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A mixing coupling scheme for spectra of singly heavy baryons with spin-1 diquarks in P-waves

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Abstract A new scheme of state classification is proposed and applied to analyze masses of the heavy baryons Ω_Q, Σ_Q and Ξ'_Q in P-waves. The results confirm all excited Ω_c and Ω_b baryons reported recently by LHCb to be bound states of a P-wave ss-diquark and a respective charm or bottom quark, and thereby predict Regge trajectories for more excited Ω_c and Ω_b baryons. We suggest one excited $J^P = 5/2^-$ Ω_b state to be unseen by LHCb around 6352 MeV, and predict P-wave masses of all spin-partners of the odd-parity baryons $\Sigma_c(2800)/\Xi'_c(2942)$ and $\Sigma_b(6097)/\Xi'_b(6227)$. A computation is further given in a relativized potential quark models to explain matched values of spin couplings of all considered baryons, by which a scaling law for these spin couplings is discussed.

1 Introduction

Scientific interest as to spectrum of strongly interacting heavy hadrons has raised again in last two decades greatly due to the discoveries of numerous heavy subatomic particles [1]. In 2017, LHCb observed five even narrower Ω_c resonances [2] (Table 1) in the $\Xi_c^+ K^-$ decay channel based on pp collision data. Very recently, LHCb, again, reported the discovery of four even narrower Ω_b resonances around 6.3 GeV decaying into $\Xi_b^0 K^-$ [3], with slightly less significance. The reported masses and decay widths are listed in Table 1 for both of them.

Observed patterns of masses of the excited $\Omega_{c,b}$ baryons turn out to be perplexing and thereby rise naturally questions as to (i) why there are four excited states of the Ω_b baryons while there are five for the Ω_c, and (ii) what their spin-parities or their inner structures are. These questions have been addressed by many authors [4–16] with recent explorations given in Refs. [17–34], where spin-parities were assigned for the Ω_c states around 3.0 ± 0.1 GeV and for the excited Ω_b states around 6.3 ± 0.2 GeV. Regarding the reported Ω_c resonances, different interpretations were suggested, which include the P-wave assignment [13,17–21], the P-wave and S/D-wave assignment [15,24,26], the charmed exotic systems [27–31], and entail more efforts to answer above two questions.

The narrowness of the five Ω_c states is suggested, in Ref. [17], to arise mainly from difficulty of pulling apart the two s quarks in a diquark ss in the $c(ss)$ binding system. During decay of Ω_c to $\Xi_c^+ K^-$, for instance, it is needed to pull apart the ss-diquark, with one s quark gone into the K^- and the other into the Ξ_c^+. This generally suppresses the respective width and is indeed in consistent with the narrowness (widths $\lesssim 60$ MeV) of the excited ordinary $\Xi = uss$ [1].

For charmed baryons, spectroscopy is known to be intricate, with many excited states expected [4–8,12,14,15]. The common practice is to utilize the heavy quark symmetry (HQS) [35], which becomes exact in the limit of heavy quark $m_Q \to \infty$. Assuming HQS by which spin $\mathbf{S}_Q$ of heavy quark Q is conserved, one can classify heavy hadrons by their total spin $\mathbf{J}$ and the angular momentum $\mathbf{j} = \mathbf{J} - \mathbf{S}_Q$ of light degrees of freedom (named jj coupling). This is analogous to the hydrogen-like atoms whose quantum states are well labeled by quantum numbers of outermost electron. While this picture works well in describing the normal charmed and bottom baryons [5–7] with strangeness $|S| < 2$, its applicability to the $\Omega_{c,b}$ baryons remains to be explored.

The purpose of this work is to explore quantum numbers and inner structures of the odd-parity bound systems of a heavy Q-quark ($Q = c, b$) and a light spin-1 qq-diquark ($qq = nn, ns, n = u, d$) and corresponding main feature of relative magnitudes of the spin coupling param-

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Table 1 Masses and widths of the LHCb-reported resonances Ω_c [2] and Ω_b [3], and excited baryons Σ_Q and Ξ'_Q [1], with the proposed values of spin-parity J^P shown also. The mass and the width are in units of MeV

State	Mass	Width	State	Mass	Width	State	Mass	Width	J^P
–	–	–	–	–	–	$\Omega_c(3000)^0$	$3000.4 \pm 0.2 \pm 0.1$	$4.5 \pm 0.6 \pm 0.3$	$1/2^-$
–	–	–	–	–	–	$\Omega_c(3050)^0$	$3050.2 \pm 0.1 \pm 0.1$	$0.8 \pm 0.2 \pm 0.1$	$1/2^-$
–	–	–	$\Xi'_b(6227)^-$	6226.9 ± 2.0	18 ± 6	$\Omega_b(6316)^-$	$6315.64 \pm 0.31 \pm 0.07^{\pm 0.5}$	< 2.8	$1/2^-$
–	–	–	–	–	–	$\Omega_b(6330)^-$	$6330.30 \pm 0.28 \pm 0.07^{\pm 0.5}$	$< 3.1,$	$1/2^-$
$\Sigma_c(2800)^{++}$	2801^{+4}_{-6}	75^{+22}_{-17}	$\Xi'_c(2930)$	2942 ± 5	36 ± 13	$\Omega_c(3066)^0$	$3065.6 \pm 0.1 \pm 0.3$	$3.5 \pm 0.4 \pm 0.2$	$3/2^-$
–	–	–	–	–	–	$\Omega_c(3090)^0$	$3090.2 \pm 0.3 \pm 0.5$	$8.7 \pm 1.0 \pm 0.8$	$3/2^-$
$\Sigma_b(6097)^-$	6098.0 ± 1.8	29 ± 4	–	–	–	$\Omega_b(6340)^-$	$6339.71 \pm 0.26 \pm 0.05^{\pm 0.5}$	< 1.5	$3/2^-$
–	–	–	–	–	–	$\Omega_b(6350)^-$	$6349.88 \pm 0.35 \pm 0.05^{\pm 0.5}$	$1.4^{+0.1}_{-0.8} \pm 1.0$	$3/2^-$
–	–	–	–	–	–	$\Omega_c(3119)^0$	$3119.1 \pm 0.3 \pm 0.9$	$1.1 \pm 0.8 \pm 0.4$[b]	$5/2^-$

[b] It is < 1.2 MeV, 95% CL totally

eters for the systems. We propose a new scheme of state classification (named the Jls mixing coupling) to perform systematic mass analysis for the newly LHCb-reported resonances of the excited Ω_c and Ω_b, which interprets all of these resonances to be the P-wave negative-parity baryons, and enables us to predict Regge trajectories of the mean-spin masses of their excited states. We update earlier computations for the negative-parity baryons Σ_Q and Ξ'_Q, and explain the matched spin-couplings of these baryons in a relativized potential quark model. Our calculation via the Jls mixing coupling indicates that an excited Ω_b state with $J^P = 5/2^-$ should exist around 6351.5 MeV, very close to the nearby $3/2^-$ state with mass of 6350.0 MeV. The $5/2^-$ state is expected to be unseen in LHCb experiment, due mainly to its near degeneracy with the nearby $3/2^-$ state and the limited data in the experiment. We hope that the updated LHCb experiment with increased data can test this observation.

To achieve the purpose, we choose the mean-spin mass $\bar{M}$ and the spin couplings (a_1, a_2, b, c) as effective parameters and determine them firstly via matching the deduced masses with the measured ones of heavy baryons considered and then via computing them dynamically in a relativized quark model. Due to close connections with interquark potential, the knowledges about these spin-couplings are useful and forms a basis for further understanding QCD interaction within hadrons.

This paper is organized as follows. In Sect. 2, we apply our proposed scheme of the Jls mixing coupling to mass analysis of the excited Ω_c baryons where preferred spin-parity are assigned for them. We examine other plausible spin-parity arrangements in Sect. 3. Similar mass analysis is given for excited Ω_b baryons in Sect. 4. In Sect. 5, the Jls mixing coupling is applied to the less strange baryons Σ_Q and Ξ'_Q. In Sect. 6, a computation is given in the relativized quark model to explain the matched spin-couplings. We end with conclusions and discussions in Sect. 7.

2 Inner structures of the $c(ss)$ systems

In heavy quark–diquark picture, two strange quarks in a $c(ss)$ system form a S-wave anti-color triplet($\bar{\mathbf{3}}_c$) diquark (ss), with spin one($S_{ss} = 1$) due to the spatial symmetry under exchange of two fermions. The diquark spin($= 1$) can couple with spin $S_c = 1/2$ of the charm quark c to form a total spin $S = 1 \pm 1/2 = 1/2, 3/2$. Let us consider the relative P-wave excitations of diquark ss with respect to the charm quark c(the relative orbital angular momentum $L = 1$). Coupling of $L = 1$ with the spin $S = 1/2$ gives states with total spin $J = 1/2, 3/2$, while coupling with $S = 3/2$ leads to states with $J = 1/2, 3/2$ and $5/2$. Symbolically, they are written as, in terms of adding of angular momentum,

$$\left(\frac{1}{2}\right)_S \otimes 1_L = \frac{1}{2} \oplus \frac{3}{2},$$
$$\left(\frac{3}{2}\right)_S \otimes 1_L = \frac{1}{2}' \oplus \frac{3}{2}' \oplus \frac{5}{2}. \tag{1}$$

In total, one has five P-wave states with $J = 1/2, 3/2, 1/2'$, $3/2'$ and $5/2$ with negative parity $P = -1$ which implies $5! = 120$ a priori possible assignments for their J^P quantum numbers.

For most heavy baryons with less strangeness, the jj coupling has been commonly used to classify their states in terms of hadron eigenstates $|J, j\rangle$, with J the eigenvalues of $\mathbf{J}$ and j of $\mathbf{j}$ respectively. In the case of doubly strange Qss baryons with the ss-diquark comparable with the heavy quark Q in mass, the finite mass effect of the heavy quark may become important and makes it appropriate to go beyond the jj coupling.

For this, we propose a new scheme of state classification(named the Jls mixing coupling) in which the eigenfunctions (bases) of spin multiplets of the $Q(ss)$ system $(Q = c, b)$ diagonalize all spin-orbit interactions between the heavy quark Q and the diquark ss. The spin-dependent interaction for the $Q(ss)$ systems, in the heavy quark–diquark picture, is [5,36]

$$H^{SD} = a_1 \mathbf{L}\cdot\mathbf{S}_{ss} + a_2\mathbf{L}\cdot\mathbf{S}_Q + bS_{12} + c\mathbf{S}_{ss}\cdot\mathbf{S}_Q,$$
$$S_{12} = 3\mathbf{S}_{ss}\cdot\hat{\mathbf{r}}S_Q\cdot\hat{\mathbf{r}} - \mathbf{S}_{ss}\cdot\mathbf{S}_Q, \tag{2}$$

where the first two terms are spin-orbit interactions, the third is the tensor energy, and the last the contact interaction between the heavy quark spin $\mathbf{S}_Q$ and the diquark spin $\mathbf{S}_{ss}$. Here, $\mathbf{L}$ is the orbital angular momentum of the system. Though four strengths $a_{1,2}$, b and c of spin-couplings will be treated, in a sense, as effective parameters in this work, they are closely related to interquark interactions within hadrons and assumed, in potential quark model [5–7,14,32], to be computable from QCD analogues of Breit-Fermi interaction in QED.

If Q is very heavy, the first spin-orbit term $\mathbf{L}\cdot\mathbf{S}_{ss}$ should dominate over the others in Eq. (2) if one assumes spin-dependent interactions enter the color hyperfine interactions through the magnetic moment $e_i\mathbf{S}_i/m_i$ of the quark $i = Q$ or ss with mass m_i, by analogy with the spin-relevant relativistic correction of the heavy quarkonium [37]. As such, the heavy quark spin $\mathbf{S}_Q$ decouples with the light degree of freedom and is conserved in the heavy quark limit, making the basis eigenfunctions of $\mathbf{L}\cdot\mathbf{S}_{ss}$(the jj coupling) appropriate to classify the baryon states. Here, the baryon spin $\mathbf{J}$ and the diquark angular momentum $\mathbf{j} = \mathbf{J} - \mathbf{S}_Q$ are both conserved, enabling baryon states to be labelled by the jj coupling states $|J, j\rangle$. In the case of the $c(ss)$ or $b(ss)$ baryons, in which the diquark mass m_{ss}(near 1 GeV) is comparable to M_Q(about 1.5 GeV), the terms like $a_2\mathbf{L}\cdot\mathbf{S}_Q$ in Eq. (2) may become important. Taking $a_1 = a_2$, for instance, the spin-orbit interaction becomes proportional to $\mathbf{L}\cdot[\mathbf{S}_{ss}+\mathbf{S}_Q] = \mathbf{L}\cdot\mathbf{S}$ and diagonal in the LS coupling, in which $\mathbf{S}_Q$ first couples $\mathbf{S}_{ss}$ to form the total quark spin $\mathbf{S} = \mathbf{S}_{ss} + \mathbf{S}_Q$ and then to $\mathbf{L}$ of the quark–diquark system to form $\mathbf{J}$, as shown in Eq. (1).

In the Jls mixing coupling considered in this work, the bases $|J, j_{LS} = j'\rangle$ diagonalize the interaction $a_1\mathbf{L}\cdot\mathbf{S}_{ss} + a_2\mathbf{L}\cdot\mathbf{S}_Q + bS_{12}$, instead of $\mathbf{L}\cdot\mathbf{S}_{ss}$ solely, in Eq. (2). The Jls scheme reduces to the jj coupling when the ratio $\epsilon = a_2/a_1$(expected to scale as m_{ss}/M_Q) tends to zero in the heavy quark limit,

$$a_1\mathbf{L}\cdot[\mathbf{S}_{ss}+\epsilon\mathbf{S}_Q] + bS_{12} \simeq a_1\mathbf{L}\cdot\mathbf{S}_{ss}, \quad \text{as } M_Q\to\infty, \tag{3}$$

where b is expected to be suppressed by $1/M_Q$ [36,38]. Finding the Jls mixing eigenstates $|J, j_{LS} = j'\rangle$ can be done by solving the linear eigenstate equation of 2×2 matrices $\Delta\mathcal{M}_J$ of the mass shift interaction (2) in the spin subspace of $J = 1/2$ and $J = 3/2$ [36,38] (see Ref. [39] also). In terms of the LS bases ${}^{2S+1}P_J = \{{}^2P_{1/2}, {}^4P_{1/2}, {}^2P_{3/2}, {}^4P_{3/2}, {}^4P_{5/2}\}$, the matrix forms of these mass shift interactions are (see appendix A of Ref. [17])

$$\Delta\mathcal{M}_{J=1/2} = \begin{bmatrix} \frac{1}{3}(a_2 - 4a_1) & \frac{\sqrt{2}}{3}(a_2 - a_1) + \frac{b}{\sqrt{2}} \\ \frac{\sqrt{2}}{3}(a_2 - a_1) + \frac{b}{\sqrt{2}} & -\frac{5}{3}(a_1 + \frac{1}{2}a_2) - b \end{bmatrix} + \begin{bmatrix} -c & 0 \\ 0 & \frac{1}{2}c \end{bmatrix}, \tag{4}$$

$$\Delta\mathcal{M}_{J=3/2} = \begin{bmatrix} \frac{2}{3}a_1 - \frac{1}{6}a_2 & \frac{\sqrt{5}}{3}(a_2 - a_1) - \frac{b}{2\sqrt{5}} \\ \frac{\sqrt{5}}{3}(a_2 - a_1) - \frac{b}{2\sqrt{5}} & -\frac{1}{3}(2a_1 + a_2) + \frac{4b}{5} \end{bmatrix} + \begin{bmatrix} -c & 0 \\ 0 & \frac{1}{2}c \end{bmatrix}, \tag{5}$$

$$\Delta\mathcal{M}_{J=5/2} = a_1 + \frac{1}{2}a_2 - \frac{b}{5} + \frac{c}{2}. \tag{6}$$

Given J^P assignments of the five Ω_c states(two states of $J^P = 1/2^-$, two of $J^P = 3/2^-$, and one of $J^P = 5/2^-$) there should, in principle, exist one unique solution for the four parameters a_1, a_2, b and c. As we shall find below, there is one solution in which all states are P-waves with reasonable values of parameters and the mass pattern as reported by LHCb for the excited Ω_c states in Table 1, where our preferred J^P assignments are shown.

Diagonalizing the mass shift operator $\mathbf{L}\cdot\mathbf{S}_{ss} + \epsilon\mathbf{L}\cdot\mathbf{S}_Q + b_1S_{12}$, with $b_1 \equiv b/a_1$, one can compute the mass shifts ΔM, the eigenvalues of Eqs. (4)–(6), by treating the contact term $c\mathbf{S}_{ss}\cdot\mathbf{S}_Q$ as a perturbation($c_1 = c/a_1$ is expected to be small for the relative $1P$-wave between diquark and Q). The lowest order perturbation theory gives (see Appendix A)

$$\Delta M(J=1/2, 0') = -\frac{a_1}{4}\left(6 + \sqrt{\Delta_1\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right)} + \frac{a_2}{a_1}\right) - \frac{b}{2} + c\Delta_3^+\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right),$$

$$\Delta M(J=1/2, 1') = -\frac{a_1}{4}\left(6 - \sqrt{\Delta_1\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right)} + \frac{a_2}{a_1}\right) - \frac{b}{2} + c\Delta_3^-\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right),$$

$$\Delta M(J=3/2, 1') = -a_1\left(\sqrt{\Delta_2\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right)} + \frac{a_2}{4a_1}\right) + \frac{2b}{5} + c\Delta_4^+\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right),$$

$$\Delta M(J=3/2, 2') = a_1\left(\sqrt{\Delta_2\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right)} - \frac{a_2}{4a_1}\right) + \frac{2b}{5} + c\Delta_4^-\left(\frac{a_2}{a_1}, \frac{b}{a_1}\right),$$

$$\Delta M(J=5/2, 2') = a_1 + \frac{a_2}{2} - \frac{b}{5} + \frac{c}{2}, \tag{7}$$

where six functions $\Delta_{1,2}(\epsilon, x)$, $\Delta_3^{\pm}(\epsilon, x)$ and $\Delta_4^{\pm}(\epsilon, x)$ are defined by

$$\Delta_1(\epsilon, x) = 4 + 12x^2 + 4x\,(5\epsilon - 2) - 4\epsilon + 9\epsilon^2,$$

$$\Delta_2(\epsilon, x) = 1 + \frac{1}{5}x^2 - \frac{x}{5}(1 + 2\epsilon) - \epsilon + \frac{9}{16}\epsilon^2. \tag{8}$$

$$\Delta_3^{+}(\epsilon, x) = \frac{4 - (2 + 6x + 7\epsilon - 3\sqrt{\Delta_1(\epsilon, x)})^2/(2\epsilon - 2 + 3x)^2}{8 + (2 + 6x + 7\epsilon - 3\sqrt{\Delta_1(\epsilon, x)})^2/(2\epsilon - 2 + 3x)^2},$$

$$\Delta_3^{-}(\epsilon, x) = \Delta_3^{+}\left(\sqrt{\Delta_1} \to -\sqrt{\Delta_1}\right). \tag{9}$$

$$\Delta_4^{+}(\epsilon, x) = \frac{10 - (40 - 24x + 5\epsilon + 60\sqrt{\Delta_2(\epsilon, x)})^2/(10 - 10\epsilon + 3x)^2}{20 + (40 - 24x + 5\epsilon + 60\sqrt{\Delta_2(\epsilon, x)})^2/(10 - 10\epsilon + 3x)^2},$$

$$\Delta_4^{-}(\epsilon, x) = \Delta_4^{+}\left(\sqrt{\Delta_2} \to -\sqrt{\Delta_2}\right), \tag{10}$$

with $\Delta_{3,4}^{-}(\epsilon, x)$ obtained from $\Delta_{3,4}^{+}(\epsilon, x)$ by merely replacing $\sqrt{\Delta_{1,2}} \to -\sqrt{\Delta_{1,2}}$. This expresses the mass shifts in terms of four parameters (a_1, a_2, b, c) nonlinearly. Expanding Eq. (7) to the leading order of ϵ, b_1 and c_1, it reduces to the linear mass shift in Ref. [17] in jj coupling. The spin-weighted sum of these mass shifts in Eqs. (4)–(6) vanishes: $\sum_J Tr_J(\Delta\mathcal{M}_J)(2J + 1) = 0$, and the same holds for the eigenvalues $\Delta M(J, j')$ in Eq. (7). Note that the sums of eigenvalues of $\Delta\mathcal{M}_{1/2}$ and $\Delta\mathcal{M}_{3/2}$ are equal to the traces of the respective matrices in Eq. (7). Note that an alternative expression similar to the mass-shifts (7) is obtained recently in Ref. [40].

Adding spin-independent mass $\bar{M}$, which equals to the spin-averaged mass of the five excited css systems, the baryon mass becomes $M(J, j_{LS} = j') = \bar{M} + \Delta M(J, j')$, with $\Delta M(J, j')$ given by Eq. (7). Confronting $M(J, j_{LS} = j')$ with the observed masses in Table 1 leads to the values of a_1 , $a_2(= \epsilon a_1)$, $b(= a_1 b_1)$ and $c(= a_1 c_1)$, with the help of the following criteria:

(i) The parameter a_1 should be positive but smaller than 119 MeV$= M(\Omega_c, 3119) - M(\Omega_c, 3000)$.
(ii) The parameter a_2 is of same order with a_1 but no more than a_1 roughly as $a_2/a_1 = \epsilon$ scales as m_{ss}/M_c .
(iii) The parameter b should be smaller than a_1 and a_2 as b scales like $1/(m_{ss}M_c)$. The parameter c should be smallest, less than b as it scales as P-wave wave function near the origin.

We carry out mass analysis for all $5! = 120$ a priori possible assignments of P-wave Ω_c states and list the preferred assignments in Table 1, corresponding to the mean mass and the parameters [17]

$$\bar{M} = 3079.94 \text{ MeV}, \tag{11}$$

$$\{a_1, a_2, b, c\} = \{26.96, 25.76, 13.51, 4.04\} \text{ (MeV)}, \tag{12}$$

with $\epsilon = 0.96, b_1 = 0.50, c_1 = 0.15$. For our preferred assignment, the inner structures of the five excited Ω_c baryons are

$$\left\{\begin{array}{ccc} \text{State}\,|J, j_{LS}\rangle & M(\Omega_c, 1P) & \text{Main comp} \\ |\frac{1}{2}, 0'\rangle & 3000.4 & {}^4P_{1/2}(97\%) \\ |\frac{1}{2}, 1'\rangle & 3050.2 & {}^2P_{1/2}(97\%) \\ |\frac{3}{2}, 1'\rangle & 3065.6 & {}^4P_{3/2}(98\%) \\ |\frac{3}{2}, 2'\rangle & 3090.2 & {}^2P_{3/2}(98\%) \\ |\frac{5}{2}, 2'\rangle & 3119.1 & {}^4P_{5/2} \end{array}\right\}, \tag{13}$$

where the third line gives the main component in terms of the normal LS coupling (${}^{2S+1}P_J$, see below). One sees that the mass degeneration within the $J = 1/2$ and $J = 3/2$ multiplets removed by the flip of the charm quark spin $\mathbf{S}_c$. Within the same multiplet, the lower state consists mainly of $S = 3/2$ configuration $c^{\uparrow}(s^{\uparrow}s^{\uparrow})$, while the higher consists mainly of that with $\mathbf{S}_c$ flipped, $c^{\downarrow}(s^{\uparrow}s^{\uparrow})$ with $S = 1/2$.

Expressing in terms of the LS eigenstates $|{}^{2S+1}P_J\rangle$, with $1/2 \le J \le 5/2$ and $S = 0, 1$, the Jls states in Eq. (13) become (see Appendix B),

$$\begin{aligned}
&|J = 1/2, j' = 0'\rangle = -0.164|{}^2P_{1/2}\rangle + 0.986|{}^4P_{1/2}\rangle, \text{ at } 3000 \\
&|J = 1/2, j' = 1'\rangle = 0.986|{}^2P_{1/2}\rangle + 0.164|{}^4P_{1/2}\rangle, \text{ at } 3050 \\
&|J = 3/2, j' = 1'\rangle = 0.129|{}^2P_{3/2}\rangle + 0.992|{}^4P_{3/2}\rangle, \text{ at } 3066 \\
&|J = 3/2, j' = 2'\rangle = -0.992|{}^2P_{3/2}\rangle + 0.129|{}^4P_{3/2}\rangle, \text{ at } 3090 \\
&|J = 5/2, j' = 2'\rangle = |{}^4P_{5/2}\rangle, \text{ at } 3119.
\end{aligned} \tag{14}$$

Dominantly, they are $|{}^4P_{1/2}\rangle$, $|{}^2P_{1/2}\rangle$, $|{}^4P_{3/2}\rangle$, $|{}^2P_{3/2}\rangle$ and $|{}^4P_{5/2}\rangle$ with the monatomically increasing mass, respectively, as shown in Eq. (13). This is in contrast with the normal state classification of heavy baryon systems via the jj coupling (see [36], for instance),

$$\begin{aligned}
|J = 1/2, j = 0\rangle &= \sqrt{\frac{1}{3}}|1^2P_{1/2}\rangle + \sqrt{\frac{2}{3}}|1^4P_{1/2}\rangle, \\
|J = 1/2, j = 1\rangle &= \sqrt{\frac{2}{3}}|1^2P_{1/2}\rangle - \sqrt{\frac{1}{3}}|1^4P_{1/2}\rangle, \\
|J = 3/2, j = 1\rangle &= \sqrt{\frac{1}{6}}|1^2P_{3/2}\rangle + \sqrt{\frac{5}{6}}|1^4P_{3/2}\rangle, \\
|J = 3/2, j = 2\rangle &= \sqrt{\frac{5}{6}}|1^2P_{3/2}\rangle - \sqrt{\frac{1}{6}}|1^4P_{3/2}\rangle,
\end{aligned} \tag{15}$$

mixing significantly between the states with $J = 1/2$ and $3/2$.

It is of interest to express the mixing states (14) in terms of the jj bases, giving (see Appendix B)

$$\begin{aligned}
\left|J = \frac{1}{2}, j' = 0'\right\rangle &= 0.711\left|\frac{1}{2}, j = 0\right\rangle + 0.703\left|\frac{1}{2}, j = 1\right\rangle, \\
\left|J = \frac{1}{2}, j' = 1'\right\rangle &= -0.703\left|\frac{1}{2}, j = 0\right\rangle + 0.711\left|\frac{1}{2}, j = 1\right\rangle, \\
\left|J = \frac{3}{2}, j' = 1'\right\rangle &= 0.958\left|\frac{3}{2}, j = 1\right\rangle + 0.286\left|\frac{3}{2}, j = 2\right\rangle,
\end{aligned}$$

$$\left|J=\frac{3}{2}, j'=2'\right\rangle = -0.286\left|\frac{3}{2}, j=1\right\rangle + 0.958\left|\frac{3}{2}, j=2\right\rangle, \tag{16}$$

which mix almost equally between the jj bases $\left|\frac{1}{2}, j=0,1\right\rangle$. Equation (16) implies that the angular momentum $\mathbf{j}=\mathbf{J}-\mathbf{S}_c$ of the diquark in our mixing states with $J=1/2$ may be observed to take values either $j=0$ or $j=1$, with almost equal probabilities (0.51 vs. 0.49), while those for the $J=3/2$ are mainly the jj eigenstates $|3/2, j=1,2\rangle$ (at 0.92), slightly mixing the respective $|3/2, j=2,1\rangle$ (at 0.08). Hence, the $J=1/2$ states do not conserve $\mathbf{j}=\mathbf{J}-\mathbf{S}_c$ of the diquark and thereby $\mathbf{S}_c$. This differs the css states from the nonstrange charmed baryons, all of which can be well classified by the jj eigenstates [5–7,17,36].

One alternative solution to assignment of the five $\Omega_c = css$ states involves identification of the five masses $M(1/2,0')$, $M(1/2,1')$, $M(3/2,1')$, $M(3/2,2')$, $M(5/2,2')$ to be that at 3000, 3066, 3050, 3090 and 3119 MeV, respectively. This gives rise to the parameters and the mean mass in Ref. [17]

$$\{a_1, a_2, b, c\} = \{21.40, 40.75, 5.67, 0.44\}\ (\text{MeV}), \tag{17}$$

$$\bar{M} = 3078.23\ \text{MeV}. \tag{18}$$

This assignment is disfavored by unreasonable relative ratio (1:2) between a_1 and a_2 in the sense of the criterion (ii).

Using Eq. (7) further, we performed systematic search for all remaining permutations among all 5! possibilities and find no acceptable solution for considered permutations: the parameters obtained either have negative (unacceptable) signs of a_1 and a_2 or unreasonable values in the light of the criteria (i)–(iii).

We also made a "recovery" test by fitting four out of the five observed Ω_c masses for the J^P assignments (13) and the alternative solution with $M(1/2,1')$ and $M(3/2,1')$ interchanged, corresponding to the parameters (17), to see if the remaining measured mass, unused in fitting inputs, can be reproduced at the site of given spin-states(marked by square bracket). The results for the "recovery" test are shown in Table 2 for all $2*5=10$ possibilities of state arrangements for chosen inputs and indicate that the test successes for both, whereas the parameter values clearly favor the assignment (13) in the light of the criterion (ii): a_2 is notably large over a_1, nearly two times, for the alternative solution.

One can also use the mass scaling of the spin-interaction couplings, from D_s mesons to the css baryons, to explain the values (12) for our preferred assignment. Using the scaling relation [36,38], one finds

$$a_1(css) = a_1(c\bar{s})\left(\frac{m_s}{m_{ss}}\right) = (89.4\ \text{MeV})\left(\frac{328}{991}\right) = 29.6\ \text{MeV}, \tag{19}$$

$$a_2(css) = \frac{a_2(c\bar{s})}{1+m_{ss}/M_c} = \frac{40.7\ \text{MeV}}{1+991/1440} = 24.1\ \text{MeV}, \tag{20}$$

which are close to the values (12), but away from that in (17). Here, $a_{1,2}(c\bar{s})$ are the parameters of the spin-orbit interactions for the P-wave $D_s = c\bar{s}$ mesons, with the respective values 89.4 MeV and 40.7 MeV [36]. The masses ($m_s = 328$ MeV and $M_c = 1440$ MeV) of the strange quark s and the charm quark are from Regge trajectory fit of the D_s mesons and the charmed baryons Σ_c/Ξ_c' [38] while the mass($m_{ss} = 991$ MeV) of the ss-diquark comes from the Regge trajectory fit of the five excited Ω_c states in Table 1 (cf. the discussion following (21) in Sect. 3).

3 Alternative possibilities examined

The possibility that must be checked is that not all of the Ω_c states in Table 1 are relative P-wave excitations between ss diquark and the charmed quark [5,23,24,41,42]. For instance, two $2S$ wave candidates are predicted for the $c(ss)$ at masses 3088 MeV ($J^P = 1/2^+$) and 3123 MeV($J^P = 3/2^+$) in Ref. [5], not far from two higher masses (3090 and 3119 MeV) reported by LHCb. This leaves possibility that some of the lower states may be that out of the five states of P-waves, with other P-wave states unseen somehow, probably due to locating below the $\Xi_c^+ K^-$ threshold ($\simeq 2962$ MeV), or invisible due to the near degeneracies [5]. To disentangle this possibility, we extend the mass analysis in Sect. 2 by assuming one of the five measured masses of the Ω_c states in Table 1 to be that of 2S state and comparing it with mass estimation of the 2S state with the help of Regge trajectory [43,44]. The results suggest that all measured masses of the excited Ω_c states are too low to be a 2S state.

We use the mass shift formula (7) to perform mass analysis for all 5! permutations of the J^P arrangement with one presumed 2S state removed from five mass inputs. Apart from the 10 arrangements already given in Table 2, we list, in Table 3, the other candidate solutions with the parameters that are not too far away from the criteria (i)–(iii), and the thereby predicted P-wave mass (presumably unseen) enclosed in square bracket. Though two arrangements in the first and second data rows recover approximately the observed masses absent in the inputs, all of these permutated arrangements are disfavored by the criteria (i)–(iii), having unreasonable values of the parameters listed: either a_2 or b is notably large over a_1 or negative, or c is abnormally large compared with one of a_2 and b, or both.

The mass estimation of the Ω_c systems in $2S$ wave can be given utilizing a Regge-like relation for the spin-averaged mass $\bar{M}$ of the charmed baryons [38,45]

Table 2 Mass and parameters for two J^P assignments (13) and the alternative arrangement of four LHCb-observed masses of the excited Ω_c systems. All $2*5=10$ possible choices of mass inputs out of the five measured masses are considered. The predicted mass at presumably unseen state is marked in square bracket. All parameters except the Regge slope a in MeV

$\lvert\frac{1}{2},0'\rangle$	$\lvert\frac{1}{2},1'\rangle$	$\lvert\frac{3}{2},1'\rangle$	$\lvert\frac{3}{2},2'\rangle$	$\lvert\frac{5}{2},2'\rangle$	a_1	a_2	b	c	$a(GeV^2)$	$\bar{M}(1P)$	$\bar{M}(2S)$
[2995.0]	3050	3066	3090	3119	27.5	27.0	15.5	3.6	0.316	3079	3244
3000	[3049.0]	3066	3090	3119	27.2	25.2	13.7	4.4	0.316	3080	3244
3000	3050	[3068.2]	3090	3119	26.7	24.8	15.4	5.0	0.317	3081	3245
3000	3050	3066	[3095.4]	3119	28.2	23.1	14.4	2.3	0.317	3081	3246
3000	3050	3066	3090	[3115.6]	26.3	23.7	14.7	3.2	0.315	3079	3243
[3000.4]	3066	3050	3090	3119	21.4	40.8	5.7	0.44	0.314	3078	3242
3000	[3067.4]	3050	3090	3119	20.4	41.9	6.4	1.2	0.315	3078	3242
3000	3066	[3051.0]	3090	3119	21.4	40.4	6.1	0.52	0.315	3078	3242
3000	3066	3050	[3090.1]	3119	21.3	40.8	5.7	0.59	0.314	3078	3242
3000	3066	3050	3090	[3117.5]	21.4	39.7	5.7	−0.57	0.314	3078	3241

Table 3 Parameters and spin-averaged masses (MeV) for the selected permutations of J^P assignments of the four states out of excited Ω_c levels. Only the arrangements that leads to the parameters near to reasonable values are listed among $5*4!=120$ possible permutations of mass inputs considered, with the mass prediction at unseen level marked by square bracket. The shown also includes the ensuing prediction for the Regge slope a (see text, in GeV^2) and spin-averaged 2S-wave mass (in MeV) of the $c(ss)$ system. $m_{ss}=991$ MeV

$\lvert\frac{1}{2},0'\rangle,\lvert\frac{1}{2},1'\rangle,\lvert\frac{3}{2},1'\rangle,\lvert\frac{3}{2},2'\rangle,\lvert\frac{5}{2},2'\rangle$	a_1	a_2	b	c	$a(GeV^2)$	$\bar{M}(1P)$	$\bar{M}(2S)$
3000, [3052.3], 3066, 3119, 3090	23.0	−2.19	35.0	−3.43	0.313	3077.0	3240.3
3000, 3066, 3050, 3119, [3093.9]	27.8	7.86	11.0	−23.9	0.312	3076.2	3239.3
3050, 3066, 3000, [3082.2], 3119	10.7	49.9	−36.2	10.6	0.307	3070.9	3231.7
3000, 3090, 3050, [3102.5], 3119	14.6	48.9	10.1	−3.20	0.320	3083.7	3249.9
3066, [3108.6], 3000, 3119, 3090	11.2	24.8	−49.1	−38.41	0.312	3076.0	3238.9
[3008.0], 3000, 3050, 3066, 3119	32.3	14.2	5.77	28.9	0.302	3066.4	3225.2
3050, 3066, 3000, [3078.7], 3119	11.1	47.6	−38.8	12.7	0.306	3074.1	3230.6

$$(\bar{M}-M_c)^2=\pi a L+\left[m_d+M_c\left(1-\frac{m_{barec}^2}{M_c^2}\right)\right]^2, \quad (21)$$

in which a is the Regge slope, m_d is the light diquark mass involved, $m_{barec}=1.275$ GeV the bare mass of charm quark and L the orbital angular momentum of the systems. The charm quark mass $M_c=1.44$ GeV is determined by confronting the relation (21) with the charmed baryons Σ_c/Ξ_c' [38]. In Ref. [43], a trajectory slope ratio π:2 for the radially and angular excitations is suggested in applying Eq. (21) to the heavy mesons B/D's and B_s/D_s's.

By analogy of the $c(ss)$ system with $D_s=c\bar{s}$ mesons, we estimate the 2S-wave mass of the $c(ss)$ state using Eq. (21) with a replaced by $a(\pi/2)$ and L by n, the radial quantum number of the $c(ss)$ system,

$$\bar{M}(2S)=M_c+\sqrt{\pi a(\pi/2)+\left(m_d+M_c\left(1-\frac{1.275^2}{M_c^2}\right)\right)^2}, \quad (22)$$

where $m_d=0.991$ GeV and the Regge slope a (listed in Tables 2 and 3) are solved from (21) applying to the spin-averaged masses $\bar{M}(1S)$ and $\bar{M}(1P)$ for a given assignment, and $M_c=1.44$ GeV. Here, the observed lowest masses of the Ω_c, $M(\Omega_c,1/2^+)=2695.2\pm1.7$ MeV and $M(\Omega_c,3/2^+)=2765.9\pm2.0$ MeV [1], lead to their spin-averaged mass in S-wave

$$\bar{M}(\Omega_c,1S)=\frac{1}{6}\left(2M(1/2^+)+4M(3/2^+)\right) = 2742.3\pm1.9\ \text{MeV}. \quad (23)$$

Usage of the mean mass (11) in P-wave($L=1$) and the mass (23) in S-wave($L=0$) in Eq. (21) gives

$$\Omega_c: m_d=m_{ss}=0.991\ \text{GeV}, a=0.316\ \text{GeV}^2. \quad (24)$$

Given trajectory parameters, one can predict Regge trajectories of the excited baryons Ω_c (Fig. 1) with radial quantum number $n=0$ and 1. In Table 4 in which n stands for $n+1=1,2$, we list corresponding spin-averaged masses of the excited Ω_c's predicted by Eqs. (21) and (22), which are quite helpful for further constructing whole families of the baryon Ω_c's.

Depending on state arrangements having different P-wave mean-mass, the above values vary slightly. The 2S-wave

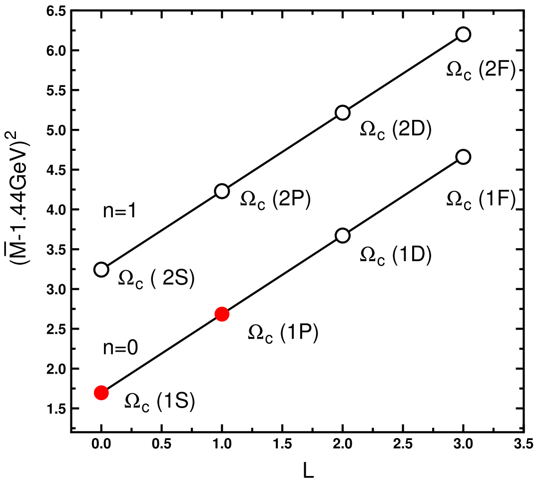

Fig. 1 Regge trajectories of the Ω_c baryons relating the shifted spin-averaged mass squared to the orbital angular momentum L of the systems, with the parameters $m_{d=ss} = 991$ MeV and the Regge slope $a = 0.316$ GeV^2 corresponding to our assignment (13). The (red) solid circles correspond to the observed (mean) masses

Table 4 Mean masses of the excited Ω_Q's predicted by Eqs. (21) and (22)

State (MeV)	$\bar{M}(nS)$	$\bar{M}(nP)$	$\bar{M}(nD)$	$\bar{M}(nF)$
$\Omega_c(n=1)$	2742.3	3078.3	3356.2	3598.7
$\Omega_c(n=2)$	3241.1	3496.5	3723.6	3930.0
$\Omega_b(n=1)$	6051.0	6342.0	6593.3	6817.7
$\Omega_b(n=2)$	6489.2	6724.1	6936.6	7132.4

masses of the $c(ss)$ system estimated via (22) are listed in Tables 2 and 3 for given assignments. Evidently, the 2S-wave mass $\bar{M}(2S) \geq 3220$ MeV, and both of the higher mass states at 3090 MeV and 3119 MeV are too low to be a $2S$ state candidates. The same conclusion holds also for other low-lying masses, as shown in Tables 2 and 3.

For other assignments searched, no solutions of the predicted unseen states, below the 2962 MeV threshold (of $\Xi_c^+ K^-$) or near degeneracies with other masses, are found up to requirements (i)–(iii) in Sect. 2.

4 P-wave masses for $\Omega_b = b(ss)$ systems

Recent observation of four Ω_b^- states [3] around 6.3 GeV (Table 1) makes it timely to apply mass analysis of the $\Omega_c = c(ss)$ states to the bottom systems $b(ss)$ which consists of a bottom quark b and a spin-1 diquark ss. Heaviness of the b quark implies that Eqs. (7) should certainly be applicable for the P-wave Ω_b^- states. We use it to search the possible assignments with the following inputs:

(i) The parameter a_1 is positive and no more than the maximum gap between the observed masses : $a_1 < 2*(6349.88 - 6315.64) = 68.48$ MeV. (ii) The parameter a_2 is of same order of a_1 roughly in magnitude, but should be smaller than a_1. (iii) The tensor parameter b should be smaller than a_1 in magnitude. (iv) The hyperfine parameter c is set to zero.

Assuming one of P-wave states is unseen experimentally the search is performed systematically via matching the observed masses of the Ω_b states in Table 1 with that in Eq. (7) for all $4! = 24$ a priori possible J^P assignments of P-wave states and 5 possible permutations of the unseen P-wave state that is expected to exist somewhere around 6.3 GeV.

The most preferred results are achieved by the identification of four masses at 6315.64, 6330.30, 6339.71 and 6349.88 MeV as $M(1/2, 0')$, $M(1/2, 1')$, $M(3/2, 1')$, $M(3/2, 2')$. The corresponding parameters and the mean mass are

$$a_1 = 8.98 \text{ MeV}, a_2 = 4.11 \text{ MeV}, b = 7.61 \text{ MeV},$$
$$\bar{M} = 6342.0 \text{ MeV}, \tag{25}$$

which lead, by Eqs. (7), to the following J^P assignment,

$$\left\{ \begin{array}{cc} \text{State } |J, j_{LS}\rangle & M(\Omega_b, 1P) \\ |\frac{1}{2}, 0'\rangle & 6315.4 \\ |\frac{1}{2}, 1'\rangle & 6332.0 \\ |\frac{3}{2}, 1'\rangle & 6337.8 \\ |\frac{3}{2}, 2'\rangle & 6350.0 \\ |\frac{5}{2}, 2'\rangle & 6351.5^{\text{Pd}} \end{array} \right\}, \tag{26}$$

In terms of the LS basis, their inner structures are

$$|1/2, 0'\rangle = -0.190|^2P_{1/2}\rangle + 0.982|^4P_{1/2}\rangle, \text{ at } 6316,$$
$$|1/2, 1'\rangle = 0.982|^2P_{1/2}\rangle + 0.190|^4P_{1/2}\rangle, \text{ at } 6330,$$
$$|3/2, 1'\rangle = 0.488|^2P_{3/2}\rangle - 0.873|^4P_{3/2}\rangle, \text{ at } 6340,$$
$$|3/2, 2'\rangle = 0.873|^2P_{3/2}\rangle + 0.488|^4P_{3/2}\rangle, \text{ at } 6350,$$
$$|5/2, 2'\rangle = |^4P_{5/2}\rangle \text{ at } 6351.5^{\text{Pd}}, \tag{27}$$

which are mainly the P-wave states $|^4P_{1/2}\rangle$, $|^2P_{1/2}\rangle$, two mixings of $|^4P_{3/2}\rangle$ and $|^2P_{3/2}\rangle$, and $|^4P_{5/2}\rangle$, respectively.

The experimental missing of the $J = 5/2$ state in Table 1 at $M(5/2, 2') = 6351.5$, which we predict, is most likely due to its degeneracy with the nearby state at $M(3/2, 2') = 6350$ MeV. If this is the case, it may be hidden in the observed peak around 6350 MeV, which, though appears consistent with a single resonance, is actually composed of two. This is compared to the recent assignment in Ref. [33], where the mass for the unseen $J = 5/2$ state ranges from 6355 MeV to 6383 MeV or from 6380 to 6407 MeV, with the favored parameters $a_1 = 10.20$ MeV, $a_2 + c = 10.04$ MeV and $b = 4.75$ MeV.

One of other plausible solutions consists of identification of the four masses at 6315.64, 6330.30, 6339.71 and

Table 5 Parameters and masses(all in MeV) for the excited Ω_b when one of four observed masses was removed from the mass inputs. The removed one (enclosed by square bracket) is reproduced by Eq. (7) approximately

$\lvert\frac{1}{2},0'\rangle$, $\lvert\frac{1}{2},1'\rangle$, $\lvert\frac{3}{2},1'\rangle$, $\lvert\frac{3}{2},2'\rangle$, $\lvert\frac{5}{2},2'\rangle$	a_1	a_2	b	$\bar{M}(1P)$
[6314.5], 6332.0, 6337.8, 6350.0, 6351.5	9.02	4.44	7.92	6341.8
6315.4, [6332.1], 6337.8, 6350.0, 6351.5	8.91	4.27	7.53	6342.0
6315.4, 6332.0, [6337.7], 6350.0, 6351.5	8.95	4.25	7.48	6341.9
6315.4, 6332.0, 6337.8, [6350.5], 6351.5	8.99	4.01	7.12	6342.1

Table 6 Parameters and spin-averaged masses(in MeV) for the selected permutations of J^P assignments of the four excited Ω_b levels, with predicted mass at unseen level(marked by square bracket). All arrangements leading to highly unreasonable or unacceptable parameters are not listed. The shown includes also ensuing prediction for the Regge slope a (see text) and spin-averaged css masses in $2S$ wave. $m_{ss} = 991$ MeV. We denoted the five states $\lvert\frac{1}{2},0'\rangle$, $\lvert\frac{1}{2},1'\rangle$, $\lvert\frac{3}{2},1'\rangle$, $\lvert\frac{3}{2},2'\rangle$, $\lvert\frac{5}{2},2'\rangle$ by $\lvert 1\rangle$, $\lvert 2\rangle$, $\lvert 3\rangle$, $\lvert 4\rangle$, $\lvert 5\rangle$ for short, respectively

$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 5\rangle$	a_1	a_2	b	$a(GeV^2)$	$\bar{M}(1P)$	$\bar{M}(2S)$
[6312.4]	6316	6329	6340	6350	21.2	26.9	42.4	0.297	6324.2	6463.3
6314	[6331.4]	6330	6341	6350	7.39	10.3	1.93	0.313	6337.4	6482.6
6316	6330	[6321.8]	6340	6350	7.17	13.3	−3.28	0.310	6335.4	6479.7
6316	6331	6338	[6388.1]	6350	12.0	−13.9	24.1	0.328	6350.1	6501.2
6313	6332	6338	6351	[6352.5]	9.4	4.87	8.37	0.318	6342.3	6489.8
[6311.5]	6330	6316	6340	6350	10.9	6.70	−10.5	0.308	6333.6	6477.1
[6304.7]	6330	6316	6350	6340	11.9	−8.98	-2.92	0.306	3331.7	6474.4

6349.88 MeV as $|1/2,0'\rangle$, $|3/2,1'\rangle$,$|3/2,2'\rangle$ and $|5/2,2'\rangle$. This assignment gives parameters and spin-averaged mass (in MeV)

$$\{a_1,a_2,b\} = \{7.39, 10.32, 1.93\}, \quad \bar{M} = 6337.4, \tag{28}$$

where the parameter $a_2 = 10.32$ MeV is larger than $a_1 = 7.39$ MeV, not compatible with the criterion (ii).

There are still two additional possibilities involving identification of the four observed Ω_b states at higher $j' = 1', 2'$. The first is to identify the states at 6330, 6316, 6340 and 6350 MeV with the respective states at $M(1/2,1')$, $M(3/2,1')$, $M(3/2,2')$, $M(5/2,2')$, giving the parameter set(in MeV)

$$\{a_1,a_2,b\} = \{10.86, 6.70, -10.54\}, \quad \bar{M} = 6333.6, \tag{29}$$

and the presumably unseen state at $M(1/2,0') = 6311.5$ MeV. This lowest state, if exist, are most likely to be visible. Together with abnormal (by the criteria (iii)) value $|b| \simeq 11$ MeV compared to $a_1 = 10.86$ MeV, we disfavor this assignment.

The second is to interchange two states at 6340 and 6350 MeV in the assignment above, leading to the parameter set(in MeV)

$$\{a_1,a_2,b\} = \{11.93, -8.98, -2.92\}, \quad \bar{M} = 6331.7, \tag{30}$$

and the five states $|1/2, j' = 0'\rangle$ at 6304.7 (prediction), $|1/2, j' = 1'\rangle$ at 6330.3, $|3/2, j' = 1'\rangle$ at 6315.6, $|3/2, j' = 2'\rangle$ at 6349.9 and $|5/2,2'\rangle = |^4P_{5/2}\rangle$ at 6339.7 MeV. Away (about 10 MeV) from the observed masses 6315 MeV, the presumed missing state, at $M(1/2,0') = 6304.7$ MeV, should not be hidden in any of the four observed peaks by LHCb. The unacceptable (negative) value of $a_2 = -8.98$ MeV also disfavors this assignment.

In addition, we carry out "recovery" test (Table 5, all in MeV) for the excited Ω_b's, in which one of four observed masses is reproduced approximately by Eq. (7) at the site enclosed by square bracket when it was removed from four mass inputs. The uncertainty is about 2 at most. This further confirms our prediction $M(5/2,2') = 6351.5$ MeV for the unseen (excited) Ω_b state.

Could it be possible that some higher states of these Ω_b peaks are the $2S$ excitations? Our answer to this question is negative. We employ Eq. (7) to perform mass fitting for the four mass inputs of the Ω_b baryons in Table 1 to find the three parameters $a_{1,2}$, b and $\bar{M}(1P)$, and thereby estimate the 2S-wave mass of the Ω_b baryons, shown in Table 6, using Eq. (22) with M_c replaced by $M_b = 4.48$ GeV (the charm bare mass 1.275 GeV by that of the bottom quark 4.18 GeV). For all assignments with the parameters not far away from the inputs (i)–(iv), we find

$$\bar{M}(\Omega_b, 2S) \gtrsim 6450 \text{ MeV}, \tag{31}$$

which disfavors identifying newly LHCb-reported Ω_b to be a $2S$ excitation. During calculating, the ss diquark mass, $m_{d=ss} = 1.001$ GeV, is determined from Eq. (21) applied to spin-averaged mass $M(\Omega_b, 1S) = 6061.4$ MeV of the

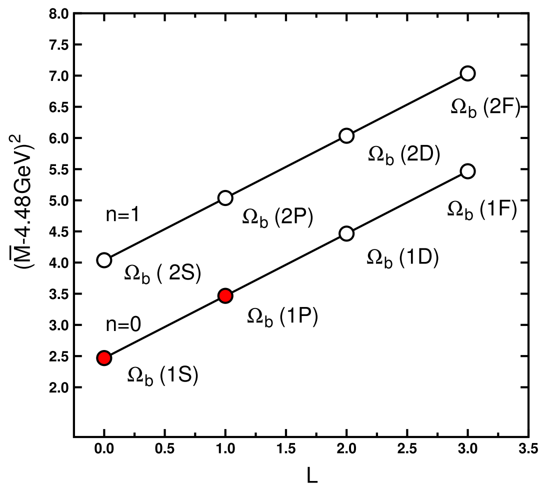

Fig. 2 Regge trajectories of the Ω_b baryons relating the shifted spin-averaged mass squared to the orbital angular momentum L of the systems, with the parameters $m_{d=ss} = 991$ MeV and the Regge slope $a = 0.318$ GeV^2 corresponding to our assignment (13). The (red) solid circles correspond to the observed (mean) masses

two lowest $1S$ states, the $\Omega_b(1/2^+)$ at 6046.1 ± 1.7 MeV and its partner $\Omega_b^*(3/2^+)$ with mass $6046.1 + \Delta E(\Omega_b, 1S)$. Here, $\Delta E(\Omega_b, 1S)$ stands for the $1S$ level-splitting between the $\Omega_b^*(3/2^+)$ and $\Omega_b(1/2^+)$, and can be estimated from the corresponding $1S$ level-splitting of the ground states Ω_c via the scaling [36],

$$\begin{aligned}\Delta E(\Omega_b) &= \left(\frac{M_c}{M_b}\right)\Delta E(\Omega_c)\\ &= \left(\frac{1.44}{4.48}\right)(71\text{ MeV})\\ &\simeq 23\text{ MeV}.\end{aligned}$$

which yields the spin-averaged mass in $1S$-wave:

$$\begin{aligned}\bar{M}(\Omega_b, 1S) &= \frac{1}{6}[6046.1 \times 2 + (6046.1 + 23) \times 4]\\ &= 6061.4\text{ MeV}.\end{aligned}$$

Hence, Eq. (31) disfavors the $2S$ state assignment for any of four observed Ω_b states, as the 2S state is too high in levels to accommodate all Ω_b states in Table 1. Similar analysis giving Eq. (24) applies to the P-wave bottom baryons Ω_b with the mean mass in Eq. (25) and gives $a = 0.318$ GeV^2. This leads to the mean masses in Table 4 for the Ω_b's, by which one can plot Regge trajectories (Fig. 2) for them.

5 Excited baryons Σ_c/Ξ'_c and Σ_b/Ξ'_b: P-wave masses

It is possible to apply the mixing Jls coupling to computation of the P-wave masses of the less strange heavy baryons (Σ_Q/Ξ'_Q), for which only a single measured mass available for each one of the odd-parity baryons, 2801 /2942.3 MeV for the $\Sigma_c(2800)/\Xi'_c(2942)$ and 6098.0/6226.9 MeV for the $\Sigma_b(6097)/\Xi'_b(6227)$ (Table 1, or see [1]). The basic idea for such a computation is to choose our previous estimates of the parameters $a_{1,2}$, $\bar{M}$ for the $1P$ states [38], as well as the following rough estimate of the parameter c(with $m_{nn} = 745$ MeV, $m_{ns} = 872$ MeV)

$$\text{Initial: } c(cqq) \approx c(css)\left(\frac{m_{ss}}{m_{qq}}\right) = \begin{cases}5.37\text{ MeV}, & \Sigma_c,\\ 4.59\text{ MeV}, & \Xi'_c,\end{cases} \tag{32}$$

with $c(css) = 4.04$ MeV given in Eq. (12), to be three initial inputs to give a b-dependent P-wave mass for each baryon using the nonlinear relation (7), and then find optimal fit of b as the initial inputs via matching b-dependent mass with the measured one for each baryons (Table 1). Finally, one evolves the obtained initial inputs via the relation (7) to find stable values of the five parameters. We list all initial parameters including $\bar{M}(1P)$ in the Table 7, where the two inputs $a_1 = 39.4$ MeV/33.6 MeV for the P-wave baryons Σ_c/Ξ'_c are scaled from that $a_2 = 26.8$ MeV/25.3 MeV for the $D_s(1P)$ masses in Ref. [38], and the mean-masses $\bar{M} = 2774.1$ MeV/2923.0 MeV for the P-wave baryons Σ_c/Ξ'_c are taken from Table 3 of Ref. [38], which are extrapolated from Regge-trajectory of the Λ_c/Ξ_c spectra.

For the P-wave bottom baryons Σ_b/Ξ'_b, corresponding initial inputs (Table 7) follow from the following estimations: the initial parameter c is estimated, by scaling from the Ω_c to the Ω_b and then to the bottom baryon bqq, to be about

$$\text{Initial: } c(bqq) \approx c(css)\left(\frac{M_c}{M_b}\right)\left(\frac{m_{ss}}{m_{qq}}\right) = \begin{cases}1.73\text{ MeV}, & \Sigma_b,\\ 1.48\text{ MeV}, & \Xi'_b,\end{cases} \tag{33}$$

with $M_c = 1.44$ GeV and $M_b = 4.48$ GeV [38]. The mass scaling $a_2(bss) \simeq a_2(css)(M_c/M_b)$ is used to estimate a_2 while the HQS relation $a_1(bqq) = a_1(cqq)$ in Ref. [36] needs to be corrected, as indicated by $a_1(bss) = 8.98 < 26.96 = a_1(css)$ and also by failure to find reasonable parameters if setting $a_1(\Sigma_b) = 39.4$ MeV$= a_1(\Sigma_c)$ or $a_1(\Xi'_b) = 33.6$ MeV$= a_1(\Xi'_c)$. We find that choosing initial values of $a_1(bqq)$ to be about one third of $a_1(cqq)$, as $a_1(bss) \simeq a_1(css)/3$ indicated, yields reasonable and stable values of the parameters during iteration of Eq. (7). In

Table 7 Initial parameters and mean masses (all in MeV) for the less strange heavy baryons ($\Sigma_{c,b}/\Xi'_{c,b}$)

Initial input	a_1	a_2	b	c	$\bar{M}(1P)$
Σ_c (MeV)	39.4	26.8	20.1	5.37	2774.1
Ξ'_c (MeV)	33.6	25.3	17.9	4.59	2923.0
Σ_b (MeV)	12.7	8.61	6.45	1.73	6088.4
Ξ'_b (MeV)	10.8	8.10	5.76	1.48	6248.2

Table 8 Masses and parameters (all in MeV) refined from the initial inputs in Table 7 via iteration of the nonlinear relation (7) for the less strange heavy baryons Σ_c/Ξ_c' and Σ_b/Ξ_b' in P-wave

State:	$\lvert\frac{1}{2},0'\rangle$, $\lvert\frac{1}{2},1'\rangle$, $\lvert\frac{3}{2},1'\rangle$, $\lvert\frac{3}{2},2'\rangle$, $\lvert\frac{5}{2},2'\rangle$	a_1	a_2	b	c	$\bar{M}\,(1P)$
Σ_c (MeV)	2668.4, 2723.1, 2757.3, 2801.0$^\diamondsuit$, 2826.6	39.96	21.75	20.70	7.85	2776.4
Ξ_c' (MeV)	2840.6, 2881.6, 2908.9, 2942.3$^\diamondsuit$, 2969.5	32.89	20.16	16.50	7.17	2925.9
Σ_b (MeV)	6053.9, 6071.8, 6082.8, 6098.0$^\diamondsuit$, 6104.8	12.99	6.42	6.45	1.73	6089.1
Ξ_b' (MeV)	6226.9$^\diamondsuit$, 6235.8, 6243.4, 6252.3, 6262.5	9.37	6.29	5.76	1.48	6249.1

the Table 8 ($\diamondsuit$ denotes mass observed), we show the results of iteration starting with the initial inputs in Table 7, with reasonable values of the parameters achieved. Note that the obtained values of b is about one half of that of a_1.

This way of parameter relaxation via the nonlinear mass relation overcome the difficulty in previous works [36,38] within the scheme of the jj coupling, that the masses of the baryons Σ_c/Ξ_c' and Σ_b/Ξ_b' remain undetermined due to unknown b: a simple comparison of the b-dependent mass predictions [36,38] with experimental masses yields unreasonable values of b except for the Σ_c. In our computation of the P-wave masses of the Σ_c/Ξ_c''s and Σ_b/Ξ_b''s, using of appropriate eigenstates, the bases of the Jls coupling here, is crucial to find stable parameters which keep them within a narrow range of a few MeV around the initial inputs in Table 7. We await upcoming measurement(mass and spin-parity) by LHCb and Belle to verify the predictions in Table 8.

Similar procedure can apply to the excited Ω_b, giving nearly same parameters and masses as Eqs. (25) and (26), only that the mass of the $J = 5/2$ state evolves to $M(5/2, 2') = 6352.2$ MeV, 0.7 MeV above that in Eq. (26), and c becomes a non-vanishing but small value of $c = 1.35$, as it should be.

6 Spin-couplings in relativized quark model

One may ask if the parameters of spin interaction (2) can be computed based on dynamics of quark interaction, as it is done for fine and hyperfine structures of the hydrogen-like atoms? In this section, we employ a QCD analogues of Breit-Fermi spin interaction (Appendix C), derived from the Lorentz-invariant quark-antiquark scattering amplitude at the tree level [37], to compute the spin-couplings (a_1, a_2, b, c) and compare them with the experimentally matched ones.

Including the spin-dependent forces (Breit-Fermi spin interaction), the quasi-static potential between quark 1 and antiquark 2 is [37,46],

$$V^{\text{quasi-static}} = V + S + \left(\frac{V'-S'}{r}\right)\mathbf{L}\cdot\left(\frac{\mathbf{S}_1}{2m_d^2}+\frac{\mathbf{S}_2}{2m_Q^2}\right) + \left(\frac{V'}{r}\right)\mathbf{L}\cdot\left(\frac{\mathbf{S}_1+\mathbf{S}_2}{m_1m_2}\right) + \frac{1}{3m_1m_2}\left(\frac{V'}{r}-V''\right)S_{12} + \frac{2}{3m_1m_2}\left(\nabla^2V\right)\mathbf{S}_1\cdot\mathbf{S}_2, \tag{34}$$

where V and S are the respective vector and scalar potentials, and V', S' and V'' their derivatives.

One way to relativize the instant potential $V^{\text{quasi-static}}$ is to use the replacements [46]

$$\begin{aligned} V(r) &\to \tilde{V}(r) = \left(\frac{m_1m_2}{E_1E_2}\right)^{1/2+\epsilon_V/2} V(r) \times \left(\frac{m_1m_2}{E_1E_2}\right)^{1/2+\epsilon_V/2}, \\ S(r) &\to \tilde{S}(r) = \left(\frac{m_1m_2}{E_1E_2}\right)^{1/2+\epsilon_S/2} S(r) \times \left(\frac{m_1m_2}{E_1E_2}\right)^{1/2+\epsilon_S/2}, \end{aligned} \tag{35}$$

where the relativistic factors $m_i/E_i = m_i/\sqrt{m_i^2+|\mathbf{p}|^2} = \sqrt{1-v_i^2}(i = 1, 2)$ rise from the length contraction $r \to r(m/E)$ when quark i moves relativistically, and these factors tend to unity in the low-momentum $(p/m_i \to 0)$ limit of the quarks. Here, $\epsilon_{V,S}$ are small homogeneous factors to be determined. Applying Eqs. (35) and (36) to a heavy quark Q in color triplet(3_c) and a diquark $d = qq(q = u, d, s)$ in the anti-color triplet($\bar{3}_c$), four spin-couplings become then

$$\begin{aligned} a_1 &= \frac{1}{2m_d}\left\langle\frac{\tilde{V}'-\tilde{S}'}{m_dr}+\frac{2\tilde{V}'}{m_Qr}\right\rangle, \\ a_2 &= \frac{1}{2m_Q}\left\langle\frac{\tilde{V}'-\tilde{S}'}{m_Qr}+\frac{2\tilde{V}'}{m_dr}\right\rangle, \\ b &= \frac{1}{3m_dm_Q}\left\langle\tilde{V}'/r-\tilde{V}''\right\rangle, \\ c &= \left\langle\frac{2\nabla^2\tilde{V}}{3m_dm_Q}\right\rangle, \end{aligned} \tag{36}$$

in which $V + S$ stands for the confining potential between Q and d, to be approximated by that for the heavy quarkonia $Q\bar{Q}$, and the quantum average $\langle\ \rangle$ is made over the P-wave wavefunction Ψ_{Qd} of the Qd system. For the spin interaction in Eq. (2), the relativistic treatment (35) implies that the factor $1/m_i$ in Eq. (36) becomes $1/E_i = (m_i^2 + p^2)^{-1/2}$, which

Table 9 Inputs for the model parameters and diquark masses defined in Hamiltonian (C1) (Appendix C) for the P-wave heavy baryons listed. The heavy quark masses $m_Q = M_Q = 1440$ MeV($Q = c$) and 4480 MeV($Q = b$), and light diquark masses m_d are taken from Ref. [38]. The confining potential parameter a is set by Regge trajectories of corresponding baryons, taken from Ref. [38] and Eq. (24) and Sect. 4

State	m_d	$a[\mathrm{GeV}^2]$	α_s	$\xi(\mathrm{GeV}^{-1})$	$\zeta(\mathrm{GeV}^{-2})$	ϵ_V	ϵ_S
Ω_c	991	0.316	0.561	0.818	0.11	-0.10	2.65
Ω_b		0.318	0.561				
Ξ_c'	872	0.255	0.590	0.820	0.12	-0.06	2.20
Ξ_b'		0.307	0.545				
Σ_c	745	0.212	0.595	0.850	0.12	-0.05	1.80
Σ_b		0.246	0.549				

makes sure, for example, that the hyperfine interactions of the light quark i do not blow up like $1/m_i$ in the chiral limit ($m_i \to 0$), but rather they should have a finite limit determined by $\langle p^{-1}\rangle$, which is in turn controlled by the radius of confinement.

To find P-wave wavefunction Ψ_{Qd} of singly heavy baryons (Qd) we consider the relativized quark model [46] with the color-coulomb plus linear potential $V + S$ (treating the spin interaction H^{SD} perturbatively)

$$H = \sqrt{m_Q^2 + p^2} + \sqrt{m_d^2 + p^2} + V(r) + S(r),$$
$$V = -k_s/r, S(r) = ar + C_0, \tag{37}$$

where $k_s = 4\alpha_s/3$ and α_s is the strong coupling.

By introducing auxiliary fields μ_d and ν, one can solve the model (37) using auxiliary field (AF) method [47] (Appendix C). The inputs of m_d, a, and α_s are given in Table 9, where the values of other parameters ξ, ζ and $\epsilon_{V,S}$ for computing the spin couplings are also listed. The solved P-wave radial wavefunction of the quark–diquark system are given in Fig. 3.

Upon using (36) to compute the spin couplings, we take into account non-point nature of the diquark d via adding a form factor $F(r) \equiv 1 - e^{-\xi r - \zeta r^2}$ [5] in the short-range potential V so that

$$V \to VF = -\frac{k_s}{r}F(r). \tag{38}$$

Putting the improved V in Eq. (38) into Eq. (36), in which the relativized potentials become now, by Eq. (35), $\tilde{V} = (m_d m_Q/(E_d^H E_Q^H))^{1+\epsilon_V} V$ and $\tilde{S} = (m_d m_Q/(E_d^N E_Q^N))^{1+\epsilon_S} S$ with $S = ar + C_0$ and

$$E_i^{H,N} = [m_i^2 + \langle p^2\rangle_{H,N}]^{1/2}, \tag{39}$$

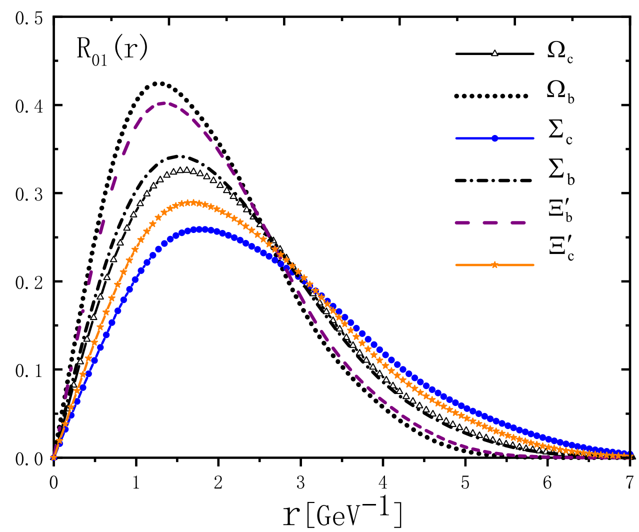

Fig. 3 The P-wave radial wavefunction $R_{nL}(r)$ with $n = 0$ and $L = 1$, numerically solved from Eq. (C8), with the inputs in Table 9

one finds

$$a_1 = \frac{1}{2m_d^2}\left\langle\left(1 + \frac{2m_d}{m_Q}\right)\left(\frac{m_d m_Q}{E_d^H E_Q^H}\right)^{1+\epsilon_V} k_s\left(\frac{F}{r^3} - \frac{F'}{r^2}\right)\right.$$
$$\left. - \left(\frac{m_d m_Q}{E_d^N E_Q^N}\right)^{1+\epsilon_S} \frac{a}{r}\right\rangle,$$
$$a_2 = \frac{1}{m_d m_Q}\left\langle\left(1 + \frac{m_d}{2m_Q}\right)\left(\frac{m_d m_Q}{E_d^H E_Q^H}\right)^{1+\epsilon_V} k_s\left(\frac{F}{r^3} - \frac{F'}{r^2}\right)\right.$$
$$\left. - \left(\frac{m_d m_Q}{E_d^N E_Q^N}\right)^{1+\epsilon_S} \frac{m_d a}{2m_Q r}\right\rangle, \tag{40}$$
$$b = \frac{k_s}{3m_d m_Q}\left(\frac{m_d m_Q}{E_d^H E_Q^H}\right)^{1+\epsilon_V}\left\langle\frac{3F}{r^3} - \frac{3F'}{r^2} + \frac{F''}{r}\right\rangle,$$
$$c = \frac{2k_s}{3m_d m_Q}\left(\frac{m_d m_Q}{E_d^H E_Q^H}\right)^{1+\epsilon_V}\left[4\pi|R_{nL}^H(0)|^2\right.$$
$$\left. - \int dr r F''|R_{nL}^H(r)|^2\right],$$

where we have used $V' = k_s(F/r^2 - F'/r)$, $V'' = k_s(2F'/r^2 - 2F/r^3 - F''/r)$, $S' = a$, and the Laplace relation $\nabla^2(F/r) = -4\pi\delta^3(\mathbf{r}) + F''/r$. Here, the average is taken over $R_{0L}(r)$, and for $n_r = 0$,

$$E_d^H = \mu_{dH},\, E_Q^H = \left[m_Q^2 + \frac{(k_s\mu_H)^2}{(L+1)^2}\right]^{1/2},$$
$$E_i^N = \left[m_i^2 + a\left(L+\frac{3}{2}\right)\sqrt{\mu/\nu}\right]^{1/2}, \quad i = Q, d, \tag{41}$$

where the average $\langle p^2\rangle_N = \alpha^2(L+3/2) = (a^2\mu/\nu)^{1/2}(L+3/2)$ in the HO wavefunction is used again.

The AF fields (μ, ν) are solvable from the nonlinear equation (C4) (Appendix C) given the inputs of the masses $m_Q = M_Q$ and m_d, and the linear potential parameter a extracted from Ref. [38], as listed in Table 9, by which Eq. (41) gives the inverse HO length α, and the relativistic factors $m_i/E_i^{H,N}$. Notice that two derivatives $F' = (2\zeta r + \xi)e^{-\xi r - \zeta r^2}$, $F'' = [2\zeta - (2\zeta r + \xi)^2]e^{-\xi r - \zeta r^2}$, one can then compute the spin-couplings in Eq. (40) via quantum averaging with $R_{nL}(r)$ solved from Eq. (C8) (Appendix C). The results are shown in Table 10, which agree reasonably with the parameters (a_1, a_2, b) given in Eqs. (12) and (25) as well as that in Table 8. One sees from Table 10 that our prediction for the spin couplings a_2 is slightly overestimated compared to that matched ones, about 4−8 MeV, and the predictions for c is anomaly larger than the matched ones, about 7−16 MeV.

One can list the resulted ratios of the spin couplings $r_{1,2} = a_{1,2}(bqq)/a_{1,2}(cqq)$ and $r_b = b(bqq)/b(cqq)$ between the bottom and charmed baryons and compare them to corresponding ratios (denoted as $[.]_{Match}$) obtained from Eqs. (12) and (25) and from Table 8:

ratio:	r_1	r_2	r_b	$[r_1]_{Match}$	$[r_2]_{Match}$	$[r_b]_{Match}$
Ω_b/Ω_c :	0.361	0.379	0.366	0.33	0.16	0.56,
Ξ'_b/Ξ'_c :	0.379	0.395	0.370	0.28	0.31	0.35,
Σ_b/Σ_c :	0.393	0.360	0.346	0.33	0.29	0.31.

(42)

One sees that the simple mass scaling law ($\sim M_b/M_c = 1440/4480 = 0.321$) holds approximately for a_2 and b. In the case of a_1, the scaling law $a_1(bqq) = a_1(cqq)$ [36] is broken somehow as $r_1 \sim 1/3$ is less than unity. Further, the matched value $r_2 = 0.16$ for the baryons Ω_b vs. Ω_c is somehow smaller than that expected by M_b/M_c and also than our computed value $r_2 = 0.379$.

From Table 11, which shows the computed values of the relevant factors and the averages in Eq. (40), one sees that the minus term $\sim -\langle m_d/m_Q\, a/r\rangle$ in Eq. (40) is notably suppressed and this gives relatively larger predictions for a_2. We remark three features of our computations:

(i) As a leading approximation, the Breit-Fermi-like interaction (34) have not considered nonperturbative corrections that may enter at long-distance regime. Assuming the spin interaction dominates at short distance, one expects that the interaction (34) includes the main spin-spin and spin-orbital forces via adding the scalar confining potential $S \sim r$ to the vector potential V, and such a nonperturbative corrections to Eq. (34) is relatively small.

(ii) Static approximation of quark–diquark potential $V+S$, which is valid in principle only for heavy quarkonia $(Q\bar{Q})$, has ignored rotation of QCD string when $L \neq 0$ [42,48,49]. In orbitally-excited state, the rotation of QCD string can change the static quark–diquark potential $V + S$ in Eq. (37), as shown in Refs. [48,49]. For instance, this corrects L-dependence of the Hamiltonian, $[m_d^2 + p_r^2 + L^2/r^2]^{1/2} \sim L$ into a Regge-like one, $H \sim \sqrt{\pi a L}$ due to addition of the orbital angular moment of the string [48].

(iii) A simple exponential form factor $F(r)$ may not be enough in the short-range to describe the three-body quark systems of $Q(qq)$. Three-body interaction [47,50] inside baryons may make $F(r)$ quite nontrivial as the effective running of the color-charge of a non-point diquark d in color-antitriplet $(\bar{3}_c)$ seen by Q in color-triplet (3_c) becomes highly distance-dependent when d is close to Q. See Ref. [50] (Eq. (18)) for other possible forms of the form factor, like $Erf[c_0 r]$ or its linear combinations. For simplicity, we used the $R_{nL}^H(r)$ in Eq. (C11) (Appendix C) to estimate c in Eq. (40) and this may overestimate values for c govern by integral of $-F''|R_{nL}^H(r)|^2$ at short-range. Note that $R_{nL}^H(0) \equiv 0$ for the P-waves.

In addition, Eq. (40) seems to imply the scaling laws $a_1 \sim \mathcal{O}(m_Q^0)$, $a_2 \sim 1/m_Q \sim b$ (see Eq. (24) in Ref. [36])), as heavy quark symmetry expected. However, the small variation of the averages within the baryons, manly in $1 + 2m_d/m_Q$ and $k_s = 4\alpha_s/3$ along with the size($\sim a_B$) of baryons, is enlarged by m_d^2 in denominator for a_1 in Eq. (40) and this makes a_1 changes notably between the bottom and charmed sectors. This kind of enlargement does not happen for other spin couplings due to their overall suppression factor $1/m_d m_Q$. Note also that the factor $1 + m_d/2m_Q$ for a_2 changes slightly with heavy flavor $Q(= c, b)$, compared to the variations of $1 + 2m_d/m_Q$:

$$\left(1+\frac{m_d}{2m_Q}\right)_{c,b} :\{1.34, 1.11\}_{ssQ}, \{1.30, 1.09\}_{snQ}, \{1.26, 1.08\}_{nnQ},$$
$$\left(1+\frac{2m_d}{m_Q}\right)_{c,b} :\{2.37, 1.44\}_{ssQ}, \{2.21, 1.39\}_{snQ}, \{2.04, 1.33\}_{nnQ},$$

As seen in Tables 10 and 11, the scaling law for the spin couplings breaks moderately, especially for a_1, due to the injured heavy quark symmetry indicated by Eq. (40).

Table 10 Spin coupling parameters a_1, a_2, b and c calculated by Eq. (40), the AF fields (μ, ν) solved from Eq. (C4) (Appendix C) and other related parameters for the P-wave states. All in MeV except for a

State	μ_d	μ	ν	μ_{dH}	μ_H	a_B[GeV^{-1}]	a_1	a_2	b	c
Ω_c	1291	681	909	1016	596	2.248	28.52	27.03	15.32	20.73
Ω_b	1379	1054	789	1034	840	1.701	10.30	10.25	5.61	9.26
Ξ'_c	1153	640	805	900	554	2.256	30.15	27.98	16.54	20.35
Ξ'_b	1273	992	787	912	757	1.882	11.42	11.06	6.14	9.82
Σ_c	1017	596	728	777	504	2.283	35.46	30.96	19.10	20.44
Σ_b	1117	894	703	781	665	2.130	13.93	11.15	6.61	8.93

Table 11 Values of the relevant factors (dimensionless) and the computed averages (in GeV^{-3}) for spin coupling $a_{1,2}$ and b in Eq. (40)

State	$\frac{m_d}{m_Q}$	$\left(\frac{m_d m_Q}{E_d^H E_Q^H}\right)^{1+\epsilon_V}$	$\left(\frac{m_d m_Q}{E_d^N E_Q^N}\right)^{1+\epsilon_S}$	$\left\langle\frac{m_d}{2m_Q}\frac{a}{r}\right\rangle$	$\left\langle\frac{F}{r^3}-\frac{F'}{r^2}\right\rangle$	$\left\langle\frac{a}{r}\right\rangle$	$\left\langle\frac{3F}{r^3}-\frac{3F'}{r^2}+\frac{F''}{r}\right\rangle$
Ω_c	0.688	0.884	0.227	0.0472	0.056	0.137	0.0994
Ω_b	0.220	0.866	0.276	0.0182	0.075	0.164	0.123
Ξ'_c	0.606	0.864	0.278	0.0309	0.049	0.102	0.090
Ξ'_b	0.195	0.861	0.278	0.0149	0.071	0.153	0.119
Σ_c	0.518	0.833	0.313	0.0204	0.044	0.079	0.085
Σ_b	0.167	0.849	0.306	0.0092	0.062	0.110	0.111

7 Conclusions and discussions

Exploring and building data set of the mean-spin mass $\bar{M}$ and the spin coupling strengths (a_1, a_2, b, c) is important as it can be a basis to explore how quarks interact within hadrons. Although baryon is a three-body system with subtle short-range interactions, it is useful to employ a simple heavy quark–diquark picture for the singly heavy baryons to understand the measured mass data of excited heavy-baryons. In this work, we have proposed a new scheme of state classification, the Jls mixing coupling, to unify analysis of masses and inner-structures of the excited baryons Ω_Q and other less-strange heavy baryons with spin-1 light diquarks. Firstly, we interpret the five Ω_c and four Ω_b excited states reported by LHCb to be the relative P-wave excitations between spin-1 ss diquark and the heavy quark, the charm quark c for the Ω_c and the bottom quark b for the Ω_b, respectively. Further analysis via Regge phenomenology for the low-lying excited states excludes the possibility that some of higher states of the resonances Ω_c and Ω_b are the $2S$ states since the $2S$ candidates are all too heavy to accommodate both of the resonances. Secondly, we applied our formalism based on the Jls mixing coupling to update the earlier computation of all P-wave masses of the less strange baryons Σ_c/Ξ'_c and Σ_b/Ξ'_b, for which only single measured mass available for each of these baryons. The masses of all spin-partners of the baryons $\Sigma_c(2800)/\Xi'_c(2942)$ and $\Sigma_b(6097)/\Xi'_b(6227)$ are then predicted. Finally, we demonstrated that the matched spin-couplings of the singly heavy baryons considered can be understood in a relativized potential quark model, based on which the scaling law for their spin coupling parameters are discussed.

With the help of the relation (7), we find that four measured masses for the excited Ω_b's are enough to make a quantitative prediction $M(5/2^-) = 6352$ MeV for one unseen state with $J^P = 5/2^-$. The missing of this $5/2^-$ state in LHCb experiment can be due to the degeneracy with the nearby state $|3/2^-, 2'\rangle$ at 6350 MeV. The same $J^P = 5/2^-$ state is predicted as the unseen state in a recent assignment [33], with two higher mass ranges predicted, about 10 MeV away or more.

One very recent prediction for the odd-parity Ω_b's is that by the LS coupling [51]:

$$
\begin{aligned}
&M(1/2^-) = 6314, 6330 \text{ MeV},\\
&M(3/2^-) = 6339, 6342 \text{ MeV},\\
&M(5/2^-) = 6352 \text{ MeV},
\end{aligned}
\tag{43}
$$

for which the explanation of near degeneracy applies only to two $J = 3/2$ states and assuming one of them to be unseen is favored. In the viewpoint of this work, with given mixing weights in Eq. (27), two LS bases $|^2P_{3/2}\rangle$ and $|^4P_{3/2}\rangle$ are not eigenstates of the Hamiltonian (2) for $J = 3/2$ and tend to mix deeply to form the true Jls eigenstates $|3/2, 1'\rangle$ and $|3/2, 2'\rangle$, with respective masses 6337.8 MeV and 6350.0 MeV in Eq. (26) both matching the measured masses(6339.7 MeV and 6349.9 MeV) nicely. Our mass predictions in Eq. (26) for other states($J = 3/2, 5/2$) should agree with that by the LS coupling because the bases for the LS and Jls couplings are same($J = 5/2$) or roughly same($J = 3/2$), as two predictions (Eq. (43) vs. Eq. (26)) indicated. This favors the state at $M(5/2^-) = 6352$ MeV to be the unseen state.

We remark that the Jls scheme of state classification is quite useful in spin-parity assignment of the excited P-wave heavy baryons as it takes into account recoil of the heavy quark properly by diagonalizing dominate part of spin interaction rigorously. Thus, the new scheme, compared to the jj coupling, is more suitable for the (J^P) quantum number assignment when the finite mass effects of heavy quark becomes important, e.g., when the heavy baryons are singly or doubly strange. Based on the "recovery" check (Tables 2 and 3) we confirm that the mass splitting (7) in terms of the Jls coupling scheme gives a reasonable mass hierarchy compatible with heavy quark symmetry.

Our computation via a simple heavy quark–diquark picture indicates that the measured masses of the excited heavy-baryons considered can be understood via a relativized potential quark model and resulted Breit-Fermi spin-orbit and tensor forces, provided that a proper relativizing is made to these spin interactions(forces). By the way, we confirm based on the relativized expression of the Breit-Fermi formula (40) for the spin coupling parameters that a mass scaling law [36] for them holds approximately for all baryons considered. Being at the order of $\mathcal{O}(1)$, the scaling relation $a_1(bqq) = a_1(cqq)$ for a_1 is broken most, in contrast with other parameters at the order of $\mathcal{O}(1/m_Q)$. Due to involved QCD dynamics of the heavy flavor baryons (see [5,35] for instance), our analysis for the spin-couplings is of quite approximative, with further quantitative corrections remained to be explored.

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Appendix A: Derivation for mass-shifts in P-waves

For a heavy quark–diquark system Qd, the bases of the mixing Jls coupling are the eigenfunctions of the mass operator $H^{mix} = \mathbf{L}\cdot\mathbf{S}_d + \epsilon\mathbf{L}\cdot\mathbf{S}_Q + b_1 S_{12}$, with $b_1 \equiv b/a_1$. They can be obtained by diagonalizing H^{mix}, that is, diagonalizing the 2×2 matrices in Eqs. (4) and (5) separately in the subspaces of $J = 1/2$, $J = 3/2$. When $J = 1/2$, the matrix

$$H_{J=1/2} = \begin{bmatrix} \frac{1}{3}(\epsilon - 4) & \frac{\sqrt{2}}{3}(\epsilon - 1) + \frac{b_1}{\sqrt{2}} \\ \frac{\sqrt{2}}{3}(\epsilon - 1) + \frac{b_1}{\sqrt{2}} & -\frac{5}{3}(1 + \frac{\epsilon}{2}) - b_1 \end{bmatrix} \tag{A1}$$

can be diagonalized to give $M_{1/2} = \langle H_{J=1/2}\rangle$. Thus, adding a perturbative correction $\sim c_1 M_c$ to $M_{1/2}$, with

$$M_c = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \tag{A2}$$

leads to the mass shifts in the $J = 1/2$ subspace:

$$\begin{aligned} \Delta M(1/2, 0')/a_1 &= -\frac{3}{2} - \frac{1}{2}b_1 - \frac{1}{4}\epsilon - \frac{1}{4}\sqrt{\Delta_1(\epsilon, b_1)} \\ &\quad + c_1\Delta_3^+, \\ \Delta M(1/2, 1')/a_1 &= -\frac{3}{2} - \frac{1}{2}b_1 - \frac{1}{4}\epsilon + \frac{1}{4}\sqrt{\Delta_1(\epsilon, b_1)} \\ &\quad + c_1\Delta_3^-, \end{aligned} \tag{A3}$$

with the functions $\Delta_1(\epsilon, x)$, $\Delta_3^{\pm}(\epsilon, x)$ defined in Eqs. (8) and (9).

In the $J = 3/2$ subspace, a similar diagonalization of $H_{J=3/2}$ which gives $M_{3/2}$, leads to, by adding a perturbative correction $\sim c_1 M_c$,

$$\begin{aligned} \Delta M(3/2, 1')/a_1 &= \frac{2}{5}b_1 - \frac{1}{4}\epsilon - \sqrt{\Delta_2(\epsilon, b_1)} + c_1\Delta_4^+, \\ \Delta M(3/2, 2')/a_1 &= \frac{2}{5}b_1 - \frac{1}{4}\epsilon + \sqrt{\Delta_2(\epsilon, b_1)} + c_1\Delta_4^-, \end{aligned} \tag{A4}$$

with the functions $\Delta_2(\epsilon, x)$ and $\Delta_4^{\pm}(\epsilon, x)$ defined in Eqs. (8) and (10). In the $J = 5/2$ subspace, $\Delta M/a_1$ is simply $\Delta M(5/2, 2')/a_1 = 1 - b_1/5 + \epsilon/2 + c_1/2$. This, combined with Eqs. (A3) and (A4), proves Eq. (7). A recent Ref. [40] gives an alternative expression similar to the mass-shifts (7).

Appendix B: Base transformations between jj and LS couplings

The coefficients of Eq. (14) form normalized vectors of the eigenstates of the operator H^{mix}. In the subspaces of $J = 1/2$, H^{mix} becomes the matrix $H_{J=1/2}$ in Eq. (A1) with the eigenstates given in Eq. (A3), which are $\mathbf{v}_1 = \{-0.208, 1\}$ and $\mathbf{v}_2 = \{4.804, 1\}$ using Eq. (12). Upon normalization, the eigenstates give rise to the coefficients in the first and second lines of Eq. (14). Similar computation leads to the coefficients in the third and fourth lines of Eq. (14).

To write the mixing bases $|J, j'\rangle$ in terms of the jj bases $|J, j\rangle$, one writes firstly the P-wave $|J, j\rangle$ in terms of the $L-S$ eigenstates $|1^{2S+1}P_J\rangle$, as given in Eq. (15). Its inverse is

$$
\begin{aligned}
|1^2P_{1/2}\rangle &= \sqrt{\frac{1}{3}}|1/2, j=0\rangle + \sqrt{\frac{2}{3}}|1/2, j=1\rangle,\\
|1^4P_{1/2}\rangle &= \sqrt{\frac{2}{3}}|1/2, j=0\rangle - \sqrt{\frac{1}{3}}|1/2, j=1\rangle. \qquad \text{(B1)}
\end{aligned}
$$

Putting it into the first two lines of Eq. (14) gives the first and second lines of Eq. (16). Similarly, one can obtain the third and fourth lines of Eq. (16) with the help of

$$
\begin{aligned}
|J=3/2, j=1\rangle &= \sqrt{\frac{1}{6}}|1^2P_{3/2}\rangle + \sqrt{\frac{5}{6}}|1^4P_{3/2}\rangle,\\
|J=3/2, j=2\rangle &= \sqrt{\frac{5}{6}}|1^2P_{3/2}\rangle - \sqrt{\frac{1}{6}}|1^4P_{3/2}\rangle, \qquad \text{(B2)}
\end{aligned}
$$

and its inverse transformation.

Appendix C: Details for solving Eq. (37) for P-wave wavefunctions

Using auxiliary field (AF) method [47], which re-expresses nonlinear operator $\sqrt{O} = \min_{\lambda>0}[O/(2\lambda) + \lambda/2]$ in terms of the AF field λ, for which the minimization is achieved when $\lambda = \sqrt{O}$, one can rewrite the model (37) as

$$
\begin{aligned}
H &= m_Q + \frac{p^2}{2m_Q} + \frac{m_d^2 + p^2}{2\mu_d} + \frac{(ar)^2}{2\nu} + \frac{\mu_d + \nu}{2}\\
&\quad - \frac{k_s}{r} + C_0, \qquad \text{(C1)}\\
\mu_d &= \left\langle \sqrt{m_d^2 + p^2} \right\rangle, \nu = a\langle r\rangle, \qquad \text{(C2)}
\end{aligned}
$$

where $p = |\mathbf{p}|$ is the 3-momentum of the heavy quark Q and the light diquark d, and the minimization conditions (C2) for the auxiliary fields (μ_d, ν) are assumed to hold at semi-classical level. In a sense, the AF method linearizes the Hamiltonian H in (37) in a manner similar to mean-field method for many-body systems. Next, we solve the P-wave radial wavefunction R_{nL} of the Qd system defined by Eq. (37) numerically in two steps.

In the first step, we find two self-consistent AF fields (μ_d, ν) via solving Eq. (C2). For this, we ignore the (color) Coulomb term $-k_s/r$ to write the Hamiltonian (C1) as that of three dimensional harmonic oscillator (HO) with reduced mass $\mu \equiv \mu_d/(1+\mu_d/m_Q)$, the HO frequency $\omega = a/\sqrt{\mu\nu}$, the inverse HO length $\alpha = \sqrt{\mu\omega} = (a^2\mu/\nu)^{1/4}$. As such, the wavefunction Ψ_{Qd} is given by that of the HO, $\psi^N = R^N_{n_rL}(r)Y_{Lm}(\theta, \varphi)$ with quantum number $N = 2n_r + L$, where Y_{Lm} is the spherical function and

$$
\begin{aligned}
R^N_{n_rL}(r) &= \alpha^{3/2}\left[\frac{2n_r!}{\Gamma(n_r + L + 3/2)}\right]^{1/2} (\alpha r)^L L^{L+1/2}_{n_r}\\
&\quad \times (\alpha^2 r^2) e^{-(\alpha r)^2/2}, \qquad \text{(C3)}
\end{aligned}
$$

with $L^{2L+1}_{n_r}$ the associated Laguerre polynomial. For the orbital states with $n_r = 0$ we address, one finds from Eq. (C2),

$$
\begin{aligned}
\mu_d &\simeq \sqrt{m_d^2 + \langle p^2\rangle_N} = [m_d^2 + \alpha^2(L + 3/2)]^{1/2},\\
\nu &\simeq a\sqrt{\langle r^2\rangle_N} = a\left[\frac{L + 3/2}{\mu\omega}\right]^{1/2},
\end{aligned}
$$

where the HO-state averages of p^2 and r^2 are used. This leads to two nonlinear equations

$$
\begin{aligned}
\mu &\equiv \frac{[m_d^2 + a(L + 3/2)(\mu/\nu)^{1/2}]^{1/2}}{1 + [m_d^2 + a(L + 3/2)(\mu/\nu)^{1/2}]^{1/2}/m_Q},\\
\nu &= \sqrt{a(L + 3/2)}(\nu/\mu)^{1/4}, \qquad \text{(C4)}
\end{aligned}
$$

which can be solved by numerical iteration. The solution for (μ, ν) and $\mu_d \equiv \mu/(1 - \mu/m_Q)$ are shown in Table 1.

The solution of the wavefunction Ψ_{Qd} can be solved from the Hamiltonian (C1) of the two-body Qd, which is

$$
\begin{aligned}
H &= \frac{p^2}{2\mu} + \frac{1}{2}m\omega^2 r^2 - \frac{k_s}{r} + m_Q + \frac{m_d^2}{2\mu_d}\\
&\quad + \frac{\mu_d + \nu}{2} + C_0, \qquad \text{(C5)}\\
\mu &= \mu_d\Big/\left[1 + \frac{\mu_d}{m_Q}\right], \omega = \frac{a}{\sqrt{\mu\nu}}. \qquad \text{(C6)}
\end{aligned}
$$

In the long-range where the color-Coulomb term $-k_s/r$ is ignorable, the radial eigenfunctions of the Hamiltonian (C5) can be given by that of the HO wavefunction,

$$
\begin{aligned}
R^N_{n_rL}(r) &= \alpha^{3/2}\left[\frac{2n_r!}{\Gamma(n_r + L + 3/2)}\right]^{1/2} (\alpha r)^L L^{L+1/2}_{n_r}\\
&\quad \times (\alpha^2 r^2) e^{-(\alpha r)^2/2}, \qquad \text{(C7)}
\end{aligned}
$$

with $L^{2L+1}_{n_r}$ the associated Laguerre polynomial.

In the second step, we use the obtained (μ, ν) to numerically solve the eigenstate equation for the singly heavy baryons Ω_Q, Ξ'_Q and $\Sigma_Q(Q=c,b)$

$$\left[-\frac{1}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{L(L+1)}{2\mu r^2}+\frac{a^2}{2\nu}r^2-\frac{k_s}{r}\right]R_{nL} = ER_{nL}, \tag{C8}$$

where $R_{nL}(r)$ are the radial wavefunctions of these systems. The results for the P-wave solutions $R_{nL}(r)$ are shown in Fig. 3.

It is useful to write the radial wavefunction $R^H_{n_rL}(r)$ in the small-r range, which is given by solution to the eigen-equation of the Hamiltonian (C1) with the HO potential($\sim r^2$) ignored. It is the solution to the hydrogen-like atoms with the effective Bohr radius $a_B = 1/(\mu_H k_s)$ and the reduced mass $\mu_H = \mu_{dH}/(1+\mu_{dH}/m_Q)$ with $\mu_{dH} = \left\langle\sqrt{m_d^2+p^2}\right\rangle_H$ averaged in the hydrogen-like wavefunction $\psi^H = R^H_{n_rL}(r)Y_{Lm}(\theta,\varphi)$, with

$$R^H_{n_rL}(r) = \left\{\left(\frac{2}{na_B}\right)^3\frac{n_r!}{2n[(n+L)!]^3}\right\}^{1/2} e^{-r/(na_B)} \times\left(\frac{2r}{na_B}\right)^L L^{2L+1}_{n_r}\left(\frac{2r}{na_B}\right), \tag{C9}$$

with $n = n_r + L + 1$. Using averaging in the hydrogen-like wavefunction ψ^H with $n_r = 0$, $\langle p^2\rangle_H = k_s\mu_H/[a_B(L+1)^2] = k_s^2\mu_H^2/(L+1)^2$, one finds that the AF parameter $\mu_{dH} \simeq \sqrt{m_d^2+\langle p^2\rangle_H}$ is subjected to the self-consistent equation,

$$\mu_{dH} = \left[m_d^2+\frac{1}{(L+1)^2}\frac{(k_s\mu_{dH})^2}{(1+\mu_{dH}/m_Q)^2}\right]^{1/2}. \tag{C10}$$

The numerical results for μ_{dH} and thereby-resulted μ_H are shown in Table 5 also.

In the short-range where the term $-k_s/r$ is dominate over the HO potential, the radial solution of wavefunction to the Hamiltonian (C5) is that of the hydrogen-like:

$$R^H_{n_rL}(r) = \left\{\left(\frac{2}{na_B}\right)^3\frac{n_r!}{2n[(n+L+1)!]^3}\right\}^{1/2} e^{-r/(na_B)} \times\left(\frac{2r}{na_B}\right)^L L^{2L+1}_{n_r}\left(\frac{2r}{na_B}\right), \tag{C11}$$

with $n = n_r + L + 1$.

References

1. P.A. Zyla et al. (Particle Data Group), Review of particle physics. Prog. Theor. Exp. Phys. **2020**, 083C01 (2020)
2. R. Aaij et al. (LHCb Collaboration), Observation of five new narrow Ω_c^0 states decaying to ΞK^-. Phys. Rev. Lett. **118**, 182001 (2017). https://doi.org/10.1103/PhysRevLett.118.182001. arXiv:1703.04639 [hep-ex]
3. R. Aaij et al. (LHCb Collaboration), First observation of excited Ω_b^- states. Phys. Rev. Lett. **124**, 082002 (2020). https://doi.org/10.1103/PhysRevLett.124.082002. arXiv:2001.00851 [hep-ex]
4. D. Ebert, R.N. Faustov, V.O. Galkin, Masses of excited heavy baryons in the relativistic quark model. Phys. Lett. B **659**, 612 (2008). https://doi.org/10.1016/j.physletb.2007.11.037. arXiv:0705.2957 [hep-ph]
5. D. Ebert, R.N. Faustov, V.O. Galkin, Spectroscopy and Regge trajectories of heavy baryons in the relativistic quark–diquark picture. Phys. Rev. D **84**, 014025 (2011). https://doi.org/10.1103/PhysRevD.84.014025. arXiv:1105.0583 [hep-ph]
6. K. Maltman, N. Isgur, Baryons with strangeness and charm in a quark model with chromodynamics. Phys. Rev. D **22**, 1701 (1980). https://doi.org/10.1103/PhysRevD.22.1701
7. W. Roberts, M. Pervin, Heavy baryons in a quark model. Int. J. Mod. Phys. A **23**, 2817(2008). https://doi.org/10.1142/S0217751X08041219. arXiv:0711.2492 [nucl-th]
8. H. Garcilazo, J. Vijande, A. Valcarce, Faddeev study of heavy baryon spectroscopy. J. Phys. G **34**, 961 (2007). https://doi.org/10.1088/0954-3899/34/5/014. arXiv:hep-ph/0703257
9. S. Migura, D. Merten, B. Metsch, H.R. Petry, Charmed baryons in a relativistic quark model. Eur. Phys. J. A **28**, 41 (2006). https://doi.org/10.1140/epja/i2006-10017-9. arXiv:hep-ph/0602153
10. A. Valcarce, H. Garcilazo, J. Vijande, Towards an understanding of heavy baryon spectroscopy. Eur. Phys. J. A **37**, 217 (2008). https://doi.org/10.1140/epja/i2008-10616-4. arXiv:0807.2973 [hep-ph]
11. Y. Yamaguchi, S. Ohkoda, A. Hosaka, T. Hyodo, S. Yasui, Heavy quark symmetry in multihadron systems. Phys. Rev. D **91**, 034034 (2015). https://doi.org/10.1103/PhysRevD.91.034034. arXiv:1402.5222 [hep-ph]
12. P. Pérez-Rubio, S. Collins, G.S. Bali, Charmed baryon spectroscopy and light flavor symmetry from lattice QCD. Phys. Rev. D **92**, 034504 (2015). https://doi.org/10.1103/PhysRevD.92.034504. arXiv:1503.08440 [hep-lat]
13. Z. Shah, K. Thakkar, A.K. Rai, P.C. Vinodkumar, Mass spectra and Regge trajectories of Λ_c, Σ_c^0, Ξ_c^0 and Ω_c^0 baryons. Chin. Phys. C **40**, 123102 (2016). https://doi.org/10.1088/1674-1137/40/12/123102. arXiv:1609.08464 [nucl-th]
14. T. Yoshida, E. Hiyama, A. Hosaka, M. Oka, K. Sadato, Spectrum of heavy baryons in the quark model. Phys. Rev. D **92**, 114029 (2015). https://doi.org/10.1103/PhysRevD.92.114029. arXiv:1510.01067 [hep-ph]
15. H.-X. Chen, W. Chen, Q. Mao, A. Hosaka, X. Liu, S.L. Zhu, P-wave charmed baryons from QCD sum rules. Phys. Rev. D **91**, 054034 (2015). https://doi.org/10.1103/PhysRevD.91.054034. arXiv:1502.01103 [hep-ph]
16. H.-X. Chen, Q. Mao, A. Hosaka, X. Liu, S.L. Zhu, D-wave charmed and bottomed baryons from QCD sum rules. Phys. Rev. D **94**, 114016 (2016). https://doi.org/10.1103/PhysRevD.94.114016. arXiv:1611.02677 [hep-ph]
17. M. Karliner, J.L. Rosner, Very narrow excited Ω_c baryons. Phys. Rev. D **95**, 114012 (2017). https://doi.org/10.1103/PhysRevD.95.114012. arXiv:1703.07774 [hep-ph]
18. M. Padmanath, N. Mathur, Quantum numbers of recently discovered Ω_c^0 baryons from lattice QCD. Phys. Rev. Lett. **119**, 042001 (2017). https://doi.org/10.1103/PhysRevLett.119.042001. arXiv:1704.00259 [hep-ph]

19. B. Chen, X. Liu, New Ω_c^0 baryons discovered by LHCb as the members of $1P$ and $2S$ states. Phys. Rev. D **96**, 094015 (2017). https://doi.org/10.1103/PhysRevD.96.094015. arXiv:1704.02583 [hep-ph]
20. W. Wang, R.L. Zhu, Interpretation of the newly observed Ω_c^0 resonance. Phys. Rev. D **96**, 014024 (2017). https://doi.org/10.1103/PhysRevD.96.014024. arXiv:1704.00179 [hep-ph]
21. Z.G. Wang, Analysis of $\Omega_c(3000)$, $\Omega_c(3050)$, $\Omega_c(3066)$, $\Omega_c(3090)$ and $\Omega_c(3119\ \ )$ with QCD sum rules. Eur. Phys. J. C **77**, 325 (2017). https://doi.org/10.1140/epjc/s10052-017-4895-5. arXiv:1704.01854 [hep-ph]
22. S.S. Agaev, K. Azizi, H. Sundu, On the nature of the newly discovered Ω_c^0 states. Eur. Phys. Lett. **118**, 61001 (2017). https://doi.org/10.1209/0295-5075/118/61001. arXiv:1703.07091 [hep-ph]
23. H.X. Chen, Q. Mao, W. Chen, A. Hosaka, X. Liu, S.L. Zhu, Decay properties of P-wave charmed baryons from light-cone QCD sum rules. Phys. Rev. D **95**, 094008 (2017). https://doi.org/10.1103/PhysRevD.95.094008. arXiv:1703.07703 [hep-ph]
24. H.Y. Cheng, C.W. Chiang, Quantum numbers of Ω_c states and other charmed baryons. Phys. Rev. D **95**, 094018 (2017). https://doi.org/10.1103/PhysRevD.95.094018. arXiv:1704.00396 [hep-ph]
25. K.L. Wang, L.Y. Xiao, X.H. Zhong, Q. Zhao, Understanding the newly observed Ω_c states through their decays. Phys. Rev. D **95**, 116010 (2017). https://doi.org/10.1103/PhysRevD.95.116010. arXiv:1703.09130 [hep-ph]
26. Z. Zhao, D.D. Ye, A. Zhang, Hadronic decay properties of newly observed Ω_c baryons. Phys. Rev. D **95**, 114024 (2017). https://doi.org/10.1103/PhysRevD.95.114024. arXiv:1704.02688 [hep-ph]
27. G. Yang, J. Ping, The structure of pentaquarks Ω_c^0 in the chiral quark model. Phys. Rev. D **97**, 034023 (2018). https://doi.org/10.1103/PhysRevD.97.034023. arXiv:1703.08845 [hep-ph]
28. H. Huang, J. Ping, F. Wang, Investigating the excited Ω_c^0 states through $\Xi_c K$ and $\Xi_c' K$ decay channels. Phys. Rev. D **97**, 034027 (2018). https://doi.org/10.1103/PhysRevD.97.034027. arXiv:1704.01421 [hep-ph]
29. C.S. An, H. Chen, The observed Ω_c^0 resonances as pentaquark states. Phys. Rev. D **96**, 034012 (2017). https://doi.org/10.1103/PhysRevD.96.034012. arXiv:1705.08571 [hep-ph]
30. Praszałowicz, Possibility of the existence of charmed exotica. Phys. Rev. D **96**, 014009(2017). https://doi.org/10.1103/PhysRevD.96.014009. arXiv:1704.04082 [hep-ph] [Erratum: Phys. Rev. D **96**, 039902(E) (2017)]
31. H.C. Kim, M.V. Polyakov, M. Praszalowicz, G.S. Yang, Strong decays of exotic and non-exotic heavy baryons in the chiral quark-soliton model. Phys. Rev. D **96**, 094021 (2017). https://doi.org/10.1103/PhysRevD.96.094021. arXiv:1709.04927 [hep-ph]
32. B. Chen, K.W. Wei, X. Liu, T. Matsuki, Low-lying charmed and charmed-strange baryon states. Eur. Phys. J. C **77**, 154 (2017). https://doi.org/10.1140/epjc/s10052-017-4708-x. arXiv:1609.07967 [hep-ph]
33. M. Karliner, J.L. Rosner, Interpretation of excited Ω_b signals. Phys. Rev. D **102**, 014027 (2020). https://doi.org/10.1103/PhysRevD.102.014027. arXiv:2005.12424 [hep-ph]
34. K. Thakkar, Z. Shahb, A. KumarRaib, P.C. Vinodkumarc, Excited state mass spectra and Regge trajectories of bottom baryons. Nucl. Phys. A **965**, 57 (2017). https://doi.org/10.1016/j.nuclphysa.2017.05.087
35. A.V. Manohar, M.B. Wise, *Heavy quark physics* (Cambridge University Press, Cambridge, 2000), p. 45
36. M. Karliner, J.L. Rosner, Prospects for observing the lowest-lying odd-parity Σ_c and Σ_b baryons. Phys. Rev. D **92**, 074026 (2015). https://doi.org/10.1103/PhysRevD.92.074026. arXiv:1506.01702 [hep-ph]
37. F.J. Ynduráin, *The Theory of Quark and Gluon Interactions* (Springer, New York, 1983), p. 261
38. D. Jia, W.-N. Liu, A. Hosaka, Regge behaviors in orbitally excited spectroscopy of charmed and bottom baryons. Phys. Rev. D **101**, 034016 (2020). https://doi.org/10.1103/PhysRevD.101.034016. arXiv:1907.04958 [hep-ph]
39. L.D. Landau, E.M. Lifshitz, *Quantum Mechanics (Nonrelativistic Theory)*, 3rd edn. (Pergamon Press, Oxford, 1977), p. 269
40. A. Ali, I. Ahmed, M.J. Aslam, A.Ya. Parkhomenko, A. Rehman, Mass spectrum of the hidden-charm pentaquarks in the compact diquark model. JHEP **10**, 256 (2019). https://doi.org/10.1007/JHEP10(2019)256. arXiv:1907.06507 [hep-ph]
41. L.Y. Xiao, K.L. Wang, M.S. Liu, X.H. Zhong, Possible interpretation of the newly observed Ω_b states. Eur. Phys. J. C **80**, 279 (2020). https://doi.org/10.1140/epjc/s10052-020-7823-z. arXiv:2001.05110 [hep-ph]
42. S.S. Agaev, K. Azizi, H. Sundu, Interpretation of the new Ω_c^0 states via their mass and width. Eur. Phys. J. C **77**, 395 (2017). https://doi.org/10.1140/epjc/s10052-017-4953-z. arXiv:1704.04928 [hep-ph]
43. D. Jia, W.-C. Dong, Regge-like spectra of excited singly heavy mesons. Eur. Phys. J. Plus **134**, 123 (2019). https://doi.org/10.1140/epjp/i2019-12474-8. arXiv:1811.04214 [hep-ph]
44. D. Jia, W.-C. Dong, A. Hosaka, Regge-like spectra of singly heavy hadrons, in *Proceedings of the 8th International Conference on Quarks and Nuclear Physics (QNP2018), JPS Conference Proceedings*, vol. 26 (2019), p. 022021
45. B. Chen, K.W. Wei, A. Zhang, Assignments of Λ_Q and Ξ_Q baryons in the heavy quark-light diquark picture. Eur. Phys. J. A **51**, 82 (2015). https://doi.org/10.1140/epja/i2015-15082-3. arXiv:1406.6561 [hep-ph]
46. S. Godfrey, N. Isgur, Mesons in a relativized quark model with chromodynamics. Phys. Rev. D **32**, 189 (1985). https://doi.org/10.1103/physrevd.32.189
47. C. Semay, F. Buisseret, F. Stancu, Charm and bottom baryon masses in the combined $1/N_c$ and $1/m_Q$ expansion versus the quark model. Phys. Rev. D **78**, 076003 (2008). https://doi.org/10.1103/PhysRevD.78.076003. arXiv:0808.3349 [hep-ph]
48. D. LaCourse, M.G. Olsson, The string potential model. 1. Spinless quarks. Phys. Rev. D **39**, 2751 (1989). https://doi.org/10.1103/PhysRevD.39.2751
49. Y.S. Kalashnikova, A.V. Nefediev, Heavy-light mesons spectrum from the nonperturbative QCD in the einbein field formalism. Phys. Lett. B **492**, 91 (2000). https://doi.org/10.1016/S0370-2693(00)01063-7. arXiv:hep-ph/0008242
50. S. Capstick, N. Isgur, Baryons in a relativized quark model with chromodynamics. Phys. Rev. D **34**, 2809 (1986). https://doi.org/10.1103/PhysRevD.34.2809
51. H. Mutuk, A study of excited Ω_b states in hypercentral constituent quark model via artificial neural network. Eur. Phys. J. A **56**, 146 (2020). https://doi.org/10.1140/epja/s10050-020-00161-5. arXiv:2002.03695 [hep-ph]