Eur. Phys. J. C (2021) 81:227
https://doi.org/10.1140/epjc/s10052-021-09025-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Generating solutions for charged stellar models in general relativity

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Received: 14 October 2020 / Accepted: 6 March 2021 / Published online: 15 March 2021

Abstract It is shown that the expressions for the tangential pressure, the anisotropy factor and the radial pressure in the Einstein–Maxwell equations may serve as generating functions for charged stellar models. The latter can incorporate an equation of state when the expression for the energy density is also used. Other generating functions are based on the condition for the existence of conformal motion (conformal flatness in particular) and the Karmarkar condition for embedding class one metrics, which do not depend on charge. In all these cases the equations are linear first order differential equations for one of the metric components and Riccati equations for the other. The latter may be always transformed into second order homogenous linear differential equations. These conclusions are illustrated by numerous particular examples from the study of charged stellar models.

1 Introduction

Gravitation is governed by the Einstein equations of general relativity in the simplest case. The Einstein–Maxwell equations are a system of highly non-linear differential second order equations in partial derivatives. In astrophysics spherical symmetry is usually used, which reduces in the static case the differential equations to ordinary ones and the derivatives are with respect to the radius. The metric is diagonal with just two components. In canonical commoving coordinates there are three Einstein equations for six unknowns – the two metric potentials and the four components of the energy-momentum tensor T_{ab}, namely, the energy density μ, the radial and the tangential pressures p_r and p_t and the charge l. Thus the fluid is anisotropic, which is backed by arguments for compact objects with very high density [1] and by a number of other reasons [2,3].

On one side these equations present expressions for the components of the energy-momentum tensor. On the other side the metric potentials enter in a rather involved way as they are obtained from the Ricci tensor and scalar. The equations remain non-linear for the metric. Durgapal and Banerjee [4] showed that in the perfect fluid case the Einstein equations are linear of first order for a function of g_{11} and the equations for p_t and the anisotropy factor Δ are linear of second order for a function of g_{00}. Later, these findings were generalized for charged anisotropic fluid. The reason for this simplification was partly clarified in a previous paper [5] and is due to the fact that the Einstein equation for p_t is a Riccati equation. It was also shown there that the Einstein equations may serve as generating functions for stellar model solutions, similar to the case of Δ [6]. The existence of an EOS leads to a relation between the metric potentials.

Something more, there are common features between the generating functions based on the equations for p_t and Δ and other ways to generate a solution, like conformal flatness, conformal motion or the possibility to embed the spacetime in a flat five-dimensional spacetime, namely they are also linear or Riccati, which in the last case is truncated to a Bernoulli equation.

In the present paper we discuss the charged anisotropic case in a systematic way. Charged anisotropic fluid is the general type of fluid in the static case. All other characteristics like shear, expansion, two types of viscosity, two types of radiation depend on time and vanish for static solutions [7]. Like in [5], we shall not study the numerous conditions for physical viability of some new solution, but concentrate on the mathematical issues and classification schemes, backing them with plenty of concrete examples from the literature, where the hard and space consuming check of viability has already been done.

There are other methods to generate static solutions in the neutral case. One is for perfect (isotropic) fluids, where isotropic and canonical coordinates are used [8,9]. Different theorems about linking the solutions were proven, or checking part of the viability conditions has been done [10,11].

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Generation procedures for finding anisotropic solutions out of known isotropic ones have also been given [12,13].

In Sect. 2 the Einstein–Maxwell equations are given, as well as some characteristics of the model and the equations for the anisotropy factor, the existence of conformal motion or flatness in particular, and the Karmarkar condition. In Sect. 3 a generating function, based on the expression for the radial pressure is discussed. When an EOS is imposed, the expression for the energy density is also necessary. Section 4 gives generating function based on the expressions for the tangential pressure. The well-known generating function, based on the anisotropy factor, is generalized to the charged case. In Sect. 5 we discuss the metric potentials as generating functions, with or without a relation between them. Section 6 deals with generating solutions when the charge is not given beforehand. Section 7 provides some discussion.

2 Einstein–Maxwell equations and definitions

The interior of static spherically symmetric stars is described by the canonical line element

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where λ and ν depend only on the radial coordinate r. The energy-momentum tensor reads

$$T_{\alpha\beta} = (\mu + p_t)\, u_\alpha u_\beta + p_t g_{\alpha\beta} + (p_r - p_t)\, \chi_\alpha \chi_\beta + E_{\alpha\beta}. \tag{2}$$

Here μ is the energy density, p_r is the radial pressure, p_t is the tangential pressure, u^α is the four-velocity of the fluid, χ^α is a unit spacelike vector along the radial direction and $E_{\alpha\beta}$ is the electromagnetic energy tensor.

We have

$$E_{\alpha\beta} = \frac{1}{4\pi}\left(F_\alpha^{\ \gamma} F_{\beta\gamma} - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta}\right), \tag{3}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor. Its only non-trivial component $F_{01} = -F_{10} = -\Phi'$ is expressed through the four-potential, which has only a time component Φ. The prime stands for a radial derivative. The Maxwell equations yield

$$\Phi' = \frac{e^{\nu/2+\lambda/2} l}{r^2}, \quad l\,(r) = 4\pi \int_0^r \sigma e^{\lambda/2} r^2 dr, \tag{4}$$

where σ is the charge density and $l\,(r)$ is the total charge up to radius r. We use relativistic units with $G = 1, c = 1, k = 8\pi$.

The Einstein equations read

$$8\pi\mu + \frac{l^2}{r^4} = \frac{1}{r^2} - \left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) e^{-\lambda}, \tag{5}$$

$$8\pi p_r - \frac{l^2}{r^4} = -\frac{1}{r^2}(1 - e^{-\lambda}) + \frac{\nu'}{r} e^{-\lambda}, \tag{6}$$

$$8\pi p_t + \frac{l^2}{r^4} = \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 + \frac{2\nu'}{r} - \nu'\lambda' - \frac{2\lambda'}{r}\right), \tag{7}$$

where μ is the matter density, p_r is the radial pressure and p_t is the tangential one.

The gravitational mass in a sphere of radius r is given by

$$\frac{2m}{r} = 1 - e^{-\lambda} + \frac{l^2}{r^2}. \tag{8}$$

which may be written also as

$$e^{-\lambda} = 1 - \frac{2m}{r} + \frac{l^2}{r^2}. \tag{9}$$

The field equations do not contain ν, but its first and second derivative. It is related to the four-acceleration a_1, namely $2a_1 = \nu'$.

As a whole, we have three field equations for six unknown functions: λ, ν, μ, p_r, p_t and l. We can choose freely three of them, but the model will be physically realistic if a number of regularity, matching and stability conditions are satisfied too. Choosing λ, ν, l means to charge a neutral solution with the same λ and ν. Then p_r and m increase, but μ and p_t decrease.

Different constraints may be imposed on the system of Einstein–Maxwell equations. One of them is the existence of an equation of state (EOS) $p_r = f\,(\mu)$.

Let us introduce the anisotropic factor $\Delta = p_t - p_r$. It measures the anisotropy of the fluid. Equations (6, 7) give

$$-8\pi\Delta - \frac{2l^2}{r^4} = e^{-\lambda}\left(-\frac{\nu''}{2} - \frac{\nu'^2}{4} + \frac{\nu'}{2r} + \frac{1}{r^2}\right) + e^{-\lambda}\frac{\lambda'}{2}\left(\frac{\nu'}{2} + \frac{1}{r}\right) - \frac{1}{r^2}. \tag{10}$$

When $\Delta = 0$ the fluid becomes perfect and all pressures are equal. Charging a neutral solution decreases its Δ.

The following two requirements may be imposed on the spacetime.

The first is conformally flat spacetime. It takes place when its Weyl tensor vanishes. This is a particular case of spacetimes with conformal motion when a Killing vector $\mathbf{K}$ exists. Then the following equation has to be satisfied

$$L_{\mathbf{K}} g_{ab} = 2\psi g_{ab}, \tag{11}$$

where $L_{\mathbf{K}}$ is the Lie derivative operator and $\psi\,(t, r)$ is the conformal factor. This implies the equation [14]

$$2\nu'' + \nu'^2 = \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{4}{r^2}(1 + s)\, e^{\lambda} - \frac{4}{r^2}, \tag{12}$$

where s is a constant of integration. The spacetime. is conformally flat when $s = 0$.

In recent years spacetimes, which are embeddings of class one, have been widely discussed. They can be embedded in a five-dimensional flat spacetime. This requires the Karmarkar relation between the components of the Riemann tensor [15]

$$R_{1010}R_{2323} - R_{1212}R_{3030} = R_{1220}R_{1330}. \tag{13}$$

It transforms into a differential equation for λ and ν:

$$2\frac{\nu''}{\nu'} + \nu' = \frac{\lambda' e^{\lambda}}{e^{\lambda} - 1}. \tag{14}$$

The charge does not enter Eqs. (12, 14), hence, the system (5–7) represents in these cases the charging of a neutral solution with conformal motion or an embedding of class one.

We have shown in the uncharged case [5] that Eqs. (5, 6, 7, 10, 12) are linear with respect to $y = e^{-\lambda}$ while Eq. (14) is linear for $y = e^{\lambda}$. Equation (5) does not contain a_1, while Eq. (6) gives an expression for it. The others belong to two types of equations with respect to a_1 – Bernoulli or Riccati. The last may be transformed into linear equation for $u = e^{\nu/2}$. Now we shall show that charging of the fluid does not alter these properties.

The Riccati equation is given by

$$g y' = f_2 y^2 + f_1 y + f_0 \tag{15}$$

and no general solution is known. In particular cases it reduces to integrable equations. Thus when $f_2 = 0$ it turns into a linear equation, which has the general solution [16]

$$y = Ce^{F} + e^{F}\int e^{-F}\frac{f_0}{g}dr, \quad F = \int \frac{f_1}{g}dr. \tag{16}$$

When $f_0 = 0$ it becomes a Bernoulli equation with $n = 2$. Then $1/y$ satisfies a linear equation and is also integrable. Every Riccati equation may be transformed into a second-order homogenous linear equation for a function u [5,16]. In the case $f_2 = -g$ we have

$$f_2 u'' + f_1 u' + f_0 u = 0, \quad u = \exp\int y dr. \tag{17}$$

The substitution $y = u'/u$ leads back to Eq. (15).

3 The energy density and the radial pressure

In the following we consider l as known. Equation (5) for the energy density does not contain a_1. It is linear with respect to $y = e^{-\lambda}$ and can be written as

$$r y' = -y + 1 - 8\pi\mu r^2 - \frac{l^2}{r^2}. \tag{18}$$

Equation (9) may be written as

$$y = 1 - \frac{2m}{r} + \frac{l^2}{r^2}. \tag{19}$$

Any equation, linear in y may be transformed into an equation, linear in m with the use of the above formula.

Equation (6) for the radial pressure may be written as

$$8\pi p_r r^2 = (2a_1 r + 1)\, y - 1 + \frac{l^2}{r^2}. \tag{20}$$

It may be regarded as an expression for p_r or y

$$y = \frac{8\pi p_r r^2 + 1 - \frac{l^2}{r^2}}{2ra_1 + 1}, \tag{21}$$

or a_1

$$2a_1 = \nu' = \frac{8\pi p_r r^2 + 1 - y - \frac{l^2}{r^2}}{ry}. \tag{22}$$

The potential ν is found by a simple quadrature.

Thus, Eq. (20), which contains p_r, y and a_1, is the simplest generating function for any of them, when the other two are known. Solutions with given y (or m) and p_r may be found in [17–21].

An EOS can be incorporated in this scheme, $p_r = f(\mu)$ or

$$2rya_1 = 1 - \frac{l^2}{r^2} - y + 8\pi r^2 f\left(-\frac{ry' + y - 1 + \frac{l^2}{r^2}}{8\pi r^2}\right), \tag{23}$$

which follows from Eqs. (18, 20). Obviously, the resulting equation is not linear in y in general, but still may be solvable by choosing an ansatz for y. Anyway, it's an expression for a_1 in terms of y and is a relation between the metric potentials. Mainly EOS with ansatz for y were used. Thus quadratic EOS is discussed in [22–24], polytropic EOS in [25–27], and other EOS in [28–30].

A special case is the linear EOS (LEOS) $p_r = a\mu - b$ with constant $0 \le a \le 1$ and the bag constant $b \ge 0$, which includes also the case $p_r = 0$. Equation (23) becomes

$$2rya_1 = (a+1)\left(1 - \frac{l^2}{r^2} - y\right) - ary' - 8\pi br^2. \tag{24}$$

This is an expression for a_1 when y and l are given and was used for concrete ansatze in [31–39].

Equation (24) is also a linear equation for y

$$ary' = -(2ra_1 + a + 1)\, y + (a+1)\left(1 - \frac{l^2}{r^2}\right) - 8\pi br^2. \tag{25}$$

It can be solved by Eq. (16) when a, b, a_1 are known. The factor $F = \int \frac{f_1}{g}dr$ is the same as in the uncharged case [5]. We have a singular e^F for $r = 0$, hence $C = 0$. Then the solution is

$$y = \frac{\int \left[(a+1)\left(1 - \frac{l^2}{r^2}\right) - 8\pi br^2\right](re^{\nu})^{1/a}dr}{a\left(r^{a+1}e^{\nu}\right)^{1/a}}. \tag{26}$$

The relation between the energy density and the mass is more complicated for a charged fluid. Integrating Eq. (5) and using formula (9) we get

$$m=\frac{1}{2}\int\left(8\pi\mu r^2+\frac{l^2}{r^2}\right)dr+\frac{l^2}{2r}. \tag{27}$$

This expression reduces to the one in the neutral case when $l=0$. It may be written also as

$$m'=4\pi\mu r^2+\frac{ll'}{r}. \tag{28}$$

This formula shows that when we pass from y to m Eq. (18) simplifies.

4 The tangential pressure and the anisotropic factor

Equation (7) is an expression for p_t and can be written as a linear equation for y

$$\frac{1}{2}\left(a_1+\frac{1}{r}\right)y'=-\left(a_1'+a_1^2+\frac{a_1}{r}\right)y+8\pi p_t+\frac{l^2}{r^4}. \tag{29}$$

Its solution from Eq. (16) reads

$$y=e^F\left(C+\int ze^{\nu}e^{2\int\frac{dr}{r^2z}}\left(16\pi p_t+\frac{l^2}{r^4}\right)dr\right), \tag{30}$$

where

$$a_1+\frac{1}{r}=\frac{\nu'}{2}+\frac{1}{r}\equiv z, \tag{31}$$

$$e^F=z^{-2}e^{-\nu}e^{-2\int\frac{dr}{r^2z}}. \tag{32}$$

The term e^F is the same as in the uncharged case. Due to Eq (9), Eq. (29) is also linear with respect to the mass.

Equation (7) is also a Riccati equation for a_1

$$ya_1'=-ya_1^2-\left(\frac{y}{r}+\frac{y'}{2}\right)a_1-\frac{y'}{2r}+8\pi p_t+\frac{l^2}{r^4} \tag{33}$$

and may be solved for particular choices of y and p_t. It can be transformed into a linear second order homogenous differential equation following Eqs. (16, 17)

$$yu''+\left(\frac{y}{r}+\frac{y'}{2}\right)u'+\left(\frac{y'}{2r}-8\pi p_t-\frac{l^2}{r^4}\right)u=0, \tag{34}$$

where

$$u=e^{\nu/2} \tag{35}$$

Sometimes it may be solved easier than the original Riccati equation, since many special functions are defined by such equations. It remains in the same time linear (and integrable) first order equation for $y=e^{-\lambda}$ or m. It can be called a double linear equation. Thus, like p_r, the expression (7) for p_t is a generating function for charged stellar models, when two of the quantities p_t, y (or m) and a_1 are known.

The generating functions based on Δ are found in a similar way. Eq (10) is linear with respect to y (or m) and may be rewritten as

$$\left(a_1+\frac{1}{r}\right)y'=-2\left(a_1'+a_1^2-\frac{a_1}{r}-\frac{1}{r^2}\right)y-2\left(\frac{1}{r^2}-8\pi\Delta-\frac{2l^2}{r^4}\right). \tag{36}$$

After some transformations it becomes

$$y'=-2\left(\frac{z'}{z}+z-\frac{3}{r}+\frac{2}{r^2z}\right)y-\frac{2}{z}\left(\frac{1}{r^2}-8\pi\Delta-\frac{2l^2}{r^4}\right). \tag{37}$$

This is the generalisation of Eq. (8) [6] to the charged case when the different definition of their Δ is taken into account and is still integrable. The result is

$$y=r^6z^{-2}e^{-\int\left(\frac{4}{r^2z}+2z\right)dr}\left[C-2\int r^{-8}z\left(1-8\pi\Delta r^2-\frac{2l^2}{r^2}\right)e^{\int\left(\frac{4}{r^2z}+2z\right)dr}dr\right]. \tag{38}$$

The generating potentials are Δ, z and l, the second, due to Eq (31), is equivalent to a_1. This generating function encompasses the important cases of charged perfect fluid when $\Delta=0$ [40] and neutral perfect fluid when $\Delta=0$, $l=0$. Solutions with given Δ, a_1 and l are discussed [41–43], where the mass is used instead of y, [44–50]. There are also solutions with $\Delta=0$ [51,52].

Equation (36) is also a Riccati one for a_1, the Riccati structure $a_1'+a_1^2$ being brought in Δ by p_t. It can be written as

$$2ya_1'=-2ya_1^2+\left(\frac{2y}{r}-y'\right)a_1+\frac{2y-2-ry'}{r^2}+16\pi\Delta+\frac{4l^2}{r^4} \tag{39}$$

and solved for particular Δ, y and l. Finally, it can be linearized following Eq. (17) into

$$-2yu''+\left(\frac{2y}{r}-y'\right)u'+\left(\frac{2y-2-ry'}{r^2}+16\pi\Delta+\frac{4l^2}{r^4}\right)u=0, \tag{40}$$

where u is given by Eq. (35). Thus, once again, Eq. (40) is doubly linear, like Eq. (34). Solutions of this equation were presented [53–56] and with $\Delta=0$ [57]. In total, Eq. (10) is a generating function for stellar models, when l and two of the quantities Δ, y (or m) and a_1 are known. The differential equations for y and u are linear.

5 The metric potentials as generating functions

The simplest way to generate solutions in the charged case is to choose independently the two generating potentials λ and ν and add to them a third potential l. Thus any neutral solution may be charged [58,59].

Some important stellar models require a relation between λ and ν, reducing the generating functions to two. For example this is the case of charged perfect fluid, when in Eq. (10) $\Delta = 0$. Similar example are spacetimes admitting conformal motion. The metric potentials of such spacetimes satisfy Eq. (12). This equation is solved by a series of transformations [14]. Surprisingly, it is also a linear equation in y (or m) and a Riccati equation for a_1. It can be written as

$$\left(\frac{1}{r} - a_1\right) y' = 2\left(a_1' + a_1^2 - \frac{a_1}{r} + \frac{1}{r^2}\right) y - \frac{2(1+s)}{r^2} \tag{41}$$

or

$$2ya_1' = -2ya_1^2 + \left(\frac{2y}{r} - y'\right) a_1 + \frac{y'}{r} + \frac{2(1+s) - 2y}{r^2}. \tag{42}$$

Once again $g = -f_2$ in Eq. (15), so it may be transformed into a linear equation, analogous to Eq. (17)

$$-2yu'' + \left(\frac{2y}{r} - y'\right) u' + \left(\frac{y'}{r} + \frac{2(1+k) - 2y}{r^2}\right) u = 0, \tag{43}$$

where u is given by Eq. (16). Its solution was found [14], possesses three branches

$$e^\nu = Ar \exp\left(\sqrt{1+s} \int \frac{e^\lambda}{r} dr\right) + Br \exp\left(-\sqrt{1+s} \int \frac{e^\lambda}{r} dr\right), \quad 1+s > 0, \tag{44}$$

$$e^\nu = Ar \int \frac{e^\lambda}{r} + Br, \quad 1+s = 0, \tag{45}$$

$$e^\nu = Ar \exp\left(\sqrt{-(1+s)} \int \frac{e^\lambda}{r} dr\right) + Br \exp\left(-\sqrt{-(1+s)} \int \frac{e^\lambda}{r} dr\right), \quad 1+s < 0. \tag{46}$$

and do not depend on the charge. Solutions with conformal motion were discussed recently [60,61], [62]. These expressions were put into Eq. (36) and another equation for y arises, which is simpler [63]. Solutions based on ψ in Eq. (12) were studied [64].

Another example is the Karmarkar condition for embedding of class one, Eq (14) [5]. It may be written as

$$a_1' = -a_1^2 + \left[\ln\left(\frac{1-y}{y}\right)\right]' \frac{a_1}{2}. \tag{47}$$

The would be Riccati equation becomes a Bernoulli one. It is also a Bernoulli equation for y

$$-\frac{a_1}{2} y' = \left(a_1' + a_1^2\right) y - \left(a_1' + a_1^2\right) y^2. \tag{48}$$

All these equations are solvable. Their integration may be done directly, without using the general formulas and we obtain the well-known results

$$e^\lambda = C\nu'^2 e^\nu + 1, \tag{49}$$

$$e^\nu = \left(A + B \int \sqrt{e^\lambda - 1} dr\right)^2. \tag{50}$$

where A, B, C are integration constants. Thus when one of the metric coefficients is given, we can find the other. The solution may be charged by introducing a known l. It only changes the system of Einstein–Maxwell equations (5–7). Solutions with given λ and l were found [65–67]. Solutions with known ν and l were also studied [68,69].

6 Solutions when the charge is not given beforehand

Up to now we have discussed cases with given l^2. However, solutions may be found when this is not so. It is clear that the LEOS Eq. (35) is also an expression for l^2

$$(a+1)\frac{l^2}{r^2} = -ary' - (2ra_1 + a + 1) y + a + 1 - 8\pi br^2. \tag{51}$$

We can find l^2 when y and a_1 are known, i.e. when λ and ν are known and a LEOS is given [70].

Equation (36) can also serve as an expression for l^2

$$\frac{4l^2}{r^4} = \left(a_1 + \frac{1}{r}\right) y' + 2\left(a_1' + a_1^2 - \frac{a_1}{r} - \frac{1}{r^2}\right) y + 2\left(\frac{1}{r^2} - 8\pi\Delta\right) \tag{52}$$

when y, a_1 and Δ are known. Thus, we can give an ansatz for λ, add the Karmarkar condition to find ν and set $\Delta = 0$ [71–75]. The isotropic condition may be written also as Eq. (40), another expression for l^2

$$-\frac{4l^2}{r^4} = -2y\frac{u''}{u} + \left(\frac{2y}{r} - y'\right)\frac{u'}{u} + \frac{2y - 2 - ry'}{r^2} + 16\pi\Delta, \tag{53}$$

where $u = e^{\nu/2}$. Fixing y, setting $\Delta = 0$ (perfect fluid) and with some simplifying assumption one can solve this equation [76].

Together, Eqs. (51, 52) give another linear equation for y which depends only on a_1 and Δ. Solving it, we find y and then l^2 from any of Eqs. (51, 52). There are particular examples of this approach [77–79].

One can use another EOS, e.g. the Chaplygin EOS

$$p_r = \alpha_1 \mu - \frac{\alpha_2}{\mu}, \tag{54}$$

where α_1, α_2 are positive constants. Summing Eqs. (5, 6) we obtain

$$p_r = G(\lambda, \nu) - \mu, \tag{55}$$

where G is some function. Replacing (55) into (54) yields a quadratic equation for μ, which is solvable

$$(\alpha_1 + 1)\mu^2 - G\mu - \alpha_2 = 0. \tag{56}$$

The metric components λ and ν may be supplied directly [80]. Another way is to fix one of them, e.g. ν and impose the Karmarkar condition to find λ [81].

Similar is the situation with the quadratic EOS

$$p_r = \alpha_1 \mu^2 + \alpha_2 \mu + \alpha_3. \tag{57}$$

Equation (55) shows that this is a quadratic equation for μ

$$\alpha_1 \mu^2 + (\alpha_2 + 1)\mu + \alpha_3 - G = 0 \tag{58}$$

and may be solved too.

Another popular EOS is the modified Van der Vaals one

$$p_r = \alpha_1 \mu^2 + \frac{\alpha_2 \mu}{1 + \alpha_3 \mu}. \tag{59}$$

It becomes

$$\alpha_1 \alpha_3 \mu^3 + (\alpha_1 + \alpha_3)\mu^2 + (\alpha_2 + 1 - \alpha_3 G)\mu - G = 0. \tag{60}$$

This is a cubic equation for μ and is still solvable.

Finally, let us discuss the polytropic EOS

$$p_r = \alpha \mu^{1+\frac{1}{N}}, \tag{61}$$

where α is a constant and N is the polytropic index. It can be written with the help of Eq. (55) as

$$\alpha^N \mu^{N+1} - (G - \mu)^N = 0. \tag{62}$$

This equation is quadratic for $N = 1$, cubic for $N = 2$ and quartic for $N = 3$ and therefore solvable for μ for these values of N.

7 Discussion

In a previous paper [5] we have studied the existence of generating functions, giving solutions for uncharged stellar models. In the present one we do the same for charged models. The addition of charge does not alter the general scheme of using the Einstein equations as generating functions. Now three of the four characteristics of the model should be given – $y = e^{-\lambda}$, $a_1 = \nu'/2$, l and either p_r, p_t or Δ. This approach is greatly simplified, because the Einstein equations with charge are still linear first order differential equations for y and linear or Riccati equations for the four-acceleration a_1. The linear equations are always integrable in quadratures, while the Riccati equations are integrable in many particular cases. There is a standard mathematical procedure to transform them into linear homogenous differential equations of second order for $u = e^{\nu/2}$ [16]. They are the "missing link" between the original form of the Einstein equations and their linear version, which appears out of nowhere in [4] for neutral perfect fluids. It holds also for anisotropic and charged fluids. The source of the Riccati structure $a_1' + a_1^2$ still comes from the component R_{0101} of the Riemann tensor, whose expression is the same in the charged case. Equation (8) shows that the mass m still satisfies a linear equation and may replace y.

There are two main ways of generating solutions. The first one accepts that l is a given function. Then the stellar models for neutral fluids are just charged. If the initial model is physically realistic this is rather probable for its charged generalization. The simplest generating function is Eq. (6) for p_r, which is an expression for p_r, a_1, y or l^2 without solving any equations. Equation (5) for the energy density cannot be used as a generating function, because it does not contain ν. However, when the model has an EOS, the combination of Eqs. (5) and (6) works as a generating function, producing a relation between the two metric potentials and l^2. The equation for Δ was used [6] to obtain λ when ν and Δ are given. It becomes a generating function for perfect fluid models when $\Delta = 0$. We have generalized it to the charged case. Equation (7) for p_t can play a similar role but is rarely used.

Of course, the simplest generating potentials are λ and ν and l. There are physical reasons that sometimes impose a relation between the metric components. This happens when an EOS exists.

A second important case is that of spacetimes admitting conformal motion (conformal flatness in particular). The surprising fact is that this relation is also a linear differential equation for y or u and a Riccati one for a_1. It does not depend on the charge.

A third well-known example are spacetimes of embedding class one, obeying the Karmarkar condition. Here there is a minor difference – the relation is a linear equation for $1/y$ and a Bernoulli equation for y, which is also integrable. Furthermore, it is a Bernoulli equation with quadratic term for a_1. The Riccati structure, discussed above, is still present but there is no free term. It is charge independent too.

The second way of generating solutions is when l is not given beforehand. In the previous section we have outlined the different ways to solve the Einstein equations in this case.

One of them relies on the existence of an EOS. It leads to algebraic equations for most of the popular EOS, which are soluble up to fourth order included. This second way more often produces models with unphysical features. However, the numerous examples of physically realistic stellar models given in Sect. 6 show that this problem may be solved successfully.

It is interesting whether in some of the alternative theories of gravitation similar simplifications occur.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical paper and this manuscript has no associated data. All the required data is already provided by the author.]

Funded by SCOAP^3.

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