Z-boson hadronic decay width up to $\mathcal{O}(\alpha_s^4)$ -order QCD corrections using the single-scale approach of the principle of maximum conformality

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Abstract In the paper, we study the properties of the Z-boson hadronic decay width by using the $\mathcal{O}(\alpha_s^4)$ -order quantum chromodynamics (QCD) corrections with the help of the principle of maximum conformality (PMC). By using the PMC single-scale approach, we obtain an accurate renormalization scale-and-scheme independent perturbative QCD (pQCD) correction for the Z-boson hadronic decay width, which is independent to any choice of renormalization scale. After applying the PMC, a more convergent pQCD series has been obtained; and the contributions from the unknown $\mathcal{O}(\alpha_s^5)$ -order terms are highly suppressed, e.g. conservatively, we have $\Delta\Gamma_Z^{\text{had}}|_{\text{PMC}}^{\mathcal{O}(\alpha_s^5)} \simeq \pm 0.004 \text{ MeV}$. In combination with the known electro-weak (EW) corrections, QED corrections, EW–QCD mixed corrections, and QED–QCD mixed corrections, our final prediction of the hadronic Z decay width is $\Gamma_Z^{\text{had}} = 1744.439_{-1.433}^{+1.390} \text{ MeV}$, which agrees with the PDG global fit of experimental measurements, $1744.4 \pm 2.0 \text{ MeV}$.

1 Introduction

In quantum chromodynamics (QCD), the Z-boson hadronic decay width plays an important role in determining the strong coupling constant (α_s). The Z-boson hadronic decay width has been measured by various collaborations at the electron–positron colliders such as LEP and SLC [1,2], which could

also be precisely measured in future high luminosity colliders such as the super Z factory [3] or CEPC [4]. Theoretically, the one-loop electroweak (EW) and the mixed EW–QCD contributions to the Z-boson hadronic decay have been investigated in Refs. [5–8], and the two-loop EW contribution has been given in Refs. [9–11]. In large- m_t limit, the higher-loop corrections have been calculated up to $\mathcal{O}(\alpha_t\alpha_s^2)$ [12,13], $\mathcal{O}(\alpha_t^2\alpha_s)$, $\mathcal{O}(\alpha_t^3)$ [14,15], and $\mathcal{O}(\alpha_t\alpha_s^3)$ [16–18], respectively, where $\alpha_t \equiv y_t^2/(4\pi)$ with y_t being the top-quark Yukawa coupling constant. The final-state QED radiations have been computed up to $\mathcal{O}(\alpha)$, $\mathcal{O}(\alpha\alpha_s)$, $\mathcal{O}(\alpha^2)$ in Ref. [19]. The non-factorizable QCD corrections have been estimated in Refs. [8,20]. The pure perturbative QCD (pQCD) corrections up to $\mathcal{O}(\alpha_s^2)$ [21,22], $\mathcal{O}(\alpha_s^3)$ [23–27], $\mathcal{O}(\alpha_s^4)$ [28–30] have also been performed in the literature. Moreover, the mass corrections to both the vector and axial vector correlators can be found in Refs. [21,22,31–35]. Those achievements give us good opportunities for precise determining of $\alpha_s(M_Z)$, e.g. a recent determination of α_s has been given in Ref. [36].

Following the renormalization group invariance, the physical observable should be independent to theoretical conventions, such as the choices of the renormalization scale and scheme, which is ensured by mutual cancellation of the scale and scheme dependence among different orders for an infinite-order pQCD prediction. However, for a fixed-order pQCD prediction, if the perturbative coefficients and the corresponding α_s do not match properly, the pQCD series may have large scale and scheme ambiguities [37]. Conventionally, the renormalization scale is taken as the “guessed” momentum flow of the process, as well as the one to eliminate the large logs or to minimize the contributions from high-order terms or to achieve the prediction in agreement with the experimental data. Those naive treatment directly causes the mismatching between the strong coupling constant and

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its coefficients and resulting in conventional renormalization scale and scheme ambiguities [38, 39]. Such guessing treatment decreases the predictive power of pQCD. In fact, predictions based on conventional scale setting are even incorrect for Abelian theory – Quantum Electrodynamics (QED); the renormalization scale of the QED coupling constant can be set unambiguously by using the Gell-Mann-Low method [40].

A correct renormalization scale-setting approach is thus important for achieving an accurate fixed-order pQCD prediction. Many ways have been suggested in the literature, most of them such as the renormalization group improved effective coupling method (FAC) [41, 42] and the principle of minimum sensitivity (PMS) [43–45] are designed to find an optimal renormalization scale of the process. On the contrary, the principle of maximum conformality (PMC) [46–50] provides a rigorous idea, whose purpose is not to find an optimal scale, but to determine the effective magnitude of α_s for a fixed-order pQCD series by using the renormalization group equation (RGE). The determined effective α_s is independent to any choice of renormalization scale, thus the conventional renormalization scale ambiguity is eliminated. Moreover, since all the scheme-dependent non-conformal $\{\beta_i\}$ -terms have been eliminated, the resultant pQCD series becomes scheme independent conformal series; thus the conventional renormalization scheme ambiguity is simultaneously eliminated.

A recent demonstration of renormalization scale-and-scheme independence of PMC prediction has been given in Refs. [51, 52], where by using the C -scheme coupling [53], it has been proven that the PMC prediction is independent of the choice of renormalization scale and scheme up to any fixed order. Generally, the convergence of the pQCD series can be improved due to the elimination of the divergent renormalon terms like $n!\beta_0^n\alpha_s^n$ or $n!\beta_0^n\alpha_s^{n+1}$ [54–56].¹

The PMC accurate renormalization scale-and-scheme independent conformal series is helpful not only for achieving precise pQCD predictions but also for a reliable prediction on the contributions of unknown higher-orders; some applications can be found in Refs. [57–61] which are estimated by using the Padé resummation approach [62–64]. In the present paper, we shall adopt the PMC single-scale approach [65]² to analyze the Z -boson hadronic decay width.

¹ The $\beta_n (n \geq 1)$ -term could be estimated by using the approximation, $\beta_n \approx \beta_0^{n+1}$, which can be adopted for transforming the $\{\beta_i\}$ -series at each order into β_0 -power series.

² In the original PMC multi-scale approach [46, 49], different types of $\{\beta_i\}$ -terms are absorbed into α_s via an order-by-order manner, and distinct PMC scales are determined at each order. Such PMC multi-scale approach has thus two kinds of residual scale dependence due to the unknown perturbative terms, which has already been pointed out in year 2013 [66]. It should be pointed out that the recently so-called ambiguities of PMC given in Ref. [67] are not the default of PMC,

It is noted that the original PMC multi-scale approach [46–50] and single-scale approach are equivalent to each other in sense of perturbative theory [65], but the residual scale dependence emerged in PMC multi-scale method can be greatly suppressed by applying the single-scale approach.

The remaining parts of the paper are organized as follows. In Sect. 2, we will give the detailed PMC treatment for a precise determination of the Z -boson hadronic decay width. In Sect. 3, we will give the numerical results. Section 4 is reserved for a summary.

2 The Z -boson hadronic decay width using the PMC

The hadronic decay width of the Z -boson can be expressed as

$$\Gamma_Z^{\text{had}} = \Gamma_0 R^{\text{nc}} + \Delta\Gamma_Z^{\text{Extra}}, \quad (1)$$

where the first term stands for the pure pQCD correction with the leading-order (LO) width $\Gamma_0 = \frac{G_F M_Z^3}{24\pi\sqrt{2}}$, and the Fermi coupling constant $G_F = 1.166 \times 10^{-5} \text{GeV}^{-2}$. The second term $\Delta\Gamma_Z^{\text{Extra}}$ contains four less important corrections, i.e.,

$$\begin{aligned} \Delta\Gamma_Z^{\text{Extra}} &= \Delta\Gamma_1 + \Delta\Gamma_2 + \Delta\Gamma_3 + \Delta\Gamma_4 \\ &= -1.577_{-0.237}^{+0.183} + 0.695_{-0.001}^{+0.000} + 6.577_{-0.560}^{+0.560} \\ &\quad + 0.609_{-0.049}^{+0.061} \text{ (MeV)} \\ &= 6.304_{-0.847}^{+0.804} \text{ (MeV)}, \end{aligned} \quad (2)$$

where the central values are for $\mu_r = M_Z$, and the errors are for $\mu_r \in [M_Z/2, 2M_Z]$. Here $\Delta\Gamma_1$ is the b - and t -quark mass corrections to the vector and axial vector correlators [31–35], $\Delta\Gamma_2$ is the quark final-state QED radiation and the mixed QED-QCD correction [19], $\Delta\Gamma_3$ is the electro-weak two-loop corrections and the higher-loop corrections in the large- m_t limit [11], $\Delta\Gamma_4$ is the mixed EW-QCD correction and nonfactorizable QCD correction [6–8, 20].

Our main concern is the perturbative QCD corrections to the dominant correlator of the neutral current, which can be divided as the following four parts:

$$R^{\text{nc}} = 3 \left[\sum_f v_f^2 r_{\text{NS}}^V + \left(\sum_f v_f \right)^2 r_S^V + \sum_f a_f^2 r_{\text{NS}}^A + r_S^A \right], \quad (3)$$

where $v_f \equiv 2I_f - 4q_f s_W^2$, $a_f \equiv 2I_f$, q_f is the f -quark electric charge, s_W is the effective weak mixing angle, and

Footnote 2 continued

but the residual scale dependence due to unknown perturbative terms. Such residual scale dependence generally suffer from both the α_s -power suppression and the exponential suppression, but could be large due to possibly poor pQCD convergence for the perturbative series of either the PMC scale or the pQCD approximant [52].

I_f is the third component of weak isospin of the left-handed component of f . $r_{\text{NS}}^V = r_{\text{NS}}^A \equiv r_{\text{NS}}$, r_S^V , and r_S^A stand for the non-singlet, the vector-singlet, and the axial-singlet part, respectively. Those contributions can be further expressed as

$$r_{\text{NS}} = 1 + \sum_{i=1}^n C_i^{\text{NS}} a_s^i, r_S^V = \sum_{i=3}^n C_i^{\text{VS}} a_s^i, r_S^A = \sum_{i=2}^n C_i^{\text{AS}} a_s^i,$$

where $a_s = \alpha_s/(4\pi)$, and the coefficients of r_{NS} , r_S^V , and r_S^A can be obtained from Refs. [28–30, 68, 69]. As for r_S^A , we adopt conventional scale setting approach to perform our analysis,³ and numerically, we obtain $\Gamma_{\text{Z}}^{\text{had}}|_S^A = [-1.725, -1.685] \text{ MeV}$ for $\mu_r \in [M_Z/2, 2M_Z]$ by using the formulas given by Ref. [28], whose magnitude is quite small in comparison to that of r_{NS} , thus fortunately, this approximate treatment will not affect our final conclusions.

The R-ratio can be rewritten as the following perturbative form by using the degeneracy relations [49, 50, 70], i.e.,

$$\begin{aligned} R^{\text{nc}} = & r_0 + r_{1,0} a_s(\mu_r) + (r_{2,0} + \beta_0 r_{2,1}) a_s^2(\mu_r) \\ & + (r_{3,0} + \beta_1 r_{2,1} + 2\beta_0 r_{3,1} + \beta_0^2 r_{3,2}) a_s^3(\mu_r) \\ & + (r_{4,0} + \beta_2 r_{2,1} + 2\beta_1 r_{3,1} + \frac{5}{2} \beta_1 \beta_0 r_{3,2} \\ & + 3\beta_0 r_{4,1} + 3\beta_0^2 r_{4,2} + \beta_0^3 r_{4,3}) a_s^4(\mu_r) + \mathcal{O}(a_s^5), \end{aligned} \quad (4)$$

where $r_0 = 3(\sum_f v_f^2 + \sum_f a_f^2)$, and the coefficients $r_{i,j}$ can be obtained from the known coefficients C^{NS} , C^{VS} , and C^{AS} of r_{NS} , r_S^V , and r_S^A . The coefficients $r_{i,0}$ are $\{\beta_i\}$ -independent conformal coefficients, and the $\{\beta_i\}$ -dependent non-conformal coefficients $r_{i,j}$ ($j \neq 0$) are generally functions of $\ln \mu_r^2/M_Z^2$, i.e.,

$$r_{i,j} = \sum_{k=0}^j C_j^k \hat{r}_{i-k,j-k} \ln^k(\mu_r^2/M_Z^2), \quad (5)$$

where the reduced coefficients $\hat{r}_{i,j} = r_{i,j}|_{\mu_r=M_Z}$, the combination coefficients $C_j^k = j!/[k!(j-k)!]$. We put the known coefficients $\hat{r}_{i,j}$ up to $\mathcal{O}(\alpha_s^4)$ -level in the Appendix.

Following the standard PMC single-scale procedures as described in detail in Ref. [65], with the help of RGE, one can determine an effective coupling $\alpha_s(Q_*)$ by absorbing all the non-conformal $\{\beta_i\}$ -terms into the running coupling, and the resultant pQCD series becomes the following conformal series,

$$\begin{aligned} R^{\text{nc}}|_{\text{PMC}} = & r_0 + r_{1,0} a_s(Q_*) + r_{2,0} a_s^2(Q_*) \\ & + r_{3,0} a_s^3(Q_*) + r_{4,0} a_s^4(Q_*) + \mathcal{O}(a_s^5), \end{aligned} \quad (6)$$

where Q_* is the PMC scale, which corresponds to the overall effective momentum flow of the process and can be deter-

mined up to next-to-next-to-leading log (NNLL) accuracy by using the present known $\mathcal{O}(\alpha_s^4)$ -order pQCD series; i.e., the $\ln Q_*^2/M_Z^2$ can be expanded as the following perturbative series,

$$\ln \frac{Q_*^2}{M_Z^2} = T_0 + T_1 a_s(M_Z) + T_2 a_s^2(M_Z) + \mathcal{O}(a_s^3), \quad (7)$$

where

$$T_0 = -\frac{\hat{r}_{2,1}}{\hat{r}_{1,0}}, \quad (8)$$

$$T_1 = \frac{\beta_0(\hat{r}_{2,1}^2 - \hat{r}_{1,0}\hat{r}_{3,2})}{\hat{r}_{1,0}^2} + \frac{2(\hat{r}_{2,0}\hat{r}_{2,1} - \hat{r}_{1,0}\hat{r}_{3,1})}{\hat{r}_{1,0}^2}, \quad (9)$$

and

$$\begin{aligned} T_2 = & \frac{3\beta_1(\hat{r}_{2,1}^2 - \hat{r}_{1,0}\hat{r}_{3,2})}{2\hat{r}_{1,0}^2} \\ & + \frac{4(\hat{r}_{1,0}\hat{r}_{2,0}\hat{r}_{3,1} - \hat{r}_{2,0}^2\hat{r}_{2,1}) + 3(\hat{r}_{1,0}\hat{r}_{2,1}\hat{r}_{3,0} - \hat{r}_{1,0}^2\hat{r}_{4,1})}{\hat{r}_{1,0}^3} \\ & + \frac{\beta_0(4\hat{r}_{2,1}\hat{r}_{3,1}\hat{r}_{1,0} - 3\hat{r}_{4,2}\hat{r}_{1,0}^2 + 2\hat{r}_{2,0}\hat{r}_{3,2}\hat{r}_{1,0} - 3\hat{r}_{2,0}^2\hat{r}_{2,1})}{\hat{r}_{1,0}^3} \\ & + \frac{\beta_0^2(2\hat{r}_{1,0}\hat{r}_{3,2}\hat{r}_{2,1} - \hat{r}_{2,1}^3 - \hat{r}_{1,0}^2\hat{r}_{4,3})}{\hat{r}_{1,0}^3}. \end{aligned} \quad (10)$$

It can be found that Q_* is exactly free of μ_r , and together with the μ_r -independent conformal coefficients $r_{i,0}$, the conventional renormalization scale ambiguity is eliminated. Therefore, the precision of R^{nc} can be greatly improved by using the PMC. Moreover, the precision of the predictions depend on the perturbative nature of both the R^{nc} and the $\ln Q_*^2/M_Z^2$, which shall be numerically analyzed in the following paragraphs.

3 Numerical results

To do the numerical calculation, we adopt the Z-boson mass $M_Z = 91.1876 \pm 0.0021 \text{ GeV}$ and top-quark pole mass $M_t = 172.9 \text{ GeV}$ [71]. We use the four-loop α_s -running behavior [52] to analyse the $\mathcal{O}(\alpha_s^4)$ -order QCD corrections. i.e.,

$$\begin{aligned} \alpha_s(\mu_r) \simeq & \frac{1}{\beta_0 t} - \frac{b_1 \ln t}{(\beta_0 t)^2} + \frac{b_1^2(\ln^2 t - \ln t - 1) + b_2}{(\beta_0 t)^3} \\ & + \frac{1}{(\beta_0 t)^4} \left[b_1^3 \left(-\ln^3 t + \frac{5}{2} \ln^2 t + 2 \ln t - \frac{1}{2} \right) \right. \\ & \left. - 3b_1 b_2 \ln t + \frac{b_3}{2} \right] + \mathcal{O} \left(\frac{1}{(\beta_0 t)^5} \right), \end{aligned}$$

Where $t = \ln(\mu_r^2/\Lambda_{\text{QCD}}^2)$, $b_i = \beta_i/\beta_0$, and the β_i ($i = 0, 1, 2, 3$)-functions have been calculated in Refs. [72–80]. Taking $\alpha_s(M_Z) = 0.1181$ [71], we obtain $\Lambda_{\text{QCD}}^{(n_f=5)} = 209.5 \text{ MeV}$.

³ From the known $\mathcal{O}(\alpha_s^4)$ -order expressions, we cannot derive the exact RG-dependent n_f -series for r_S^A , which is however very important for using the PMC scale-setting; so we have to take this approximation.

Fig. 1 The Z-boson hadronic decay width Γ_Z^{had} under the conventional scale-setting approach versus the renormalization scale μ_r . The dotted, the dashed-dot, the dashed and the solid lines are for the predictions up to $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^3)$, $\mathcal{O}(\alpha_s^4)$ order levels, respectively

First, by setting all input parameters to be their central values, we present the Z-boson hadronic decay width Γ_Z^{had} up to different known α_s -orders under conventional scale-setting approach in Fig. 1. It shows that in agreement of the conventional wisdom, the renormalization scale dependence becomes small when we have known more loop terms. For examples, we obtain $\Gamma_Z^{\text{had}}|_{\text{Conv.}} = [1744.378, 1744.587]$ MeV for $\mu_r \in [M_Z/2, 2M_Z]$, and $\Gamma_Z^{\text{had}}|_{\text{Conv.}} = [1744.378, 1745.008]$ MeV for $\mu_r \in [M_Z/3, 3M_Z]$; e.g., the net scale errors are only $\sim 0.01\%$, and $\sim 0.04\%$, respectively. We should point out that as has been mentioned in the Introduction, such small net scale dependence for the $\mathcal{O}(\alpha_s^4)$ -order prediction is due to good convergence of the perturbative series, e.g., the relative magnitudes of the α_s -terms: α_s^2 -terms: α_s^3 -terms: α_s^4 -terms=1: 2.9%: -2.2%: -0.4% for the case of $\mu_r = M_Z$; and also due to the cancellation of the scale dependence among different orders.⁴ The scale errors for each order term remain unchanged and large, e.g. the Γ_Z^{had} has the following perturbative feature up to $\mathcal{O}(\alpha_s^4)$ -order:

$$\begin{aligned} \Gamma_Z^{\text{had}}|_{\text{Conv.}} &= 1681.262 + 62.966_{+4.268}^{-5.925} + 1.802_{-4.078}^{+4.838} \\ &\quad - 1.382_{-0.505}^{+1.311} - 0.230_{+0.275}^{-0.055} \\ &= 1744.418_{-0.040}^{+0.169} \text{ (MeV)}, \end{aligned} \quad (11)$$

where the central values are for $\mu_r = M_Z$, and the errors are obtained by varying $\mu_r \in [M_Z/2, 2M_Z]$. It shows that the absolute scale errors are 16%, 495%, 131%, and 143% for the α_s -terms, α_s^2 -terms, α_s^3 -terms, and α_s^4 -terms, respectively;

⁴ In cases when each perturbative terms varies synchronously with the changes of μ_r , there will have no cancellations of scale dependence among different orders and the pQCD approximant could be still large even for higher-orders. A recent example can be found in a two-loop QCD correction for $\gamma + \eta_c$ production in electron-positron collisions [61].

Fig. 2 The Z-boson hadronic decay width Γ_Z^{had} under the PMC scale-setting approach versus the renormalization scale μ_r . The dotted, the dashed-dot, the dashed and the solid lines are for the predictions up to $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^3)$, $\mathcal{O}(\alpha_s^4)$ order levels, respectively

and there do have large scale cancellations among different orders.

Second, we present the Z-boson hadronic decay width Γ_Z^{had} up to different known α_s -orders under the PMC scale-setting approach in Fig. 2. At the $\mathcal{O}(\alpha_s)$ -order level, the perturbative series of Γ_Z^{had} does not have $\{\beta_i\}$ -terms to fix the α_s value, thus the prediction of $\Gamma_Z^{\text{had}}|_{\text{PMC}}^{\mathcal{O}(\alpha_s)}$ is the same as the conventional one. The PMC starts to work at the $\mathcal{O}(\alpha_s^2)$ and higher order levels. It shows that after applying the PMC, the pQCD convergence can be greatly improved, e.g., the relative magnitudes of the α_s -terms: α_s^2 -terms: α_s^3 -terms of the pQCD series changes 1: 4.34%: -0.49% by applying the PMC scale-setting to the perturbative series up to $\mathcal{O}(\alpha_s^3)$, whose PMC scale $Q_* = 113.0$ GeV is fixed up to the NLL accuracy. The relative magnitudes of the α_s -terms: α_s^2 -terms: α_s^3 -terms: α_s^4 -terms of the pQCD series becomes to 1: 4.33%: -0.49%: 0.01% by applying the PMC up to $\mathcal{O}(\alpha_s^4)$, whose PMC scale $Q_* = 114.9$ GeV is fixed up to NNLL accuracy. And there is no renormalization scale dependence for Γ_Z^{had} at any fixed order, i.e.,

$$\begin{aligned} \Gamma_Z^{\text{had}}|_{\text{PMC}} &= 1681.262 + 60.838 + 2.634 - 0.299 \\ &\quad + 0.004 \\ &= 1744.439 \text{ (MeV)}, \end{aligned} \quad (12)$$

where each perturbative terms and the net total decay width are unchanged for any choice of μ_r . This behavior is consistent with that of the previous PMC multi-scale approach analysis on R^{nc} [81]. The PMC single scale Q_* is an effective scale which effectively replaces the individual PMC scales introduced in the PMC multi-scale approach in the sense of a mean value theorem, which can be regarded as the overall effective momentum flow of the process; it shows stability and convergence with increasing order in pQCD via the pQCD approximates. More explicitly, we obtain $Q_* = 114.9$

$\text{GeV} \sim 1.3M_Z$, which can be fixed up to NNLL accuracy by using the present known $\mathcal{O}(\alpha_s^4)$ -order pQCD series, i.e.,

$$\begin{aligned} \ln \frac{Q_*^2}{M_Z^2} &= 0.2249 + 21.7363a_s(M_Z) + 376.287a_s^2(M_Z) \\ &= 0.2249 + 0.2043 + 0.0332. \end{aligned} \quad (13)$$

One may observe that the relative magnitudes of each order terms in Q_* perturbative series are 1 : 91% : 15%, which also shows a good convergence behavior.

Third, it is helpful to predict the magnitude of the “unknown” higher-order pQCD corrections. The renormalization scale independent PMC series is helpful for such purpose. Because the PMC series has a good perturbative convergence, e.g., the magnitude of $\mathcal{O}(\alpha_s^4)$ -order term is only 0.01% of $\mathcal{O}(\alpha_s)$ -order term, it is reasonable to take the magnitude of the last known term $\pm|r_{4,0}a_s^4(Q_*)|$ as a conservative prediction of the uncalculated higher-order terms [39]. By further taking the variation of $\Delta Q_* \simeq \pm 1.9 \text{ GeV}$, which is the difference between the NLL and NNLL PMC scales, as the magnitude of its unknown NNNLL term into consideration,⁵ we obtain

$$\Delta\Gamma_Z^{\text{had}}|_{\text{PMC}}^{\text{High order}} \simeq \pm 0.004 \text{ (MeV)}. \quad (14)$$

Finally, after eliminating the renormalization scale uncertainty by applying the PMC, we still have uncertainties from the α_s fixed-point error $\Delta\alpha_s(M_Z)$ and the Z -boson mass error ΔM_Z . As for the α_s fixed-point error, by using $\Delta\alpha_s(M_Z) = 0.0011$ [71] together with the four-loop α_s -running behavior, we obtain $\Lambda_{\text{QCD}}^{n_f=5} = 209.5^{+13.2}_{-12.6} \text{ MeV}$ and

$$\Delta\Gamma_Z^{\text{had}}|_{\text{PMC}}^{\Delta\alpha_s(M_Z)} = \pm 0.574 \text{ (MeV)}. \quad (15)$$

And for the error of Z -boson mass $\Delta M_Z = \pm 0.0021 \text{ GeV}$, we obtain

$$\Delta\Gamma_Z^{\text{had}}|_{\text{PMC}}^{\Delta M_Z} = \pm 0.120 \text{ (MeV)}. \quad (16)$$

Here, when discussing one uncertainty, the other input parameters shall be set as their central values.

As a whole, the squared average of the above mentioned three errors leads to a net error, $\pm 0.586 \text{ MeV}$, to the PMC prediction of the total decay width Γ_Z^{had} , among which the magnitude of $\Delta\alpha_s(M_Z)$ dominates the error sources. Thus more precise measurements on the reference point $\alpha_s(M_Z)$ is important for a more precise pQCD prediction.

⁵ This is a conservative estimation of unknown contributions for the PMC scale Q_* , since as shown by Eq. (13), Q_* suffers from both the exponential suppression and α_s suppression, and it shows good convergence and is already at a high precision.

Fig. 3 The PMC prediction of the Z -boson hadronic decay in comparison with the experimental values given by the PDG global fit of experimental data [71], and by the OPAL [82], DELPHI [83], L3 [84], and ALEPH [85] collaborations

4 Summary

Under conventional scale-setting approach, the fixed-order scale-setting ambiguity could be softened by including enough higher-order loop terms due to large cancellation among different orders; for the present considered decay width up to $\mathcal{O}(\alpha_s^4)$ -order, the net scale uncertainty is $\left(\begin{smallmatrix} +0.169 \\ -0.040 \end{smallmatrix} \right) \text{ MeV}$ for $\mu_r \in [M_Z/2, 2M_Z]$; and by further including the mentioned other error sources, we have $\Gamma_Z^{\text{had}}|_{\text{Conv.}} = 1744.418^{+1.595}_{-1.621} \text{ (MeV)}$.

In the paper, we have presented an accurate prediction on the Z -boson hadronic decay width by applying the PMC single-scale approach to eliminate the conventional renormalization scale ambiguity. We obtain

$$\Gamma_Z^{\text{had}}|_{\text{PMC}} = 1744.439^{+1.390}_{-1.433} \text{ (MeV)}. \quad (17)$$

where the errors are the sum of two parts, one is the squared average of those from $\Delta\alpha_s(M_Z)$, ΔM_Z , and the uncalculated higher-order terms, another is the error from $\Delta\Gamma_Z^{\text{Extra}}$ as given in Eq. (2). After applying the PMC single-scale approach, the pQCD series becomes scale independent and more convergent, thus a reliable pQCD prediction can be achieved. Due to the perturbative terms have been known up to enough high-orders, the predictions under the PMC and conventional scale-setting approaches are consistent with each other. We present the PMC prediction of the Z -boson hadronic decay width in Fig. 3, where the experimental data are presented as a comparison. The PMC prediction agrees with the PDG global fit of the experimental measurements. Thus, one obtains optimal fixed-order predictions for the Z -boson hadronic decay width by applying the PMC, enabling high precision test of the Standard Model.

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Appendix: The PMC reduced perturbative coefficients $\hat{r}_{i,j}$

In this appendix, we give the required PMC reduced coefficients $\hat{r}_{i,j}$ for the perturbative series of the Z-boson hadronic decay width, which can be obtained from Refs. [28–30, 68, 69] with proper transformations, i.e.,

$$\hat{r}_{1,0} = 9\gamma_1^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right), \quad (18)$$

$$\hat{r}_{2,0} = 4 \left[9\gamma_2^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right) - 37 + 12 \ln \frac{M_Z^2}{M_t^2} \right], \quad (19)$$

$$\hat{r}_{2,1} = 9\Pi_1^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right), \quad (20)$$

$$\begin{aligned} \hat{r}_{3,0} = & 144 \left[\gamma_3^{NS} + \frac{\gamma_3^S \left(\sum_f q_f \right)^2}{\sum_f q_f^2} \right] \left(\sum_f a_f^2 + \sum_f v_f^2 \right) \\ & + \left(\sum_f v_f \right)^2 \left(\frac{440}{9} - \frac{320\zeta_3}{3} \right) + \frac{2144}{3} \ln \frac{M_Z^2}{M_t^2} \\ & + 368 \ln^2 \frac{M_Z^2}{M_t^2} + 192\zeta_3 + \frac{368\pi^2}{3} - \frac{40600}{9}, \end{aligned} \quad (21)$$

$$\hat{r}_{3,1} = 36\Pi_2^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right), \quad (22)$$

$$\hat{r}_{3,2} = -3\pi^2 \gamma_1^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right), \quad (23)$$

$$\begin{aligned} \hat{r}_{4,0} = & 576 \left[\gamma_4^{NS} + \frac{\gamma_4^S \left(\sum_f q_f \right)^2}{\sum_f q_f^2} \right] \left(\sum_f a_f^2 + \sum_f v_f^2 \right) \\ & + \left(\sum_f v_f \right)^2 \left(\frac{3980}{9} - \frac{28960\zeta_3}{9} + \frac{11200\zeta_5}{9} \right) \\ & + \left(\frac{356000}{9} - 44192\zeta_3 \right) \ln \frac{M_Z^2}{M_t^2} + \frac{33776}{3} \ln^2 \frac{M_Z^2}{M_t^2} \\ & + \frac{8464}{3} \ln^3 \frac{M_Z^2}{M_t^2} - \frac{13083735979}{56700} + \frac{35934334\zeta_3}{525} \\ & + \frac{12328\zeta_5}{9} - \frac{170272 \ln^4 2}{405} + \frac{512 \ln^5 2}{45} \\ & + \left(11748 + \frac{512 \ln 2}{3} + \frac{170272 \ln^2 2}{405} - \frac{512 \ln^3 2}{27} \right) \pi^2 \\ & + \left(\frac{4425752}{6075} + \frac{1424 \ln 2}{135} \right) \pi^4, \end{aligned} \quad (24)$$

$$\begin{aligned} \hat{r}_{4,1} = & 144 \left[\Pi_3^{NS} + \frac{\Pi_3^S \left(\sum_f q_f \right)^2}{\sum_f q_f^2} \right] \left(\sum_f a_f^2 + \sum_f v_f^2 \right) \\ & + \left(\sum_f v_f \right)^2 \left(\frac{5960}{27} - \frac{1040\zeta_3}{3} - \frac{320\zeta_3^2}{3} + \frac{800\zeta_5}{3} \right), \end{aligned} \quad (25)$$

$$\hat{r}_{4,2} = -12\pi^2 \gamma_2^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right), \quad (26)$$

$$\hat{r}_{4,3} = -9\pi^2 \Pi_1^{NS} \left(\sum_f a_f^2 + \sum_f v_f^2 \right), \quad (27)$$

where the expressions for the coefficients γ_i and Π_i can be found in Refs. [69, 86].

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