was made for the magnetic moments of the octet baryons [54–57], the decuplet baryons [58–61] and the ρ meson [62]. In Refs. [63] and [64], the authors calculated the magnetic moment of $Z_c(3900)$ as an axialvector tetraquark state and an axialvector molecular state, respectively.

In the present work, we extend this method to the investigation of the magnetic moment of the $P_c(4312)$ state viewed as a $\bar{D}\Sigma_c$ molecular state with quantum number $J^P = \frac{1}{2}^-$. Electromagnetic multipole moments are the major and meaningful parameters of hadrons. Analysis of the electromagnetic multipole moments of the exotic states can help us get valuable knowledge about the electromagnetic properties of these states, the charge distributions inside them, their charge radius and geometric shapes and finally their internal sub-structures.

The rest of the paper is organized as follows. In Sect. 2, the sum rule for the magnetic moment of the $P_c(4312)$ state is given. Section 3 is devoted to the numerical analysis and a short summary is given in Sect. 4. In Appendix B, the spectral densities are shown.

2 The derivation of the sum rules

The starting point of our calculation is the time-ordered correlation function in the QCD vacuum in the presence of a constant background electromagnetic field $F_{\mu\nu}$,

$$\Pi(p) = i\int dx^4 e^{ipx}\langle 0 \mid T[J^{P_c}(x)\bar{J}^{P_c}(0)] \mid 0\rangle_F = \Pi^{(0)}(p) + \Pi^{(1)}_{\mu\nu}(p)F^{\mu\nu} + \cdots, \tag{1}$$

where

$$J^{P_c}(x) = [\bar{c}(x)i\gamma_5 d(x)][\epsilon^{abc}(u_a^t(x)C\gamma_\mu u_b(x))\gamma^\mu\gamma_5 c_c(x)], \tag{2}$$

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is the interpolating current of $P_c(4312)$ considered as a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$ with t denoting the matrix transposition on the Dirac spinor indices, C meaning charge conjugation operator, and a, b, c being color indices. In the present work, we shall consider the linear response term, $\Pi^{(1)}_{\mu\nu}(p)F^{\mu\nu}$, from which the magnetic moment will be extracted.

The external electromagnetic field can interact directly with the quarks inside the hadron and also polarize the QCD vacuum. As a consequence, the vacuum condensates involved in the operator product expansion of the correlation function in the external electromagnetic field $F_{\mu\nu}$ are,

- dimension-2 operator,

$$F_{\mu\nu}, \tag{3}$$

- dimension-3 operator,

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F, \tag{4}$$

- dimension-5 operators,

$$\langle 0|\bar{q}q|0\rangle F_{\mu\nu},\ \langle 0|\bar{q}g_s G_{\mu\nu}q|0\rangle_F,\ \epsilon_{\mu\nu\alpha\beta}\langle 0|\bar{q}g_s G^{\alpha\beta}q|0\rangle_F, \tag{5}$$

- dimension-6 operators,

$$\langle 0|\bar{q}q|0\rangle\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F,\ \langle 0|g_s^2 GG|0\rangle F_{\mu\nu}, \ldots, \tag{6}$$

- dimension-7 operators,

$$\langle 0|g_s^2 GG|0\rangle\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F,\ \langle 0|g_s\bar{q}\sigma\cdot Gq|0\rangle F_{\mu\nu}, \ldots, \tag{7}$$

- dimension-8 operators,

$$\begin{aligned}&\langle 0|\bar{q}q|0\rangle^2 F_{\mu\nu},\ \langle 0|g_s\bar{q}\sigma\cdot Gq|0\rangle\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F,\\ &\langle 0|\bar{q}q|0\rangle\langle 0|\bar{q}g_s G_{\mu\nu}q|0\rangle_F,\\ &\epsilon_{\mu\nu\alpha\beta}\langle 0|\bar{q}q|0\rangle\langle 0|\bar{q}g_s G^{\alpha\beta}q|0\rangle_F, \ldots,\end{aligned} \tag{8}$$

and so on. The new vacuum condensates induced by the external electromagnetic field $F_{\mu\nu}$ can be described by introducing new parameters, χ, κ and ξ, called vacuum susceptibilities as follows,

$$\begin{aligned}&\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = ee_q\chi\langle 0|\bar{q}q|0\rangle F_{\mu\nu},\\ &\langle 0|\bar{q}g_s G_{\mu\nu}q|0\rangle_F = ee_q\kappa\langle 0|\bar{q}q|0\rangle F_{\mu\nu},\\ &\epsilon_{\mu\nu\alpha\beta}\langle 0|\bar{q}g_s G^{\alpha\beta}q|0\rangle_F = iee_q\xi\langle 0|\bar{q}q|0\rangle F_{\mu\nu}.\end{aligned} \tag{9}$$

In order to express the two-point correlation function (1) physically, we expand it in powers of the electromagnetic interaction Hamiltonian $H_{int} = -ie\int d^4y j^{em}_\alpha(y)A^\alpha(y)$,

$$\begin{aligned}\Pi(p) &= i\int dx^4 e^{ipx}\langle 0 \mid T[J^{P_c}(x)\bar{J}^{P_c}(0)] \mid 0\rangle\\ &+i\int dx^4 e^{ipx}\langle 0 \mid T\{J^{P_c}(x)\\ &\times\left[-ie\int d^4y j^{em}_\mu(y)A^\mu(y)\right]\bar{J}^{P_c}(0)\} \mid 0\rangle + \cdots,\end{aligned} \tag{10}$$

where $j^{em}_\mu(y)$ is the electromagnetic current and $A^\mu(y)$ is the electromagnetic four-vector.

Inserting two complete sets of physical intermediate states with the same quantum numbers as the current operator $J^{P_c}(x)$ into the second term of (10), we have

$$\begin{aligned}\Pi(p) &= e\int d^4x d^4y \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}e^{ipx}A^\mu(y)\\ &\times\sum_{P_c,P_c'}\sum_{s,s'}\frac{-i}{k^2-m^2_{P_c}}\frac{-i}{k'^2-m^2_{P_c'}}\\ &\times\langle 0|J^{P_c}(x)|P_c'(k',s')\rangle\langle P_c'(k',s')|j^{em}_\mu(y)|P_c(k,s)\rangle\\ &\times\langle P_c(k,s)|\bar{J}^{P_c}(0)|0\rangle,\end{aligned} \tag{11}$$

where the sum $\sum_{P_c,P_c'}$ is over all possible intermediate states including the ground state $P_c(4312)$ we are interested in, higher resonances and continuum. One can translate the coordinates of the operators in (11) to the origin, carry out the integrals over x and k' and then finds that

$$\begin{aligned}\Pi(p) &= -e\int d^4y \frac{d^4k}{(2\pi)^4}e^{i(p-k)y}A^\mu(y)\\ &\times\sum_{P_c,P_c'}\sum_{s,s'}\frac{1}{\left(p^2-m^2_{P_c'}\right)\left(k^2-m^2_{P_c}\right)}\\ &\times\langle 0|J^{P_c}(0)|P_c'(p,s')\rangle\langle P_c'(p,s')|j^{em}_\mu(0)|P_c(k,s)\rangle\\ &\times\langle P_c(k,s)|\bar{J}^{P_c}(0)|0\rangle.\end{aligned} \tag{12}$$

The sum $\sum_{P_c,P_c'}$ can be divided into three parts, the ground-ground term, the ground-excited (continuum) term and the excited (continuum)-excited (continuum) term.

After standard manipulation, the ground-ground term can be written as

$$\begin{aligned}\Pi^{(1)}_{\mu\nu}(p)F^{\mu\nu} = &-\frac{\lambda^2_{P_c}}{4(p^2-m^2_{P_c})^2}\Big[2m_{P_c}\mu_{P_c}\sigma^{\mu\nu}\\ &+\frac{\mu_{P_c}-1}{m_{P_c}}(p^2-m^2_{P_c})\sigma^{\mu\nu}\\ &+\mu_{P_c}(\not{p}\sigma^{\mu\nu}+\sigma^{\mu\nu}\not{p})\\ &+2i\frac{\mu_{P_c}-1}{m_{P_c}}(p^\mu\gamma^\nu-p^\nu\gamma^\mu)\not{p}\Big]F_{\mu\nu},\end{aligned} \tag{13}$$

where we make use of the following formulas,

$$\langle 0 \mid J^{P_c}(0) \mid P_c(p,s)\rangle = \lambda_{P_c}u(p,s), \tag{14}$$

and

$$\langle P_c'(k',s')|j_\mu^{em}(0)|P_c(k,s)\rangle = \bar{u}(k',s')\left[F_1(Q^2)\gamma_\mu + F_2(Q^2)i\sigma_{\mu\nu}\frac{q^\nu}{2m_{P_c}}\right]u(k,s), \tag{15}$$

with $q = k' - k$ and $Q^2 = -q^2$. λ_{P_c} and $u(k,s)$ are the pole residue and Dirac spinor of the $P_c(4312)$ state, respectively. The Lorentz invariant form factors $F_1(Q^2)$ and $F_2(Q^2)$ are related to the charge and magnetic form factors by

$$G_C(Q^2) = F_1(Q^2) - \frac{Q^2}{4m_{P_c}^2}F_2(Q^2),$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2). \tag{16}$$

The magnetic moment μ_{P_c} is given by $G_M(0)$.

Now, we consider the ground-excited (continuum) and the excited (continuum)-excited (continuum) parts. For each Lorentz structure in (13), the hadronic representation of the the second term in (10) is

$$\frac{A_{P_cP_c}}{(p^2-m_{P_c}^2)^2} + \sum_{P_c^*}\frac{A_{P_cP_c^*}}{(p^2-m_{P_c}^2)(p^2-m_{P_c^*}^2)} + \sum_{P_c^*}\frac{A_{P_c^*P_c^*}}{(p^2-m_{P_c^*}^2)^2}, \tag{17}$$

where $A_{P_cP_c}$, $A_{P_cP_c^*}$ and $A_{P_c^*P_c^*}$ are constants, and the symbol $\sum_{P_c^*}$ means the sum over the excited states and the integral over continuum. The first term in the above equation is the ground state pole which contains the desired magnetic moment μ_{P_c}. The second term represents the transition between the ground sate and the excited states (continuum) induced by the external electromagnetic field. The last term is the contributions from pure excited states (continuum). Making Borel transform, one has

$$\frac{A_{P_cP_c}}{M_B^2}e^{-m_{P_c}^2/M_B^2} + e^{-m_{P_c}^2/M_B^2}\left[\sum_{P_c^*}\frac{A_{P_cP_c^*}}{m_{P_c^*}^2-m_{P_c}^2}(1-e^{-(m_{P_c^*}^2-m_{P_c}^2)/M_B^2})\right] + \sum_{P_c^*}\frac{A_{P_c^*P_c^*}}{M_B^2}e^{-m_{P_c^*}^2/M_B^2}, \tag{18}$$

where M_B^2 is the Borel parameter. It is obvious that the transition between the ground sate and the excited states (continuum) gives a contribution which is not suppressed exponentially relative to the ground state. We can approximate the quantity in the square brackets by a constant. The third term is suppressed exponentially relative to the ground state and can be modeled in the usual way by introducing the continuum model and threshold parameter.

On the other hand, $\Pi(p)$ can be calculated theoretically via OPE method at the quark-gluon level. To this end, one can substitute the interpolating current $J^{P_c}(x)$ (2) into the correlation function (1), contract the relevant quark fields by Wick's theorem and find

$$\Pi^{OPE}(p) = -2i\epsilon_{abc}\epsilon_{a'b'c'}\int d^4x e^{ipx}\{\gamma^\mu\gamma_5 S_{cc'}^{(c)}(x)\gamma^\nu\gamma_5 \times Tr[(i\gamma_5)S_{dd'}^{(d)}(x)(i\gamma_5)S_{d'd}^{(c)}(-x)] \times Tr[\gamma_\mu S_{bb'}^{(u)}(x)\gamma_\nu C S_{aa'}^{(u)T}(x)C]\}_F, \tag{19}$$

where $S^{(c)}(x)$ and $S^{(q)}(x)$, $q = u, d$ are the full charm- and up (down)-quark propagators, whose expressions are given in Appendix A. Through dispersion relation, $\Pi^{OPE}(p)$ can be written as

$$\Pi^{OPE}(p) = \sigma^{\mu\nu}F_{\mu\nu}\int_{4m_c^2}^\infty ds\frac{\rho_1(s)}{s-p^2} + (\not{p}\sigma^{\mu\nu} + \sigma^{\mu\nu}\not{p})F_{\mu\nu}\int_{4m_c^2}^\infty ds\frac{\rho_2(s)}{s-p^2} + i(p^\mu\gamma^\nu - p^\nu\gamma^\mu)\not{p}F_{\mu\nu} \times\int_{4m_c^2}^\infty ds\frac{\rho_3(s)}{s-p^2} + \cdots, \tag{20}$$

where $\rho_i(s) = \frac{1}{\pi}\mathrm{Im}\Pi_i^{OPE}(s)$, $i = 1, 2, 3$ are the spectral densities. We will choose the Lorentz structure $i(p^\mu\gamma^\nu - p^\nu\gamma^\mu)\not{p}F_{\mu\nu}$ to obtain our sum rule for the magnetic moment μ_{P_c} because of its better convergence. The spectral density $\rho_3(s)$ is given in Appendix B.

Finally, with the help of the quark-hadron duality and the above discussion, we match the phenomenological side (13) and the QCD representation (20) for the Lorentz structure $i(p^\mu\gamma^\nu - p^\nu\gamma^\mu)\not{p}F_{\mu\nu}$

$$-\frac{\lambda_{P_c}^2}{2(p^2-m_{P_c}^2)^2}\frac{\mu_{P_c}-1}{m_{P_c}} + \frac{a}{m_{P_c}^2-p^2} + \int_{s_0^{P_c}}^\infty ds\frac{\rho_3(s)}{s-p^2} + \text{subtractions} = \int_{4m_c^2}^\infty ds\frac{\rho_3(s)}{s-p^2}, \tag{21}$$

where the constant a is introduced to parameterize the contributions of the ground-excited states (continuum) transition and $s_0^{P_c}$ is the threshold parameter. Subtracting the contributions of pure excited states (continuum), one gets

$$-\frac{\lambda_{P_c}^2}{2(p^2-m_{P_c}^2)^2}\frac{\mu_{P_c}-1}{m_{P_c}} + \frac{a}{m_{P_c}^2-p^2} + \text{subtractions} = \int_{4m_c^2}^{s_0^{P_c}} ds\frac{\rho_3(s)}{s-p^2}. \tag{22}$$

In order to eliminate the subtractions, it is necessary to make a Borel transform which can also improve the convergence of the OPE series and suppress the contributions from the excited and continuum states. As a result, we have

$$\left(-\frac{\mu_{P_c}-1}{2m_{P_c}M_B^2}+A\right)\lambda_{P_c}^2 e^{-m_{P_c}^2/M_B^2}=\int_{4m_c^2}^{s_0^{P_c}} ds e^{-s/M_B^2}\rho_3(s), \tag{23}$$

where $A=\frac{a}{\lambda_{P_c}^2}$ and M_B^2 is the Borel parameter.

Table 1 Some input parameters needed in the calculations

Parameter	Value
$\langle\bar{q}q\rangle$	$-(0.24\pm0.01)^3$ GeV^3
$\langle g_s\bar{q}\sigma Gq\rangle$	$(0.8\pm0.1)\langle\bar{q}q\rangle$ GeV^2
$\langle g_s^2 GG\rangle$	0.88 ± 0.25 GeV^4
m_c	$1.275^{+0.025}_{-0.035}$ GeV [66]
m_{P_c}	$4311.9\pm0.7^{+6.8}_{-0.6}$ MeV [2]
λ_{P_c}	$1.91^{+0.12}_{-0.13}\times10^{-3}$ GeV^6 [30]

3 Numerical analysis

The input parameters needed in numerical analysis are presented in Table 1. For the vacuum susceptibilities χ, κ and ξ, we take the values $\chi=-(3.15\pm0.30)\text{GeV}^{-2}$, $\kappa=-0.2$ and $\xi=0.4$ determined in the detailed QCD sum rules analysis of the photon light-cone distribution amplitudes [65]. Besides these parameters, we should determine the working intervals of the threshold parameter $s_{P_c}^0$ and the Borel mass M_B^2 in which the magnetic moment is stable. The continuum threshold is related to the square of the mass of the first exited states having the same quantum number as the interpolating field and we use the value determined in Ref. [30], while the Borel parameter is determined by demanding that both the contributions of the higher states and continuum are sufficiently suppressed and the contributions coming from higher dimensional operators are small.

We define two quantities, the ratio of the pole contribution to the total contribution (RP) and the ratio of the highest dimensional term in the OPE series to the total OPE series (RH), as followings,

$$RP\equiv\frac{\int_{4m_c^2}^{s_0^{P_c}} ds\rho_3(s)e^{-\frac{s}{M_B^2}}}{\int_{4m_c^2}^{\infty} ds\rho_3(s)e^{-\frac{s}{M_B^2}}},$$
$$RH\equiv\frac{\int_{4m_c^2}^{s_0^{P_c}} ds\rho_3^{(d=9)}(s)e^{-\frac{s}{M_B^2}}}{\int_{4m_c^2}^{s_0^{P_c}} ds\rho_3(s)e^{-\frac{s}{M_B^2}}}. \tag{24}$$

Firstly, we determine the working region of the M_B^2. In Fig. 1a, we compare the various OPE contributions as functions of M_B^2 with $\sqrt{s_0^{P_c}}=4.8$GeV. From it one can see that the OPE has good convergence. Figure 1b shows RP and RH varying with M_B^2 at $\sqrt{s_0^{P_c}}=4.8$GeV. The figure shows that the requirement $RP\geq50\%$ ($RP\geq40\%$) gives $M_B^2\leq4.3\text{GeV}^2$ ($M_B^2\leq4.9\text{GeV}^2$).

Figure 2a shows the dependence of the magnetic moment μ_{P_c} on the Borel mass M_B^2 in the interval of $2\text{GeV}^2\leq M_B^2\leq$

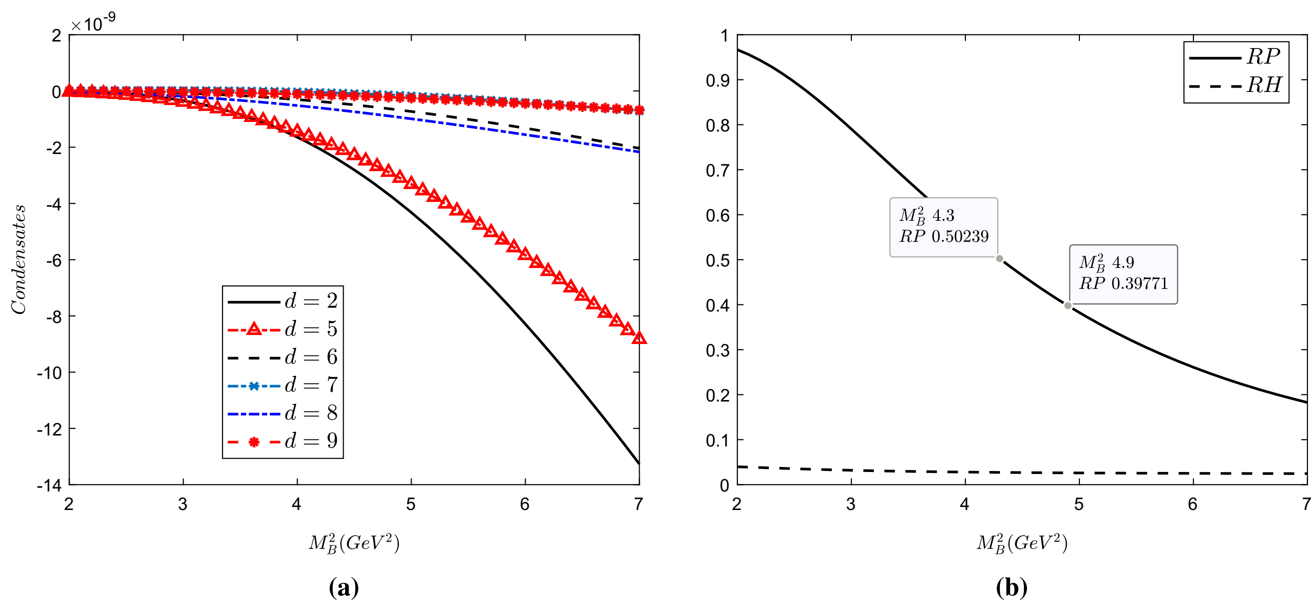

Fig. 1 **a** denotes the various condensates as functions of M_B^2 with $\sqrt{s_0^{P_c}}=4.8$GeV; **b** represents RP and RH varying with M_B^2 at $\sqrt{s_0^{P_c}}=4.8$GeV

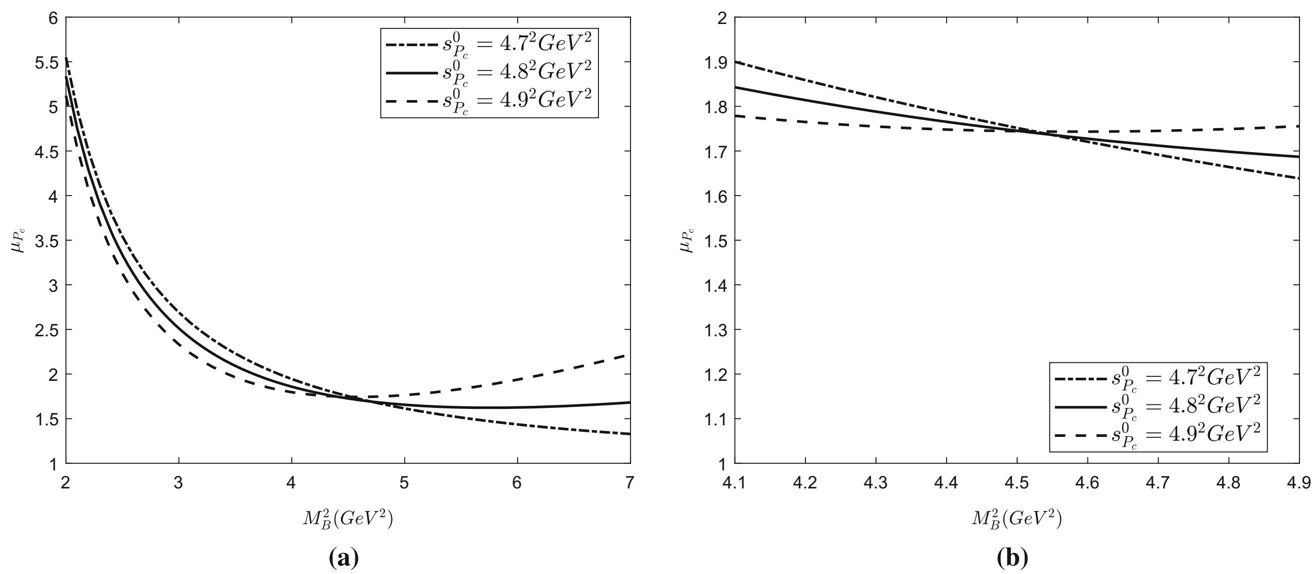

Fig. 2 The dependence of the magnetic moment μ_{P_c} on M_B^2 at three different values of $s_0^{P_c}$

7GeV^2. From the figure we can see that μ_{P_c} depends strongly on M_B^2 and $s_0^{P_c}$ as 4GeV$^2 \leq M_B^2$ or $M_B^2 \geq$ 5GeV^2. In order to have a larger working interval of the Borel mass M_B^2, we require $RP \geq 40\%$. As a result, we limit M_B^2 from 4.1GeV^2 to 4.9GeV^2. The result is shown in Fig. 2b, from which we can read reliably the value of the magnetic moment, $\mu_{P_c} = 1.75^{+0.15}_{-0.11}$. Because the hadron's magnetic moments contain important information on the charge distributions inside them, their geometric shapes and finally their quark configuration. This value can be confronted to the experimental data in the future and tell us whether it is reasonable to view the $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state with quantum number $J^P = \frac{1}{2}^-$.

4 Conclusion

In this paper, we tentatively assign the $P_c(4312)$ to be a $\bar{D}\Sigma_c$ molecular state with quantum number $J^P = \frac{1}{2}^-$, calculate its magnetic moment using the QCD sum rule method in the external weak electromagnetic field. Starting with the two-point correlation function in the external electromagnetic field and expanding it in power of the electromagnetic interaction Hamiltonian, we extract the magnetic moment from the linear response to the external electromagnetic field. The numerical value of the magnetic moment of the $P_c(4312)$ state is $\mu_{P_c} = 1.75^{+0.15}_{-0.11}$. The prediction can be confronted to the experimental data in the future and give important information about the inner structure of the $P_c(4312)$ state.

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Data Availability Statement This manuscript has associated data in a data repository. [Authors' comment: All data included in this manuscript are available upon request by contacting with the corresponding author.]

Appendix A: The quark propagators

The full quark propagators are given as

$$\begin{aligned}
S_{ij}^q(x) = {} & \frac{i\not{x}}{2\pi^2 x^4}\delta_{ij} - \frac{m_q}{4\pi^2 x^2}\delta_{ij} - \frac{\langle\bar{q}q\rangle}{12}\delta_{ij} \\
& + i\frac{\langle\bar{q}q\rangle}{48}m_q \not{x}\delta_{ij} - \frac{x^2}{192}\langle g_s\bar{q}\sigma Gq\rangle\delta_{ij} \\
& + i\frac{x^2\not{x}}{1152}m_q\langle g_s\bar{q}\sigma Gq\rangle\delta_{ij} - i\frac{g_s t_{ij}^a G_{\mu\nu}^a}{32\pi^2 x^2}(\not{x}\sigma^{\mu\nu} + \sigma^{\mu\nu}\not{x}) \\
& + i\frac{\delta_{ij}e_q F_{\mu\nu}}{32\pi^2 x^2}(\not{x}\sigma^{\mu\nu} + \sigma^{\mu\nu}\not{x}) - \frac{\delta_{ij}e_q\chi\langle\bar{q}q\rangle\sigma^{\mu\nu}F_{\mu\nu}}{24} \\
& + \frac{\delta_{ij}e_q\langle\bar{q}q\rangle F_{\mu\nu}}{288}(\sigma^{\mu\nu} - 2\sigma^{\alpha\mu}x_\alpha x^\nu) \\
& + \frac{\delta_{ij}e_q\langle\bar{q}q\rangle F_{\mu\nu}}{576} \\
& \times[(\kappa+\xi)\sigma^{\mu\nu}x^2 - (2\kappa-\xi)\sigma^{\alpha\mu}x_\alpha x^\nu] + \cdots \qquad \text{(A1)}
\end{aligned}$$

for light quarks, and

$$S_{ij}^{Q}(x) = i\int \frac{d^4k}{(2\pi)^4}e^{-ikx}\left[\frac{\not k + m_Q}{k^2-m_Q^2}\delta_{ij} - \frac{g_s t_{ij}^a G_{\mu\nu}^a}{4}\frac{\sigma^{\mu\nu}(\not k + m_Q)+(\not k + m_Q)\sigma^{\mu\nu}}{(k^2-m_Q^2)^2} + \frac{\langle g_s^2 GG\rangle}{12}\delta_{ij}m_Q\frac{k^2+m_Q \not k}{(k^2-m_Q^2)^4} + \frac{\delta_{ij}e_Q F_{\mu\nu}}{4}\frac{\sigma^{\mu\nu}(\not k + m_Q)+(\not k + m_Q)\sigma^{\mu\nu}}{(k^2-m_Q^2)^2}+\cdots\right] \tag{A2}$$

for heavy quarks. In these expressions $t^a = \frac{\lambda^a}{2}$ and λ^a are the Gell–Mann matrix, g_s is the strong interaction coupling constant, and i, j are color indices, $e_{Q(q)}$ is the charge of the heavy (light) quark and $F_{\mu\nu}$ is the external electromagnetic field.

Appendix B: The spectral densities

In this appendix, we will give the explicit expression of the spectral density $\rho_3(s)$.

$$\rho_3(s) = \rho_3^{(d=2)} + \rho_3^{(d=3)}(s) + \rho_3^{(d=5)}(s) + \rho_3^{(d=6)}(s) + \rho_3^{(d=7)}(s) + \rho_3^{(d=8)}(s) + \rho_3^{(d=9)}(s), \tag{B1}$$

with

$$\rho_3^{(d=2)}(s) = -\frac{m_c}{12288\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{a^2b^4}(1-a-b)^4(m_c^2(a+b)-abs)^3 + \frac{m_c}{3072\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{a^2b^3}(1-a-b)^3(m_c^2(a+b)-abs)^3, \tag{B2}$$

$$\rho_3^{(d=3)}(s) = 0, \tag{B3}$$

$$\rho_3^{(d=5)}(s) = \frac{m_c^2\langle 0|\bar q q 0|\rangle}{384\pi^6}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{ab^2}(1-a-b)^3(m_c^2(a+b)-abs) - \frac{m_c^2\langle 0|\bar q q|0\rangle}{128\pi^6}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{ab}(1-a-b)^2(m_c^2(a+b)-abs), \tag{B4}$$

$$\begin{aligned}\rho_3^{(d=6)}(s) = &-\frac{m_c^3\langle 0|g_s^2 GG|0\rangle}{147456\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{ab^2}(1-a-b)^4\\ &-\frac{m_c\langle 0|g_s^2 GG|0\rangle}{12288\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{ab^2}(1-a-b)^3(m_c^2(a+b)-abs)\\ &-\frac{m_c\langle 0|g_s^2 GG|0\rangle}{24576\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{b^2}(1-a-b)^2(m_c^2(a+b)-abs)\\ &+\frac{m_c\langle 0|g_s^2 GG|0\rangle}{12288\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{b^3}(1-a-b)^3(m_c^2(a+b)-abs)\\ &+\frac{m_c\langle 0|g_s^2 GG|0\rangle}{8192\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{ab^2}(1-a-b)^2(a+2b)(m_c^2(a+b)-abs)\\ &+\frac{m_c^3\langle 0|g_s^2 GG|0\rangle}{36864\pi^8}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{b^3}(1-a-b)^3(a+b),\end{aligned} \tag{B5}$$

$$\rho_3^{(d=7)}(s) = \frac{m_c^2\langle 0|g_s\bar{q}\sigma\cdot Gq0|\rangle}{512\pi^6}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{b}(1-a-b)^2$$
$$-\frac{m_c^2\langle 0|g_s\bar{q}\sigma\cdot Gq0|\rangle}{768\pi^6}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{ab}(1-a-b)^3$$
$$-\frac{m_c^2\langle 0|g_s\bar{q}\sigma\cdot Gq0|\rangle}{256\pi^6}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db(1-a-b)$$
$$+\frac{m_c^2\langle 0|g_s\bar{q}\sigma\cdot Gq0|\rangle}{256\pi^6}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} db\frac{1}{b}(1-a-b)^2, \quad \text{(B6)}$$

$$\rho_3^{(d=8)}(s) = -\frac{m_c(2\kappa-\xi)\langle 0|\bar{q}q|0\rangle^2}{576\pi^4}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} dba$$
$$-\frac{m_c\langle 0|\bar{q}q|0\rangle^2}{144\pi^4}\int_{a_{min}}^{a_{max}} da\int_{b_{min}}^{1-a} dba, \quad \text{(B7)}$$

$$\rho_3^{(d=9)}(s) = \frac{m_c^4\langle 0|\bar{q}q|0\rangle\langle 0|g_s^2GG|0\rangle}{27648\pi^6 M_B^2}\int_{a_{min}}^{a_{max}} da\frac{(1-a-b_{min})^3}{(as-m_c^2)b_{min}^2}$$
$$-\frac{m_c^2\langle 0|\bar{q}q|0\rangle\langle 0|g_s^2GG|0\rangle}{9216\pi^6}\int_{a_{min}}^{a_{max}} da\frac{(1-a-b_{min})^3}{(as-m_c^2)b_{min}}$$
$$-\frac{m_c^2\langle 0|\bar{q}q|0\rangle\langle 0|g_s^2GG|0\rangle}{9216\pi^6}\int_{a_{min}}^{a_{max}} da\frac{a(1-a-b_{min})}{(as-m_c^2)}$$
$$-\frac{m_c^4\langle 0|\bar{q}q|0\rangle\langle 0|g_s^2GG|0\rangle}{4608\pi^6 M_B^2}\int_{a_{min}}^{a_{max}} da\frac{a(1-a-b_{min})^2}{(as-m_c^2)b_{min}^2}$$
$$+\frac{m_c^2\langle 0|\bar{q}q|0\rangle\langle 0|g_s^2GG|0\rangle}{1536\pi^6}\int_{a_{min}}^{a_{max}} da\frac{a(1-a-b_{min})^2}{(as-m_c^2)b_{min}}$$
$$+\frac{m_c^2\langle 0|\bar{q}q|0\rangle\langle 0|g_s^2GG|0\rangle}{3072\pi^6}\int_{a_{min}}^{a_{max}} da\frac{a(1-a-b_{min})}{(as-m_c^2)}. \quad \text{(B8)}$$

In the above equations, $a_{max} = \frac{1+\sqrt{1-\frac{4m_c^2}{s}}}{2}$, $a_{min} = \frac{1-\sqrt{1-\frac{4m_c^2}{s}}}{2}$ and $b_{min} = \frac{am_c^2}{as-m_c^2}$.

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