



No short hair behaviors of ultra-compact stars

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Abstract In the black hole spacetime, a no short hair theorem was proved, which states that the effective radius of black hole hairs must extend beyond the null circular orbit. In the present paper, in the horizonless gravity, we find a similar no short hair behavior that the effective radius of matter fields must also extend beyond the null circular orbit of ultra-compact stars. We should emphasize that the theorem presented in this manuscript is valid only for spatially regular matter configurations whose energy-momentum traces are non-negative.

1 Introduction

One of the most interesting predictions of general relativity is the existence of null circular orbits outside compact astrophysical objects, such as black holes and horizonless ultra-compact stars [1, 2]. These null circular geodesics provide the way that massless particles (photons, gravitons) can orbit a central compact objects, which gives important information about the highly curved spacetime. At present, the null circular orbits have been widely studied

Recently, it was proved that the null circular orbit can be used to describe the distribution of exterior matter fields outside black holes. In spherically symmetric hairy black hole spacetimes, it was found that the effective radius of hairs must extend beyond the null circular orbit [22, 23]. The authors in [22] have proposed a nice heuristic physical picture that the null circular orbit divides matter fields into two parts (fields below the orbit tending to be swallowed by black holes and fields above the orbit tending to be radiated away to infinity) and the self-interaction between these two parts bind together fields in different regions leading to the existence of black hole hairs. And it was further conjectured that the region above the null circular orbit contains at least half of the total hair mass [23–26].

The main goal of this paper is to study properties of horizonless ultra-compact spherical stars with null circular orbits. We shall analytically prove a no short hair theorem that the effective radius of matter fields must extend beyond the null circular orbit.

2 Investigations on effective radius of matter fields

We consider static horizonless stars with null circular orbits. In Schwarzschild coordinates, these ultra-compact spherically symmetric objects take the following form [22, 23]

$$ds^2 = -g(r)e^{-2\chi(r)}dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

The metric functions $\chi(r)$ and $g(r)$ only depend on the Schwarzschild radial coordinate r . Asymptotical flatness of the spacetime at the infinity requires that $g(r \rightarrow \infty) = 1$ and $\chi(r \rightarrow \infty) = 0$. Near the origin, regularity conditions of the gravity are [27, 28]

$$g(r \rightarrow 0) = 1 + O(r^2) \quad \text{and} \quad \chi(0) < \infty. \quad (2)$$

We take $\rho = -T_t^t$, $p = T_r^r$ and $p_\tau = T_\theta^\theta = T_\phi^\phi$, where ρ , p and p_τ are interpreted as the energy density, the radial pressure and the tangential pressure respectively [23]. Einstein differential equations $G_\nu^\mu = 8\pi T_\nu^\mu$ yield the metric equations

$$g' = -8\pi r\rho + \frac{1-g}{r}, \quad (3)$$

$$\chi' = \frac{-4\pi r(\rho + p)}{g}. \quad (4)$$

We label $m(r)$ as the gravitational mass contained within a sphere of radial radius r . It can be expressed by the integral relation

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'. \quad (5)$$

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Considering relations (3) and (5), one can express the metric function $g(r)$ in the form [28]

$$g = 1 - \frac{2m(r)}{r}. \tag{6}$$

Following approaches in [23], we obtain equations of null circular orbits. As the metric (1) is independent of the time t and angular coordinates ϕ , the geodesic trajectories are characterized by conserved energy E and conserved angular momentum L . And the null circular orbits are determined by an effective potential

$$V_r = (1 - e^{2\chi})E^2 + g \frac{L^2}{r^2} \tag{7}$$

with the characteristic relations

$$V_r = E^2 \text{ and } V'_r = 0. \tag{8}$$

Substituting Einstein equations (3) and (4) into (7) and (8), we deduce the null circular orbit equation [27]

$$N(r_\gamma) = 3g(r_\gamma) - 1 - 8\pi(r_\gamma)^2 p(r_\gamma) = 0, \tag{9}$$

where we have introduced a new function $N(r) = 3g(r) - 1 - 8\pi r^2 p(r)$. The discrete roots of (9) correspond to the radii of the null circular orbit.

With the relation (2) and the regular condition $p(0) < \infty$, one obtains

$$N(0) = 2. \tag{10}$$

We label r_γ^{in} as the radius of the innermost null circular orbit. That is to say r_γ^{in} is the smallest positive root of $N(r) = 0$. Also considering relation (10), one deduces that

$$N(r) \geq 0 \text{ for } r \in [0, r_\gamma^{in}]. \tag{11}$$

The conservation equation $T_{v;\mu}^\mu = 0$ has only one non-trivial component

$$T_{r;\mu}^\mu = 0. \tag{12}$$

Substituting Eqs. (3) and (4) into (12), one gets equation of the pressure

$$p'(r) = \frac{1}{2rg} [(3g - 1 - 8\pi r^2 p)(\rho + p) + 2gT - 8gp] \tag{13}$$

with $T = -\rho + p + 2p_\tau$ standing for the trace of the energy momentum tensor.

With a new pressure function $P(r) = r^4 p$, the Eq. (13) can be expressed as

$$P'(r) = \frac{r}{2g} [N(\rho + p) + 2gT], \tag{14}$$

where $N = 3g - 1 - 8\pi r^2 p$.

The energy condition usually plays a central role in determining the spacetime geometry. We assume the dominant energy condition

$$\rho \geq |p|, \quad |p_\tau| \geq 0, \tag{15}$$

which means that the energy density ρ is non-negative and bounds the pressures [22,23]. We also assume the non-negative trace of the energy momentum tensor expressed as

$$T = -\rho + p + 2p_\tau \geq 0. \tag{16}$$

In this work, we take the non-negative trace conditions [27–30]. A well known example for such horizonless configurations is the gravitating Einstein-Yang-Mills solitons, which are characterized by the identity $T = 0$ [31]. In contrast, the usual non-positive trace condition was imposed in the black hole spacetime [22,23].

From (11) and (14–16), one deduces that

$$P'(r) \geq 0 \text{ for } r \in [0, r_\gamma^{in}]. \tag{17}$$

We impose the condition that ρ goes to zero faster than r^{-4} [22,23]. Also considering (15) and $P(r) = r^4 p(r)$, there is the asymptotical behavior

$$P(r \rightarrow \infty) = 0. \tag{18}$$

Near the origin, the pressure function $P(r)$ has the asymptotical behavior

$$P(r \rightarrow 0) = 0. \tag{19}$$

Relations (18) and (19) imply that $|P(r)|$ must have a local maximum value at some extremum point r_0 . We can define $r_m = r_0$ as the effective radii of matter fields [22,23]. According to (17), $P(r)$ is an increasing function of r before it reaches the innermost null circular orbit. So the effective radii have a lower bound

$$r_m \geq r_\gamma^{in}, \tag{20}$$

which is the same as the no short hair theorem of black holes [22,23].

3 Conclusions

We investigated distributions of matter fields in the background of horizonless ultra-compact spherically symmetric stars. We assumed the dominant energy condition and the non-negative trace condition. We defined an effective matter field radius at an extremum point where the pressure function $|P(r)|$ possesses a local maximum value. Using analytical methods, we obtained a lower bound on the effective matter field radius expressed as $r_m \geq r_\gamma^{in}$ with r_m as the effective matter field radius and r_γ^{in} corresponding to the innermost null circular orbit radius. So we found a no

short hair behavior that the effective matter field radius must extend beyond the innermost null circular orbit. We also emphasized that the theorem presented in this manuscript is valid only for spatially regular matter configurations whose energy-momentum traces are non-negative $T \geq 0$.

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