Mimetic-metric-torsion with induced axial mode and phantom barrier crossing

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Abstract We extend the basic formalism of mimetic-metric-torsion gravity theory, in a way that the mimetic scalar field can manifest itself geometrically as the source of not just the trace mode of torsion, but its axial (or, pseudo-trace) mode as well. Specifically, we consider the mimetic field to be (i) coupled explicitly to the well-known Holst extension of the Riemann–Cartan action, and (ii) identified with the square of the associated Barbero–Immirzi field, presumably a pseudo-scalar. The conformal symmetry originally prevailing in the theory would still hold, as the associated Cartan transformations do not affect the torsion pseudo-trace, and hence the Holst term. Demanding the theory to preserve the spatial parity symmetry as well, we show a geometric unification of the cosmological dark sector, and the feasibility of a super-accelerating regime in the course of evolution of the universe. From the observational perspective, assuming the cosmological evolution profile to be very close to that for Λ CDM, we further illustrate a smooth crossing of the so-called phantom barrier at a low red-shift, albeit with a restricted parametric domain. Subsequently, we determine the extent of the super-acceleration by examining the evolution of the relevant torsion parameters.

1 Introduction

Emergence of the theory of mimetic gravity [1–3] has marked a fascinating recent advancement in modified gravity researches purporting to a geometric description of the cosmic dark sector constituents, viz. *dark energy* (DE) and *dark matter* (DM) [4–18]. Such a theory has attracted a lot of attention not only for providing an exact mimicry of a dust-like cold dark matter (CDM) component in the standard framework of Friedmann–Robertson–Walker (FRW) cosmology,

but also for arousing, via simple extensions, the very prospect of the unification of the entire dark sector. In fact, the original Chemseddine–Mukhanov (CM) model [1] of mimetic gravity¹ have had a plethora of extensions proposed and studied from various perspectives till date [2,3,22–57], with many interesting implications in both cosmology and astrophysics [58–88].

Strictly speaking, the basic CM theory amounts to a scalar-tensor reformulation of General Relativity (GR), of a specific sort which exploits the diffeomorphism invariance to reparametrize the physical metric $g_{\mu\nu}$ by a fiducial metric $\hat{g}_{\mu\nu}$ and a scalar field ϕ , in a way that $g_{\mu\nu}$ remains invariant under a conformal transformation of $\hat{g}_{\mu\nu}$. The gravitational conformal degree of freedom thus gets encoded by the field ϕ , known as the ‘mimetic field’ [1], which ostensibly has no prior relevance to geometry however. It therefore seems legitimate to look for a generalized mimetic scenario in which ϕ can manifest itself geometrically, say for instance, as the source of *torsion*, which is often regarded as important a space-time characteristic as curvature.² Such a generalization essentially implies contemplating on a conceivable way of expanding the purview mimetic gravity to the metric-compatible Riemann–Cartan (U_4) geometry that admits torsion in addition to curvature. An endeavour pertaining to this has resulted in our mimetic-metric-torsion (MMT) gravity formulation in a recent paper (henceforth, ‘Paper 1’ [191]), with the eventual enunciation of a viable unified cosmological dark sector paradigm. Nevertheless, scope remains for looking into the subtler aspects of the evolving dark sector,

¹ See also some relevant precursors [19–21].

² Refer to the hefty literature on the long history torsion gravity theories (from the pioneering Einstein–Cartan formulation to the plethora of scenarios that emerged from supergravity, string theory, brane-worlds, etc., or the modified *tele-parallelism*, Poincaré gauge gravity, and so on) [89–153]. See also the literature on the predicted observable effects of torsion, and their extensive searches [154–190].

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by suitably extending the MMT theory, which we intend to do in this work.

From the technical point of view, a consistent MMT formulation primarily amounts to a proper ratification of the principle of mimetic gravity in presence of torsion. This specifically implies the isolation of the conformal (scalar) degree of freedom of gravity via not only the parametrization of the physical metric $g_{\mu\nu}$ by a fiducial metric $\hat{g}_{\mu\nu}$ and the mimetic field ϕ , but also that of the physical torsion $T^\alpha_{\mu\nu}$ by a corresponding fiducial space torsion $\hat{T}^\alpha_{\mu\nu}$ and ϕ . We have been able to assert this torsion parametrization in Paper 1 by examining carefully the form of the fiducial space Cartan transformations under which, and a conformal transformation of $\hat{g}_{\mu\nu}$, the physical fields $g_{\mu\nu}$ and $T^\alpha_{\mu\nu}$ are to be preserved. Considering next, from certain standpoints, a contact coupling $\beta(\phi)$ of the mimetic field and the torsional part of the U_4 Lagrangian, it has been shown that the trace mode $\mathcal{T}_\mu$ of torsion gets sourced by ϕ , which thereby manifests itself geometrically [191]. The resulting energy-momentum tensor resembles that of a perfect fluid,³ dubbed the ‘MMT fluid’, which characterizes *dust*, albeit with a non-zero pressure. The latter is attained by virtue of an effective potential $W(\phi)$, culminating from $\beta(\phi)$ and its first derivative, in an equivalent formulation illustrated in Paper 1. Any desired expansion history of the universe can therefore be reconstructed in the standard FRW cosmological framework, by suitably choosing the form of the ‘MMT coupling’ function $\beta(\phi)$. However, in most circumstances such choices lack proper physical motivation, and are merely phenomenological. In Paper 1 though, the consideration of a well-known (and well-motivated) coupling $\beta(\phi) \sim \phi^2$ has made the potential W identifiable as a cosmological constant Λ (modulo a dimensional factor). The ultimate outcome has therefore been an effective Λ CDM evolution in the MMT cosmological setup, even when the torsion strength diminishes to its unobtrusiveness at late times (thus corroborating to its miniscule experimental signature till date [184–190]).

Nevertheless, the entire MMT formalism in Paper 1 is in some sense *minimalistic*, since it does not reflect on the actual (intriguing) characteristics of torsion in shaping up the cosmological evolution profile. Specifically, the latter has no direct influence of the antisymmetry property of torsion, unless one assigns an external source for the *axial* (or *pseudo-trace*) mode $\mathcal{A}_\mu$ of torsion. Such an external source, supposedly a non-gravitational field, is anyhow not much interesting, since its admittance may not only affect the predictability of the theory (due to the additional degree(s) of freedom), but also imply the redundancy in aspiring for a geometric unification of the cosmological dark sector. It is imperative therefore

to look for extending the basic MMT formalism in such way that apart from torsion’s trace $\mathcal{T}_\mu$, its axial mode $\mathcal{A}_\mu$ (with no assigned external source) can actively influence the dynamical solutions of equations of motion in a given setup (that of FRW cosmology for instance). A simple and natural way to do so is to consider the mimetic field’s explicit coupling with not only the torsional part of the U_4 curvature scalar $\tilde{R}$, but also its Hodge dual. The latter is commonly known as the ‘Holst extension’ of the U_4 Lagrangian, describing modified versions of the conventional metric-torsion theories of Einstein–Cartan (EC) type, or more generally, the refined editions of the canonical (Hilbert–Palatini) connection-dynamic gravitational theories [196–206]. In fact, it had been a generalized Hilbert–Palatini formulation of gravity which had accounted for the Holst term in Holst’s original paper in 1996 [197]. Nevertheless, the equivalent term, viz. the Hodge dual of $\tilde{R}$, in the Einstein–Hilbert formulation had long been known for [196]. In either formulation though, the Holst extension being merely *topological* (at least, at the minimal level), does not affect the classical dynamics. However, apart from a potential significance in canonical quantum gravity (particularly, relating to Ashtekar–Barbero formulation) [198–201, 206–211], the Holst term’s presence may reveal intriguingly via *gravitational parity violation*, possible consequences of which have been studied in various contexts, such as that of an axial torsion induced by the string theoretic Kalb–Ramond field [212–217].

A majority of recent works on the formulations of metric-torsion or connection-dynamic theories have pondered on relaxing the topological characterization of the Holst extension, in which case it can in principle affect the classical dynamics. An appropriate methodology, which has been motivated from various standpoints (such as the chiral anomaly cancellation), is to promote the associated coupling parameter γ , known as the Barbero–Immirzi (BI) parameter, to the status of a scalar or a pseudo-scalar field [211, 217–222]. The latter of course is the most suitable option if the theory is to preserve spatial parity symmetry – a consideration amenable to our endeavour on extending the basic MMT formalism by incorporating the Holst term, coupled to the mimetic field ϕ , in this paper. In fact, a pseudo-scalar (or, an *axion*-like) BI field γ is reasonable from the point of view of ensuring not only that the Holst term would affect the MMT cosmological solutions irrespective of the ϕ -coupling, but also that such solutions would conform to the general non-perception of the effect(s) of gravitational parity violation, at least at the background level.⁴ It is also worth mentioning here that the admittance of the Holst term does not disturb

³ Or an imperfect fluid, if one also considers incorporating term(s) containing higher order derivatives of ϕ , such as $(\square\phi)^2$, in the effective Lagrangian [3, 59, 192–195].

⁴ Note for e.g. that the recent Planck observations do not even provide a reasonably clear indication of the E - B and T - B cross-correlations in the Cosmic Microwave Background (CMB) polarization power spectra [223].

the overall conformal symmetry of the MMT theory, subject to the torsion parametrization suggested in Paper 1, as the associated Cartan transformations leave the axial mode $\mathcal{A}_\mu$ of torsion unaffected [191].

Following are the additional steps we take in our extended MMT formulation:

1. Using a Lagrange multiplier, we make an identification of ϕ with γ^2 , modulo some constant. This of course is meant for a simplification, not only from the point of view of dealing effectively with just a solitary scalar field ϕ , but also in letting ϕ manifest itself geometrically as the source of both the modes $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$ of torsion. Moreover, note that for such simplification, alongside the preservation of the gravitational parity symmetry, it suffices to identify ϕ with any even function of the pseudo-scalar BI field γ , the γ^2 being just the simplest possible choice.
2. We retain the overall MMT coupling $\beta(\phi)$, but now between ϕ and the entire torsional part of the extended U_4 Lagrangian (the Holst term inclusive). This is essential in the sense that we require to get an equation of motion leading to $\mathcal{T}_\mu \propto \partial_\mu \phi$ (as in Paper 1), so that the MMT fluid continues to be dust-like⁵ even in the extended setup. Furthermore, we consider the same form of the coupling function, viz. $\beta(\phi) \sim \phi^2$, as in Paper 1, for the motivations cited therein. In fact, such a quadratic coupling can have a natural appearance in MMT gravity, as demonstrated in our subsequent paper [229].
3. We also consider augmenting the effective MMT action with a higher derivative $(\square\phi)^2$ term, which leads to a non-zero sound speed c_s of mimetic matter density perturbations, side-by-side preserving the form of the background cosmological solution one obtains in its absence [2, 3].

With this setup, we derive the MMT field equations and constraints, working out in due course the form of the effective potential $W(\phi)$ in Sect. 2.

Subsequently, in Sect. 3 we carry out the study of the qualitative aspects of the scenario that emerges in the realm of the standard FRW cosmology. In particular, we show that the equation of state parameter w_x of the effective DE component, construed in such a scenario, always tends to fall below the Λ CDM value -1 , thereby exhibiting a *super-accelerating* phase of cosmic expansion. The transition from $w_x > -1$ to $w_x < -1$, or the so-called *phantom barrier* crossing, is nonetheless an intriguing pathology, with a fair amount of observational support. For instance, the results of

the Planck 2018 combined analysis [230] do conform to a certain extent the pre-existing signature of such a crossing at an epoch t_c in the recent past phase of evolution of the universe, and that the super-accelerated expansion is still continuing (albeit mildly) at the present epoch t_0 . However, commonly known scalar field DE models (quintessence, k-essence, etc.) cannot usually account for this without the involvement of ghost (or phantom) degree(s) of freedom. There is no such concern though, with our MMT extension scheme outlined above, as the corresponding Lagrangian we propose does not ostensibly consist of any ghost-like term, and in a reduced form looks precisely the same as that in the existing literature on mimetic gravity [2, 3]. It is the specific form of the potential $W(\phi)$ we get as a consequence of our weird constraining of the system, that leads to the phantom crossing. Actually, the total (mimetic + external) energy-momentum content of the system does not violate any of the energy conditions, except the strong one. It is the mathematical artifact of $W(\phi)$ we interpret as the effective DE constituent in analogy with conventional multi-component cosmological models, that behaves as a phantom. Such an interpretation, although not meant for pinpointing any physical DE candidate, helps us immensely in a bit by bit understanding of the emerging cosmological scenario.

In Sect. 4, we demonstrate the physical realizability of the phantom crossing, i.e. its occurrence at an epoch t_c close to but earlier than t_0 . However, this requires our MMT model parametric values to be kept within very narrow domains, which we illustrate after explicitly working out $w_x(t)$, albeit for an approximation justifiable from the point of view of the observational concordance on Λ CDM. Specifically, we presume the ϕ -coupled Holst term is not capable of producing much distortion of the Λ CDM solution we have had in its absence in Paper 1. We determine the validity of the approximation (and subsequently the extent of the super-acceleration) in Sect. 5, by examining the evolution profiles of the torsion parameters, viz. the norms of $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$, for specific parametric choices.

We conclude in Sect. 6 with a summary and a discussion on some relevant areas open for future investigations, and thereafter demonstrate in the Appendix, the stringent problem of the Ostrogradsky instabilities due to the higher derivative(s), and the possible avoidance of the same, via an alternative Lagrangian extension.

As to the notation and conventions, we adopt those in Paper 1, except using for brevity the units in which the gravitational coupling factor $\kappa^2 \equiv 8\pi G_N = 1$.

2 General formalism

To begin with, let us refer the reader to the main tenets of Paper 1 – in particular, the sections 2 and 3 therein, starting with the definition of the torsion tensor

⁵ That is, the fluid velocity u_μ (which is taken to be equal to $\partial_\mu \phi$, in analogy with k-essence cosmologies [224–228]) has to be tangential to the time-like affine geodesics (or the *auto-parallels*) in the U_4 space-time. More specifically, the fluid acceleration must vanish, and thereby confirm the auto-parallel equation (see the Appendix of Paper 1 [191]).

$$T_{\mu\nu}^{\alpha} := \tilde{\Gamma}_{\mu\nu}^{\alpha} - \tilde{\Gamma}_{\nu\mu}^{\alpha}, \quad (2.1)$$

in four-dimensional Riemann–Cartan (U_4) space-time, characterized by an asymmetric (but metric compatible) affine connection $\tilde{\Gamma}_{\mu\nu}^{\alpha}$. We look to formulate an extended MMT theory in which the (dimensionless) mimetic scalar field ϕ can drive the classical dynamics, while inducing both the vector modes of torsion, viz. its trace $T_{\mu}^{\nu} := T_{\mu\nu}^{\nu}$ and pseudo-trace $\mathcal{A}^{\sigma} := \epsilon^{\alpha\beta\gamma\sigma} T_{\alpha\beta\gamma}$. For this purpose we resort to the following minimal level generalization of the basic MMT action (of Paper 1):

$$S = S^{(m)} + \frac{1}{2} \int d^4x \sqrt{-g} \left[R(g_{\mu\nu}) + \alpha (\Box\phi)^2 + \beta(\phi) \left\{ \Theta(g_{\mu\nu}, T_{\mu\nu}^{\alpha}) + \frac{1}{2\gamma} {}^*\Theta(g_{\mu\nu}, T_{\mu\nu}^{\alpha}) \right\} + \lambda \cdot (X - 1) + \nu \cdot (\phi - s\gamma^2) \right], \quad (2.2)$$

where Θ and ${}^*\Theta$ denote the torsion-dependent part of the U_4 curvature scalar $\tilde{R}$ and the Hodge dual of the U_4 curvature tensor $\tilde{R}_{\alpha\beta\gamma\delta}$, respectively. In explicit forms, they are expressed in terms of T_{μ} , $\mathcal{A}_{\mu}$, and the (pseudo)tracefree mode $\mathcal{Q}_{\mu\nu}^{\alpha}$ of torsion, as

$$\begin{aligned} \Theta(g_{\mu\nu}, T_{\mu\nu}^{\alpha}) &:= \tilde{R}(g_{\mu\nu}, T_{\mu\nu}^{\alpha}) - R(g_{\mu\nu}) \\ &= -2\nabla_{\mu} T^{\mu} - \frac{2}{3} T_{\mu} T^{\mu} + \frac{1}{24} \mathcal{A}_{\mu} \mathcal{A}^{\mu} \\ &\quad + \frac{1}{2} \mathcal{Q}_{\alpha\mu\nu} \mathcal{Q}^{\alpha\mu\nu}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} {}^*\Theta(g_{\mu\nu}, T_{\mu\nu}^{\alpha}) &:= \epsilon^{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}(g_{\mu\nu}, T_{\mu\nu}^{\alpha}) \\ &= -\nabla_{\mu} \mathcal{A}^{\mu} - \frac{2}{3} T_{\mu} \mathcal{A}^{\mu} + \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \mathcal{Q}_{\alpha\beta}^{\mu} \mathcal{Q}_{\mu\gamma\delta}, \end{aligned} \quad (2.4)$$

with R being the usual Riemannian (R_4) curvature scalar, and the tensor indices being raised or lowered using the physical metric $g_{\mu\nu}$.

The other constituent terms and symbols in Eq. (2.2) are as explained as follows:

- $S^{(m)}$ is the external matter action.
- $\Box \equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$, with ∇_{μ} denoting the R_4 covariant derivative.
- α is a dimensionless constant parameter measuring the strength of coupling of $(\Box\phi)^2$.
- $\beta(\phi)$ denotes the overall contact coupling of ϕ with Θ , and its *Holst* extension, viz. $\frac{{}^*\Theta}{2\gamma}$, with γ being the (dimensionless) Berbero–Immrizi (BI) field, which is taken to

be a pseudo-scalar (or an *axion*) in order that the intrinsic parity in the theory is preserved.⁶

- λ and ν are scalar Lagrange multiplier fields, used respectively for enforcing the *mimetic constraint*⁷:

$$X \equiv -g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi = 1, \quad (2.5)$$

and for identifying ϕ with the squared BI field, i.e.

$$\phi \equiv s\gamma^2, \quad (2.6)$$

where s is a dimensionless coupling constant.

Note also the following:

1. The above action (2.2) has merits in itself from the point of view that it comprises of curvature and torsion constructs of only the lowest orders (linear and quadratic respectively). So there is apparently no scale dependence in the theory.
2. The overall conformal symmetry of the theory could be retained even in presence of the ϕ -coupled Holst term, viz. $\frac{\beta(\phi)}{2\gamma} {}^*\Theta \sqrt{-g}$, in the Lagrangian. In principle, just as in Paper 1, we may resort to the physical metric and torsion parametrizations:

$$g_{\mu\nu} = \hat{X} \hat{g}_{\mu\nu}, \quad (2.7)$$

$$T_{\mu\nu}^{\alpha} = \hat{T}_{\mu\nu}^{\alpha} + q \delta_{[\mu}^{\alpha} \partial_{\nu]} (\ln \hat{X}), \quad (2.8)$$

where $\hat{X} = -\hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi$ and q is a real numerical parameter, which although arbitrary in general, had been set to unity from certain standpoints in Paper 1 (see the section 2 therein [191]). Under the conformal and Cartan transformations of the fiducial metric and torsion, viz.

$$\hat{g}_{\mu\nu} \rightarrow e^{2\sigma} \hat{g}_{\mu\nu}, \quad (2.9)$$

$$\hat{T}_{\mu\nu}^{\alpha} \rightarrow \hat{T}_{\mu\nu}^{\alpha} + q (\delta_{\mu}^{\alpha} \partial_{\nu} \sigma - \delta_{\nu}^{\alpha} \partial_{\mu} \sigma), \quad (2.10)$$

$g_{\mu\nu}$ and $T_{\mu\nu}^{\alpha}$ remain invariant, for any given real scalar function of coordinates σ . Now, the conformal covariance of the physical torsion (i.e. its conformal invariance in the mixed form $T_{\mu\nu}^{\alpha}$) implies the same for its irreducible modes as well. In particular, the transformations (2.9) and (2.10) leave invariant such modes in the respective forms T_{μ} , $\mathcal{A}_{\mu}$ and $\mathcal{Q}_{\mu\nu}^{\alpha}$. Moreover, a given conformal

⁶ In particular, following the convention of Rovelli et al. [199, 200, 206], we keep γ in the denominator of the coefficient of ${}^*\Theta$. The factor of 2, also appearing in the denominator, is just for the purpose of the eliminating certain cumbersome numerical factors in our subsequent derivations.

⁷ Or, the condition one gets by demanding the physical metric $g_{\mu\nu}$ to be non-singular, and hence invertible.

transformation would not affect the scalar field ϕ , as it is dimensionless.⁸ By the same token, a conformal transformation would leave the pseudo-scalar γ invariant, since γ is dimensionless as well. Therefore, expressing the ϕ -coupled Holst term explicitly as

$$\frac{\beta(\phi)}{2\gamma} \star \Theta \sqrt{-g} = -\frac{\beta(\phi)}{2\gamma} \left[\partial_\mu (\sqrt{-g} g^{\mu\nu} \mathcal{A}_\nu) + \sqrt{-g} \left\{ \frac{2}{3} g^{\mu\nu} \mathcal{T}_\mu \mathcal{A}_\nu - \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} g_{\mu\nu} \mathcal{Q}_{\alpha\beta}^\mu \mathcal{Q}_{\gamma\delta}^\nu \right\} \right], \quad (2.11)$$

we can see its invariance under (2.9) and (2.10), since all of its elements, including the contravariant Levi-Civita tensor density $\epsilon^{\alpha\beta\gamma\delta} \equiv \sqrt{-g} \epsilon^{\alpha\beta\gamma\delta}$, remain unchanged.

3. The explicit ϕ -coupling with the Holst term may not seem to be necessary for the MMT extension we are seeking. In fact, it may seem much convenient to consider the action

$$S = S^{(m)} + \frac{1}{2} \int d^4x \sqrt{-g} \left[R + \alpha (\Box \phi)^2 + \beta(\phi) \Theta + \frac{1}{2\gamma} \star \Theta + \lambda (X-1) + \nu \{ \phi - f(\gamma) \} \right]. \quad (2.12)$$

This, unlike (2.2), does not contain any such coupling, and invokes, instead of (2.6), the constraint $\phi = f(\gamma)$, where $f(\gamma)$ needs to be an even function of the axionic BI field γ , in order to preserve intrinsic parity. We however, prefer to stick to the action (2.2), not only for some technical simplifications, but also for clarity in understanding certain results. Consider for instance the torsion mode $\mathcal{A}_\mu$, which is the most crucial ingredient in this extended MMT setup. As shown below (see Eqs. (2.15)), the action (2.2) leads to the result that $\mathcal{A}_\mu$ is determined completely by the dynamical solution of the BI field γ , regardless of the chosen form of the coupling function $\beta(\phi)$. On the other hand, it can be easily verified that $\mathcal{A}_\mu$ derived from the action (2.12) would depend not only on γ (and $\partial_\mu \gamma$), but also on $\beta(\phi)$. With either of the actions though, the determination of $\mathcal{A}_\mu$ would ultimately require the solution for ϕ , once the latter is pre-assigned to be identified with some even function of γ (for e.g. γ^2). Nevertheless, the clarity of wherefrom a given observable (such as the norm of $\mathcal{A}_\mu$) emerges is somewhat obscured if one uses (2.12).

Eliminating surface terms, we have the action (2.2) expressed in the form

$$S = S^{(m)} + \frac{1}{2} \int d^4x \sqrt{-g} \left[R(g_{\mu\nu}) + \alpha (\Box \phi)^2 + \lambda (X-1) + \nu (\phi - s\gamma^2) + \beta(\phi) \left\{ 2 (\ln \beta)_\phi \mathcal{T}_\mu \partial^\mu \phi - \frac{2}{3} \mathcal{T}_\mu \mathcal{T}^\mu + \frac{1}{24} \mathcal{A}_\mu \mathcal{A}^\mu + \frac{1}{2} \mathcal{Q}_{\alpha\mu\nu} \mathcal{Q}^{\alpha\mu\nu} \right\} + \frac{\beta(\phi)}{2\gamma} \left\{ [(\ln \beta)_\phi \partial_\mu \phi - \partial_\mu (\ln \gamma)] \mathcal{A}^\mu - \frac{2}{3} \mathcal{T}_\mu \mathcal{A}^\mu + \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \mathcal{Q}_{\alpha\beta}^\mu \mathcal{Q}_{\mu\gamma\delta} \right\} \right], \quad (2.13)$$

where $(\ln \beta)_\phi \equiv \frac{d}{d\phi} (\ln \beta)$, and the indices have been raised and lowered using the physical metric $g_{\mu\nu}$.

Assuming that no external sources of the torsion modes $\mathcal{T}_\mu$, $\mathcal{A}_\mu$ and $\mathcal{Q}_{\alpha\mu\nu}$ exist in the matter action $S^{(m)}$, the variation of (2.13) with respect to the each of them leads us to the constraints

$$\mathcal{T}_\mu = \frac{3}{2} \left[(\ln \beta)_\phi \partial_\mu \phi - \frac{\partial_\mu \gamma}{\gamma (\gamma^2 + 1)} \right], \quad (2.14)$$

$$\mathcal{A}_\mu = \frac{6 \partial_\mu \gamma}{\gamma^2 + 1}, \quad (2.15)$$

$$\mathcal{Q}_{\alpha\mu\nu} = 0 \quad (\forall \alpha, \mu, \nu), \quad (2.16)$$

showing that $\mathcal{A}_\mu$ is fully dependent on the dynamics of the BI field γ , whereas $\mathcal{T}_\mu$ is not so. Ultimately of course, the identification $s\gamma^2 = \phi$ [cf. Eq. (2.6)] would leave both these modes, $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$, as functions of ϕ and $\partial_\mu \phi$ only. In fact, we may justify such an identification from the point of view of what it entails, viz. the condition $\mathcal{T}_\mu \propto \partial_\mu \phi$ necessary for the MMT fluid to remain dust-like even in presence of torsion (as proved explicitly in the Appendix of Paper I [191]). Accordingly, the norms of $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$ can finally be expressed as

$$\mathcal{T}^2 \equiv -g^{\mu\nu} \mathcal{T}_\mu \mathcal{T}_\nu = \frac{9X}{4\phi^2} \left[\phi (\ln \beta)_\phi - \frac{s}{2(\phi + s)} \right]^2, \quad (2.17)$$

$$\mathcal{A}^2 \equiv -g^{\mu\nu} \mathcal{A}_\mu \mathcal{A}_\nu = \frac{9sX}{\phi(\phi + s)^2}, \quad (2.18)$$

where $X \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$.

Furthermore, choosing the same quadratic coupling as in Paper I (for the motivations cited therein), viz.

$$\beta(\phi) = \beta_0 \left(\frac{\phi}{\phi_0} \right)^2, \quad (2.19)$$

⁸ Otherwise, it would be $\phi \rightarrow e^{-\sigma} \phi$, if ϕ happens to be a unit mass dimension scalar field, as in most theories [231].

with ϕ_0, β_0 as some constant reference values, we may reduce the action (2.2) to

$$S = S^{(m)} + \frac{1}{2} \int d^4x \sqrt{-g} \left[R + \alpha (\Box\phi)^2 + \lambda (X - 1) - W(\phi) X \right]. \quad (2.20)$$

This is of the same form as in Paper 1, albeit with the effective potential

$$W(\phi) = 2\Lambda \left[1 - Y(\phi) \right], \quad (2.21)$$

where

$$Y(\phi) = \frac{s}{16(\phi + s)} \quad \text{and} \quad \Lambda = \frac{3\beta_0}{\phi_0}. \quad (2.22)$$

Because of the mimetic constraint $X = 1$ [cf. Eq. (2.5)], the gravitational field equations are therefore,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \left[T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(d)} \right], \quad (2.23)$$

where $R_{\mu\nu}$ is the usual (R_4) Ricci tensor, $T_{\mu\nu}^{(m)}$ is the matter energy-momentum tensor and $T_{\mu\nu}^{(d)}$ is its mimetic extension. The latter is given by

$$T_{\mu\nu}^{(d)} = \left[R + T^{(m)} - 2W \right] \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} W + 2\alpha (J_{\mu\nu} + J \partial_\mu \phi \partial_\nu \phi), \quad (2.24)$$

where $T^{(m)}$ denotes the trace of $T_{\mu\nu}^{(m)}$, and

$$J_{\mu\nu} := \frac{1}{2} \left[\partial_\mu (\Box\phi) \partial_\nu \phi + \partial_\nu (\Box\phi) \partial_\mu \phi \right] - \frac{1}{2} g_{\mu\nu} \left[\partial_\sigma (\Box\phi) \partial^\sigma \phi + \frac{1}{2} (\Box\phi)^2 \right], \quad (2.25)$$

$$J = g^{\mu\nu} J_{\mu\nu} = - \left[\partial_\sigma (\Box\phi) \partial^\sigma \phi + (\Box\phi)^2 \right]. \quad (2.26)$$

On the other hand, the variation of the action (2.20) with respect to ϕ leads to the field equation:

$$\nabla_\mu \left[\left(R + T^{(m)} - 2W \right) \partial^\mu \phi \right] = \frac{1}{2} W_\phi - \alpha \left[\Box(\Box\phi) + 2 \nabla_\mu (J \partial^\mu \phi) \right], \quad (2.27)$$

where

$$W_\phi \equiv \frac{dW}{d\phi} = -2\Lambda \frac{dY}{d\phi} = \frac{32\Lambda}{s} Y^2(\phi), \quad (2.28)$$

for the form of $Y(\phi)$ given above in Eq. (2.22).

3 Extended MMT cosmological evolution in the standard setup

In the standard spatially flat FRW space-time background, the mimetic constraint $X = 1$ as usual implies that the mimetic field could be treated as a dimensionally rescaled *cosmic clock* (viz. $\phi \equiv t$, the co-moving time),⁹ without loss of generality [1]. This obviously means $\phi_0 \equiv t_0$, the present age of the universe. Henceforth, we shall use t only, to refer to the mimetic field, as well as the comoving time coordinate, and always denote the total derivative with respect to t by an overhead dot (e.g. $\dot{Y} \equiv dY/dt$).

3.1 Extended MMT cosmological equations

Note first that the vanishing of the (pseudo-)tracefree torsion mode $\mathcal{Q}^\alpha_{\mu\nu}$ [cf. Eq. (2.16)] is perfectly consistent with the stringent constraints on the torsion tensor required for the FRW metric structure in the U_4 space-time [232]. In fact, such constraints also imply that only the temporal components of the other torsion modes $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$ should exist, and hence by Eqs. (2.14) and (2.15), the scalar fields ϕ and γ should only depend on time. Secondly, because of the form of $J_{\mu\nu}$ [cf. Eq. (2.25)], that stems out of the $\alpha(\Box\phi)^2$ term in the action (2.20), the expression (2.24) for $T_{\mu\nu}^{(d)}$ cannot in general be recast in the form of a perfect fluid energy-momentum tensor. Nevertheless, in the FRW space-time one has the relationship

$$\Box\phi = -3H, \quad (3.1)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, with $a(t)$ denoting the scale factor. Therefore, as is well-known [2,3,192], the Friedmann and Raychaudhuri equations (that could be derived from Eqs. (2.23) and (2.27)) reduce to their standard forms (viz. the ones in presence of a perfect fluid matter) only with the effective potential $W(t)$ rescaled by a constant factor that depends on the coefficient α of the $(\Box\phi)^2$ term in the mimetic action. Given the form of our MMT potential (2.21), this simply implies a rescaling of the constant parameter Λ (which we shall consider as a *cosmological constant*):

$$\Lambda \rightarrow \left(1 + 3c_s^2 \right) \Lambda, \quad \text{with} \quad c_s^2 = \frac{\alpha}{2 - 3\alpha}. \quad (3.2)$$

The quantity c_s here is nothing but the sound speed of the mimetic field perturbations, which the $\alpha(\Box\phi)^2$ term in the mimetic action famously leads to [2,3,192].

⁹ Actually, $\phi \equiv \kappa t$. The dimensional factor $\kappa = \sqrt{8\pi G_N}$ is however set to unity in this paper.

Consider now the cosmological matter to be the (pressureless) baryonic dust, specifically for the study of the late-time evolution of the universe:

$$T^{(m)\mu}_{\nu} = \text{diag}[-\rho^{(m)}, 0, 0, 0], \quad (3.3)$$

with the corresponding energy density given by

$$\rho^{(m)}(t) = \frac{\rho_0^{(m)}}{a^3(t)}, \quad \text{where } \rho_0^{(m)} \equiv \rho^{(m)}|_{t=t_0}. \quad (3.4)$$

The total effective energy density and pressure, ρ and p , which satisfy the standard Friedmann and conservation equations, viz.

$$H^2 = \frac{\rho}{3} \quad \text{and} \quad \dot{\rho} = -3H(\rho + p), \quad (3.5)$$

are then, respectively,

$$\rho = \rho^{(m)} + \rho^{(d)} = R + 3p, \quad (3.6)$$

$$p = p^{(d)} = -\frac{1}{2}W(t) = -\Lambda[1 - Y(t)]. \quad (3.7)$$

These of course reduce to that for the effective Λ CDM solution in Paper 1, once the BI field is turned off, i.e. in the limit the parameter $s \rightarrow 0$, whence $Y(t) \rightarrow 0$. In general however, (for $s \neq 0$) it is reasonable to consider the MMT energy density $\rho^{(d)}$ to consist of a pressureless (dust-like) cold dark matter (CDM) density $\rho^{(c)}$, and a left-over ρ_X , which may be treated as the effective dark energy (DE) density. In other words, while seeking a MMT cosmological dark sector, we may conveniently decompose its energy density as

$$\rho^{(d)}(t) = \rho^{(c)}(t) + \rho_X(t), \quad (3.8)$$

with

$$\rho^{(c)}(t) = \frac{\rho_0^{(c)}}{a^3(t)}, \quad \text{where } \rho_0^{(c)} \equiv \rho^{(c)}|_{t=t_0}. \quad (3.9)$$

The effective MMT fluid pressure $p^{(d)}$ (that equals the total pressure p), on the other hand, may inevitably be attributed to that of the effective DE constituent, i.e.

$$p_X(t) = p(t) = p^{(d)}(t) = -\Lambda[1 - Y(t)]. \quad (3.10)$$

Hence, from Eq. (3.5) it follows that

$$\dot{\rho}_X(t) = -3H(t)[1 + w_X(t)]\rho_X(t), \quad (3.11)$$

for a barotropic DE equation of state (EoS):

$$w_X(t) := \frac{p_X(t)}{\rho_X(t)} = -\frac{\Lambda[1 - Y(t)]}{\rho_X(t)}. \quad (3.12)$$

3.2 Cosmic super-acceleration in the extended MMT scenario

Eqs. (3.11) and (3.12) lead to the first order differential equation

$$\dot{w}_X(t) := -\left[\frac{\dot{Y}(t)}{1 - Y(t)} - 3\{1 + w_X(t)\}H(t)\right]w_X(t). \quad (3.13)$$

Now, given the form

$$Y(t) = \frac{s}{16(t + s)}, \quad (3.14)$$

we have

$$\dot{Y}(t) = -\frac{s}{16(t + s)^2} = -\frac{16}{s}Y^2(t), \quad (3.15)$$

$$\frac{1}{1 - Y(t)} = \frac{16(t + s)}{16t + 15s}. \quad (3.16)$$

Hence, for the constant parameter s to be presumably positive-valued,¹⁰ $\dot{Y}$ is negative definite, whereas $(1 - Y)^{-1}$ is positive definite. Also, in an expanding universe, the Hubble parameter H is positive definite. Therefore in Eq. (3.13), the first term within the square brackets is negative definite, and so is the second term, as long as $w_X > -1$. Since, in any viable DE model, the corresponding EoS parameter w_X is negative-valued, at least from reasonably high red-shifts (~ 100 or so) all the way to the distant future, we may safely say that $\dot{w}_X < 0$ whenever $w_X > -1$. The inferences that can be drawn from this are as follows:

- (i) If we have a *non-phantom* (i.e. accelerating, but not *super-accelerating*) regime in a moderately distant past, i.e. $w_X > -1$, then with the progress of time w_X would decrease and tend to attain the value -1 , as the slope $\dot{w}_X$ is always negative. In fact, at $t = t_C$, the epoch at which w_X equals -1 , the slope $\dot{w}_X < 0$. Thereafter (i.e. with the advancement of time further beyond t_C), $\dot{w}_X$ would continue to remain negative, albeit with progressively lesser magnitude, at least for a certain period after which it may become positive. Hence, given the circumstances, we would definitely have a *crossing* of the *phantom barrier* (or the $w_X = -1$ point) at the epoch $t = t_C$. There of course remain issues of sorting out finer details, for e.g. the condition

¹⁰ For $s < 0$, the identification $\phi \equiv t \equiv s\gamma^2$ implies an *imaginary* BI field. We prefer to keep that possibility out of the scope of this paper, although it is not unphysical, and in some sense reminiscent of Ashtekar's original works on the canonical quantum gravity formulation [207, 208].

for such a crossing to happen in the past (i.e. $t_C < t_0$), the possibility of the phantom regime to be *transient* (i.e. $\dot{w}_X$ becomes positive after a certain lapse of time beyond t_C , and increases to the extent that $w_X \geq -1$ once more), and so on. However, addressing to these issues requires us to solve Eq. (3.13) explicitly by setting up the appropriate boundary conditions, or(and) assert somehow the appropriate range of values of the parameter s . Of course, appropriateness here implies satisfying the requirement $t_C > t_T$, where t_T denotes the epoch of transition from a decelerating phase to an accelerating phase. Otherwise, our presumption of having a non-phantom regime in the past, and consequently the phantom barrier crossing, would be invalidated.

- (ii) If, on the contrary, we have a phantom (or, super-accelerating) regime in a moderately distant past, i.e. $w_X < -1$, then $\dot{w}_X$ would be < 0 (> 0) depending on whether the first term within the square brackets in Eq. (3.13) is bigger (smaller) than the second term therein. Evidently, there would be no phantom barrier crossing for $\dot{w}_X < 0$, whereas for $\dot{w}_X > 0$ such a crossing can happen, albeit from a phantom phase to a non-phantom phase. In either case though, no realistic cosmological evolution could be extracted unless we have a prior knowledge of a decelerating phase, followed by an accelerating but non-phantom phase, to precede the super-accelerating regime that we assume. This means that our consideration must have to be diverted back to the point (i) above, in order that a viable phantom crossing MMT model can emerge.

The bottomline is therefore that the extended MMT setup we are considering here would definitely give rise to a super-accelerating cosmological scenario, via a fractional reduction in the effective potential $W(t)$ by an amount $Y(t)$, over and above the cosmological constant Λ we have had in Paper I. Most importantly, this super-acceleration is not apparently stemming out of any ghost (or phantom) degree(s) of freedom in the effective action (2.2). It is the result of the specific inverse time-dependence $Y(t)$ acquires [cf. Eq. (3.14)], due to our chosen coupling $\beta(\phi) \sim \phi^2$, of the mimetic field and the Holst term, as well as the identification (2.6). In fact, as we shall explain below, there is nothing unusual with the energy conditions, at least as long as the strength of $Y(t)$, measured by the parameter s in Eq. (3.14), is reasonably weak. After all, it is this parameter s which decides not only the extent of the super-acceleration, but also the physical realizability of the same (see the next section for further clarification). A large value of s compared to the original MMT coupling strength $\beta_0 (= \Lambda t_0^3)$ necessarily means a strong super-acceleration, culminating from a large distortion of the Λ CDM solution, that we had in Paper I because of the constant potential $\overline{W} =$

2Λ therein. Since any major deviation from Λ CDM is not even remotely supported by the observations, we may safely assume the parameter s to be small, and for all practical purposes treat $Y(t)$ as a perturbation over $\overline{W} = 2\Lambda$. This may also be reasoned from a purely theoretical standpoint, as follows:

Recall that by Eq. (3.14), the fractional correction $Y(t)$ is a monotonically decreasing function of time. Its maximum value is only $\frac{1}{16}$, and that too is attained at $t = 0$. Therefore, at least for the late time cosmology, we may expect $Y(t)$ not to make any strong impact on the Λ CDM solution one gets in its absence. Such an expectation would possibly surmount to the level of a conviction once we consider the smallness of s . More specifically, a small s would imply that the solution of Eq. (3.11) may not possibly change its limiting value, viz. $\rho_X|_{s=0} = \Lambda$, by an amount of the order of Λ itself. Hence, Λ being the dominant term in it, ρ_X would be positive-valued, and so would be the total density $\rho (= \rho^{(m)} + \rho^{(c)} + \rho_X)$. The positivity of ρ_X may nonetheless seem a bit surprising, since it implies that our effective DE component is not characteristically phantom-like, yet it leads to a phantom barrier crossing. However, remember that such a component is liable to show some weirdness, as it is merely a mathematical artifact, meant only for facilitating our understanding of the cosmological dynamics in analogy with the conventional scalar field induced DE models.

Now, at least for small s , the phantom barrier would not be breached to that extent which would imply that the quantity $|1 + w_X| \sim \mathcal{O}(1)$, i.e. $|p_X|$ exceeding ρ_X by an amount of the order of ρ_X itself. This can be inferred right away from Eqs. (3.10) and (3.14) which lead to

$$\rho_X(t) + p_X(t) = \delta\rho_X(t) + \frac{\Lambda s}{16(t+s)}, \quad (3.17)$$

where $\delta\rho_X(t) = \rho_X(t) - \Lambda$ denotes the correction in the DE density over its limiting value Λ . The second term on the right hand side of Eq. (3.17) is positive definite (under our presumption $s > 0$ of course). Therefore, $(\rho_X + p_X)$ would become negative only when $\delta\rho_X$ becomes negative and exceeds that term in magnitude. However, as argued above, $\delta\rho_X$ cannot be of the order of Λ . So there is no question of $|\rho_X + p_X|$ reaching up to the order of ρ_X . By all means then, the total equation of state parameter of the system, $w = p/\rho$, would not get reduced to a value < -1 , even in the phantom regime. The reason is obvious – in order to make $w < -1$, i.e. $(\rho + p) < 0$, the correction $|\delta\rho_X|$ would need to overcome not only the second term in Eq. (3.17), but also $\rho^{(m)}$ and $\rho^{(c)}$, which is impracticable at least for small s . We would thus have $\rho + p > 0$, in addition to $\rho > 0$. Therefore, both the *null* and *weak* energy conditions would hold [233–235]. Moreover, w not crossing -1 means that it would have a fractional value, i.e. $|p| < \rho$. So the *dominant* energy

condition would hold as well. Nevertheless, the *strong* energy condition, viz. $\rho + 3p > 0$, or $w > -\frac{1}{3}$, would be violated as usual for an accelerated cosmic expansion.

4 Viable MMT cosmology with phantom barrier crossing

Let us, for convenience, treat the inverse squared BI field as an effective coupling function, viz.

$$\sigma(\tau) := \gamma^{-2}(\tau), \quad \text{where } \tau = \frac{t}{t_0}. \quad (4.1)$$

Since $\gamma^{-2} = s\phi^{-1} \equiv st^{-1}$, by Eq. (2.6), we can write

$$\sigma(\tau) = \frac{\sigma_0}{\tau}, \quad \text{where } \sigma_0 \equiv \sigma|_{\tau=1} = \frac{s}{t_0}. \quad (4.2)$$

Accordingly, Eq. (3.13) can be recast in the form

$$\frac{d\mathbf{w}_x}{d\tau} := - \left[\frac{1}{1-Y(\tau)} \frac{dY}{d\tau} - 3t_0(1+\mathbf{w}_x(\tau))H(\tau) \right] \mathbf{w}_x(\tau), \quad (4.3)$$

where

$$Y(\tau) = \frac{\sigma(\tau)}{16[1+\sigma(\tau)]} = \frac{\sigma_0}{16(\tau+\sigma_0)}. \quad (4.4)$$

Therefore,

$$\frac{dY}{d\tau} = -\frac{16}{\sigma_0} Y^2(\tau), \quad (4.5)$$

$$\frac{1}{1-Y(\tau)} = \frac{16(\tau+\sigma_0)}{16\tau+15\sigma_0}, \quad (4.6)$$

which when substituted back in Eq. (4.3) gives

$$\frac{d\mathbf{w}_x}{d\tau} := \left[\frac{\sigma_0}{(\tau+\sigma_0)(16\tau+15\sigma_0)} + 3t_0\{1+\mathbf{w}_x(\tau)\}H(\tau) \right] \mathbf{w}_x(\tau). \quad (4.7)$$

4.1 Effective dark energy state parameter in a linear approximation

In order to solve Eq. (4.7), for the effective dark energy EoS parameter $\mathbf{w}_x$, one first requires to find the functional form of $H(\tau)$ by solving the cosmological equations (3.5). While getting an exact analytical solution is always a hard proposition, numerical methods may be applied, with a lot of intuition though, from the point of view of setting priors on the parameters σ_0 , t_0 and so on. We may however take a shorter

course, since for the reasons given earlier, it suffices us to look for small deviations from the Λ CDM scenario. Let us therefore resort to the following expansions:

$$\mathbf{w}_x(\tau) = \mathbf{w}_x^{(0)}(\tau) + \sigma_0 \mathbf{w}_x^{(1)}(\tau) + \sigma_0^2 \mathbf{w}_x^{(2)}(\tau) + \dots, \quad (4.8)$$

$$H(\tau) = H^{(0)}(\tau) + \sigma_0 H^{(1)}(\tau) + \sigma_0^2 H^{(2)}(\tau) + \dots, \quad (4.9)$$

by assuming the parameter s to be small enough, so that $\sigma_0 = s/t_0$ can be treated as a perturbation parameter. The unperturbed quantities correspond to those for Λ CDM, viz.¹¹

$$\mathbf{w}_x^{(0)} = -1 \quad \text{and} \quad H^{(0)}(\tau) = \overline{H}(\tau), \quad (4.10)$$

whereas the linear order perturbation equation, obtained from Eqs. (4.7)–(4.9), is given by

$$\frac{d\mathbf{w}_x^{(1)}}{d\tau} = - \left[\frac{1}{16\tau^2} + 3t_0 \mathbf{w}_x^{(1)}(\tau) \overline{H}(\tau) \right]. \quad (4.11)$$

For simplicity (and also the adequacy justified later on), we shall confine ourselves to this order of perturbation in what follows.

Consider first the zeroth order scenario, i.e. the effective Λ CDM evolution in the MMT cosmological setup. The total pressure being $\overline{p} = -\Lambda$, the corresponding Friedmann and conservation equations lead to following equation for the total energy density

$$\frac{d\overline{\rho}}{d\tau} = -t_0 \sqrt{3\overline{\rho}(\tau)} \left[\overline{\rho}(\tau) - \Lambda \right], \quad (4.12)$$

with solution

$$\overline{\rho}(\tau) = \Lambda \coth^2 \zeta(\tau), \quad (4.13)$$

where

$$\zeta(\tau) = \zeta_0 \tau, \quad \zeta_0 = \zeta|_{\tau=1} = \frac{t_0 \sqrt{3\Lambda}}{2}. \quad (4.14)$$

So, by the Λ CDM Friedmann equation, the corresponding Hubble parameter assumes the form

$$\overline{H}(\tau) = \sqrt{\frac{\overline{\rho}(\tau)}{3}} = \frac{2\zeta_0}{3t_0} \coth(\zeta_0 \tau). \quad (4.15)$$

Using this in (4.14) we have, more conveniently,

$$\zeta(\tau) = \tanh^{-1} \sqrt{\overline{\Omega}^{(\Lambda)}(\tau)}, \quad \zeta_0 = \tanh^{-1} \sqrt{\overline{\Omega}_0^{(\Lambda)}}, \quad (4.16)$$

¹¹ Throughout this paper, we denote the Λ CDM parameters and functions by placing an overbar, for e.g. the Λ CDM total energy density denoted by $\overline{\rho}$, the corresponding total pressure denoted by $\overline{p}$, and so on.

where $\bar{\Omega}^{(\Lambda)} = \Lambda/\rho$ is the Λ -density parameter (specific to the zeroth order case), with value $\bar{\Omega}_0^{(\Lambda)}$ at the present epoch $t = t_0$ (or $\tau = 1$).

Resort now to the linear (or first) order approximation, in which case we can determine

$$w_x(\tau) \approx -1 + \sigma_0 w_x^{(1)}(\tau), \quad (4.17)$$

by plugging Eq. (4.15) back in Eq. (4.11), and solving the resulting equation. This gives

$$w_x^{(1)}(\tau) = \frac{1}{16} \left[\frac{1}{\tau} - \left(1 - 16 w_{x0}^{(1)} \right) \frac{\sinh^2 \zeta_0}{\sinh^2(\zeta_0 \tau)} - \frac{\zeta_0 \{ \text{Shi}(2\zeta_0 \tau) - \text{Shi}(2\zeta_0) \}}{\sinh^2(\zeta_0 \tau)} \right], \quad (4.18)$$

where $w_{x0}^{(1)} \equiv w_x^{(1)}|_{\tau=1}$, and

$$\text{Shi}(x) := \int_x^\infty \frac{\sinh x'}{x'} dx' = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2 (2k)!}, \quad (4.19)$$

is the hyperbolic sine integral function.

In order to get a quantitative measure of the extent to which w_x deviates from the Λ CDM value $w_x^{(0)} = -1$ at different epochs of time, it is convenient for us to figure out the circumstances in which the time-evolution of the percentage change

$$\begin{aligned} \mu(\tau) &:= \frac{w_x(\tau) - w_x^{(0)}(\tau)}{w_x^{(0)}(\tau)} \times 100 \\ &= -100 \left[1 + w_x(\tau) \right], \end{aligned} \quad (4.20)$$

would comply with the observations. From Eqs. (4.17), (4.18) and (4.20) we see that at the present epoch $\tau = 1$ (or $t = t_0$),

$$\sigma_0 w_{x0}^{(1)} \approx -\frac{\mu_0}{100}, \quad \text{where } \mu_0 = \mu(t_0). \quad (4.21)$$

Observational estimations of the value of $w_x(t_0)$, considering say, the well-known Chevallier–Polarski–Linder (CPL) ansatz [239, 240], imply that the percentage correction $\mu(t_0) \equiv \mu_0$ to be typically $\mathcal{O}(1)$ or lesser, up to the 1σ error limits. For instance, the Planck 2018 results of the analysis of CMB TT,TE,EE+lowE combined with Lensing, Baryon Acoustic Oscillation (BAO) and type Ia Supernovae (SNe) data show [230]

$$w_x(t_0) = -1.028 \pm 0.032, \quad (4.22)$$

at 68% confidence level. This means $\mu_0 \in (-0.04, 6)$, which indicates not only a cosmological evolution very close to

Λ CDM, but also a tendency of the universe to be in a super-accelerating (or phantom) state at least at low red-shifts. Therefore, while seeking a viable model of a dynamical dark phase of the universe, one requires to ponder heavily on prioritizing the demand that the time-evolution of w_x must turn out to be slow enough, so that the Λ CDM solution is not distorted too much at least at the later epochs. So our linear approximation is duly justified. Moreover, it is desirable to have a provision for a phantom barrier crossing at an epoch in the recent past regime, in order that the expansion of the universe super-accelerates, albeit mildly, at the present epoch. After all, allowing for such a crossing provides flexibility in making statistical estimations of the model parameters (using the observational data), as opposed to say, the quintessence scenarios in which one has to comply with the condition $w_x > -1$ beforehand. As we shall see in the next subsection, our approximated solution for w_x can indeed lead to the perception of a viable phantom crossing scenario, however at the expense of restricting the domain of the parameters σ_0 and μ_0 severely.

4.2 Parametric bounds from observational results

For illustrative purposes, let us resort to the results of Planck 2018 TT,TE,EE+lowE+Lensing+BAO combined analysis for the base Λ CDM model [230], as in Paper 1. In particular, we shall use the best fit value of the estimated Λ -density parameter at the present epoch, viz. $\bar{\Omega}_0^{(\Lambda)} = 0.6889$, which implies

$$\zeta_0 = \tanh^{-1} \sqrt{\bar{\Omega}_0^{(\Lambda)}} = 1.1881. \quad (4.23)$$

Now, it is obvious that the linear approximation would not lead to a significant effect on the deceleration-to-acceleration transition epoch t_T . So to determine the realistic parametric range of σ_0 or μ_0 with reasonable precision, it would suffice us to approximate t_T to be that for Λ CDM,¹² which can be obtained simply from the condition for cosmic acceleration, viz. $\bar{w}(\tau_T) = -\frac{1}{3}$, where $\tau_T \equiv t_T/t_0$ and

$$\bar{w}(\tau) := \frac{-\Lambda}{\bar{\rho}(\tau)} = -\bar{\Omega}^{(\Lambda)}(\tau) = -\tanh^2(\zeta_0 \tau) \quad (4.24)$$

is the Λ CDM total EoS parameter. With $\zeta_0 = 1.1881$, it follows that

$$\tau_T \equiv \frac{t_T}{t_0} = \frac{1}{\zeta_0} \tanh^{-1} \sqrt{\frac{1}{3}} = 0.554. \quad (4.25)$$

¹² In fact, the deviation of the actual t_T from its Λ CDM value turns out to be about $0.056 \sigma_0$, as shown rigorously by using the Planck 2018 TT,TE,EE+lowE+Lensing+BAO results in a subsequent paper [236]. With a presumably small σ_0 value, this deviation is really quite insignificant.

Moreover, from the parametric relation (4.21), we see that the solution (4.18) can admit a viable phantom barrier crossing only when, for a fixed σ_0 , the range of μ_0 is restricted, or vice versa. Of course, by ‘viable’ we mean that such a crossing has to take place at an epoch t_c in the near past (or low redshift) era of the evolving universe, when the latter’s expansion is already accelerating, after the end of the decelerating phase at $t = t_T$. In other words, t_c must lie within the temporal range (t_T, t_0) .

Again, since the phantom crossing implies

$$0 = 1 + w_x(\tau_c) \approx w_x^{(1)}(\tau_c), \quad \text{where } \tau_c \equiv \frac{t_c}{t_0}, \quad (4.26)$$

we have from the above Eqs. (4.18) and (4.21),

$$\mu_0 \approx \frac{25\sigma_0}{4} \frac{\zeta_0}{\sinh^2 \zeta_0} \left[\frac{\sinh^2(\zeta_0 \tau_c)}{\zeta_0 \tau_c} - \frac{\sinh^2 \zeta_0}{\zeta_0} - \text{Shi}(2\zeta_0 \tau_c) + \text{Shi}(2\zeta_0) \right]. \quad (4.27)$$

This relationship is crucial for determining the parametric bounds, as we shall see below.

I. For a fixed σ_0 , the requisite parametric range of μ_0 that meets the criterion $t_c \in (t_T, t_0)$ is by no means broad, especially given the condition that σ_0 has to be small in order to make the linear approximation tenable. In particular, even if we consider fixing σ_0 at a reasonably significant value 0.05, the plots of the percentage correction $-\mu(\tau)$ [cf. Eq. (4.20)] in Fig. 1a, for various parametric values of μ_0 , make it evident that the latter cannot go much further beyond 0.1. Otherwise (for e.g. when $\mu_0 = 0.15$), t_c would fall below t_T , thus making the phantom crossing unrealistic. Analytically, this can be easily seen from Eq. (4.27), since with $\zeta_0 = 1.1881$ and $t_T = 0.554 t_0$ [cf. Eqs. (4.23) and (4.25) respectively], the criterion $t_c \in (t_T, t_0)$ implies

$$0 < \mu_0 \lesssim 2.367 \sigma_0 \quad (= 0.118, \text{ for } \sigma_0 = 0.05). \quad (4.28)$$

The solid curve in Fig. 1a shows the $-\mu(\tau)$ variation for the optimum value $\mu_0 = 0.118$. The minimum (or turning) point of this curve, at $\tau = \tau_m = 0.945$, is of particular significance since it marks the earliest possible occurrence of the minimum of $-\mu(\tau)$ corresponding to a μ_0 value legitimate for a realistic phantom crossing. More specifically, at the crossing point $\tau = \tau_c$ the function $-\mu(\tau)$ requires to be decreasing at such a rate that the minimum is reached very close to the present epoch, or afterwards. This is in fact a *general* requirement, irrespective of the chosen fixed value $\sigma_0 = 0.05$, as verified below.

From Eqs. (4.17) and (4.20) we see that the minimization of $-\mu(\tau)$ implies that of $w_x^{(1)}(\tau)$. Therefore, using Eqs. (4.11), (4.15), (4.18) and (4.21), we get

(a) $-\mu$ vs t/t_0 for $\sigma_0 = 0.05$ and $\mu_0 \in (0, 0.15)$.

(b) $-\mu$ vs Y/Y_0 for $\sigma_0 = 0.05$ and $\mu_0 \in (0, 0.15)$.

Fig. 1 Variation of the percentage change in the effective DE equation of state: $-\mu = 100(1 + w_x)$, with **a** the rationalized cosmic time $\tau = t/t_0$, and **b** the rationalized correction $y = Y/Y_0$ in the MMT potential, for a fixed parametric value $\sigma_0 = 0.05$ and discrete values of the parameter $\mu_0 \in (0, 0.15)$. The thick dots indicate the minima of the plots

$$\begin{aligned} \frac{\tanh(\zeta_0 \tau_m)}{2 \zeta_0 \tau_m} &= -16 \tau_m w_x^{(1)}(\tau_m) \\ &\approx \frac{\zeta_0 \tau_m}{\sinh^2(\zeta_0 \tau_m)} \left[\left(1 + \frac{4\mu_0}{25\sigma_0} \right) \frac{\sinh^2 \zeta_0}{\zeta_0} \right. \\ &\quad \left. + \text{Shi}(2\zeta_0 \tau_m) - \text{Shi}(2\zeta_0) \right] - 1, \end{aligned} \quad (4.29)$$

or, equivalently, by virtue of the relationship (4.27),

$$\left[1 + \frac{\tanh(\zeta_0 \tau_m)}{2 \zeta_0 \tau_m}\right] \frac{\sinh^2(\zeta_0 \tau_m)}{\zeta_0 \tau_m} - \text{Shi}(2\zeta_0 \tau_m) \approx \frac{\sinh^2(\zeta_0 \tau_c)}{\zeta_0 \tau_c} - \text{Shi}(2\zeta_0 \tau_c). \quad (4.30)$$

Solving this equation numerically, for $\zeta_0 = 1.1881$, we find $\tau_m = 0.945$ in the optimal case $\tau_c = \tau_r = 0.554$ (or equivalently $\mu_0 = 2.367 \sigma_0$), without any allusion to the typical fixation of μ_0 or σ_0 .

It is also worth examining how the percentage correction $-\mu$ varies with the quantity $y(t) \equiv Y(t)/Y_0$, where $Y_0 \equiv Y(t_0)$. After all, it is the specific form of the fractional correction in the potential, $Y(t)$, given by Eq. (4.4), which makes the cosmic super-acceleration plausible. Now, as we see from Fig. 1b, the criterion $t_c \in (t_r, t_0)$, and correspondingly the viable range of μ_0 (for a fixed σ_0), could be ascribed to the valid zone of occurrence of $y = y_m$, i.e. the point at which $-\mu$ is minimized. In particular, the above bound $\mu_0 \lesssim 2.367 \sigma_0$ holds only for $y_m \lesssim 1.057$, thus implying that the $-\mu(y)$ variation should be such that the minimum point y_m is reached no earlier than an epoch just surpassing the present epoch. This is nonetheless a general criterion for the viability of the phantom crossing (not specific to the choice $\sigma_0 = 0.05$). Figure 1b also illustrates the *non-transience* of the phantom regime, i.e. the super-acceleration is ever-lasting (although it slows down progressively after reaching y_m). Analytically, this follows from Eq. (4.4) which implies $y \equiv Y/Y_0 \approx \tau^{-1}$. Therefore, y tends to vanish in the asymptotic limit ($\tau \rightarrow \infty$), and so does the correction μ .

II. If, on the other hand, we resort to a fixed value of μ_0 , say a fairly low one ($= 0.05$), then as illustrated in Fig. 2a, and as derived from Eq. (4.27), a realistic phantom barrier crossing requires a lower limit on the parameter σ_0 , close to (but greater than) 0.02, for the ζ_0 and τ_r estimated above.

However, if we aspire to get a much larger μ_0 , say ~ 0.5 (i.e. about $\frac{1}{2}\%$ correction in w_x), then by Eq. (4.27), we require $\sigma_0 \gtrsim 0.211$, which may nonetheless compel us to go beyond the linear approximation. The above arguments concerning the criterion $t_c \in (t_r, t_0)$ being ascribed to the domain of appearance of the turning points $\tau = \tau_m$ would apply here as well. Also for completeness, it is worth examining the variation of $-\mu$ with $y = Y/Y_0$, for μ_0 fixed at 0.05 and various parametric values of σ_0 . The corresponding plots shown in Fig. 2b display the same non-transient phantom regime as illustrated in Fig. 1b above.

5 Evolving torsion parameters and the extent of the Super-acceleration

Let us recall the expressions (2.17) and (2.18) for the relevant torsion parameters, viz. the norms of the torsion mode vectors

(a) $-\mu(t)$ vs t/t_0 for $\mu_0 = 0.05$ and $\sigma_0 \in (0.02, 0.05)$.

(b) $-\mu(t)$ vs Y/Y_0 for $\mu_0 = 0.05$ and $\sigma_0 \in (0.02, 0.05)$.

Fig. 2 Variation of the percentage change in the effective DE equation of state: $-\mu = 100(1 + w_x)$, with **a** the rationalized cosmic time $\tau = t/t_0$, and **b** the rationalized correction $y = Y/Y_0$ in the MMT potential, for a fixed parametric value $\mu_0 = 0.05$ and discrete values of the parameter $\sigma_0 \in (0.02, 0.05)$. The thick dots indicate the minima of the plots

$\mathcal{T}_\mu$ and $\mathcal{A}_\mu$. With the MMT coupling function $\beta(\phi)$ given by Eq. (2.19), and the identification $s\gamma^2 = \phi \equiv t$, such expressions reduce to

$$\mathcal{T}^2 \equiv -g^{\mu\nu} \mathcal{T}_\mu \mathcal{T}_\nu = \frac{9}{t^2} \left[1 - \frac{s}{4(t+s)} \right]^2, \quad (5.1)$$

$$\mathcal{A}^2 \equiv -g^{\mu\nu} \mathcal{A}_\mu \mathcal{A}_\nu = \frac{9s}{t(t+s)^2}. \quad (5.2)$$

Our interest, of course, is in getting a quantitative measure of the extent to which the scenario in Paper 1 is modified when the parameter $s \neq 0$. For this we require a careful examination of the time-evolution of

$$U_T := \frac{\mathcal{T}^2 - \bar{\mathcal{T}}^2}{\bar{\mathcal{T}}^2} \quad \text{and} \quad U_A := \frac{\mathcal{A}^2}{\bar{\mathcal{T}}^2}, \quad (5.3)$$

which are the fractional changes over the *only* existent torsion parameter in Paper 1, viz.

$$\bar{\mathcal{T}}^2 \equiv \left[-g^{\mu\nu} \mathcal{T}_\mu \mathcal{T}_\nu \right]_{s=0} = \frac{9}{t^2}, \quad (5.4)$$

that had led to the Λ CDM solution therein [191].

Using Eqs. (5.1), (5.2) and (5.4), along with the definitions (4.2), we can write

$$U_T(\tau) = -\frac{U_A(\tau)}{2} \left(1 + \frac{7\sigma_0}{8\tau} \right), \quad (5.5)$$

$$U_A(\tau) = \frac{\sigma_0 \tau}{(\tau + \sigma_0)^2}, \quad (5.6)$$

and consequently re-express the fractional correction to the effective potential $\bar{W} = 2\Lambda$ of Paper 1 as

$$Y(\tau) = \frac{\sigma_0}{16(\tau + \sigma_0)} = -\frac{U_A(\tau) + 16U_T(\tau)}{112}. \quad (5.7)$$

Note that so far there is no approximation – all the relations (5.5)–(5.7) are exact. Nevertheless, from the perspective of our analysis in the previous section (anticipating per se, mild deviations from Λ CDM due to the MMT extension), we shall only consider small values of the parameter σ_0 from here on.

Now for small σ_0 , the fractional corrections U_A and U_T , although decrease with time, remain of the same order of magnitude as σ_0 itself, in the temporal range of interest $t \in (t_T, t_0)$, or $\tau \in (\tau_T, 1)$. However, the resulting fractional change in the potential, Y , turns out be comparatively much smaller, as illustrated with two fiducial settings $\sigma_0 = 0.05$ and $\sigma_0 = 0.1$ in Fig. 3a, b respectively. This smallness of Y , and as a consequence the weak time-variation of the effective DE equation of state parameter w_x , of course justifies the linear approximation in the preceding section. Note also that the peak values of U_A , $|U_T|$ and Y are independent of the value of σ_0 . In particular, the above expression for U_A implies that the latter attains its maximum value ($= 0.25$) at $\tau = \sigma_0$, as illustrated in Fig. 3a, b. On the other hand, $|U_T|$ and Y being monotonically decreasing functions of τ , have their maximum values (equal to 0.4375 and 0.0625 respectively) at $\tau = 0$.

As to the values of U_A , $|U_T|$ and Y at the present epoch $t = t_0$, it is evident from Eqs. (5.5)–(5.7) that all of them are

(a) U_A , U_T and Y vs t/t_0 for $\sigma_0 = 0.05$.

(b) U_A , U_T and Y vs t/t_0 for $\sigma_0 = 0.1$.

Fig. 3 Variations of the fractional changes in the torsion parameters, viz. $U_A(\tau)$ and $U_T(\tau)$, and their resultant $Y(\tau)$, for fixed parametric values: **a** $\sigma_0 = 0.05$ and **b** $\sigma_0 = 0.1$

smaller than σ_0 , for $0 < \sigma_0 \lesssim 1$:

$$U_0^A \equiv U_A(t_0) = \sigma_0 \left[1 + \varepsilon_0^A \right], \quad (5.8)$$

$$|U_0^T| \equiv |U_T(t_0)| = \frac{\sigma_0}{2} \left[1 + \varepsilon_0^T \right], \quad (5.9)$$

$$Y_0 \equiv Y(t_0) = \frac{\sigma_0}{16} \left[1 + \varepsilon_0^Y \right], \quad (5.10)$$

(a) σ_0 , U_0^A , U_0^T and Y_0 vs t_c/t_0 for $\mu_0 = 0.05$.

(b) σ_0 , Y_0/σ_0 and $|\varepsilon_0^A|$ vs t_c/t_0 for $\mu_0 = 0.05$.

Fig. 4 Approximated variations of the parametric values of **a** σ_0 and the fractional changes in the torsion parameters at the present epoch, viz. U_0^A and U_0^T , as well as their resultant Y_0 , and **b** σ_0 , Y_0/σ_0 and $|\varepsilon_0^A|$, i.e. the fractional deviation of U_0^A from σ_0 , with the rationalized phantom crossing time $\tau_c = t_c/t_0$ in the viable range (0.55, 1), for a fixed $\mu_0 = 0.05$

where

$$\varepsilon_0^A = -\frac{\sigma_0(2 + \sigma_0)}{(1 + \sigma_0)^2} = -2\sigma_0 + \mathcal{O}(\sigma_0^2), \quad (5.11)$$

$$\varepsilon_0^T = -\frac{\sigma_0(9 + 8\sigma_0)}{8(1 + \sigma_0)^2} = -\frac{9\sigma_0}{8} + \mathcal{O}(\sigma_0^2), \quad (5.12)$$

$$\varepsilon_0^Y = -\frac{\sigma_0}{1 + \sigma_0} = -\sigma_0 + \mathcal{O}(\sigma_0^2). \quad (5.13)$$

Now, using Eq. (4.27) one can determine how σ_0 , and hence U_0^A , $|U_0^T|$ and Y_0 , vary approximately with the rationalized phantom crossing time $\tau_c (= t_c/t_0)$, for a fixed value of the percentage correction μ_0 in w_X at $t = t_0$ (or $\tau = 1$). Figure 4a shows such parametric variations, within the viable range $t_c \in (t_T, t_0)$, or $\tau_c \in (0.55, 1)$, for a fiducial setting $\mu_0 = 0.05$ (which nonetheless is well within the 1σ error estimates from recent observations [230]).

A stringent upper limit can in principle be placed on t_c , and hence on σ_0 , for a given value of μ_0 , from the argument that Y_0 would be of significance only when it is $\gtrsim \sigma_0|\varepsilon_0^A|$, where $|\varepsilon_0^A|$ is the absolute fractional deviation of U_0^A from σ_0 . Specifically, this could be seen as a consequence of a stringent limit being imposed on the extent of the linear approximation, for which $U_0^A \approx \sigma_0$, by Eq. (5.8). The higher order change, i.e. the deviation of the exact U_0^A from σ_0 , given by the amount $\sigma_0|\varepsilon_0^A|$, need to be $\lesssim Y_0$, because otherwise Y_0 would also be a higher order effect, which would in turn imply the invalidity of the linear approximation¹³. As shown in Fig. 4b, the effect of Y_0 of a very small amount ($\lesssim 0.061\sigma_0$), could yet be perceptible in the linear approximation (i.e. satisfy the condition $Y_0 \gtrsim \sigma_0|\varepsilon_0^A|$) and lead to a reasonably significant amount of μ_0 ($= 0.05$), only if $t_c \lesssim 0.72t_0$, or $\sigma_0 \lesssim 0.032$ correspondingly. In principle, a lesser upper bound on σ_0 , culminating to an even smaller amount of Y_0 , may provide a more significant μ_0 , however at the expense of reducing the upper limit of t_c . One can therefore infer the following:

- As the phantom crossing time t_c cannot be smaller than the deceleration-to-acceleration transition time t_T , the percentage correction μ_0 (in the effective DE equation of state $w_X(t_0)$) cannot get enhanced much further beyond the value 0.05, at least in the linear approximation.
- Nevertheless, the extended MMT-cosmological scenario seems more attractive for a t_c value closer to t_T , i.e. for an earlier commencement of the super-accelerating regime, from the point of view of having larger value of μ_0 with lesser strength of coupling σ_0 of the mimetic field with the Holst term. In other words, the weaker the MMT extension, the stronger is the effect in $w_X(t_0)$. There is a limit to this of course, since t_c must not fall short of t_T .

¹³ One can alternatively consider imposing $Y_0 \gtrsim \sigma_0|\varepsilon_0^T|$ or $Y_0 \gtrsim \sigma_0|\varepsilon_0^Y|$, however the resulting upper bound on σ_0 (or t_c) would then be less stringent, since both $|\varepsilon_0^T|$ and $|\varepsilon_0^Y|$ are smaller than $|\varepsilon_0^A|$, by Eqs. (5.11)–(5.13).

6 Conclusion

A viable phantom crossing evolution of a unified cosmological dark sector is thus demonstrated by extending the basic mimetic-metric-torsion (MMT) formalism with an explicit coupling of the mimetic field ϕ and the Holst term, motivated from the following:

Firstly, its compliance with the basic precept of MMT gravity, viz. the preservation of conformal symmetry while letting ϕ to manifest geometrically as the source of torsion (or certain mode(s) thereof).

Secondly, and most crucially, its instrumentality in making torsion's main characteristic, viz. the anti-symmetry, accountable for the dynamical evolution of a given physical system.

To be more specific, the reason why we have incorporated the ϕ -coupled Holst term is that, apart from keeping the conformal symmetry of the MMT theory unaffected, it ideally meets our objective of perceiving a plausible effect of the axial (or pseudo-trace) mode $\mathcal{A}_\mu$ of torsion, in addition to that of the latter's trace mode $\mathcal{T}_\mu$, on the cosmic evolution profile. In particular, for clarity in interpreting results, irrespective of the explicit ϕ -coupling, we have preferred to resort to a non-topological characterization of the *bare* Holst term by promoting its coefficient, viz. the Barbero–Immirzi (BI) parameter γ , to the status of a field. Considering this field to be a (presumably *primordial*) pseudo-scalar (or, an axion), we could avoid gravitational parity violation, and hence a direct confrontation with the latter's miniscule observational signature at cosmological scales. Consequently, an identification of ϕ with γ^2 (i.e. the simplest possible even function of γ), using a scalar Lagrange multiplier field, had made it evident that both the (a priori independent) torsion modes $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$ are induced by ϕ . In the paradigm of the standard FRW cosmology, the outcome is in some sense, a 'geometric unification' of the dark sector, as both the effective DM and DE components of the universe, being purportedly the artefacts of $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$, have had their dynamics driven by the same scalar field ϕ .

Technically of course, the MMT theory (introduced in Paper 1) is designed to have almost all of its weight resting on the pre-assigned contact coupling, $\beta(\phi)$, between the mimetic field and the entire torsion-dependent part of the Riemann–Cartan (U_4) Lagrangian. In the extended formulation, such a coupling obviously carries to the Holst term, which augments the U_4 Lagrangian. The further postulation of a pseudo-scalar BI field, and subsequently the identification $\phi = s\gamma^2$ (where s is a dimensionless constant), are only expected to lead to a subtle modification of what $\beta(\phi)$ alone can inflict on a given system configuration. Nevertheless, there arises the important question as to whether such a modification, despite its suppressed quantitateness, can turn out to be significant in the cosmological context. This is

what we have addressed specifically via our analysis in the preceding sections.

In particular, taking the MMT coupling function in the form $\beta(\phi) \sim \phi^2$, we have brought the time-variance of the effective mimetic potential $W(\phi)$ in the reckoning, as opposed to its constancy in Paper 1, viz. $W = \bar{W} = 2\Lambda$, due to the only existent torsion mode $\mathcal{T}_\mu$ (in absence of the Holst term) therein. While the coupling $\beta(\phi) \sim \phi^2$ has had its motivation from the point of view of its natural appearance in metric-torsion theories involving scalar field(s) [97, 148–150], the effect of the typical functional form it ascribes to the fractional modification $Y(\phi)$ of the constant potential $\bar{W} = 2\Lambda$ (of Paper 1) has turned out to be quite fascinating in the cosmological context. Specifically, such a ϕ^2 -coupling implies an inverse functional dependence of Y on ϕ , or equivalently, on the cosmic time t in the FRW framework. The consequence of this, by virtue of the mimetic constraint, is a *super-accelerating* phase of cosmic evolution, marked by the diminution of the effective DE equation of state parameter $w_x(t)$ below the Λ CDM value -1 . In fact, as we have inferred via a set of arguments, the super-acceleration is always plausible in our extended MMT scenario, which most importantly, does not appear to have any theoretical obstacles, such as those concerning ghost or phantom degrees of freedom. This may be understood from the point of view that none of the energy conditions, except the strong one, is violated in our entire formalism. The violation of the strong energy condition is of course nothing unusual – it is mandated for any attempt of explaining the late-time cosmic acceleration (regardless of an even later super-acceleration) in the standard FRW framework.

Nevertheless, despite having the theoretical consistency, there remained the task for us to assert whether the transition from $w_x > -1$ to $w_x < -1$, or the *phantom barrier crossing*, is at all physically realizable, and if so, under what condition(s). In specific terms, a physically realistic (or viable) phantom crossing implies that the epoch of its occurrence, $t = t_c$, has to be within the time-span (t_T, t_0) , where t_T denotes the deceleration-to-acceleration transition epoch and t_0 denotes the present epoch. While the lower bound $t_c > t_T$ is obvious, the upper bound $t_c < t_0$ actually pertains to a fair amount of observational signification of a *mild* super-acceleration to have commenced in the near past phase of evolution of the universe, and continuing to the present epoch and beyond [230]. The mildness of course alludes to the gross observational concordance on Λ CDM cosmology. That is to say, in whichever way the DE equation of state parameter w_x may deviate from the Λ CDM value -1 , observations disfavour the latter's error limits to be breached to any perceivable extent. Now, to get the condition(s) for $t_c \in (t_T, t_0)$ to hold (if at all), it is imperative to determine the functional form of $w_x(t)$ by explicitly solving the corresponding cosmological equations. However, instead of attempting an exact

analytic solution, we have adopted an approximation method, upon treating the effect of the function $Y(\phi)$ as a small perturbation over that due to the constant potential $\bar{W} = 2\Lambda$, viz. the Λ CDM solution in Paper 1. Admittedly, the strong observational support for Λ CDM has made it reasonable for us to assume that $Y(\phi)$ would always inflict a small deviation of w_X from the value -1 .

Working out $w_X(t)$ in the approximated form, we have shown that the phantom crossing is indeed physically realizable, however for a severely restricted range of values of our MMT model parameters, viz. σ_0 and μ_0 , where $\sigma_0 = \dot{s}/t_0$ and μ_0 is the fractional amount by which w_X differs from -1 at $t = t_0$. In particular, limiting our analysis to the *linear* order of the (presumably small) parameter σ_0 , we have illustrated the time-evolution of w_X for certain fiducial settings of either a fixed σ_0 or a fixed μ_0 . Strikingly enough, all such settings have made the revelation of the phantom regime being *non-transient*, i.e. the super-acceleration is ever-lasting, even though it slows down progressively after reaching an optimum point. A close inspection of the evolution profiles of the torsion parameters, viz. the norms of the torsion trace and pseudo-trace mode vectors $\mathcal{T}_\mu$ and $\mathcal{A}_\mu$, have enabled us to determine the validity of the linear approximation, and hence the extremity of the phantom crossing time t_c , as well as that of $w_X(t_0)$ (or equivalently, μ_0). In fact, the latter is found to be not exceeding a value typically well within the Planck 2018 TT,TE,EE+lowE+Lensing+BAO error estimates for Λ CDM [230]. This nonetheless signifies a very low extent of the super-acceleration, in well accord with our presumption of its smallness.

On the whole, we have seen some really appealing outcomes of our entire program of extending the basic MMT formalism of Paper 1. Not only that it consolidates the tantalizing picture of a geometrically unified dark sector we have had therein, but also discerns the viability of the cosmic super-acceleration, or the phantom phase, without any impending danger of dealing with a ghost-like entity. This may nonetheless be counted towards the robustness of the emergent cosmological model, since after all, from a purely theoretical standpoint, a provision for the phantom barrier crossing is desirable on account of the flexibility thus comprehended while making statistical estimations of the model parameters using the observational data.

Nevertheless, apart from the technical difficulties the extreme mildness of the cosmic super-acceleration may pose in its detection with the current generation of observational probes, there remain several issues to be pondered on. The foremost, of course, are those concerning the Ostrogradsky ghost or (and) gradient instabilities at the perturbative level, due to the presence of higher derivative ($\sim (\Box\phi)^2$) extension of the MMT Lagrangian [25–27]. A plausible removal of such instabilities is demonstrated, via a proposed alternative extension, in the Appendix below. Secondly, there is the

associated problem of the caustic singularities, which in fact affects not only the mimetic theory, but also any scalar-tensor model exhibiting the mimicry of a dust-like fluid description. Caustic formation is unavoidable for any constrained scalar field, such as the mimetic one, that leads to a dust velocity flow tangential to the time-like geodesics [58, 193–195]. Within the scope of mimetic gravity, this problem has been circumvented via vectorial (gauge field) extensions [52–54]. However, in the MMT scenario the presence of the torsion coupling with the mimetic field ϕ may be of some aid, since it implies that apart from being a fundamental mode of gravity, the field ϕ manifests itself geometrically as the source of one (or more) vector mode(s) of torsion. Furthermore, the geodesics of actual importance here are the affine geodesics (or, the *auto-parallel* curves) which are in general not identical to the metric geodesics in the Riemann–Cartan (U_4) space-time. Therefore, a careful inspection resulting in a chosen specific form of the ϕ -coupling with the torsion vector mode $\mathcal{T}_\mu$ had enabled us to get an effective dust-like (MMT) fluid description in Paper 1 (see the Appendix therein). Admittedly however, a more rigorous contemplation of the focusing theorem and caustics is needed in this context, by taking note of the existent references [237, 238], which is one of the assignments we have undertaken presently.

Other pertinent questions are, for instance, (i) can there be any obvious way of realizing the ϕ -coupling with torsion, with or without involving the Holst term? (ii) would that be perceptible via the Hilbert–Palatini formulation of mimetic gravity? (iii) how would the magnitude of the super-acceleration be affected if there be a constraint more complicated than (2.6)? (iv) can that constraint be imposed automatically (i.e. without the aid of a Lagrange multiplier) via some mechanism? Attempts of addressing to some of these are currently underway, and hopefully be reported soon.

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Appendix: A note on the possible aversion of ghost and gradient instabilities

Let us refer back to the action (2.2). Apart from the matter action $S^{(m)}$, it has the pure MMT action with the general structure

$$S_{MMT} = \int d^4x \sqrt{-g} [\mathcal{L}^{(g)} + \mathcal{L}^{(c)} + \mathcal{L}^{(h)}], \quad (6.1)$$

that includes the purely gravitational part (alongwith the ϕ -coupled torsional contributions) described by the Lagrangian

$$\mathcal{L}^{(g)} = \frac{1}{2} \left[R + \beta(\phi) \left(\Theta + \frac{\star\Theta}{2\gamma} \right) \right], \quad (6.2)$$

the constraint part with Lagrangian

$$\mathcal{L}^{(c)} = \frac{1}{2} \left[\lambda \cdot (X - 1) + \nu \cdot (\phi - s\gamma^2) \right], \quad (6.3)$$

and the higher derivative (HD) augmentation, described by the Lagrangian

$$\mathcal{L}^{(h)} = \frac{\alpha}{2} (\Box\phi)^2. \quad (6.4)$$

As is well-known, such an augmentation crucially leads to a non-zero sound speed of the mimetic matter density perturbations, given by [2]

$$c_s = \sqrt{\frac{\alpha}{2 - 3\alpha}}, \quad (6.5)$$

however, at the expense of making the mimetic field dynamical. While a non-zero c_s is essential for defining the usual quantum fluctuations due to ϕ , so that the latter can provide the seeds of the observed large scale structure of the universe [2,3], the presence of a HD term in the action is in general problematic as it leads to Ostrogradsky instabilities. Remarkably, the mimetic constraint $X = 1$ comes to the rescue here, as it eliminates the dynamical mode arising from $(\Box\phi)^2$, and in turn the Ostrogradsky ghost, for a suitable range of values of the parameter α [2]. However, a rather critical inspection reveals that there remains a stringent problem of a gradient instability, which requires to be confronted with substantial rigour [25–27]. As demonstrated by several authors in recent times, such a problem could be remedied by introducing an explicit contact coupling of an appropriate function of $\Box\phi$ (and possible other HD constructs, e.g. $\phi_{\mu\nu}\phi^{\mu\nu}$ where $\phi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi$) with curvature invariants,

such as R [26–29]. Nevertheless, this still implies a dynamical ϕ – the assumption which we endeavor to disregard in what follows.

To be specific, we consider a scenario in which the pure MMT action (6.1) has the same gravitational and constraint Lagrangians given respectively by (6.2) and (6.3), but the HD Lagrangian (6.4) replaced by

$$\mathcal{L}^{(h)} = \frac{1}{2} [\phi \Box\psi - m^2 \psi^2 + n\psi R - \ell^2 R^2], \quad (6.6)$$

where ψ is a dynamical scalar field (considered dimensionless), and n, ℓ, m are constants (n is dimensionless, whereas ℓ and m have unit length and mass dimensions, respectively). Eq. (6.6) essentially means that we are considering a theory with two scalar fields – one is the non-dynamical mimetic field ϕ whereas the other is a (presumably primordial) dynamical field with a non-canonical kinetic term and having explicit contact couplings with both ϕ and R . The purpose of incorporating the R^2 term is to eliminate any scale dependence in the theory, as we shall see below. Note also that although the proposed Lagrangian (6.6) seems to have nothing to do with the presence of torsion, a correspondence could be traced, albeit with certain additional subtleties which are being examined in our subsequent works (in preparation) [229,241]. For the time being, however, in this paper, we consider the Lagrangian (6.6) to be just an exemplary one, that replaces Eq. (6.4) in an attempt of providing a non-zero c_s , and side-by-side removing the gradient instability, without requiring the mimetic field to be dynamical, to begin with.

Eliminating a surface term one can make ψ an auxiliary field, whence the corresponding variation of the full action would lead to the constraint

$$\psi = \frac{\Box\phi + nR}{2m^2}. \quad (6.7)$$

Hence, implementing this and the other constraint (2.6), and proceeding in the same way as in Sect. 2, we get

$$S_{MMT} = \frac{1}{2} \int d^4x \sqrt{-g} \left[(1 + \epsilon \Box\phi) R + \alpha (\Box\phi)^2 + \lambda (X - 1) - W(\phi) X \right], \quad (6.8)$$

where $W(\phi)$ is as given by Eq. (2.21), and we denote

$$\alpha \equiv \frac{1}{4m^2} \quad \text{and} \quad \epsilon \equiv \frac{\ell}{2m}, \quad (6.9)$$

upon choosing $n = 2m\ell$, so that the R^2 contribution gets cancelled (thereby removing the scale-dependence).

Let us now resort to the standard Arnowitt–Deser–Misner (ADM) decomposition of the line element for the perturbed

FRW space-time

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (6.10)$$

where h_{ij} is the induced metric, and N and N^i are respectively the lapse function and the shift vector. It is convenient to choose the unitary gauge in which the scalar (mimetic) field perturbations $\delta\phi = 0$, and the induced metric assumes the form

$$h_{ij}(\mathbf{x}, t) = a^2(t) \exp \left[2I_{ij}\zeta(\mathbf{x}, t) + \gamma_{ij}(\mathbf{x}, t) \right], \quad (6.11)$$

where ζ and γ_{ij} respectively denote the scalar (or curvature) mode and the elements of the (transverse and trace-free) tensor mode of perturbations. As ϕ is purely a function of time in this gauge, the mimetic constraint $X = 1$ implies

$$N = \dot{\phi}, \quad \partial_i N = 0. \quad (6.12)$$

Setting therefore, without loss of generality, $N = 1$, as in ref. [27], we get

$$\square\phi = -K, \quad (6.13)$$

where $K = h^{ij} K_{ij}$ denotes the trace of the extrinsic curvature $K_{ij} = \frac{1}{2} [\dot{h}_{ij} - (\nabla_i N_j + \nabla_j N_i)]$.

The perturbed MMT action can hence be written, after certain simplifications, as

$$S_{MMT} = \frac{1}{2} \int d^4x \sqrt{h} \left[(1 - \epsilon K) \left\{ {}^{(3)}R + K_{ij} K^{ij} \right\} - (1 - \alpha) K^2 - W(\phi) \right], \quad (6.14)$$

where h is the determinant of the induced metric h_{ij} and ${}^{(3)}R$ denotes the corresponding curvature scalar. Varying this action with respect to the shift vector N^i we get the corresponding equation of motion, given up to the first order in the perturbations, as

$$\partial_i N^i = \frac{2\epsilon \partial^2 \zeta + [2(1 - \epsilon K) - 3(\alpha - 5\epsilon H)] a^2}{(\alpha - 5\epsilon H) a^2}, \quad (6.15)$$

which shows that N^i only has a longitudinal part [27]. Following then the steps of ref. [27], and the general methodology of ref. [242], we obtain the quadratic order action (after

a great deal of simplifications) as¹⁴

$$S_{MMT}^{(2)} = \int d^4x a^3 \left[f^{(T)} \left\{ \dot{\gamma}_{ij} \gamma^{ij} - e^{(T)} \partial_i \gamma_{jk} \partial^i \gamma^{jk} \right\} + f^{(S)} \left\{ \dot{\zeta}^2 - e^{(S)} \partial_i \zeta \partial^i \zeta \right\} + \tilde{f}^{(S)} \left(\partial^2 \zeta \right)^2 - \frac{1}{2} W(\phi) \right], \quad (6.16)$$

where

$$f^{(T)} = \frac{1 - 3\epsilon H}{8}, \quad (6.17)$$

$$e^{(T)} = a^{-2}, \quad (6.18)$$

$$f^{(S)} = \frac{(1 - 3\epsilon H) [3(\alpha - 3\epsilon H) - 2]}{\alpha - 5\epsilon H}, \quad (6.19)$$

$$e^{(S)} = -[3(\alpha - 3\epsilon H) - 2]^{-1} \left[(\alpha - 3\epsilon H) - 2\epsilon^2 \left(\frac{3}{1 - 3\epsilon H} - \frac{5}{\alpha - 5\epsilon H} \right) \dot{H} \right] a^{-2}, \quad (6.20)$$

$$\tilde{f}^{(S)} = -\frac{2\epsilon^2 a^{-4}}{\alpha - 5\epsilon H}, \quad (6.21)$$

are the purely time-dependent coefficients of the tensor (T) and scalar (S) mode kinetic terms.

Ghost-freeness of the theory requires $f^{(T)}$, $f^{(S)}$ and $\tilde{f}^{(S)}$ to be positive definite at all epochs, and the gradient instabilities would be removed if $e^{(T)}$ and $e^{(S)}$ remain positive definite as well. Moreover, from Eq. (6.5) we see that $0 \leq c_s^2 \leq 1$ implies $0 \leq \alpha \leq \frac{1}{2}$. Therefore, severe restrictions are bound to get imposed on the Hubble parameter $H(t)$, after being compounded with the demand that $\dot{H}(t) < 0$ ($\forall t$), so that by the Raychaudhuri–Friedmann equation $\rho + p > 0$, i.e. the null and weak energy conditions hold (see the discussion in the last paragraph of Sect. 3). A close inspection of Eqs. (6.17)–(6.21) nonetheless reveal that all the above criteria are fulfilled for the following bounds on the dimensionless Hubble rate

$$\frac{\alpha}{5\epsilon H_0} \leq \frac{H}{H_0} \leq \frac{\alpha}{3\epsilon H_0}. \quad (6.22)$$

So, to have the range of values of H/H_0 enhanced for a given $\alpha \in [0, \frac{1}{2}]$ and H_0 , so as to bring more flexibility in the model, we require the parameter $\epsilon \ll \alpha$, in units of H_0 . The smallness of ϵ implies a very weak coupling of $\square\phi$ with R , for which of course the background MMT cosmological evolution would not be affected much. Hence, our analysis in Sects. 3, 4 and 5 would still be valid, by and large.

¹⁴ The detailed calculations could be found in the subsequent papers [229, 241], which we hope to report soon.

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