THE EUROPEAN
PHYSICAL JOURNAL C

Superintegrability of Kontsevich matrix model

Andrei Mironov[1,2,3,a], **Alexei Morozov**[2,3,4,b]

[1] Lebedev Physics Institute, Moscow 119991, Russia
[2] ITEP, Moscow 117218, Russia
[3] Institute for Information Transmission Problems, Moscow 127994, Russia
[4] MIPT, Dolgoprudny 141701, Russia

Abstract Many eigenvalue matrix models possess a peculiar basis of observables that have explicitly calculable averages. This explicit calculability is a stronger feature than ordinary integrability, just like the cases of quadratic and Coulomb potentials are distinguished among other central potentials, and we call it *superintegrability*. As a peculiarity of matrix models, the relevant basis is formed by the Schur polynomials (characters) and their generalizations, and superintegrability looks like a property $\langle character \rangle \sim character$. This is already known to happen in the most important cases of Hermitian, unitary, and complex matrix models. Here we add two more examples of principal importance, where the model depends on *external fields*: a special version of complex model and the cubic Kontsevich model. In the former case, straightforward is a generalization to the complex tensor model. In the latter case, the relevant characters are the celebrated Q Schur functions appearing in the description of spin Hurwitz numbers and other related contexts.

In the both cases, one can prove that the partition functions are actually τ functions of KP/Toda integrable theories, see [1–6] for details, we do not need them in the present text.

However, besides being just a τ-function, the partition function of matrix model can be often presented as an explicit combinatorial power sum [7,8]. This property is much similar to superintegrability of mechanical systems [9,10], hence the term.

The superintegrability has been explicitly formulated so far for the matrix models of the first kind, and, in this paper, our goal is to extend formulation to the second kind. So far the known examples include:

- **Rectangular complex model**
  Correlators in the rectangular complex model are expressed [7,8,11,12] through the Schur polynomials (see earlier results in [13,14]), which are characters of linear groups:

$$\left\langle \chi_R\{\mathrm{Tr}\,(X\bar{X})^k\}\right\rangle := \int \chi_R\{\mathrm{Tr}\,(X\bar{X})^k\} \cdot \exp\left(-\mathrm{Tr}\,X\bar{X}\right) d^2X = \frac{\chi_R\{N_1\}\,\chi_R\{N_2\}}{\chi_R\{\delta_{k,1}\}} \tag{1}$$

1 Introduction

In the original definition, matrix models are defined as averages over matrix ensembles, often described by integrals over matrices or eigenvalues. Generating functions of all correlators are then described by integrals with arbitrary potentials in the action (matrix models of the first kind) or, alternatively, with background fields (matrix models of the second kind).

The matrix X here is $N_1 \times N_2$ rectangular matrix, $\chi_R\{p_k\}$ is the Schur polynomial (which is defined to be a symmetric function $\chi_R(z_i)$ of the variables z_i, or a polynomial $\chi_R\{p_k\}$ of the power sums $p_k := \sum_i z_i^k$), and the formula explains what is special about the Schur polynomials. At the l.h.s. (1) the role of z_i is played by the eigenvalues of the matrix $X\bar{X}$, while at the r.h.s. $p_k = N_1$, N_2 or $\delta_{k,1}$. Averages of $\mathrm{tr}\,(X\bar{X})^k$ *per se* are more involved than those of characters and actually contain an additional summation over Young diagrams at the level $|R|$, for examples of the corresponding Harer-Zagier formulas, see [15–17]. In (1) and below, all the integration measures are normalized

[a] e-mails: mironov@lpi.ru; mironov@itep.ru (corresponding author)

[b] e-mail: morozov@itep.ru

to unity, $\langle 1 \rangle := 1$; in other words, we omit normalization integrals in denominators to simplify formulas.

For the quantity $\chi_R\{\delta_{k,1}\}$, there is a hook formula [19]

$$\chi_R\{\delta_{k,1}\} = \frac{1}{\prod_{(\alpha,\beta)\in R} h_{\alpha,\beta}} = \frac{\prod_{i<j}^{l_R}(R_i - i - R_j + j)}{\prod_i^{l_R}(l_R + R_i - i)!} \tag{2}$$

where l_R is the number of lines R_i in the Young diagram R: $R_1 \geq R_2 \geq \cdots \geq R_{l_R}$, and $|R|$ is the size of the Young diagram R: $|R| := \sum_{i=1}^{l_R} R_i$, and $h_{\alpha,\beta}$ is the hook length for a box $(\alpha, \beta) \in R$. For other similar loci we have, up to a coefficient, equal to ± 1 and 0,

$$\chi_R\{\delta_{k,s}\} \sim \prod_{(\alpha,\beta)} \frac{1}{[[h_{\alpha,\beta}]]_{s,0}} \tag{3}$$

where we use the notation

$$[[n]]_{s,a} = n \text{ if } n = a \bmod(s) \text{ and } 1 \text{ otherwise} \tag{4}$$

which will be also useful beyond Gaussian models. Also simple in terms of the box coordinates is the expression on the topological locus:

$$\chi_R\{N\} = \chi_R\{\delta_{k,1}\} \prod_{(\alpha,\beta)\in R}(N + \alpha - \beta) \tag{5}$$

Using Eq. (1) and the Cauchy formula

$$\exp\left(\sum_k \frac{p_k}{k} \operatorname{Tr}(X\bar{X})^k\right) = \sum_R \chi_R\{p_k\} \cdot \chi_R\{\operatorname{Tr}(X\bar{X})^k\} \tag{6}$$

one can immediately obtain a combinatorial expression for the generating function of all correlators:

$$\int \exp\left(-\operatorname{Tr} X\bar{X} + \sum_k \frac{p_k}{k} \operatorname{Tr}(X\bar{X})^k\right) d^2 X$$
$$= \sum_R \frac{\chi_R\{p_k\}\,\chi_R\{N_1\}\,\chi_R\{N_2\}}{\chi_R\{\delta_{k,1}\}} \tag{7}$$

- **Gaussian Hermitian model**

  Correlators in the Gaussian Hermitian model are [7,8,12] (cf. also with a Fourier expansion of [20, Eq. (2.18)] in terms of characters)

$$\boxed{\left\langle \chi_R\{\operatorname{Tr} X^k\}\right\rangle := \int \chi_R\{\operatorname{Tr} X^k\} \cdot \exp\left(-\frac{1}{2}\operatorname{Tr} X^2\right) dX = \frac{\chi_R\{N\}\,\chi_R\{\delta_{k,2}\}}{\chi_R\{\delta_{k,1}\}}} \tag{8}$$

where X is an $N \times N$ matrix, and the generating function of all correlators is

$$\int \exp\left(-\frac{1}{2}\operatorname{Tr} X^2 + \sum_k \frac{p_k}{k} \operatorname{Tr} X^k\right) dX$$
$$= \sum_R \frac{\chi\{p_k\}\chi_R\{N\}\,\chi_R\{\delta_{k,2}\}}{\chi_R\{\delta_{k,1}\}} \tag{9}$$

- **Trigonometric (unitary type) matrix model**

  This model is given by the eigenvalue integral, and the correlators are [21]

$$\boxed{\left\langle \chi_R(e^{m_i})\right\rangle := \int \chi_R(e^{m_i}) \prod_{i<j}^N \sinh^2\left(\frac{m_i - m_j}{2}\right) \prod_{i=1}^N \exp\left(-\frac{m_i^2}{2g^2}\right) dm_i = A^{|R|} \cdot q^{2\varkappa_R} \cdot \chi_R\{p^*\}} \tag{10}$$

When averaging, the argument of character in the integrand is the diagonal matrix with the entries e^{m_i}, and, at the r.h.s., the parameters are $q = e^{g^2/2}$, $A = q^N = e^{Ng^2/2}$, the exponent $\varkappa_R = \sum_{(x,y)\in R}(y - x)$, and the time variables $p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}}$ in the argument of the character at the r.h.s. lie in the "topological locus" obtained by the q-deformation of $p_k = N$ in (8).

- **Knot matrix model**

  Further deformation of the measure to the knot eigen-

value model gives basically the same expression for the correlators [22,23]:

$$\left\langle \chi_R(e^{\frac{m_i}{a}}) \right\rangle := \int \chi_R(e^{\frac{m_i}{a}}) \prod_{i<j} \sinh \frac{m_i - m_j}{2a} \sinh \frac{m_i - m_j}{2b} \prod_{i=1}^{N} e^{-\frac{m_i^2}{2g^2}} dm_i = \left(A^{|R|} \cdot q^{2\varkappa_R} \right)^{b/a} \cdot \chi_R\{p^*\} \quad (11)$$

with $q = e^{\frac{g^2}{2ab}}$.

The correlators of $\langle \chi_R(e^{m_i}) \rangle$ are more involved this time, but are related to the HOMFLY-PT polynomials of the torus knots $\mathcal{H}_R^{\text{Torus}_{a,b}}(A, q)$ q1 [22,23]:

$$\left\langle \chi_R(e^{m_i}) \right\rangle = \mathcal{H}_R^{\text{Torus}_{a,b}}(A, q) = \chi_R\{N\} \cdot H_R^{\text{Torus}_{a,b}}(A, q) \quad (12)$$

where

$$\mathcal{H}_R^{\text{Torus}_{a,b}}(A, q) = \left\langle \sum_{Q \vdash a|R|} c_{R,Q} \chi_Q(e^{\frac{m_i}{a}}) \right\rangle$$

$$= A^{\frac{b|R|}{a}} \sum_{Q \vdash a|R|} c_{R,Q} \cdot q^{\frac{2b\varkappa_Q}{a}} \cdot \chi_Q\{p^*\} \quad (13)$$

where $c_{R,Q}$ are peculiar Adams coefficients [24,25]. Generalization to non-torus knots, for example, like that in [26,27], remains to be analyzed.

- **q, t deformed Gaussian Hermitian model**

The eigenvalue q, t-deformed Gaussian Hermitian model is associated with a deformation of Schur polynomials: all of them in the formula (8) are replaced by the corresponding Macdonald polynomials M_R [28]:

$$\langle M_R(z_i) \rangle := \int_{-1}^{1} M_R(z_i) \prod_{j \neq i} \frac{\left(\frac{z_i}{z_j}; q\right)_\infty}{\left(t \frac{z_i}{z_j}; q\right)_\infty} \prod_i z_i^{(N-1)\log_q t} (q^2 z_i^2; q^2)_\infty d_q z_i = \frac{M_R\left\{\frac{1-t^{Nk}}{1-t^k}\right\} M_R\left\{\frac{1+(-1)^k}{1-t^k}\right\}}{M_R\left\{\frac{1}{1-t^k}\right\}} \quad (14)$$

where

$$(z; q)_\infty := \prod_{k=0}^{\infty} (1 - q^k z) \quad (15)$$

and the integral is defined to be the Jackson integral,

$$\int_{-1}^{1} d_q z f(z) := (1 - q) \sum_{k=0}^{\infty} q^k \left(f(\xi q^k) + f(-\xi q^k) \right) \quad (16)$$

The measure $d\mu_{q,t}(z) = (q^2 z^2; q^2)_\infty d_q z$ gives rise to the Gaussian measure $d\mu_G(z) = \exp\left(-\frac{1}{2}z^2\right) dz$ in the limit of $q \to 1$.

One can expect also a further extension of similar character identities to the elliptic q, t model [29,30].

- **Monomial non-Gaussian models**

Superintegrability is in no way restricted to the Gaussian integration measures or their deformations, though beyond them the knowledge is still restricted. The most important example is provided by *monomial* non-Gaussian actions:

$$\langle \chi_R\{\text{Tr} X^k\} \rangle_a := \int_{C_{s,a}^{\otimes N}} \chi_R\{\text{Tr} X^k\} e^{-\text{Tr} X^s/s} dX$$

$$= \chi_R\{\delta_{k,s}\} \cdot \prod_{(\alpha,\beta) \in R} [[N + \alpha - \beta]]_{s,0} \cdot [[N + \alpha - \beta]]_{s,a}$$

$$N = 0 \text{ or } a \bmod s \quad (17)$$

where we used the notation from (4). The new trick here is the use of a special star-like (closed) integration contour $C_{s,a}$:

$$\int_{C_{s,a}} F(x) e^{-x^s/s} dx = \sum_{b=1}^{s} e^{-2\pi i (a-1)b/s}$$

$$\cdot \int_0^\infty F\left(e^{2\pi i b/s} x\right) e^{-x^s/s} dx \quad (18)$$

which picks up only the powers of x, which are equal to $a - 1 \bmod s$, in particular,

$$\int_{C_{s,a}} x^k e^{-x^s/s} dx = \delta_{k+1-a}^{(s)} \cdot \Gamma\left(\frac{k+1}{s}\right) \quad (19)$$

$\delta_k^{(s)}$ is defined to be 1 if $k = 0 \bmod s$ and to vanish otherwise. This makes the answer depending on an additional parameter $a = 0, \ldots s - 1$. The r.h.s. of (17) contains some factors $N + j$ from $\chi_R\{N\}$, (5), that is, those with $N + i = 0, a \bmod(s)$, and, hence, its vanishing depends also on the value of N. Note that if the condition $N = 0$ or $a \bmod s$ is not satisfied, one can not define the correlator by the condition $\langle 1 \rangle = 1$ because of zeroes in the denominator. Note also that the Vandermonde determinant in the integrand is independent of s and a. For more details, see the original paper [31].

In this paper, we extend these results to the simplest background field models: the matrix integral over $N_1 \times N_2$ complex matrices X with the measure

$$\mu_C(X) \sim e^{-\operatorname{tr} A X B \bar{X}} d^2 X \tag{20}$$

where $N_1 \times N_1$ matrix A and $N_2 \times N_2$ matrix B are fixed external matrices, and the Hermitian generalized Kontsevich model [32,33] in background matrix field Λ, i.e. the matrix integral over $N \times N$ Hermitian matrix X with the measure

$$\mu_K(X) \sim e^{\operatorname{tr}(W(X) + \Lambda X)} dX \tag{21}$$

and Λ is an external $N \times N$ matrix. In this paper, we consider these two examples, the potential in the second one being $W(X) = -\frac{X^3}{3}$, which is just the original Kontsevich model [34]. Remarkably, the difference between these two models is severely increased in this case: in particular, in the Hermitian case, only one background field can be easily handled.

The question we address is what are the functions of X-variables that have simple and explicitly calculable averages. As we demonstrate in this paper, in the complex case, these are still the Schur polynomials $\chi_R\{\operatorname{tr}(X\bar{X})^k\}$, however, the r.h.s. of (1) now contains traces of A and B instead of N_1 and N_2:

$$\boxed{\left\langle \chi_R\{\operatorname{Tr}(X\bar{X})^k\} \right\rangle = \frac{\chi_R\{\operatorname{tr} A^{-k}\} \chi_R\{\operatorname{tr} B^{-k}\}}{\chi_R\{\delta_{k,1}\}}} \tag{22}$$

see Sect. 2 below for details, and for generalization to the complex *tensor* model. In the Hermitian (Kontsevich) case, the story makes a new twist: the relevant functions are more restricted Q Schur polynomials, see Sect. 3.3 for the correlators in this case,

$$\boxed{\left\langle Q_R\{\operatorname{Tr} X^k\} \right\rangle = \delta_{R,even} \cdot \frac{Q_{R/2}\{\operatorname{Tr} \Lambda^{-k}\} Q_{R/2}\{\delta_{k,1}\}}{Q_R\{\delta_{k,1}\}}} \tag{23}$$

and Sect. 4.2 for the combinatorial expression for the Kontsevich model

$$\boxed{Z_K = \sum_{R \in \mathrm{SP}} \frac{1}{4^{|R|}} \frac{Q_R\{\operatorname{Tr} \Lambda^{-k}\} Q_R\{\delta_{k,1}\} Q_{2R}\{\delta_{k,3}\}}{Q_{2R}\{\delta_{k,1}\}}} \tag{24}$$

Contributing here are only diagrams R with all lines of even length. Equation (24) can be considered as the main new result of this paper. Important feature of formulas (22)–(24) is that they depend on the matrix size only implicitly, through the traces of powers of background fields. This is the crucial feature, which allows one to forget about the matrix-integral origin/realization of these models, in particular to treat them in terms of τ-functions of integrable hierarchies [32,33].

2 Rectangular complex model

2.1 Correlators in terms of permutations

We start with calculating the correlators in the complex matrix model with the external matrix (see [35] for a square matrix example), and use the symmetry group technique worked out in [8]. In this section, we closely follow that paper.

We consider the rectangular complex model

$$Z_C := \int d^2 X \exp\left(-\operatorname{Tr} A X B \bar{X}\right) \tag{25}$$

with correlators defined as

$$\langle \mathcal{O}(X) \rangle := \int \mathcal{O}(X) \exp\left(-\operatorname{Tr} A X B \bar{X}\right) d^2 X \tag{26}$$

where X is $N_1 \times N_2$ rectangular matrix, A and B are square matrices of sizes $N_1 \times N_1$ and $N_2 \times N_2$ respectively, and (notice that, with this definition, the matrix multiplication is defined by convolution of the first indices with each other, and of the second indices with each other)

$$\operatorname{Tr} A X B \bar{X} := \sum_{ijk} A_k^i X_{ij} B_l^j \bar{X}^{kl} \tag{27}$$

The pair correlator is equal to

$$\left\langle X_{ij} \bar{X}^{kl} \right\rangle = \left(A^{-1}\right)_{ik} \left(B^{-1}\right)_{jl} \tag{28}$$

The $2m$-point correlator can be labelled by a permutation of indices σ that belongs to the symmetric group S_m:

$$\langle \mathcal{O}_\sigma \rangle = \left\langle \prod_{p=1}^{l} \operatorname{Tr}(X\bar{X})^{m_p} \right\rangle = \left\langle \prod_{i=1}^{m} X_{a_i \alpha_i} \bar{X}^{a_i \alpha_{\sigma(i)}} \right\rangle \tag{29}$$

In order to calculate this correlator, we apply the Wick theorem and use formula (28):

$$\left\langle \prod_{i=1}^{m} X_{a_i \alpha_i} \bar{X}^{b_i \beta_i} \right\rangle = \sum_{\gamma \in S_m} \prod_{i=1}^{m} \left(A^{-1}\right)_{a_i}^{b_{\gamma(i)}} \left(B^{-1}\right)_{\alpha_i}^{\beta_{\gamma(i)}} \tag{30}$$

and obtain

$$\langle \mathcal{O}_\sigma \rangle = \sum_{\gamma \in S_m} p_\gamma^A p_{\gamma \circ \sigma}^B \tag{31}$$

where

$$p_k^A := \text{Tr } A^{-k}, \qquad p_k^B := \text{Tr } B^{-k},$$

$$p_\gamma^{A,B} := \prod_{i=1}^{l_\gamma} p_{i(\gamma)}^{A,B} \qquad (32)$$

and l_γ is the number of cycles in the permutation γ, with $i(\gamma)$ denoting the i-th cycle in the permutation γ.

Now we use the standard identity [19]

$$p_\gamma = \sum_{R \vdash |\gamma|} \psi_R(\gamma) \chi_R\{p\} \qquad (33)$$

Here the sum goes over all Young diagrams R with $|\gamma|$ boxes, $\chi_R\{p\}$ is the Schur polynomial (the character of the linear group $GL(N)$), $p_\gamma = \prod_{i=1}^{l_\gamma} p_{\gamma_i}$ and $\psi_R(\gamma)$ is the character of the symmetric group $S_{|R|}$. Now using the orthogonality relation

$$\sum_\gamma \psi_R(\gamma) \psi_Q(\gamma \circ \sigma) = \sum_\gamma \psi_R(\gamma^{-1}) \psi_Q(\gamma \circ \sigma)$$
$$= \frac{\psi_R(\sigma)}{d_R} \delta_{QR} \qquad (34)$$

where $d_R := \chi_R\{\delta_{k,1}\}$, we finally come to

$$\langle \mathcal{O}_\sigma \rangle = \sum_{R \vdash m} \frac{\chi_R\{p^A\} \chi_R\{p^A\}}{\chi_R\{\delta_{k,1}\}} \psi_R(\sigma) \qquad (35)$$

where the sizes of R and σ coincide.

2.2 Complex model in background fields

Let us now use the Frobenius formula [19]

$$\chi_R\{p_k\} = \frac{1}{|R|!} \sum_\gamma \psi_R(\gamma) p_\gamma \qquad (36)$$

where $|\gamma|$ is the size of the Young diagram R, and the orthogonality relation (34) with the particular value $\psi_R(id) = \psi_R([1^{|R|}]) = |R|! d_R$. Then, one immediately obtains from (35)

$$\boxed{\left\langle \chi_R\{\text{Tr } (X\bar{X})^k\} \right\rangle = \frac{\chi_R\{\text{tr } A^{-k}\} \chi_R\{\text{tr } B^{-k}\}}{\chi_R\{\delta_{k,1}\}}} \qquad (37)$$

This generalizes the answer (1) to the case of $A \neq I$, $B \neq I$.

Similarly to (7), we also can obtain the combinatorial expression for the generating function

2.3 Tensor model in background fields

The results of this section for the complex matrix model can be straightforwardly extended to the Gaussian tensor model in background fields. To this end, we follow [36,37] and again apply the technique of [8].

The Gaussian tensor model is a model of complex r-tensors $X_{a^1,\ldots a^r}$ where each subscript runs through its interval $a_i = 1, \ldots, N_i$, in the background of r external square matrices $A_{(i)}$ of the sizes N_i with the Gaussian action

$$S := \sum_{a^1,b^1=1}^{N_1} \cdots \sum_{a^r,b^r=1}^{N_r} X_{a^1,\ldots a^r} \bar{X}^{b^1,\ldots b^r} \prod_{i=1}^r \left(A_{(i)}\right)_{b_i}^{a_i} \qquad (39)$$

Gauge invariant operators at the level m in any (not obligatory Gaussian) tensor model are the tensorial counterparts of "multi-trace" operators

$$\mathcal{O}_{\sigma_1,\ldots,\sigma_r} = \sum_{\vec{a}^1=1}^{N_1} \cdots \sum_{\vec{a}^r=1}^{N_r} \left(\prod_{p=1}^m X_{a_p^1,\ldots a_p^r} \bar{X}^{a_{\sigma_1(p)}^1,\ldots,a_{\sigma_r(p)}^r}\right) \qquad (40)$$

labeled by the set of m elements σ_i of the permutation group S_m. In fact, the labeling is reduced to a double coset $S_m \backslash S_m^{\otimes r} / S_m$ [38–42], but we do not need these details here.

There is also a distinguished set of operators called generalized characters in [36,37] that are defined as

$$\chi_{R_1,\ldots,R_r}(X, \bar{X}) = \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_r \in S_n} \psi_{R_1}(\sigma_1) \ldots \psi_{R_r}(\sigma_r) \cdot \mathcal{O}_{\sigma_1,\ldots,\sigma_r} \qquad (41)$$

These generalized characters do not form a full basis in the space of all gauge invariant operators, instead they form an over-complete basis in the space of all gauge invariant operators with non-vanishing Gaussian averages [36,37].

The pair correlator is now $\left\langle X_{a_p^1,\ldots a_p^r} \bar{X}^{b_p^1,\ldots b_p^r} \right\rangle = \left(A_{(1)}^{-1}\right)_{a_1}^{b_1}$ $\cdots \left(A_{(r)}^{-1}\right)_{a_r}^{b_r}$ and, in complete analogy with what we were doing in the complex matrix model case, we obtain the averages of these generalized characters

$$\boxed{\langle < \chi_{R_1,\ldots,R_r} \rangle = \frac{C_{R_1,\ldots,R_r}}{d_{R_1} \cdot \cdots \cdot d_{R_r}} \cdot \chi_{R_1}\left\{\text{Tr } A_{(1)}^{-k}\right\} \cdot \cdots \cdot \chi_{R_r}\left\{\text{Tr } A_{(r)}^{-k}\right\}} \qquad (42)$$

where

$$\boxed{\int \exp\left(-\text{Tr } AXB\bar{X} + \sum_k \frac{p_k}{k} \text{Tr } (X\bar{X})^k\right) d^2X = \sum_R \frac{\chi_R\{p_k\} \chi_R\{\text{tr } A^{-k}\} \chi_R\{\text{tr } B^{-k}\}}{\chi_R\{\delta_{k,1}\}}} \qquad (38)$$

$$C_{R_1,\ldots,R_r} := \sum_{\Delta \vdash n} \frac{\prod_{i=1}^{r} \psi_{R_i}(\Delta)}{z_\Delta} \tag{43}$$

and $z_\Delta := \prod_k k^{m_k} m_k!$ is the standard symmetric factor of the Young diagram (order of the automorphism). In the case of $r = 3$, C_{R_1,R_2,R_3} are the Clebsch-Gordan coefficients of the three irreducible representations R_1, R_2, R_3 of the symmetric group.

3 Correlators in Hermitian model

3.1 Examples of correlators

Let us calculate the correlators in the Gaussian Hermitian matrix model with external fields

Since the matrices are square, we return to the standard convention of convolution of indices in the matrix multiplication. One could expect that, similarly to coming from the rectangular complex model to the Gaussian Hermitian one, the results will not be too much different from those in Sect. 2. It turns out, however, not to be the case, and both the calculations in the Hermitian case with the external field are much more tedious even in the case of one external matrix, and the results remarkably differ from those of Sect. 2. We consider here only the case of one external matrix, i.e. the correlators of the form

$$\langle F(X) \rangle := \int F(X) \exp\left(-\operatorname{Tr} X^2 \Lambda\right) dX \tag{44}$$

Since, the partition function is an invariant function of the external matrix Λ, i.e. depends only on its traces, it suffices to choose Λ diagonal. Similarly to (32), we define $p_k := \operatorname{Tr} \Lambda^{-k}$.

In this model, the pair correlator in terms of eigenvalues λ_i of the (diagonal) matrix Λ is equal to

$$\langle X_{ij} X_{kl} \rangle = \frac{2\delta_{il}\delta_{jk}}{\lambda_k + \lambda_l} \tag{45}$$

Here is the difference with the case of two external matrices: in that case one can not work in terms of eigenvalues, and the pair correlator is much more involved.

Now we consider examples of simple correlators in order to demonstrate the structure of answers.

First consider $\langle \operatorname{Tr} X^3 \cdot \operatorname{Tr} X^3 \rangle = \langle X_{ij} X_{jk} X_{ki} X_{\alpha\beta} X_{\beta\gamma} X_{\gamma\alpha} \rangle$. This average is equal to the sum of products of pairings: $\langle X_{ij} X_{jk} \rangle \langle X_{ki} X_{\alpha\beta} \rangle \langle X_{\beta\gamma} X_{\gamma\alpha} \rangle$ plus all possible other pairings. Totally, there are 15 different possibilities: 3 terms of the form

$$C_1 = \langle X_{ij} X_{\alpha\beta} \rangle \langle X_{jk} X_{\beta\gamma} \rangle \langle X_{ki} X_{\gamma\alpha} \rangle$$
$$= \sum_{\alpha} \frac{1}{\lambda_\alpha^3} = p_3 \tag{46}$$

three terms of the form

$$C_2 = \langle X_{ij} X_{\alpha\beta} \rangle \langle X_{jk} X_{\gamma\alpha} \rangle \langle X_{ki} X_{\beta\gamma} \rangle$$
$$= \sum_{\alpha,\beta,\gamma} \frac{8}{(\lambda_\alpha + \lambda_\beta)(\lambda_\beta + \lambda_\gamma)(\lambda_\gamma + \lambda_\alpha)} \tag{47}$$

and nine terms of the form

$$C_3 = \langle X_{ij} X_{jk} X_{ki} X_{\alpha\beta} X_{\beta\gamma} X_{\gamma\alpha} \rangle$$
$$= \sum_{\alpha,\beta,\gamma} \frac{4}{\lambda_\alpha(\lambda_\alpha + \lambda_\beta)(\lambda_\alpha + \lambda_\gamma)} \tag{48}$$

Note that $C_2 + 3C_3 = p_1^3/2$. Thus, we finally obtain $\langle \operatorname{Tr} X^3 \cdot \operatorname{Tr} X^3 \rangle = 3p_3 + 12p_1^3$.

Consider examples of all correlators up to the level 6 (for the sake of brevity, we use the notation $\xi_k := \sum_{i,j} \frac{1}{(\lambda_i + \lambda_j)^k}$):

$$\langle \operatorname{Tr} X^2 \rangle = 2\xi_1$$

$$\boxed{\langle \operatorname{Tr} X \cdot \operatorname{Tr} X \rangle = p_1}$$

$$\langle \operatorname{Tr} X^4 \rangle = 8\sum_{i,j,k} \frac{1}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)} + p_2$$

$$\boxed{\langle \operatorname{Tr} X^3 \cdot \operatorname{Tr} X \rangle = 3p_1^2}$$

$$\langle \operatorname{Tr} X^2 \cdot \operatorname{Tr} X^2 \rangle = 4\xi_1^2 + 8\xi_2$$

$$\langle \operatorname{Tr} X^2 \cdot \operatorname{Tr} X \cdot \operatorname{Tr} X \rangle = 2p_1\xi_1 + 2p_2$$

$$\boxed{\langle \operatorname{Tr} X \cdot \operatorname{Tr} X \cdot \operatorname{Tr} X \cdot \operatorname{Tr} X \rangle = 3p_1^2}$$

$$\langle \operatorname{Tr} X^6 \rangle = 16 \sum_{i,j,k,l} \frac{1}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)(\lambda_i + \lambda_l)}$$

$$+ 24 \sum_{i,j,k,l} \frac{1}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)(\lambda_k + \lambda_l)} + p_3 + 6p_1p_2$$

$$\boxed{\langle \operatorname{Tr} X^5 \cdot \operatorname{Tr} X \rangle = 5p_3 + 10p_1^3}$$

$$\langle \operatorname{Tr} X^4 \cdot \operatorname{Tr} X^2 \rangle = 8p_1 \sum_{i,j,k} \frac{1}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)}$$

$$+ 64 \sum_{i,j,k} \frac{1}{(\lambda_i + \lambda_j)^2(\lambda_i + \lambda_k)} + p_1p_2 + p_3$$

$$\langle \operatorname{Tr} X^4 \cdot \operatorname{Tr} X \cdot \operatorname{Tr} X \rangle = 8p_1 \sum_{i,j,k} \frac{1}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)}$$

$$+ 8 \sum_{i,j} \frac{1}{\lambda_i^2(\lambda_i + \lambda_j)} + 9p_1p_2$$

$$\boxed{\langle \operatorname{Tr} X^3 \cdot \operatorname{Tr} X^3 \rangle = 3p_3 + 12p_1^3}$$

$$\langle \operatorname{Tr} X^3 \cdot \operatorname{Tr} X^2 \cdot \operatorname{Tr} X \rangle = 6p_1^2\xi_1 + 12p_1p_2$$

$$\boxed{\langle \operatorname{Tr} X^3 \cdot \operatorname{Tr} X \cdot \operatorname{Tr} X \cdot \operatorname{Tr} X \rangle = 3(2p_3 + 3p_1^3)}$$

$$\langle \operatorname{Tr} X^2 \cdot \operatorname{Tr} X^2 \cdot \operatorname{Tr} X^2 \rangle = 8\xi_1^3 + 48\xi_1\xi_2 + 64\xi_3$$

$$\langle \text{Tr } X^2 \cdot \text{Tr } X^2 \cdot \text{Tr } X \cdot \text{Tr } X \rangle = 4p_1\xi_1^2 + 8p_1\xi_2 + 8p_2\xi_1 + 8p_3$$

$$\langle \text{Tr } X^2 \cdot \text{Tr } X \cdot \text{Tr } X \cdot \text{Tr } X \cdot \text{Tr } X \rangle = 6p_1^2\xi_1 + 12p_1p_2$$

$$\boxed{\langle \text{Tr } X \cdot \text{Tr } X \cdot \text{Tr } X \cdot \text{Tr } X \cdot \text{Tr } X \cdot \text{Tr } X \rangle = 15p_1^3} \tag{49}$$

and all odd level correlators evidently vanish.

These examples demonstrate that the correlators that involve traces of only odd degrees of the matrix X (they are boxed in (49)) are expressed only through the times (32), moreover, through the odd times. We will return to other correlators elsewhere, and here we consider only this type of correlators. We will need a set of polynomials of odd times that gives a complete basis and forms a closed ring.

3.2 Q Schur polynomials

Emergence of only the odd times immediately gives one a hint to use the Q Schur polynomials instead of the standard Schur polynomials, as we did in the previous sections. These polynomials[1] depend only on *odd* time-variables p_{2k+1} and only on *strict* Young diagrams $R = \{r_1 > r_2 > \cdots > r_{l_R} > 0\} \in \text{SP}$. They were introduced by I. Schur [44] in the study of projective representations of symmetric groups, and were later identified by I. Macdonald [45] with the Hall-Littlewood polynomials $\text{HL}_R\{p\} := \text{Mac}_R\{p\}\big|_{q=0}$ at $t = -1$:

$$Q_R := \begin{cases} 2^{l_R/2} \cdot \text{HL}_R(t = -1) & \text{for } R \in \text{SP} \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

Macdonald's observations were that $\text{HL}_R\{p\}$ for $R \in \text{SP}$ depend only on odd time-variables p_{2k+1}, and that $\text{HL}_R\{p\}$ for $R \in \text{SP}$ form a sub-ring, i.e. the Littlewood-Richardson coefficients $\mathcal{N}_{R_1,R_2}^R$ in the ring

$$\text{HL}_{R_1}\{p\} \cdot \text{HL}_{R_2}\{p\} = \sum_{\substack{R \in R_1 \otimes R_2 \\ R \in \text{SP}}} \mathcal{N}_{R_1,R_2}^R \text{HL}_R\{p\} \tag{51}$$

vanish for $R \notin \text{SP}$, provided $t = -1$ and $R_1, R_2 \in \text{SP}$. Note that $\text{HL}_R\{p\}$ do not vanish for $R \notin \text{SP}$, and then they can also depend on even p_{2k}, thus the set of $Q_R\{p\}$ is not the same as the *set* of HL_R, it is a *sub-set*, and a *sub-ring*.

There is also a manifest way to construct the Q Schur polynomial as a Pfaffian (see, e.g., [43, Eq. (74)]).

The Q Schur polynomials form a system, which has very close properties to the standard Schur functions, only they form a basis in a *sub*space of time-variables. In particular, there is a counterpart of the Frobenius formula (36), for the Q Schur polynomials it states

$$Q_R\{p_k\} = \sum_{\Delta \in \text{OP}} \frac{\Psi_R(\Delta)}{z_\Delta} p_\Delta \tag{52}$$

where OP (odd partitions) is a set of Young diagrams with all lengths of lines odd, and $\Psi_R(\Delta)$ are the characters of the Sergeev group [46,48]. As any characters, they satisfy the orthogonality conditions:

$$\sum_{\Delta \in \text{OP}} \frac{\Psi_R(\Delta)\Psi_{R'}(\Delta)}{2^{l_\Delta} z_\Delta} = \delta_{RR'},$$

$$\sum_{R \in \text{SP}} \frac{\Psi_R(\Delta)\Psi_R(\Delta')}{2^{l_\Delta} z_\Delta} = \delta_{\Delta\Delta'} \tag{53}$$

Actually relevant for the Q Schur polynomials is the restriction to odd times, i.e. the Young diagram Δ in (52), which defines the monomial $p_\Delta = \prod_i^{l_\Delta} p_{\Delta_i}$ should have all the lines of odd length: $\Delta \in \text{OP}$. Therefore of crucial importance is the celebrated **one-to-one correspondence between the sets of SP and OP.**

At last, the Q Schur polynomials satisfy the Cauchy formula, which we write in a form similar to (6), because this is how it will be used in Sect. 4.2 below:

$$\sum_{R \in \text{SP}} Q_R\{p\} Q_R\{\text{Tr } X^k\} = \exp\left(\sum_k \frac{p_{2k+1}\text{Tr } X^{2k+1}}{k + 1/2}\right) \tag{54}$$

3.3 Combinatorial expression for correlators

We are now ready to formulate a nice result for the correlators (44) in terms of the Q Schur polynomials. Let us denote through $2R$ the Young diagram produced from the diagram R by doubling all line lengths, and denote through $R|2$ the Young diagram with all even line lengths (in this case, $R/2$ denotes the Young diagram with all line lengths being half of those of R)[2]. Then (compare with [49]),

$$\boxed{\left\langle Q_R\{\text{Tr } X^k\}\right\rangle = \begin{cases} \dfrac{Q_{R/2}\{\text{Tr } \Lambda^{-k}\}Q_{R/2}\{\delta_{k,1}\}}{Q_R\{\delta_{k,1}\}} & \text{if } R|2 \\ 0 & \text{otherwise} \end{cases}} \tag{55}$$

For the quantity $Q_R\{\delta_{k,1}\}$, there is a counterpart of the standard hook formula (2):

$$Q_R\{\delta_{k,1}\} = 2^{|R|-\frac{l_R}{2}} \left(\frac{1}{\prod_j^{l_R} R_j!}\right) \prod_{k<m} \frac{R_k - R_m}{R_k + R_m} \tag{56}$$

---

[1] More detailed review of these polynomials can be found, e.g., in [43].

[2] For instance, an example of $R|2$ is $R = [6, 4, 2]$. In this case, $R/2 = [3, 2, 1]$ and $2R = [12, 8, 4]$.

It is non-zero only for $R \in \mathrm{SP}$, and

$$\left\langle Q_{2R}\{\mathrm{Tr}\, X^k\}\right\rangle = \prod_i^{l_R} \frac{(2R_i)!}{2^{|R|}\cdot R_i!}\cdot Q_R\{\mathrm{Tr}\,\Lambda^{-k}\} \tag{57}$$

Note that, unlike most formulas in the Introduction, the r.h.s. in (55) is sensitive to relative normalization of two different Q-Schur functions Q_R and $Q_{R/2}$, which is actually well defined, because these quantities are characters of a group and they form an algebra with integer Littlewood-Richardson coefficients. Another property that distinguishes this normalization is that, with the scalar product

$$\left\langle p_{2k+1}\middle| p_{2l+1}\right\rangle = (k+1/2)\cdot \delta_{k,l} \tag{58}$$

the Q-polynomials are orthonormal:

$$\left\langle Q_R\middle| Q_{R'}\right\rangle = \delta_{R,R'} \tag{59}$$

and the Cauchy formula (54) acquires an especially simple form with unit coefficients in the sum.

4 Kontsevich model

Now we are ready to consider the Kontsevich model given by the integral [34]

$$Z = \int \exp\left(-\frac{\mathrm{Tr}\, X^3}{3} + \mathrm{Tr}\,\Lambda^2 X\right) dX \tag{60}$$

and it can be further generalized to non-cubic potentials [32,33], including the quadratic one, when the model is equivalent to the Hermitian matrix model [3]. In this paper, we concentrate on the original cubic case, and we denote the background field by Λ^2 to simplify the formulas in Sect. 4.2.

4.1 Kontsevich model in the character phase

The normalization of measure of the Kontsevich integral depends on the phase of the model [50]. In the character phase, the integral is understood as a formal series in positive powers of $\mathrm{Tr}\,\Lambda^k$, and the generating function of the correlators is defined to be

$$Z_{ch} = \frac{\int dX \exp\left(-\frac{\mathrm{Tr}\, X^3}{3} + \mathrm{Tr}\,\Lambda^2 X\right)}{\int dX \exp\left(-\mathrm{Tr}\,\frac{X^3}{3}\right)} \tag{61}$$

Since the cubic part does not depend on the angular part of the matrix X, one can first perform the angular integration using the expansion [51] of the Itzykson-Zuber integral [52,53]

$$\int \exp\left(\mathrm{Tr}\,\Lambda^2 U X U^\dagger\right)[dU]$$
$$= \sum_R \frac{\chi_R\{\delta_{k,1}\}}{\chi_R\{N\}}\cdot \chi_R\{\mathrm{Tr}\,\Lambda^{2k}\}\chi_R\{\mathrm{Tr}\, X^k\} \tag{62}$$

so that one remains with

$$Z_{ch} = \sum_R \frac{\chi_R\{\delta_{k,1}\}}{\chi_R\{N\}}\cdot \chi_R\{\mathrm{Tr}\,\Lambda^{2k}\}\left\langle\chi_R\{\mathrm{Tr}\, X^k\}\right\rangle \tag{63}$$

and then calculate the correlators $\left\langle\chi_R\{\mathrm{Tr}\, X^k\}\right\rangle$ for the star-like integration contours directly applying formulas like (17). Thus, in this phase, the expression in terms of characters is straightforward and nearly obvious.

4.2 Kontsevich model in the Kontsevich phase

Much more interesting is the case of Kontsevich phase, where the character calculus is much less trivial. In this phase, the model is given by the integral

$$Z_K = \exp\left(-\frac{2}{3}\mathrm{Tr}\,\Lambda^3\right)\frac{\int dX \exp\left(-\frac{\mathrm{Tr}\, X^3}{3} + \mathrm{Tr}\,\Lambda^2 X\right)}{\int dX \exp\left(-\mathrm{Tr}\, X^2 \Lambda\right)} \tag{64}$$

and is understood as a formal series in $\mathrm{Tr}\,\Lambda^{-k}$. The simplest way to deal with this integral is to shift the integration variable to the saddle point $X \to X + \Lambda$ so that the integral takes the form

$$Z_K = \frac{\int dX \exp\left(-\mathrm{Tr}\,\frac{X^3}{3} - \mathrm{Tr}\, X^2\Lambda\right)}{\int dX \exp\left(-\mathrm{Tr}\, X^2\Lambda\right)} \tag{65}$$

and one calculates this integral perturbatively using the Cauchy formula (54),

$$\exp\left(-\frac{\mathrm{Tr}\, X^3}{3}\right) = \sum_{R\in\mathrm{SP}} Q_R\{-\mathrm{Tr}\, X^k\}\cdot Q_R\left\{\frac{1}{2}\delta_{k,3}\right\} \tag{66}$$

Since, for even sizes of Young diagrams $|R|$, $Q_R\{-p_k\} = Q_R\{p_k\}$ and averages are non-vanishing only for even sizes, one finally obtains the Kontsevich partition function:

$$Z_K = \sum_{R\in\mathrm{SP}} \frac{1}{2^{|R|}}\left\langle Q_R\{\mathrm{Tr}\, X^k\}\right\rangle \cdot Q_R\{\delta_{k,3}\} = \sum_{R\in\mathrm{SP}} \frac{1}{4^{|R|}}\frac{Q_R\{\mathrm{Tr}\,\Lambda^{-k}\}Q_R\{\delta_{k,1}\}Q_{2R}\{\delta_{k,3}\}}{Q_{2R}\{\delta_{k,1}\}} \tag{67}$$

where we used (55) for the average. Note that, because of the factor $Q_{2R}\{\delta_{k,3}\}$, this sum effectively runs over R of sizes divisible by 3.

As expected in Kontsevich phase, this partition function depends on the matrix size N only through the variables $\mathrm{Tr}\,\Lambda^{-k}$. This is exactly the same phenomenon which we observed in (22).

Derivation of the formula (67) could be one of the goals in [49], but at that time the Q Schur polynomials were even less known then now, and an explicit answer like this was unavailable.

4.3 Integrability and character expansion

Note that expansion of the partition function in characters typically allows one to check if it is a τ-function of the KP hierarchy immediately by checking that the expansion coefficients satisfy the Plücker relations. For instance, this is the case for the partition functions (7) and (9). Indeed, any partition function of the form

$$Z_w = \sum_R S_R\{p\} S_R\{\bar{p}\} w_R \tag{68}$$

with the function w_R being the product

$$w_R = \prod_{i,j \in R} f(i - j) \tag{69}$$

and $\bar{p}_k$ just arbitrary parameters, is a KP τ-function [11, 54–56], since arbitrary Schur polynomial $S_R\{\bar{p}\}$ satisfies the Plücker relations, and multiplying any solution to the Plücker relations by w_R of the form (69) preserves the solution. Now using (5) and fixing $\bar{p}_k = N_2$, one obtains that (7) is of the form (68). Similarly, fixing $p_k = \delta_{k,1}$, one obtains that (9) is of the form (68).

As we explained in [43, Sect. 8] (see also [57–60]), similarly looking at expansion in the Q Schur polynomials, one can expect that the partition function is a τ-function of the BKP hierarchy. Moreover, we proposed an example of expansion literally repeating formula (7) but with the Schur polynomials substituted by the Q Schur polynomials, though have not managed to construct a matrix model that possesses such an expansion. Hence, it is surprising to realize that the Kontsevich model, which is a τ-function of the KP hierarchy [32,33], not BKP have a similar character expansion in the Q Schur polynomials. This realization in terms of matrix model is especially important as the Q Schur polynomials are related to the spin Hurwitz numbers and to the corresponding cut-and-join operators [43].

Let us emphasize that, as soon as the Kontsevich partition function is a KdV τ-function [32,33], and the KdV τ-function depends on odd times only, this is quite natural to expand the Kontsevich partition function in the Q Schur polynomials, which, in variance with the ordinary Schur polynomials, depends only on odd times. One may ask why people usually consider expansions of KdV τ-functions into the ordinary Schur polynomials and not into the Q Schur polynomials (see, however, [61]), which could look more natural. The reason is that the standard technique of the KdV hierarchy as an embedding into the KP hierarchy uses the formalism of free fermions [62], hence the ordinary Schur polynomials, and it is only using the neutral fermions that gives rise to the Q Schur polynomials, but this type of fermions is associated with the BKP hierarchy [57–60,63,64]. We will consider this issue in detail elsewhere.

5 Conclusion

In this paper, we extended the superintegrability relation $\langle character \rangle \sim$ character to the second kind of matrix models, depending on background fields. This provides a very nice generalization for the complex matrix model, where dimensions of matrices are lifted to more general traces of background matrices. The best known example of the background-field model is, however, different: this is the Kontsevich model, which is rather analogous to the Hermitian, not complex model. In the Hermitian case, the simple formulas are known to get a little more involved: they include characters at a peculiar locus $\delta_{k,s}$, where s is the power of the potential. Thus, it was an intriguing question what happens in background fields. Our result in this paper is for the ordinary Kontsevich model with cubic potential, and it is that the very set of characters is changed from the Schur polynomials to the very interesting set of Q Schur polynomials, and then their averages are given by a rather transparent formula (55). The formula is, however, somewhat non-trivial, because it requires a correlation between normalization of different Q Schur polynomials, in most other examples this does not matter because just a single character appears in the answer which is made from its values at three different loci. In the Kontsevich case, this is different, and this adds new colors to the yet-no-so-well-known subject of Q Schur polynomials.

It is important that the superintegrability-related expansion (67) of the Kontsevich partition function in Q Schur polynomials is much simpler than the expansion through the ordinary Schur polynomials, which could seem natural from the point of view of KP integrability [4–6,65]. Indeed, the coefficients of the latter expansion were found in [66,67] and turned out to be quite complicated and do not expose any clear structure. Even the character nature of the coefficients is not evident in this KP induced expansion. Equation (67) is obviously free from all such drawbacks, in full accordance with the expectations. It, however, remains a problem to lift (67) to generalized Kontsevich model [32,33], while the ordinary Schur expansion of them should be a direct lifting of [66,67].

References

1. A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov, A. Orlov, Nucl. Phys. B **357**, 565–618 (1991)
2. S. Kharchev, A. Marshakov, A. Mironov, A. Orlov, A. Zabrodin, Nucl. Phys. B **366**, 569–601 (1991)
3. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, Nucl. Phys. B **397**, 339–378 (1993). arXiv:hep-th/9203043
4. A. Morozov, Phys.Usp.(UFN) **37** (1994) 1; arXiv:hep-th/9502091; arXiv:hep-th/0502010
5. A. Mironov, Int. J. Mod. Phys. A **9**, 4355 (1994)
6. A. Mironov, Phys. Part. Nucl. **33**, 537 (2002). arXiv:hep-th/9409190
7. A. Mironov, A. Morozov, Phys. Lett. B **771**, 503 (2017). arXiv:1705.00976
8. A. Mironov, A. Morozov, Phys. Lett. B **774**, 210 (2017). arXiv:1706.03667
9. A. Mishchenko, A. Fomenko, Funct. Anal. Appl. **12**, 113 (1978)
10. W. Miller Jr., S. Post, P. Winternitz, J. Phys. **A46**, 423001 (2013). arXiv:1309.2694
11. A. Alexandrov, A. Mironov, A. Morozov, S. Natanzon, JHEP **11**, 080 (2014). arXiv:1405.1395
12. H. Itoyama, A. Mironov, A. Morozov, JHEP **1706**, 115 (2017). arXiv:1704.08648
13. C. Kristjansen, J. Plefka, G.W. Semenoff, M. Staudacher, Nucl. Phys. B **643**, 3–30 (2002). arXiv:hep-th/0205033
14. S. Corley, A. Jevicki, S. Ramgoolam, Adv. Theor. Math. Phys. **5**, 809–839 (2002). arXiv:hep-th/0111222
15. J. Harer, D. Zagier, Invent. Math. **85**, 457–485 (1986)
16. C. Itzykson, J.-B. Zuber, Comm. Math. Phys. **134**, 197–208 (1990)
17. S.K. Lando, A.K. Zvonkin, *Embedded graphs*, Max-Plank-Institut für Mathematik, Preprint 2001 (63)
18. A. Morozov, S. Shakirov, JHEP **0904**, 064 (2009). arXiv:0902.2627
19. W. Fulton, *Young tableaux: with applications to representation theory and geometry*, LMS (1997)
20. R. de Mello Koch, S. Ramgoolam, arXiv:1002.1634
21. A. Mironov, A. Morozov, JHEP **1808**, 163 (2018). arXiv:1807.02409
22. M. Tierz, Mod. Phys. Lett. A **19**, 1365–1378 (2004). arXiv:hep-th/0212128
23. A. Brini, B. Eynard, M. Mariño, Annales Henri Poincaré. Vol. 13. No. 8. SP Birkhäuser Verlag Basel (2012). arXiv:1105.2012
24. M. Rosso, V.F.R. Jones, J. Knot Theory Ramifications **2**, 97–112 (1993)
25. X.-S. Lin, H. Zheng, Trans. Am. Math. Soc. **362**, 1–18 (2010). arXiv:math/0601267
26. A. Alexandrov, A. Mironov, A. Morozov, An. Morozov, JETP Lett. **100**, 271–278 (2014) (Pis'ma v ZhETF **100** (2014) 297–304). arXiv:1407.3754
27. A. Alexandrov, D. Melnikov, JETP Lett. **101**, 51–56 (2015). arXiv:1411.5698
28. A. Morozov, A. Popolitov, S. Shakirov, Phys. Lett. B **784**, 342 (2018). arXiv:1803.11401
29. A. Mironov, A. Morozov. arXiv:2011.02855
30. A. Mironov, A. Morozov. arXiv:2011.01762
31. C. Cordova, B. Heidenreich, A. Popolitov, S. Shakirov, Commun. Math. Phys. **361**, 1235 (2018). arXiv:1611.03142
32. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, A. Zabrodin, Phys. Lett. B **275**, 311 (1992). arXiv:hep-th/9111037
33. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, A. Zabrodin, Nucl. Phys. B **380**, 181 (1992). arXiv:hep-th/9201013
34. M. Kontsevich, Commun. Math. Phys. **147**, 1 (1992)
35. E. Langmann, R.J. Szabo, K. Zarembo, JHEP **01**, 017 (2004). arXiv:hep-th/0308043
36. H. Itoyama, A. Mironov, A. Morozov, JHEP **1912**, 127 (2019). arXiv:1909.06921
37. H. Itoyama, A. Mironov, A. Morozov, Phys. Lett. B **802**, 135237 (2020). arXiv:1910.03261
38. J. Ben Geloun, S. Ramgoolam, arXiv:1307.6490
39. R. de Mello Koch, D. Gossman, L. Tribelhorn, JHEP **2017**, 011 (2017). arXiv:1707.01455
40. P. Diaz, S.J. Rey,. arXiv:1706.02667
41. J. Ben Geloun, S. Ramgoolam, arXiv:1708.03524
42. H. Itoyama, A. Mironov, A. Morozov, Nucl. Phys. B **932**, 52 (2018). arXiv:1710.10027
43. A. Mironov, A. Morozov, S. Natanzon, Eur. Phys. J. C **80**, 97 (2020). arXiv:1904.11458
44. I. Schur, J. Reine Angew. Math. **139**, 155–250 (1911)
45. I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edn. (Oxford University Press, Oxford, 1995)
46. A. Sergeev, Math. Sb. USSR **51**, 419–427 (1985)
47. M. Yamaguchi, J. Algebra **222**, 301–327 (1999). arXiv:math/9811090
48. A. Kleshchev, *Linear and projective representations of symmetric groups, Cambridge Tracts in Mathematics 163* (Cambridge Univ. Press, Cambridge, 2005)
49. P. Di Francesco, C. Itzykson, J.B. Zuber, Commun. Math. Phys. **151**, 193 (1993). arXiv:hep-th/9206090
50. A. Mironov, A. Morozov, G.W. Semenoff, Int. J. Mod. Phys. A **11**, 5031 (1996). arXiv:hep-th/9404005
51. A. Y. Morozov, Theor. Math. Phys. **162** (2010) 1 [Teor. Mat. Fiz. **161** (2010) 3]. arXiv:0906.3518
52. Am Harish-Chandra, J. Math. **79**, 87 (1957)
53. C. Itzykson, J.B. Zuber, J. Math. Phys. **21**, 411 (1980)
54. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, Int. J. Mod. Phys. A **10**, 2015 (1995). arXiv:hep-th/9312210
55. A. Orlov, D.M. Shcherbin, Theor. Math. Phys. **128**, 906–926 (2001)
56. A. Orlov, Theor. Math. Phys. **146**, 183–206 (2006)
57. A. Orlov, Theor. Math. Phys. **137**, 1574–1589 (2003). math-ph/0302011
58. J. Harnad, J.W. van de Leur, AYu. Orlov, Theor. Math. Phys. **168**, 951–962 (2011). arXiv:1101.4216
59. J.W. van de Leur, A.Yu. Orlov, arXiv:1404.6076. arXiv:1611.04577
60. A. Yu. Orlov, T. Shiota, K. Takasaki. arXiv:1201.4518. arXiv:1611.02244
61. J.J.C. Nimmo, A. Orlov, Glasgow Math. J. **47 (A)**, 149–168. arXiv:nlin/0405009 (2005)

62. E. Date, M. Jimbo, M. Kashiwara, T. Miwa, R.I.M.S. Symp, *Nonlinear integrable systems: classical theory and quantum theory* (World Scientific, Singapore, 1983)
63. E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Phys. D **4**, 343–365 (1982)
64. M. Jimbo, T. Miwa, Publ. RIMS Kyoto Univ. **19**, 943–1001 (1983)
65. Y. Ohta, J. Satsuma, D. Takahashi, T. Tokihiro, Prog. Theor. Phys. Suppl. **94**, 210 (1988)
66. J. Zhou. arXiv:1306.5429
67. F. Balogh, D. Yang, Lett. Math. Phys. **107**, 1837 (2017). arXiv:1412.4419