Approximate neutrino oscillations in the vacuum

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Abstract It is well known that neutrino oscillations may damp due to decoherence caused by the separation of mass eigenstate wave packets or by a baseline uncertainty of order the oscillation wave length. In this note we show that if the particles created together with the neutrino are not measured and do not interact with the environment, then the first source of decoherence is not present. This demonstration uses the saddle point approximation and also assumes that the experiment lasts longer than a certain threshold. We independently derive this result using the external wave packet model and also using a model in which the fields responsible for neutrino production and detection are treated dynamically. Intuitively this result is a consequence of the fact that the neutrino emission time does not affect the final state and so amplitudes corresponding to distinct emission times must be added coherently. This fact also implies that oscillations resulting from mass eigenstates which are detected simultaneously arise from neutrinos which were not created simultaneously but are nonetheless coherent, realizing the neutrino oscillation paradigm of Kobach, Manohar and McGreevy.

1 Introduction

Surprisingly, neutrino oscillations in the vacuum have received less attention than those in a medium. This is because interactions of the particles responsible for neutrino production with the environment are implicit in standard treatments in a sense that we will now review. Historically, the first neutrino oscillation calculations treated neutrinos as monochromatic plane waves. Clearly this approach is inconsistent as plane waves are homogeneous and so oscillation minima and maxima will be superimposed. However it was eventually understood that the plane waves are just a mnemonic for a true description in

terms of wave packets, which are no longer monochromatic and so may be spatially localized. The spatial localization allows for oscillations and decoherence as desired. But how does the spatial localization arise?

The spatial localization of a neutrino wave packet is equivalent to a space-time localization of the neutrino production [1]. This can be done if the trajectory of the source particle, say a pion, is known as well as the trajectories of the other particles produced along with the neutrino, such as the muon. In practice, these source final state particles are never measured, so how does the localization arise? The answer is given in Refs. [2,3]. The source final state particles need not be measured by the experimenter, it is sufficient that they interact with the environment. If they interact with the environment, then the final state of the environment will depend upon the neutrino emission time, and so neutrinos emitted at different times are not coherent and so cannot interfere.¹ As a result, neutrino oscillations are washed out, as described in Refs. [5,6].

Localized wave packets are then simply a proxy for the fact that, as a result of environmental interactions, neutrinos produced at different times contribute to different final states and so are not coherent. More precisely, the calculation of the neutrino probability reduces to a sum over distinct final environmental states and the neutrinos contributing to each final state are indeed localized. Conversely, in the absence of an environment, neutrinos are not localized into wave packets, as was found in Ref. [7].

One could object that the decoherence of neutrino oscillations is an extraordinarily old subject [5], and so ask why should one write or read more papers about it. The subject of decoherence is timely because JUNO will provide the most sensitive test of neutrino decoherence to date [8], but is it already understood? There is certainly no consensus in the literature regarding neutrino decoherence. Some papers

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¹ Indeed, following Ref. [4], distinct emission times will belong to distinct superselection sectors.

[9,10] find no decoherence at all. Others [2] find that there is so much decoherence that there is a maximum distance after which oscillations are necessarily unobservable. Other papers [11], using the same framework, claim that there is no maximum distance. In the popular external wave packet approach, different neutrino flavors are created in the same region and then separate, following the original paradigm of [5]. However in a direct quantum field theory calculation, Ref. [12] found that instead they are created separately and then coalesce.

The wave packet approach is rooted in the following picture, described clearly in Ref. [13]. Measurement, in practice by the environment, restricts the neutrino creation and detection to two specific regions in space time. If all neutrino mass eigenstates have similar enough speeds that they travel from one region to the other, then the final state will be independent of the mass eigenstate and so the various eigenstates will contribute coherently to the oscillation probability and there will be no decoherence. A physically equivalent description employs a neutrino wave packet of size equal to that the measured production region. The wave packet description is easily connected to observables such as oscillation probabilities.

The trouble is that different approaches disagree even qualitatively on the nature of these regions. In the external wave packet approach [11] the regions have a universal dependence on the kinematics of the particles and are ellipsoids. In the approach of Ref. [6] and earlier, instead the regions have an infinite spatial extent and are time windows with size related to environmental interactions. These two pictures, needless to say, lead to entirely distinct phenomenology.

The goal of our program is to derive neutrino oscillation probabilities from the microphysics of the environmental interactions together with a consistent treatment of all fields involved using quantum field theory. This program is divided into two steps. First, one should understand neutrino oscillations in the vacuum, where there is no environment with which to interact. Next, in practice one knows the environmental interactions but the calculation requires the size of the wave packet. One must understand how to convert the environmental interactions into a wave packet size. There have been many estimates [5,14,15] but derivations only exist in the simplest of cases [16]. In fact, in the case of reactor neutrinos it is still a matter of debate whether the relevant interactions are atomic or nuclear, leading to estimates which differ by many orders of magnitude.

We feel that to approach the second step, one needs to first complete the first. That is the goal of the present work. To be certain of our response, we use multiple independent derivations. Actually, in the context of nonrelativistic quantum mechanics, the first question has already been answered in Ref. [15]. However, as has been reviewed in Ref. [11], there

are a number of reasons to not trust a quantum mechanical treatment of such an ultrarelativistic system. That said, our results will in fact agree with those of Ref. [15].

Our goal is to eliminate this ambiguity by starting from the simplest possible physical principles. We consider an initial condition controlled by and known to the experimenter, and evolve it consistently using the Hamiltonian imposing no constraints. As has been stressed by [17] it is essential that all entanglement is kept. Thus we do not use external wave packets. However for the question of decoherence, the approximation of [17] that the initial state is a momentum eigenstate is too crude and so we begin with a known wave packet. Keeping the entanglements is necessarily complicated, and so in this paper we stick to the vacuum case, leaving realistic experiments to future work.

The paradigm which we propose, derived entirely within quantum field theory, is as follows. Begin with a fixed initial state and use the Hamiltonian to evolve to a fixed time in the Schrodinger picture. This evolution can be interpreted as usual as a sum over different histories. Only those histories which lead to the same final state are added coherently. Therefore one can *derive* the coherent production and detection regions of [13]. Intuitively, they are the regions for which one ends in the same final state.

We will first treat this problem using an external wave packet model [18] in Sect. 2. We will need the slightly more sophisticated model of Ref. [2] because our source and detector are treated asymmetrically, as the first is not measured but the second is. In this model, one considers an experiment which lasts for an infinite time. In an infinite time, the source and detector would spread to infinity, unless additional interactions are introduced to stabilize them. Furthermore the source, being unstable, would disappear. Therefore in such an approach the source and detector are not treated as dynamical fields, but rather as rigid external sources which do not spread, back react, dissipate or entangle. Given the relevance of entanglement between the source and the neutrino to oscillation physics which has been highlighted in Ref. [17] and also the relevance of the finite time nature of such experiments which may in principle invalidate an S-matrix treatment as was claimed in Ref. [19], one may also not trust the results of the external wave packet model. Motivated by these concerns, in Sect. 3 we provide a manifestly finite time treatment of the problem in quantum field theory, treating all fields responsible for production and detection as fully dynamical quantum fields, using the model introduced in Ref. [20]. In Sect. 4 we apply our result to the old question of whether there exists a maximum coherence length beyond which neutrino oscillations become unobservable.

What is the range of validity of our results? We describe a Universe which contains a single unstable particle and a detector. All calculations are done to leading order in perturbation theory, consisting of a single vertex for neutrino

creation and detection. As described at length in Ref. [20], the leading order only provides a good approximation to the amplitude at time scales much less than the lifetime of the unstable particle. As a result, we only expect our analysis and conclusions to be valid at time scales much less than the lifetime of our neutrino source. We are motivated by reactor neutrinos, as these are believed to be the most likely to manifest observable decoherence effects. While it is true that high energy reactor neutrinos often come from short-lived steps in decay chains, nevertheless the lifetime of the parent nucleus in the vacuum is billions of years. Thus we believe that at least in some settings our approximation is reasonable. More generally, we suspect the decoherence effects will be a competition between finite source lifetime and source interactions, both of which limit the wave packet size, but that source interactions will dominate over finite lifetime effects in the case of reactor neutrinos. Therefore we do not believe that finite lifetime effects will prove to be relevant in this line of research.

While this manuscript was in preparation, Ref. [9] appeared which also considers neutrino oscillations in the vacuum, following Refs. [7, 10]. These papers, like us, are concerned with neutrino oscillations in the vacuum and also do not find any intrinsic decoherence effect due to a separation of mass eigenstates. Curiously, Refs. [9, 10] also do not find decoherence due to uncertainty in the distance traveled by a neutrino, perhaps as a result of their localization of the source and detector, but this source of decoherence does appear in Ref. [7].

2 External wave packet model

In this section we will treat neutrino oscillations in the vacuum using the approach of Ref. [2]. While naively the results of that paper contrast with ours, we will see that this is a result of a very specific assumption which is violated in the case of vacuum neutrino production.

In Ref. [2] electroweak interactions are used to create and destroy the neutrino inside of a rigid, but moving, external source. The computation is in 3+1 dimensions and extends over an infinite time. All initial state and final state particles are described by rigid wave packets. We will argue that the case of interest for us, corresponding to an experiment in the vacuum, is in fact described by the same setup as Ref. [2] with the widths of unobserved wave packets taken to be infinite. Therefore we will not need any modification of the setup in Ref. [2]. As a result Sect. 2.1 will be simply a review of results in that paper. However, as was already noted there, the case of infinite width wave packets is inconsistent with an approximation made in an intermediate step of the calculation of Ref. [2]. The later steps of the calculation will therefore be different in our case. These are reported in Sect. 2.2.

2.1 The setup

Neutrinos are created in the process

$$P_I \longrightarrow P_F + l_\alpha^+ + \nu_\alpha \quad (2.1)$$

where P_I and P_F are the initial and final source particles, for example, P_I may be a nucleus which β^+ decays. Here l^+ is a charged lepton. The neutrino can oscillate while propagating, and ν_β is detected in the process

$$\nu_\beta + D_I \longrightarrow D_F + l_\beta^- \quad (2.2)$$

where D_I and D_F are the initial and final detector particles, for example D_I may be a free proton and D_F a neutron. It is assumed that the particles P_F , D_F and the charged leptons are observed, either by the experimenter or by the environment. This observation is incorporated into the calculation by imposing that these particles are described by wave functions whose width is the precision with which their positions are measured and whose velocities are determined by the measurement. These wave functions extend through all time, with fixed width, velocity and normalization. They are taken to be sharply peaked around their average momenta, which are called $\langle \vec{p}_k \rangle$, while the corresponding average energies are given by $\langle E_k \rangle = \sqrt{\langle \vec{p}_k \rangle^2 + m_k^2}$.

The neutrino is created in the overlap of the external particles in Eq. (2.1) and is destroyed in the intersection of the external particles in Eq. (2.2). Reference [2] starts by considering the transition amplitude for the process (2.1)+(2.2), which is computed in a previous work [18]. The amplitude is calculated by folding the neutrino propagator into these intersections. Each wave packet has a center at each moment in time. The centers of the production (detection) particles intersect at the average location of production (detection) in space-time. The time difference between these two space time points is called T and represents the average neutrino propagation time, while the spatial displacement is given by the 3-vector $\vec{L}$.

This yields the oscillation amplitude

$$\begin{aligned} A_{\alpha\beta}(\vec{L}, T) &\propto \sum_a \mathcal{U}_{\alpha a}^* \mathcal{U}_{\beta a} \int \frac{d^4 q}{(2\pi)^4} \frac{\phi(p)}{q^2 - m_a^2 + i\epsilon} \\ &\times \exp \left[-i q_0 T + i \vec{q} \cdot \vec{L} \right] \\ &\times \int d^4 x_1 \exp \left[-i (E_P - q_0) t_1 + i (\vec{p}_P - \vec{q}) \cdot \vec{x}_1 \right. \\ &\quad \left. - \frac{\vec{x}_1^2 - 2\vec{v}_P \cdot \vec{x}_1 t_1 + \Sigma_P t_1^2}{4\sigma_{xP}^2} \right] \\ &\times \int d^4 x_2 \exp \left[-i (E_D + q_0) t_2 + i (\vec{p}_D + \vec{q}) \cdot \vec{x}_2 \right. \\ &\quad \left. - \frac{\vec{x}_2^2 - 2\vec{v}_D \cdot \vec{x}_2 t_2 + \Sigma_D t_2^2}{4\sigma_{xD}^2} \right] \end{aligned} \quad (2.3)$$

where E_k is related to the energy particle k (see Ref. [2] for details). σ_{xP} is defined as

$$\frac{1}{\sigma_{xP}^2} = \frac{1}{\sigma_{xP_I}^2} + \frac{1}{\sigma_{xP_F}^2} + \frac{1}{\sigma_{x\alpha}^2} \quad (2.4)$$

where σ_{xP_I} , σ_{xP_F} and $\sigma_{x\alpha}$ are the widths of the wave packets of P_I , P_F and l_α^+ respectively. Here and in the following definitions the equivalent quantities for the detector can be obtained by substituting the index P with D and α with β . The velocities are denoted similarly by $\vec{v}_{P_I}$, $\vec{v}_{P_F}$ and $\vec{v}_\alpha$. $\vec{v}_P$ and Σ_P are defined as

$$\begin{aligned} \vec{v}_P &= \sigma_P^2 \left(\frac{\vec{v}_{P_I}}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}}{\sigma_{xP_F}^2} + \frac{\vec{v}_\alpha}{\sigma_{x\alpha}^2} \right) \\ \Sigma_P &= \sigma_P^2 \left(\frac{\vec{v}_{P_I}^2}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}^2}{\sigma_{xP_F}^2} + \frac{\vec{v}_\alpha^2}{\sigma_{x\alpha}^2} \right). \end{aligned} \quad (2.5)$$

The integral over x_1 and x_2 is trivial, while the integral over $\vec{q}$ can be performed using the Grimus–Stockinger theorem [10], which states that

$$\int d^3q \frac{\phi(\vec{q}) e^{i\vec{q} \cdot \vec{L}}}{q_a^2 - \vec{q}^2 + i\epsilon} \xrightarrow{L \rightarrow \infty} -\frac{2\pi^2}{L} \phi(q_a \vec{L}/L) e^{iq_a L} \quad (2.6)$$

where $L = |\vec{L}|$. The remaining integral over q^0 can be performed using a saddle-point approximation around $q^0 = E_a$, that can be interpreted as the effective energy of the neutrino.

This leads to the final formula for the transition amplitude, reported in Eq. (18) of Ref. [2] (published version). The contribution from each neutrino mass eigenstate a is equal to

$$\begin{aligned} A_a &= C \exp \left[-\frac{(L - v_a T)^2}{2v_a^2 \Omega_a} \right], \\ \Omega_a &= \frac{2\sigma_P^2 (v_a - \vec{L} \cdot \vec{v}_P/L)^2}{v_a^2 \lambda_P} + F \end{aligned} \quad (2.7)$$

where v_a is the expected speed of the neutrino mass eigenstate v_a . The term C is independent of T . λ_P is defined as

$$\lambda_P = \Sigma_P - \vec{v}_P^2. \quad (2.8)$$

In all neutrino experiments of which we are aware, the particles P_F and l_α^+ are not observed. However they do interact with the environment. If we perform our experiment in the vacuum, so that there is no environment, then their positions will be unconstrained and so

$$\sigma_{xP_F} = \sigma_{x\alpha} = \infty \quad (2.9)$$

which implies

$$\sigma_P = \sigma_{P_I}, \quad \Sigma_P = \vec{v}_{P_I}^2 = \vec{v}_{P_I}^2, \quad \lambda_P = 0. \quad (2.10)$$

In Ref. [2], the authors write that this case “corresponds to a different physical process from the one under consideration, which can be discussed modifying the calculation presented

here in the appropriate way.” In the next subsection we will do just this.

2.2 Working in a vacuum

For brevity, we will not repeat the full computation of Ref. [2], but will describe how $\lambda_P = 0$ affects it.

Note that in Eq. (2.7) the expression $v_a - \vec{L} \cdot \vec{v}_P/L$ does not vanish without infinite fine tuning, even at $\lambda_P = 0$, as v_a depends on the neutrino masses but the other terms do not.² Therefore

$$\lim_{\lambda_P \rightarrow 0} \Omega_a = \infty \quad (2.11)$$

and so in our case A_a is independent of T . This is in fact necessary for the consistency of our calculation, because the expected production time is not well defined when the produced particles are not measured, therefore T is the center of a homogeneous distribution, which is arbitrary. In other words, T is not defined in our case and it is an important consistency check that our final results do not depend on T .

Next, following the logic of Ref. [18], one sums the amplitude over the neutrino mass eigenstates and takes the absolute value squared to obtain a probability. The probability is then integrated over the unmeasured quantity T . The only dependence on T arose from the term in Eq. (2.7). In Ref. [2] the integration of this term led to the following term in the probability, given in their Eq. (22)

$$P \sim \exp \left[-\frac{L^2}{2} \frac{(v_a - v_b)^2}{v_a^2 v_b^2 (\Omega_a + \Omega_b)} \right] \quad (2.12)$$

where a and b are neutrino mass eigenstate indices which must be summed over. In our case, the amplitude is independent of T and so the integration over T must simply yield an infinite constant, which can be normalized as usual by considering a production rate. This result is in fact consistent with Eq. (2.12) because in our case $\Omega_a = \Omega_b = \infty$ and so the term shown is unity.

The term in Eq. (2.12) is the only term in the exponential which is proportional to L^2 . The coherence length L_{coh} of neutrino oscillations is defined by the proportionality of the oscillation probability

$$P \propto \exp \left[-\frac{L^2}{L_{coh}^2} \right]. \quad (2.13)$$

In our case there is no L^2 term in the exponential, and so the coherence length is infinite. This is our main result. Formally it may be obtained from Eq. (28) of [2] by noting that their ω is infinite.

This is not to say that there is no decoherence. Decoherence due to the uncertainty in the baseline does not arise from

² F is finite when $\lambda_P = 0$, so that term can be ignored.

the T dependent terms and in fact it is independent of both L and T , and so it persists even in this case. Indeed, in our case (2.9) the uncertainty in the location of the production and detection point, and so the distance traveled by the neutrino, is even larger and so one expects more decoherence in the vacuum.

3 Fully dynamical model

3.1 Review of analytic results

We will now present a second, independent derivation of this result using the model of Ref. [20]. In this model, neutrinos in the flavor eigenstate i are produced by a two body decay

$$\phi_{SH} \longrightarrow \phi_{SL} + \psi_i \quad (3.1)$$

where ϕ_{SH} and ϕ_{SL} are the initial and final source particles, for example, ϕ_{SH} may be a pion and ϕ_{SL} an antimuon. We will call the particle ψ_i a neutrino of mass eigenstate i . It is detected in the process

$$\psi_i + \phi_{DL} \longrightarrow \phi_{DH}. \quad (3.2)$$

The indices H and L denote in each case the heavier and lighter particle respectively. The fields of all particles will be evolved consistently in the Schrodinger picture of quantum field theory, using the Hamiltonian H , for a fixed time t . We remind the reader that in the Schrodinger picture, H is time-independent.

We will keep track of the full quantum states, together with all entanglements. To keep such a computation tractable, instead of electroweak interactions we use a simplified scalar model in 1+1 dimensions. In the ultrarelativistic limit, neutrinos may be approximated by scalars as described in Ref. [11]. Despite this brutal approximation, we will continue to refer to the field ψ as a neutrino. The Hamiltonian H is the sum of the standard free massive scalar Hamiltonian H_0 and an interaction term

$$H_I = \int dx : \mathcal{H}_I(x) :, \quad \mathcal{H}_I(x) = \sum_{\alpha=\{S,D\}} \phi_{\alpha H}(x) \phi_{\alpha L}(x) (\psi_1(x) + \psi_2(x)). \quad (3.3)$$

The sum $\psi_1 + \psi_2$ represents a flavor eigenstate and colons denote the standard normal ordering.

Working in the Schrodinger picture, we will calculate the amplitude

$$\mathcal{A}(k, l) = \langle H, k; L, l | e^{-iHt} | 0 \rangle \quad (3.4)$$

where $|0\rangle$ is our initial state

$$|0\rangle = \int dp_1 e^{-\frac{p_1^2}{2\sigma_s^2}} \int dp_2 e^{-\frac{p_2^2}{2\sigma_d^2}} e^{-ixp_2} |L, p_2; H, p_1\rangle. \quad (3.5)$$

The state $|L, p_2; H, p_1\rangle$ consists of a ϕ_{DL} with momentum p_2 and a ϕ_{SH} with momentum p_1 , where L and H run over the indices S and D . The constants σ_s and σ_d are the initial wave packet momentum spreads of the source and detector, which are fixed by the experimenter. The source is centered at the position 0 whereas the position of the center of the detector is x . Note that momentum conservation implies that, before integration over p_1 and p_2 , the amplitude (3.4) is proportional to $\delta(l + q - p_1)\delta(k - q - p_2)$ where q is the momentum transfer.

To second order in H_I , there are two possible processes with final states with no neutrinos. First, a neutrino may travel from the source to the detector. Second, a neutrino may travel from the detector to the source. If we let the masses M_{IH} of the H particles be more massive than those M_{IL} of the L particles, where I runs over $\{S, D\}$ then the second process will be far off shell and will have a negligible contribution for macroscopic baselines [19]. Therefore we will consider only the contribution from the first process.

The amplitude can be written as a sum over contributions from each neutrino mass eigenstate ψ_1 and ψ_2 with mass m_1 and m_2

$$\mathcal{A}(k, l) = \sum_{i=1}^2 \mathcal{A}_i(k, l). \quad (3.6)$$

These in turn are given by integrals over the neutrino momentum q [20]

$$\begin{aligned} \mathcal{A}_i(k, l) &= \int \frac{dq}{2\pi} \frac{F_i(q)}{c_i(q)} \\ &\times \text{Exp} \left[-\frac{(l+q)^2}{2\sigma_s^2} - \frac{(k-q)^2}{2\sigma_d^2} - ix(k-q) \right] \\ c_i(q) &= 8e_i(q) \sqrt{E_{SL}(l)E_{SH}(l+q)E_{DL}(k-q)E_{DH}(k)} \\ F_i &= -\int_0^t dt_1 \int_0^{t-t_1} dT \\ &\times \text{Exp} [-i(t_1\mathcal{E}_0(l+q, k-q) \\ &+ T\mathcal{E}_{1i}(l, k-q, q) + (t-T-t_1)\mathcal{E}_2(l, k)]. \end{aligned} \quad (3.7)$$

Here t_1 and T are naturally interpreted as the neutrino creation time and propagation time. We remind the reader that our experiment begins at time 0 in the state $|0\rangle$ and concludes at time t when the interactions are switched off.

The on-shell energies are

$$E_{\alpha I}(p) = \sqrt{M_{\alpha I}^2 + p^2}, \quad e_i(p) = \sqrt{m_i^2 + p^2} \quad (3.8)$$

which are summands in the eigenvalues of the free Hamiltonian H_0 before neutrino production, during neutrino propagation and after neutrino absorption respectively

$$\begin{aligned} \mathcal{E}_0(p_1, p_2) &= E_{SH}(p_1) + E_{DL}(p_2), \\ \mathcal{E}_{1i}(p_1, p_2, q) &= E_{SL}(p_1) + E_{DL}(p_2) + e_i(q) \end{aligned}$$

$$\mathcal{E}_2(p_1, p_2) = E_{SL}(p_1) + E_{DH}(p_2). \quad (3.9)$$

We note that, as always in a Lorentz-invariant quantum field theory, momentum and energy are exactly conserved at each vertex. However energy is the eigenvalue of H . On the other hand $\mathcal{E}_i$ is the eigenvalue of H_0 and so in general $\mathcal{E}_0$, $\mathcal{E}_{1i}$ and $\mathcal{E}_2$ will not be equal except when all particles are exactly on-shell.

3.2 Saddle point approximation to the amplitude

The result of the measurement is defined to be the state of the detector at time t . More precisely, only the final momentum k of the detector is measured. Once the conservation of momentum has been imposed, the only other two momenta in the problem are the neutrino momentum q and the final source momentum l . Once k is fixed, if one demands that all particles be on-shell then q and l will be fixed to the on-shell values q_i and l_i , where the i index reminds the reader that these depend on the neutrino mass eigenstate i . The dependence of q_i and l_i on k will be left implicit. On-shell the energies $\mathcal{E}$ agree

$$\mathcal{E}_0(l_i + q_i, k - q_i) = \mathcal{E}_{1i}(l_i, k - q_i, q_i) = \mathcal{E}_2(l_i, k) = \epsilon_i. \quad (3.10)$$

Our particles will not be on-shell. But they will nearly be on-shell. Therefore we may expand the energies $\mathcal{E}$ about the on-shell value. For example, to linear order in $q - q_i$ but zeroth order in $l - l_i$ one finds

$$\begin{aligned} \mathcal{E}_0(l + q, k - q) &= \epsilon_i + v_{0i}(q - q_i), \\ \mathcal{E}_{1i}(l, k - q, q) &= \epsilon_i + v_{1i}(q - q_i), \\ \mathcal{E}_2(l, k) &= \epsilon_i \end{aligned} \quad (3.11)$$

where we have defined the on-shell velocities

$$\begin{aligned} v_{0i} &= v_{SH,i} - v_{DL,i} = \frac{l_i + q_i}{E_{SH}(l_i + q_i)} - \frac{k - q_i}{E_{DL}(k - q_i)}, \\ v_{1i} &= v_{\psi_i} - v_{DL,i} = \frac{q_i}{E_{\psi_i}(q_i)} - \frac{k - q_i}{E_{DL}(k - q_i)}. \end{aligned} \quad (3.12)$$

This expansion of the energy allows us to expand the amplitude (3.7) and perform the integral over q

$$\begin{aligned} \mathcal{A}_i(k, l) &= -\frac{B_i}{c_i(q_i)} \\ B_i &= \int_0^t dt_1 \int_0^{t-t_1} dT e^{\gamma_i} \int \frac{dq}{2\pi} \text{Exp} \left[-\frac{\sigma_x^2 q^2}{2} + \beta_i q \right] \\ &= \frac{1}{\sigma_x \sqrt{2\pi}} \int_0^t dt_1 \int_0^{t-t_1} dT \text{Exp} \left[\gamma_i + \frac{\beta_i^2}{2\sigma_x^2} \right] \\ \sigma_x &= \left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_d^2} \right)^{1/2} \end{aligned}$$

$$\begin{aligned} \beta_i &= i(\delta_i - d_i), \quad \delta_i = x + i \left(\frac{l}{\sigma_s^2} - \frac{k}{\sigma_d^2} \right), \\ d_i &= t_1 v_{0i} + T v_{1i} \\ \gamma_i &= -\frac{l^2}{2\sigma_s^2} - \frac{k^2}{2\sigma_d^2} + i(-xk - t\epsilon_i + q_i d_i). \end{aligned} \quad (3.13)$$

To perform the T integral, one need only complete the square

$$\begin{aligned} &\int_0^{t-t_1} dT \text{Exp} \left[\gamma_i + \frac{\beta_i^2}{2\sigma_x^2} \right] \\ &= e^{-\rho_i} \int_0^{t-t_1} dT \text{Exp} \left[-\mu_i (T - T_{0i})^2 \right] \\ \rho_i &= \frac{(l + q_i)^2}{2\sigma_s^2} + \frac{(k - q_i)^2}{2\sigma_d^2} + i(x(k - q_i) + t\epsilon_i) \\ \mu_i &= \frac{v_{1i}^2}{2\sigma_x^2}, \quad T_{0i} = \frac{\delta_i - t_1 v_{0i} + i\sigma_x^2 q_i}{v_{1i}}. \end{aligned} \quad (3.14)$$

Again, to avoid clutter, the dependences on k are left implicit.

Now we come to the key simplification. If μ is much larger than $1/\sqrt{t - T}$, and the imaginary part of T_0 is small enough, then the Gaussian on the first line is essentially a Dirac delta function

$$\begin{aligned} \text{Exp} \left[-\mu_i (T - T_{0i})^2 \right] &\sim \sqrt{\frac{\pi}{\mu_i}} \delta(T - T_{0i}) \\ &= \sqrt{2\pi} \frac{\sigma_x}{v_{1i}} \delta(T - T_{0i}) \end{aligned} \quad (3.15)$$

and the integral gives $\sqrt{\pi/\mu}$ if $\text{Re}(T_0)$ is in the range of integration $[0, t - t_1]$ and otherwise gives zero

$$\begin{aligned} &\int_0^{t-t_1} dT \text{Exp} \left[-\mu_i (T - T_{0i})^2 \right] \\ &= \sqrt{2\pi} \frac{\sigma_x}{v_{1i}} \theta(\text{Re}(T_0)) \theta(t - t_1 - \text{Re}(T_0)) \end{aligned} \quad (3.16)$$

where θ is the Heaviside step function. The product of step functions is nonzero whenever

$$0 \leq x - t_1 v_{0i} \leq v_{1i}(t - t_1). \quad (3.17)$$

This is just the condition that the on-shell neutrino can travel as far as the detector if it is emitted at time t_1 . As $v_{1i} > v_{0i}$ for ultrarelativistic neutrinos, this implies

$$t_1 \leq \frac{tv_{1i} - x}{v_{1i} - v_{0i}}. \quad (3.18)$$

If the neutrino is emitted after this time, it will not arrive at the detector before it is measured at time t . The t_1 integral is then easily computed. So long as $x > v_{0i}t$, which means that the source and detector have not moved past one another, one finds

$$\begin{aligned} &\int_0^t dt_1 \int_0^{t-t_1} dT \text{Exp} \left[\mu_i (T - T_{0i})^2 \right] \\ &= \sqrt{2\pi} \frac{\sigma_x}{v_{1i}} \int_0^t dt_1 \theta \left(\frac{tv_{1i} - x}{v_{1i} - v_{0i}} - t_1 \right) \end{aligned}$$

$$= \sqrt{2\pi} \frac{\sigma_x}{v_{1i}} \frac{tv_{1i} - x}{v_{1i} - v_{0i}} \theta(tv_{1i} - x). \quad (3.19)$$

Putting this all together

$$\mathcal{A}_i(k, l) = -\frac{(tv_{1i} - x)\theta(tv_{1i} - x)}{c_i(q_i)v_{1i}(v_{1i} - v_{0i})} e^{-\rho_i}. \quad (3.20)$$

3.3 Ultrarelativistic and small mass splitting approximations

Each q_i may be obtained from the defining relation

$$\begin{aligned} 0 &= \epsilon_i - \epsilon_i = \mathcal{E}_{1i}(l_i, k - q_i, q_i) - \mathcal{E}_2(l_i, k) \\ &= \sqrt{M_{DL}^2 + (k - q_i)^2} + \sqrt{q_i^2 + m_i^2} - \sqrt{M_{DH}^2 + k^2} \end{aligned} \quad (3.21)$$

which happens to be independent of l_i . We will consider the leading term in two expansions. First, the ultrarelativistic expansion is a series in m_i^2/q_i^2 and second we will fix an arbitrary q_0 and consider a power series in $q_i - q_0$. This second expansion obviously is valid for q_0 close enough to q_i , but we will use the same expansion for every flavor i and so the expansion is valid for some q_0 if $q_2 - q_1$ is smaller than the other momenta in the problem, which we will see occurs when the mass splitting is small. The ultrarelativistic expansion is performed first, and so $q_2 - q_1$ need not be smaller than m_i .

The leading term in the double expansion of Eq. (3.21) is

$$\begin{aligned} &-\sqrt{M_{DL}^2 + (k - q_0)^2} - q_0 + \sqrt{M_{DH}^2 + k^2} \\ &= \frac{m_i^2}{2q_0} + (q_i - q_0) \left[1 - \frac{m_i^2}{2q_0} - \frac{k - q_0}{\sqrt{M_{DL}^2 + (k - q_0)^2}} \right]. \end{aligned} \quad (3.22)$$

The left hand side is independent of the flavor i . Subtracting the right hand sides at two flavors i and j one finds

$$q_i - q_j = \frac{(m_j^2 - m_i^2)/(2q_0)}{1 - \frac{m_i^2}{2q_0} - \frac{k - q_0}{\sqrt{M_{DL}^2 + (k - q_0)^2}}} \sim \frac{m_j^2 - m_i^2}{2q_0}. \quad (3.23)$$

The leading term in the expansion is a good approximation if $q_0 \sim q_i$ for all i . In the ultrarelativistic approximation, these in turn are roughly equal to the on-shell neutrino energy, which we will call e . Thus we arrive at our final expression

$$q_2 - q_1 = \frac{m_1^2 - m_2^2}{2e}. \quad (3.24)$$

This does not imply that we are approximating the neutrinos to actually have the same energy, the neutrino momenta for each mass eigenstate have been integrated over all values of q and no cross-terms have been dropped. Recall that q_1 and q_2 are just the on-shell values of the momenta.

Similarly we will approximate

$$c = c_i(q_i), \quad v_0 = v_{0i}, \quad v_1 = v_{1i}, \quad \epsilon = \epsilon_i. \quad (3.25)$$

These approximations may be justified via double expansions such as that above. In addition, given the kinematics of our process, the on-shell neutrino energies for the two eigenstates are nearly equal

$$\begin{aligned} e_2(q_2) - e_1(q_1) &= \sqrt{q_2^2 + m_2^2} - \sqrt{q_1^2 + m_1^2} \sim q_2 \\ &+ \frac{m_2^2}{2q_2} - q_1 - \frac{m_1^2}{2q_1} \sim q_2 - q_1 \\ &- \frac{m_1^2 - m_2^2}{2e} \end{aligned} \quad (3.26)$$

which vanishes in our approximation by Eq. (3.24).

3.4 The probability density

The unnormalized probability density is

$$P(k, l) = |\mathcal{A}_1(k, l) + \mathcal{A}_2(k, l)|^2. \quad (3.27)$$

It may be normalized as in Ref. [21], but we will not normalize it here. It is now easily computed

$$\begin{aligned} P(k, l) &= \left(\frac{(tv_1 - x)}{cv_1(v_1 - v_0)} \right)^2 \theta(tv_1 - x) \\ &\times \left[2\text{Exp} \left[-\frac{(l + q_1)^2 + (l + q_2)^2}{2\sigma_s^2} \right. \right. \\ &\quad \left. \left. - \frac{(k - q_1)^2 + (k - q_2)^2}{2\sigma_d^2} \right] \right. \\ &\quad \times \cos \left(\frac{x(m_1^2 - m_2^2)}{2e} \right) \\ &\quad \left. + \sum_{i=1}^2 \text{Exp} \left[-\frac{(l + q_i)^2}{\sigma_s^2} - \frac{(k - q_i)^2}{\sigma_d^2} \right] \right]. \end{aligned} \quad (3.28)$$

We are interested in the case in which l is not observed, and so we must integrate over l , yielding

$$\begin{aligned} P(k) &= \int dl P(k, l) = \left(\frac{(tv_1 - x)}{cv_1(v_1 - v_0)} \right)^2 \theta(tv_1 - x) \\ &\times \left[2\text{Exp} \left[-\frac{(k - q_1)^2 + (k - q_2)^2}{2\sigma_d^2} - \left(\frac{(m_1^2 - m_2^2)}{4e\sigma_s} \right)^2 \right] \right. \\ &\quad \left. \times \cos \left(\frac{x(m_1^2 - m_2^2)}{2e} \right) + \sum_{i=1}^2 \text{Exp} \left[-\frac{(k - q_i)^2}{\sigma_d^2} \right] \right]. \end{aligned} \quad (3.29)$$

This is the function that would be determined by an experiment which perfectly measures k .

It is clearly wrong. At large t it is proportional to t^2 . However the source strength is constant at this leading order in perturbation theory, and so the probability $P(k)$ of having absorbed a neutrino by time t should be proportional to t [1]. We will see in the Sect. 3.6 that this is an artifact of the approximations used in this section: indeed, if the contribution from the off-shell momenta of the source is taken into

account, $P(k)$ grows linearly in t . A more rigorous demonstration of this statement can be found in Appendix A. Let us ignore this problem for the moment, as it does not affect the exponent, which is the part of interest to us.

The total, unnormalized oscillation probability can be found by integrating (3.29) over k , yielding

$$P = \int dk P(k) = 2 \left(\frac{(tv_1 - x)}{cv_1(v_1 - v_0)} \right)^2 \theta(tv_1 - x) \times \left[1 + \text{Exp} \left[- \left(\frac{(m_1^2 - m_2^2)}{4e} \right)^2 \sigma_x^2 \right] \cos \left(\frac{x(m_1^2 - m_2^2)}{2e} \right) \right]. \quad (3.30)$$

Here we have assumed that σ_d is very narrow, and so have ignored the dependence of q_i on k . Had we not done this, we would have found an additional source of decoherence due to the uncertain momentum k . However this would not be an intrinsic source of decoherence, as an ideal detector can measure k as precisely as desired given enough time. For example, one can wait until the detector smears as closely as desired to a plane wave.

Note that when the arguments of the exponentials are small, there is no decoherence and we recover the standard oscillation formula

$$P \sim 2 \left(\frac{(tv_1 - x)}{cv_1(v_1 - v_0)} \right)^2 \theta(tv_1 - x) \cos^2 \left(\frac{\pi x}{L_{osc}} \right), \quad L_{osc} = \frac{4\pi e}{|m_1^2 - m_2^2|}. \quad (3.31)$$

The absolute value in Δm^2 was introduced in order to avoid any ambiguities in the sign, however it should be noted that, since we assumed $m_1 < m_2$, $|m_1^2 - m_2^2| = m_2^2 - m_1^2$.

We see in Eq. (3.30) that more generally the oscillations decohere due to the factor $\exp(-\sigma_x^2/L_{osc}^2)$. This is just the usual decoherence due to an uncertain neutrino travel distance reported, for example, in Refs. [2, 18].

On the other hand, at the order considered in our various expansions, there is no sign of intrinsic³ decoherence due to the width of the distribution of the neutrino energy or momentum, despite the fact that the oscillation pattern depends on the neutrino momentum and energy. This is also despite the fact that the detector has a large momentum spread σ_d , which one might expect to wash out oscillations with a momentum difference beneath this threshold. The reason that such oscillations are not washed out is that, given a perfect

determination of k , for each neutrino mass eigenstate the on-shell condition fixes the momenta q which contribute to the amplitudes to a very narrow range, much narrower than σ_d . This counterintuitive fact is a result of the very long time integration in the definition of our amplitude, reflecting the fact that the neutrino may be produced at any time. This long time integration forces the neutrinos to be very close to on-shell. When we approximated the T integral by that of a delta function, we effectively imposed an infinite time integration and so fixed the neutrino momenta for each value of k and mass eigenstate. That is not to say that the detector can measure the neutrino momentum q more precisely than its intrinsic scatter σ_d , on the contrary it cannot tell which mass eigenstate arrived and the two mass eigenstates have very different on-shell momenta.

3.5 Numerical results

In this subsection we will test the above results numerically. Following Ref. [20] we tune the masses to optimize the sensitivity of the detector

$$M_{DH} = M_{SH}, \quad M_{SL} = M_{DH}(1 - \epsilon), \quad M_{DL} = M_{SH} (1 - \epsilon + \epsilon^2) \quad (3.32)$$

and we then fix the parameters

$$M_{DH} = 10, \quad \epsilon = 0.1. \quad (3.33)$$

In addition we fix the neutrino masses

$$m_1 = 0, \quad m_2 = 0.1. \quad (3.34)$$

With these choices we obtain the on-shell conditions

$$q_1 = 0.9498, \quad q_2 = 0.9444, \quad l_1 = -0.9525, \quad l_2 = -0.9528 \quad (3.35)$$

which lead to the on-shell velocities

$$v_{00} = -0.0058, \quad v_{01} = -0.0069, \quad v_{10} = 0.9945, \quad v_{11} = 0.9883. \quad (3.36)$$

At time $t = 10^4$ we plot the unnormalized oscillation probability density $P(k, l)$ at $k = 1$ as a function of baseline x with two different sizes $1/\sigma_s$ and $1/\sigma_d$ for the source and detector, one of order the oscillation length and one much smaller. These are plotted in three approximations. First we use the exact second-order formula for the amplitude (3.7). Next we use our saddle point approximation amplitude (3.20) and finally we use directly use our formula for the probability (3.28) which used the approximations (3.25). Our results are shown in Fig. 1. As expected, one can see that the oscillation amplitude is about 100% when the source and detector sizes are much smaller than the oscillation wavelength, but is reduced when the sizes are comparable as a result of decoherence. The total probability also decreases in the case of a

³ As noted above, an imperfect measurement of k does reduce the observed amplitude of the neutrino oscillations. However k is part of the final state of the detector and so in principle may be determined to any desired precision, given sufficient time, after the experiment is complete.

Fig. 1 The unnormalized probability density $P(k, l)$ at $k = 1$ at various values of l , computed using the exact expression (3.7) (black curve) and also the approximations (3.20) (red curve) and (3.28) (blue curve). One can see that in the top panels, where l is near the on-shell value given in (3.35), the approximations are quite accurate, but they drop too slowly when l differs from the on-shell value in the lower panels. In the left we chose $\sigma_s = \sigma_d = 0.1$, on the right $\sigma_s = \sigma_d = 0.015$. Thus only on the right the source and detector sizes are of order the oscillation wavelength, and so the oscillation amplitude is reduced

larger source and detector as a result of our normalization of the source and detector wave functions in Eq. (3.7).

One can see that the approximations are quite reliable when l is close to the on-shell values l_i given in Eq. (3.35), but do not capture the correct fall-off when $l - l_i$ is increases. This is caused by our very crude expansion of the energy in Eq. (3.11), which was to zeroth order in $l - l_i$. We will now expand it to first order in $l - l_i$.

3.6 The source momentum

To first order in $l - l_i$ the energy may be expanded

$$\begin{aligned}\mathcal{E}_0(l + q, k - q) &= \epsilon_i + u_{0i}(l - l_i) + v_{0i}(q - q_i) \\ \mathcal{E}_{1i}(l, k - q, q) &= \epsilon_i + u_{1i}(l - l_i) + v_{1i}(q - q_i), \\ \mathcal{E}_2(l, k) &= \epsilon_i + u_{2i}(l - l_i)\end{aligned}\quad (3.37)$$

where the new on-shell velocities are

$$\begin{aligned}u_{0i} &= v_{SH,i} = \frac{l_i + q_i}{E_{SH}(l_i + q_i)}, \\ u_{1i} &= u_{2i} = v_{SL,i} = \frac{l_i}{E_{SL}(l_i)}.\end{aligned}\quad (3.38)$$

Note that at $l = l_i$ the two expansions (3.37) and (3.11) agree, and so one expects the probability density $P(k, l_i)$ in (3.27) to be correct, as can be seen in the top panels of Fig. 1.

The calculation above proceeds similarly to the zeroth order case studied above. As $u_{1i} = u_{2i}$, the on-shell neutrino propagation time $\text{Re}(T_0)$ is unaffected and Eq. (3.13) is also the same except for a shift in γ_i

$$\gamma_i \longrightarrow \gamma'_i = \gamma_i - i(l - l_i)(t_1 u_{0i} + (t - t_1)u_{2i}). \quad (3.39)$$

As the T -dependent terms are unaffected, the T integral leads to the same Heaviside step function as before, but now with a t_1 -dependent phase $e^{\gamma'_i}$ from (3.39). As a result, the t_1 integral is no longer trivial.

This is to be expected on physical grounds. Holding t_1 fixed and varying T , one multiplies the amplitude by a phase $e^{iT_0(\mathcal{E}_2 - \mathcal{E}_1)}$ which is independent of l . The T integral therefore fixes q to be near its on-shell value but does not constrain l . On the other hand, varying t_1 one multiplies by phase $e^{it_1(\mathcal{E}_1 - \mathcal{E}_0)}$ which depends on l . Therefore it is the t_1 integral which forces l to be on-shell.

Incorporating the new phase from (3.39), together with the old step function, the t_1 integral is

$$\int_0^t dt_1 \theta(t - t_1 - \text{Re}(T_0)) e^{-it_1(l - l_i)(u_{0i} - u_{2i})}$$

$$= i \frac{e^{i\alpha(l-l_i)} - 1}{(l-l_i)(u_{0i} - u_{2i})} \quad (3.40)$$

where

$$\alpha = \frac{(u_{0i} - u_{2i})(tv_{1i} - x)}{v_{1i} - v_{0i}}. \quad (3.41)$$

This new factor multiplies $P(k, l)$. It tends to unity if

$$|l - l_i| \ll \left| \frac{v_{1i} - v_{0i}}{(tv_{1i} - x)(u_{0i} - u_{2i})} \right| \quad (3.42)$$

but approaches zero for higher values. One integrates $P(k, l)$ over l to obtain $P(k)$. The restricted range (3.42) causes $P(k)$ to lose one power of α , or equivalently one power of $tv_1 - x$. As a result, now $P(k)$ grows only linearly with respect to $tv_1 - x$, not quadratically as in Eq. (3.29). In Appendix A, where we evaluate the integral exactly and calculate $P(k)$, we prove this claim. This linear dependence was shown, in Ref. [1], to be a general consequence of energy conservation.

Equation (3.42) has a simple, physical interpretation. $u_0 - u_2$ is the velocity recoil of the source when it admits the neutrino. $t - x/v_1$ is the time difference between the first allowed emission time $t_1 = 0$ and the last time at which the neutrino may be emitted and arrive at the detector by time t . Therefore the product $(u_0 - u_2)(t - x/v_1)$ is the size of range of possible centroids of the source when the neutrino is emitted. Eq. (3.42) is then just the uncertainty principle, the source momentum l cannot be constrained more tightly than this range in source positions. Interestingly the intrinsic source size, $1/\sigma_s$, does not play any role in this manifestation of the uncertainty principle. Of course were σ_s too small, then σ_x would be large and so the step function approximation of the T integral would be invalid. However this corresponds to the case in which the run time of the experiment is comparable to the uncertainty in the neutrino travel time, which is never realized in practice. The fact that the momentum uncertainties depend on the velocity recoil and not the momentum smearing of the source and detector is in contradiction with the usual intuition [22], but it is a robust implication of our dynamical treatment of the source and detector fields.

3.7 Effect of the source, detector and neutrino masses

In order to simplify the numerical calculations, we have considered very unrealistic values of the masses involved: the ratio between the source (or detector) mass and the difference between the neutrino masses is around 100. What would happen if we consider more realistic values for the masses? As we will see in this subsection, if we change the masses in our model, this will affect the range of off-shell momenta whose contribution is non-negligible, however except for this very little will change.

Following Eq. (3.32), we can see that once we fix M_{DH} and ϵ , all the masses of the source and detector particles are

fixed. For simplicity, let us consider the case where $q_{i,s} \simeq 1$ ($q_{i,s}$ are the on-shell value of q for each neutrino mass eigenstate). This can be easily obtained by considering $\epsilon = 1/M_{DH}$, as in Eq. (3.33). Moreover we will still take $m_1 = 0$.

For consistency, we will continue to consider $k = 1$. Once k is fixed, $q_{i,s}$ can be determined from Eq. (3.10) by requiring $\mathcal{E}_{1i}(l, k - q, q) = \mathcal{E}_2(l, k)$. As can be seen from Eq. (3.9), the l -dependent term is the same in both sides of the equation, so $q_{i,s}$ does not depend on l . The on-shell value of the source momentum $l_{i,s}$ can be calculated again from Eq. (3.10), this time considering $\mathcal{E}_0(l - q_{i,s}, k - q_{i,s}) = \mathcal{E}_{1i}(l, k - q_{i,s}, q_{i,s})$ and solving for l . It will be useful to define the deviation from the on-shell momenta as

$$\delta l_i = l - l_{i,s} \quad \delta q_i = q - q_{i,s}. \quad (3.43)$$

The function $F_i(q, l)$ defined in Eq. (3.7) can be rewritten as

$$F_i(q, l) = \frac{t^2}{2} \tilde{F}(q) = \frac{t^2}{2} \frac{2e^{-i\mathcal{E}_2 t}}{(\mathcal{E}_0 - \mathcal{E}_{1i})t} \times \left(\frac{e^{-i(\mathcal{E}_0 - \mathcal{E}_2)t} - 1}{(\mathcal{E}_0 - \mathcal{E}_2)t} - \frac{e^{-i(\mathcal{E}_{1i} - \mathcal{E}_2)t} - 1}{(\mathcal{E}_{1i} - \mathcal{E}_2)t} \right). \quad (3.44)$$

For readability, we left implicit the dependence of $\mathcal{E}_j$ on l, k , and q , however we remind the reader that, as per Eq. (3.7)

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}_0(l + q, k - q, q) & \mathcal{E}_{1i} &= \mathcal{E}_{1i}(l, k - q, q) \\ \mathcal{E}_2 &= \mathcal{E}_2(l, k). \end{aligned} \quad (3.45)$$

We factor out the $t^2/2$ to clarify the t -dependence. Indeed

$$\begin{aligned} \lim_{q, l \rightarrow q_{i,s}, l_{i,s}} |F(q, l)| &= \frac{t^2}{2} \\ \lim_{q, l \rightarrow q_{i,s}, l_{i,s}} |\tilde{F}(q, l)| &= 1. \end{aligned} \quad (3.46)$$

$F(q, l)$ will rapidly go to zero if either q or l are off-shell. In order to get a better understanding of how the value of the mass will affect the transition probability, we can expand the energies around the on-shell momenta (which is always justified for large values of t). We have

$$F_i(\delta q_i, \delta l_i) = \frac{t^2}{2} \frac{e^{-i\mathcal{E}_2 t}}{(u_{0i} - u_{1i})\delta l_i + (v_{0i} - v_{1i})\delta q_i} \times \left(\frac{e^{-i((u_{0i} - u_{1i})\delta l_i + v_{0i}\delta q_i)t} - 1}{((u_{0i} - u_{1i})\delta l_i + v_{0i}\delta q_i)t} - \frac{e^{iv_{1i}\delta q_i t} - 1}{v_{1i}\delta q_i t} \right) \quad (3.47)$$

where v_{ji} and u_{ji} are defined as in Eq. (3.12) and Eq. (3.38), respectively. We remind the reader that in Eq. (3.47) we used the equivalence $u_{2i} = u_{1i}$.

- If we change m_2 , we need to rescale L as well. Since it runs from 0 to t , this is equivalent to a rescaling of t . Indeed

$$m_2 \rightarrow \mathcal{R}m_2 \quad \Delta m^2 \rightarrow \mathcal{R}^2 \Delta m^2 \quad L \rightarrow \frac{L}{\mathcal{R}^2} \quad t \rightarrow \frac{t}{\mathcal{R}^2}.$$

(3.48)

From Eq. (3.47) we notice that in order to have the same behaviour of $F(q, l)$ we need to rescale both δl_i and δq_i by a factor $\mathcal{R}^2$

$$\begin{aligned} m_2 \rightarrow \mathcal{R}m_2 &\Rightarrow t \rightarrow \frac{t}{\mathcal{R}^2} \\ &\Rightarrow \delta l_i \rightarrow \delta l_i \mathcal{R}^2 \quad \delta q_i \rightarrow \delta q_i \mathcal{R}^2. \end{aligned} \quad (3.49)$$

This is not surprising. It simply means that if Δm^2 is lower, the neutrinos need to travel for more time before they oscillate, and as a consequence the off-shell contribution of the momenta will be suppressed.

- For large values of M_{DH} , the u_{ji} velocities go like l/M_{DH} , meaning that

$$M_{DH} \rightarrow \alpha M_{DH} \Rightarrow \delta l_i \rightarrow \alpha \delta l_i. \quad (3.50)$$

In Fig. 2 we can see $|F(q, l)|$ plotted as a function of δq_1 for different values of m_2 , M_{SH} and δl_i . As we decrease m_2 , the range of δq shrinks. Similarly, if we increase the value of M_{SH} , the same suppression can be obtained with a lower value of δl_i . In Fig. 3, we can see the transition probability plotted as a function of L where, if the source momentum was considered to be off-shell, δl_i was suitably rescaled: we can see that the oscillating behavior is roughly the same. We should stress that black curves in Fig. 3 are obtained using the exact expression of Eq. (3.7), no approximations were used (except for the tree-level approximation used to compute the transition amplitude).

A difference that can be noted is that, when $m_2 = 0.1$ and $M_{SH} = 10$, at low L the probability density $P(k, l)$ at the oscillation minimum is nonzero, while at lower (higher) values of m_2 (M_{SH}), even at low L the oscillations reach 0. The reason for this is to be found in the shape of the wavepackets. Indeed, with the values we are considering, the peak of $F_i(q, l)$ and of the wavepackets (which we will call $w(l, q)$ and that, when $k = 1$, are peaked for $q = 1, l = -1$) are not exactly at the same point. If we increase M_{SH} the values of $q_{i,s}$ ($l_{i,s}$) approach 1 (-1), while if we decrease m_2 the range of relevant values of δq_i becomes so small that the wavepacket cannot change significantly, however in the aforementioned case we can appreciate the difference, as can be seen in Fig. 4. The same effect can be obtained by increasing the value of σ_s and σ_d , as can be seen in Fig. 1.

4 Maximal coherence length?

In Ref. [23], as in our Sect. 3, the authors consider a 1+1 dimensional model of scalar neutrinos in the Schrodinger picture of quantum field theory. The neutrinos are described

by Gaussian wave packets, characterized by a single spatial width σ_x , and the source and detector particles are not explicitly considered. The authors find the same two contributions to decoherence as in Ref. [2]. Decoherence due to the baseline uncertainty provides an upper bound on σ_x beyond which oscillations cannot be observed, while that due to the momentum uncertainty of the wave packet provides a lower bound. Oscillations cannot be measured for any σ_x if the upper bound is less than the lower bound, which the authors find always occurs beyond some distance $L_{\max}$ which depends only on the neutrino mass splitting and energy

$$L_{\max} = \frac{16\pi^2 e^3}{(m_2^2 - m_1^2)^2}. \quad (4.1)$$

Needless to say, our results are in stark contradiction with those of Ref. [23], even those which are found in Sect. 2 using an external wave packet model⁴ introduced by the same authors a few months earlier in Ref. [2]. This contradiction stems from the fact that we, like Ref. [9], do not find any intrinsic source of decoherence to the momentum uncertainty, and so we have no lower bound on σ_x . This, in turn, is due to the entanglement of the detector with the neutrino, which is not considered in Ref. [23] as the detector wave function is not considered explicitly in that work. In our case, neutrinos with different emission times t_1 are summed coherently, while t_1 varies over a macroscopic interval whose size is of order t . In the language of wave packets, this would naively imply that our wave packet width σ_w is essentially infinite (of order ct , which is literally astronomical) as one cannot say where the neutrino is located. Wave packet intuition would then suggest that neutrino oscillations should be washed out, as they are damped by $\text{Exp}[(\sigma_x/L_{osc})^2]$. However this is not the case, as the σ_w mentioned here is the uncertainty in the neutrino's position at a fixed time, it is not the uncertainty in the baseline σ_x . The uncertainty in the baseline σ_x is bounded by the size of the source and detector, which is certainly finite and much smaller than ct . Considering the neutrino in isolation, as in Ref. [23], one apparently cannot distinguish σ_x from σ_w . It is their identification which leads to an apparent maximum coherence length.

The absence of a maximum coherence length does not appear to be an artifact of the approximations used to go from (3.7) to (3.28). We have checked numerically that the full probability density obtained from (3.7) manifests coherent oscillations for an arbitrary time. For example, consider

$$\begin{aligned} \sigma_d = \sigma_s = 0.5, \quad \epsilon = 10^{-4}, \quad m_{SH} = m_{DH} = 10^4, \\ m_{SL} = m_{DH}(1 - \epsilon) \\ m_{DL} = m_{SH}(1 - \epsilon + \epsilon^2), \quad m_1 = 0.5, \quad m_2 = 0. \end{aligned} \quad (4.2)$$

⁴ Reference [11] also finds that there is no maximum coherence length in an external wave packet model.

Fig. 2 $F(q, l)$ plotted for different values of m_2 and M_{SH} . The same values of l should be used for both mass eigenstates. For this reason we defined $\delta l = l - \tilde{l}$, where $\tilde{l} = (l_{1,s} + l_{2,s})/2$

The kinematics dictates that the neutrino energy will be about 1 when $k \sim 1$ and so

$$L_{\max} \sim 256\pi^2 \sim 3 \times 10^3. \quad (4.3)$$

In Fig. 5 we plot the unnormalized oscillation probability density $P(k)$ at time $t = 4 \times 10^4$. One can observe oscillations with amplitude of order unity at $L \gg L_{\max}$.

5 Conclusions

Our main result is that in an idealized setting, with no environmental interactions, there is no intrinsic decoherence due to the uncertain neutrino momentum. More precisely, neutrino oscillations are the result of the coherence between processes with the same neutrinoless initial state but in which the intermediate state contained distinct neutrino mass eigenstates. The kinematics dictate that these distinct neutrino mass eigenstates will have distinct velocities. As a result the moment of neutrino production or annihilation, will differ. Decoherence occurs if this difference in production or annihilation point causes a distinct final state, as one only coher-

ently sums amplitudes leading to the same final state. The neutrino oscillations are also damped if the absolute values of the amplitudes corresponding to these processes differ.

There are several distinct physical mechanisms which could lead to such effects. First, environmental interactions, for example of a particle created simultaneously with the neutrino, can cause the final state to depend upon the space-time region in which the neutrino is created and so lead to decoherence. Second, if for some reason the transition amplitude of one of the mass eigenstate is suppressed with respect to the other, the oscillation will be damped. This could happen, for example, if kinematic constraints impose that one of the eigenstates cannot be on shell, as we have seen at the end of Sect. 3.7. Similarly, a large difference in the neutrino velocities may affect the ratio of the kinematic factors in the amplitudes. Third, the source particle may be unstable on the time-scale relevant to an event. In this case, as explained in footnote 4 of [24], there would be loop corrections to the ϕ_{SH} propagator contributing to the self-energy. The imaginary part of the self-energy enforces the finite life time in quantum field theory while a branch cut leads to the quantum Zeno effect and the power law decay at late times [25]. All of

Fig. 3 $P(k, l)$ as a function of L , plotted for different values of m_2 , M_{SH} and δl (as defined in Fig. 2). The legend for the colors is the same as in Fig. 1. For larger values of M_{SH} , the difference between the red and blue curve is negligible. $\sigma_d = \sigma_s = 0.015$

these effects change the relative amplitudes of the processes with the two neutrino mass eigenstates, reducing that with the lighter neutrino relative to the heavier neutrino as the lighter neutrino is produced later. This is considered in Ref. [2] and also in the case of pion decay in Ref. [26]. Needless to say, these loop effects occur at a subleading order in perturbation theory and so are not included in our leading order approach. Finally, of course, in a real detector there will be an experimental imprecision that will lead to decoherence,

which can easily be found by folding the energy resolution into Eq. (3.29).

One may measure the momentum l with arbitrary precision, leading to the oscillation probability $P(k, l)$. Such a measurement gives no information concerning the position or time of the neutrino creation, indeed by the uncertainty principle it can destroy such information. A perfect measurement of l , for example, takes an infinite time and so yields no time constraint. Also a measurement of l does not kine-

Fig. 4 As in Fig. 2, but here the wavepackets were taken into account as well ($\sigma_d = \sigma_s = 0.015$). Notice that, when $m_2 = 0.1$ (top panels), the range of q considered was broader than in Fig. 2, to better show the effect of the wavepackets. In the other cases (bottom panels), the range of q is the same as in Fig. 2

matically exclude a neutrino mass eigenstate, and so it does not compromise the coherence between contributions to the amplitude with distinct neutrino mass eigenstates. However this measurement does lead to kinematic constraints. If k is known precisely, then only one value of l is consistent with an on-shell initial and final state. As t is finite, the initial and final states need not be on shell. However, if the measured value of l differs from the on-shell value by $O(1/t)$ then this will cause the usual amplitude oscillations that occur away from a saddle point of the path integral, leading to a distortion of the probability density as a function of distance. Of course as usual the probability density of l is peaked at the on-shell value as interference washes out the probability of off-shell results.

Another way to explain the fact that the uncertain momentum does not, by itself, lead to decoherence is as follows. The initial momentum spread of the source and detector are not in fact lower bounds on the momentum uncertainty of the neutrino. On the contrary, in the 2-body interactions considered here, a precise determination of the final momentum of the detector is sufficient to determine the momenta of

all other particles in the problem if they are on-shell. They will of course not be exactly on-shell, but the deviation from the on-shell momentum can be much smaller than the initial momentum smearing.

The precise enforcement of the on-shell condition is a result of the fact that the amplitude is a coherent integral over the entire runtime of the experiment, as we have not considered environmental interactions which ruin this coherence. The final state is independent of the neutrino emission time, as is obvious when working in the basis $|H, k; L, l\rangle$ of definite momenta k and l as the emission time affects neither k nor l . As a result, processes in which neutrinos are emitted at very different times lead to the same final state, and so are added coherently. Therefore neutrino mass eigenstates which arrive at the same time but are nonetheless coherent were emitted at different times and then coalesced following the paradigm of Ref. [12].

Environmental interactions are crucial for the decoherence only because they constrain the neutrino creation time. Indeed it would be possible to measure with an arbitrary precision the final momentum of the source particle, with-

P(k) at t=40000

Fig. 5 The unnormalized probability density $P(k)$ at $k = 0$ (black), 0.2 (red), 0.4 (blue), 0.6 (green), 0.8 (brown), 1.0 (purple), 1.2 (grey), 1.4 (yellow), 1.6 (orange) and 1.8 (pink) is plotted as a function of the baseline. It is obtained from Eq. (3.7) via numerical integration, and so does not use the approximations introduced in this note. One can see oscillations well beyond the putative maximum $L_{\max} \sim 3 \times 10^3$

out introducing any decoherence, just by waiting a sufficient amount of time after the end of the experiment. In the computation of $P(k, l)$, we used this fact, considering the final state of the source particle to be an asymptotic state whose momentum can be known exactly.

This leads to another conclusion of our study: the uncertainty of the momentum of the source and detector particles do not lead to decoherence, at least as long as they are non-relativistic. This is naively in contradiction with the usual intuition [22] that, in order to have a localized detector, the uncertainty principle requires that the momentum cannot be known precisely. The key realization is that the on-shell condition fixes the neutrino momentum quite precisely for each mass eigenstate, and the uncertainty principle guarantees that these two small windows cannot be distinguished by the detector and so contribute coherently. This coherence is necessary for the existence of neutrino oscillations.

In fact, if the detector or source is nonrelativistic, a change in the measured k or l hardly affects the neutrino momentum q which, for a given mass eigenstate, provides the dominant contribution to the amplitude. The resulting shift in q is suppressed by the recoil velocity. Thus while intrinsic smearing in k and l do lead to some smearing in q , and it is true that smearing in q for each mass eigenstate damps neutrino oscillations, the suppression of the smearing of q means that neutrino oscillations persist in a regime where they would not in a simplified model which equates the width of neutrino momentum with some weighted average of the widths of the external particle momenta.

In our study we also found evidence for a new quantum effect, even if the requirements for its observation are very likely to be unachievable with current technology. A key step in our calculation was the replacement of our Gaussian inte-

gration over T with the integral of a delta function. This approximation seems to be easily justified in experimental setups that can be realized with present technology. It would require an incredible time resolution to actually probe the shape of this Gaussian. However, could it be done, the oscillations measured would not obey the usual formula. Instead, one would observe that in a very short time window after the neutrinos arrive, they have not yet oscillated. The neutrino detection probability actually then decreases with time at the oscillation minima, as the oscillations turn on. This is due to destructive interference in the time integrals in our amplitude. This interesting, but probably unobservable phenomenon, will be the subject of our next project.

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Appendix A: Contribution from source momentum

Here we will explicitly calculate the amplitude taking into account also the off-shell momentum of the source, as discussed generally in Sect. 3.6. We will use the same approximations employed in the previous sections, namely we will neglect the dependence of the velocities, the neutrino energy and the numerical factor $c(q)$ on the neutrino mass eigenstate following Eq. (3.25). This will be true also for the velocities u_0 , u_1 and u_2 defined in Eq. (3.38). Moreover we will also consider $l_i = l_0 \quad \forall i$: this can be justified noticing that, while $q_i - q_j \simeq (m_i^2 - m_j^2)/2e$, $l_i - l_j \simeq (m_i^2 - m_j^2)/2m_{SH}$, i.e. the difference between the on-shell momenta of the source considering two different neutrino mass eigenstates is suppressed by an additional factor of m_{SH} . From Eq. (3.40) we

have

$$\mathcal{A}_i(k, l) = \frac{i e^{-\rho_i}}{c(q) v_1 (u_0 - u_2)} \frac{e^{-i t u_2 \Delta l} (1 - e^{i \alpha \Delta l})}{\Delta l} \theta(t v_1 - x) \quad (\text{A.1})$$

where ρ_i and α are defined as in Eqs. (3.14) and (3.41), respectively, and $\Delta l = l - l_0$. Following the same procedure that lead to Eq. (3.28) we can write

$$P(k, l) = \frac{2}{(c(q) v_1 (u_0 - u_2))^2} \frac{1 - \cos(\alpha \Delta l)}{\Delta l^2} P_{nl}(k, l) \quad (\text{A.2})$$

where $P_{nl}(k, l)$ is proportional to Eq. (3.28), up to some multiplicative factors that do not depend on l . We can write

$$P_{nl}(k, l) = \sum_{j=1}^3 \xi_j e^{-(\Delta l + \delta q_j)^2 / 2 \sigma_j^2} \quad (\text{A.3})$$

where $\delta q_j = l_0 + \tilde{q}_j$ and

$$\begin{aligned} \tilde{q}_j &= \left(q_1, q_2, \frac{q_1 + q_2}{2} \right) \\ \sigma_j^2 &= \left(\sigma_s^2, \sigma_s^2, \sigma_s^2 / 2 \right) \\ \xi_j &= \left(\text{Exp} \left[-\frac{(k - \tilde{q}_1)^2}{2 \sigma_d^2} \right], \text{Exp} \left[-\frac{(k - \tilde{q}_2)^2}{2 \sigma_d^2} \right], \text{Exp} \right. \\ &\quad \left. \times \left[-\frac{(k - \tilde{q}_3)^2}{\sigma_d^2} - \frac{(q_1 - q_2)^2}{\sigma_x^2} \right] \cos \left(\frac{x(m_1^2 - m_2^2)}{2e} \right) \right). \end{aligned}$$

In order to compute $P(k)$ we need to integrate over Δl (technically speaking the integral is over l , however it differs from Δl simply by a shift, so the two integrals are equivalent). We define

$$I_j(\alpha) = \int d(\Delta l) \frac{e^{-(\Delta l + \delta q_j)^2 / 2 \sigma_j^2} (1 - \cos(\alpha \Delta l))}{\Delta l^2}. \quad (\text{A.4})$$

We differentiate twice with respect to α , obtaining

$$\begin{aligned} I_j''(\alpha) &= - \int d(\Delta l) (e^{-(\Delta l + \delta q_j)^2 / 2 \sigma_j^2} (1 - \cos(\alpha \Delta l))) \\ &= \sqrt{2\pi} \sigma_j e^{-\alpha^2 \sigma_j^2 / 2} \cos(\alpha \delta q_j) \end{aligned} \quad (\text{A.5})$$

Solving the differential equation with respect to α we have

$$\begin{aligned} I_j(\alpha) &= c_0 + c_1 \alpha + \frac{\sqrt{2\pi} \sigma_j e^{-\alpha^2 \sigma_j^2 / 2} \cos(\alpha \delta q_j)}{\sigma_j^2} \\ &\quad + \frac{e^{-\frac{\delta q_j^2}{2 \sigma_j^2}} \left(\pi (\alpha \sigma_j^2 - i \delta q_j) \text{Erf} \left(\frac{\alpha \sigma_j^2 - i \delta q_j}{\sqrt{2} \sigma_j} \right) - \pi (\delta q_j - i \alpha \sigma_j^2) \text{Erfi} \left(\frac{\delta q_j - i \alpha \sigma_j^2}{\sqrt{2} \sigma_j} \right) \right)}{2 \sigma_j^3}. \end{aligned} \quad (\text{A.6})$$

The coefficients c_0 and c_1 can be easily determined using the following argument

- Since the integrand in Eq. (A.4) is invariant under $\alpha \rightarrow -\alpha$, only even powers of α can appear, hence $c_1 = 0$
- c_0 can be determined imposing $I_j(0) = 0$, since the integrand in (A.4) identically vanishes if $\alpha = 0$.

We then have

$$\begin{aligned} I_j(\alpha) &= \frac{\sqrt{\pi} \left(2 \delta q_j F \left(\frac{\delta q_j}{\sqrt{2} \sigma_j} \right) - \sqrt{2} \sigma_j \right)}{\sigma_j^3} \\ &\quad + \frac{e^{-\frac{\delta q_j^2}{2 \sigma_j^2}} \left(\pi (\alpha \sigma_j^2 - i \delta q_j) \text{Erf} \left(\frac{\alpha \sigma_j^2 - i \delta q_j}{\sqrt{2} \sigma_j} \right) - \pi (\delta q_j - i \alpha \sigma_j^2) \text{Erfi} \left(\frac{\delta q_j - i \alpha \sigma_j^2}{\sqrt{2} \sigma_j} \right) \right)}{2 \sigma_j^3} \end{aligned} \quad (\text{A.7})$$

where $F(x)$ is the Dawson function. In the limit $\alpha \rightarrow \infty$ we obtain

$$\begin{aligned} I_j(\alpha) &= \frac{\pi \alpha \sigma^2 e^{-\frac{\delta q_j^2}{2 \sigma^2}} - \sqrt{2\pi} \sigma + 2\sqrt{\pi} \delta q_j F \left(\frac{\delta q_j}{\sqrt{2} \sigma} \right)}{\sigma^3} \\ &\quad + \mathcal{O} \left(\frac{1}{\alpha} \right). \end{aligned} \quad (\text{A.8})$$

Notice that the leading term scales like α , which is proportional to $t v_1 - x$, as was claimed in Sect. 3.6.

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