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Surface terms in effective action of O-plane at order α'^2

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Abstract The effective action of string theory has both bulk and boundary terms if the spacetime is an open manifold. Recently, the known classical effective action of string theory at the leading order of α' and its corresponding boundary action have been reproduced by constraining the effective actions to be invariant under gauge transformations and under string duality transformations. In this paper, we use this idea to find the classical effective action of the O-plane and its corresponding boundary terms in type II superstring theories at order α'^2 and for NS–NS couplings. We find that these constraints fix the bulk action and its corresponding boundary terms up to one overall factor. They also produce three multiplets in the boundary action that their coefficients are independent of the bulk couplings under the string dualities.

1 Introduction

Perturbative string theory is a quantum theory of gravity with a finite number of massless fields and a tower of infinite number of massive fields reflecting the stringy nature of the gravity at the weak coupling. String theory on the spacetime manifolds with boundary is conjectured to be dual to a gauge theory on the boundary [1,2]. The string theory and its non-perturbative objects are usually explored by studying their low-energy effective actions which include the massless fields and their covariant derivatives. For the open spacetime manifolds, the effective actions have both bulk and boundary terms, i.e., $\mathbf{S}_{\rm eff}+\partial\mathbf{S}_{\rm eff}$. They should be produced by specific techniques in string theory.

There are various approaches for calculating the bulk effective action $\mathbf{S}_{\rm eff}$, e.g., the S-matrix approach [3,4], the sigma-model approach [5,6], the Double Field Theory approach [7,8] and the duality approach [9–14]. In the duality approach, the consistency of the effective actions with gauge transformations and with T- and S-duality transformations are imposed to find the higher derivative couplings. The Double Field Theory and T-duality approaches are based on the observation made by Sen in the context of closed string field theory [15] that the classical effective action of bosonic string theory should be invariant under T-duality to all orders in α'. Similar observation has been made for the hetrotic string theory in [16].

In the T-duality approach, by removing total derivative terms, using field redefinitions and using Bianchi identities, one first find the minimum number of independent and gauge invariant couplings in the string frame action $\mathbf{S}_{\rm eff}$ at each order of α'. Then one reduces the spacetime on a circle, i.e., $M^{(D)}=S^{(1)}\times M^{(D-1)}$. The T-duality [17,18] is imposed as a constraint on the reduction of the effective action on the circle to find the coefficients of the independent couplings, i.e., the effective action satisfies the following constraint:

$$S_{\rm eff}(\psi)-S_{\rm eff}(\psi')=\mathrm{TD} \tag{1}$$

where $S_{\rm eff}$ is the reduction of the effective action on the circle, ψ represents all massless fields in the base space $M^{(D-1)}$ and ψ' represents their transformations under the T-duality transformations which are the Buscher rules [19,20] and their higher derivative corrections. They form a Z_2-subgroup of $O(1,1;R)$. On the right-hand side, TD represents some total derivative terms in the base space which may not be invariant under the T-duality. They become zero for the closed spacetime manifolds using the Stokes's theorem. This approach has been used in [21,22] to find effective action of the bosonic string theory at orders α', α'^2. This approach has been also used in [23,24] to construct NS–NS couplings in type II superstring effective action at order α'^3

The constraint (1) for the effective action of the non-perturbative D_p-brane/O_p-plane objects is such that $S_{\rm eff}(\psi)$ represents the reduction of $(p-1)$-brane action along the circle transverse to the brane, i.e., $M^{(D)}=M^{(p)}\times M^{(D-p)}$ where $(p-1)$-brane is in the subspace $M^{(p)}$ and $M^{(D-p)}=S^{(1)}\times M^{(D-p-1)}$, and $S_{\rm eff}(\psi')$ represents T-duality transfor-

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mation of the reduction of p-brane action along the circle tangent to the brane, i.e., $M^{(D)} = M^{(p+1)} \times M^{(D-p-1)}$ where p-brane is in the subspace $M^{(p+1)}$ and $M^{(p+1)} = S^{(1)} \times M^{(p)}$. This approach has been used to construct the O_p-plane effective action at order α'^2 in type II superstring theory for zero R–R field in [25,26], and for linear R–R field in [27]. The latter couplings include the well-known anomalous coupling $C \wedge \mathrm{Tr}(R \wedge R)$, as well as some non-anomalous couplings involving the R–R field strengths.

The type IIB superstring theory has S-duality, hence, its effective action should be invariant under the S-duality as well. To have an S-duality invariant effective action one should include to the tree-level effective action the non-perturbative and string loop effects [11,12]. They are required to make the tree-level effective action to be invariant under the S-duality group $SL(2, Z)$. Even the tree-level effective action at a given order of α' should be also consistent with S-duality in the sense that up to an overall dilaton factor, the action should be invariant under the S-duality group $SL(2, R)$. To study the S-duality, one first should change the string-frame metric to the Einstein-frame metric, i.e., $G_{\mu\nu} = e^{\phi/2} G^{(E)}_{\mu\nu}$. Then up to some total derivative terms the effective action should be written as an S-duality invariant form, i.e.,

$$\begin{aligned} &\mathbf{S}_{\rm eff}(G, \phi, B, C^{(0)}, C^{(2)}, C^{(4)}) \\ &\quad = \mathbf{S}_{\rm eff}(G^E, \tau, \bar{\tau}, \mathcal{H}, C^{(4)}) + \mathbf{TD} \end{aligned} \tag{2}$$

where the Einstein-frame metric and R–R four-form are invariant under the S-duality, $\mathcal{H}$ which includes the B-field and the R–R two-form, transforms as doublet and τ which includes the dilaton and the R–R scalar, transforms as modular transformation. On the right-hand side of above equation, **TD** again represents some total derivative terms which may not be invariant under the S-duality. They however become zero for the closed spacetime manifolds using the Stokes's theorem. Since the R–R four-form couples to the non-perturbative D_3-brane and O_3-plane objects, up to some total derivative terms, the effective action of these objects should be also invariant under the S-duality [28].

When the spacetime manifold has boundary $\partial M^{(D)}$, the total derivative terms on the right-hand sides of (1) and (2) can not be ignored. If one ignores them then the effective action would not be invariant under the T-duality and S-duality. In fact, for the open spacetime manifold, the total derivative terms in the original spacetime and in the base space have physical effects and, hence, should not be ignored. On the other hand, there might be some couplings $\partial\mathbf{S}_{\rm eff}$ at the boundary of the spacetime that one should take into account to have a fully duality invariant effective action. At the leading order of α', requiring the effective action to be invariant under the gauge transformations and under the T- and S-duality transformations, one can fix the couplings up to an overall normalisation factor [29]. In fact, the total derivative terms on the right-hand sides of the duality constraints (1) and (2) at the leading order of α' are cancelled if one includes the Gibbons–Hawking–York boundary term [30,31] in the boundary action. At the higher orders of α', the gauge and duality constraints may also fix both the bulk and the boundary actions.

Using the Stokes's theorem, the total derivative terms in the bulk action $\mathbf{S}_{\rm eff}$ can be transferred to the boundary action $\partial\mathbf{S}_{\rm eff}$ to produce couplings that are proportional to the unit vector of the boundary. As a result, one can write the bulk action without total derivative terms even for the spacetime manifolds which have boundary. Hence the extension of the T-duality constraint (1) to the spacetime with boundary has two parts. One part is exactly as in (1) in which the bulk action $\mathbf{S}_{\rm eff}$ has no total derivative term, however, the total derivative terms in the base space $M^{(D-1)}$ which appear on the right-hand side of (1), are transferred to the boundary $\partial M^{(D-1)}$ in the base space using the Stokes's theorem. We call them ∂TD. In the second part one first write all independent gauge invariant couplings in the boundary action $\partial\mathbf{S}_{\rm eff}$ including the couplings which are proportional to the unit vector. Then one should add ∂TD to the T-duality constraint on the boundary action, i.e.,

$$\partial \mathrm{TD} + \partial S_{\rm eff}(\psi) - \partial S_{\rm eff}(\psi') = \mathcal{TD} \tag{3}$$

where $\mathcal{TD}$ represents some boundary total derivative terms. Since the boundary of boundary, i.e., $\partial\partial M^{(D-1)}$, is zero $\mathcal{TD}$ becomes zero after using the Stokes's theorem. The sum of the bulk constraint (1) and the boundary constraint (3) means the total bulk and boundary actions are invariant under the T-duality, i.e.,

$$S_{\rm eff}(\psi) + \partial S_{\rm eff}(\psi) = S_{\rm eff}(\psi') + \partial S_{\rm eff}(\psi') \tag{4}$$

up to some total derivative terms in the boundary of base space $\partial M^{(D-1)}$ which are zero by the Stokes's theorem. The above T-duality constraint has been used in [29] to reproduce the bulk and boundary couplings at the leading order of α'. It reproduces the known bulk couplings and its corresponding the Gibbons–Hawking–York boundary term. However, it produces an extra T-dual multiplet in the boundary as well.

The T-duality constraint (4) for D_p-brane/O_p-plane is such that when spacetime has boundary $\partial M^{(D)}$, the p-branes may end on the boundary, i.e., $\partial M^{(D)} = \partial M^{(p+1)} \times M^{(D-p-1)}$. Hence their corresponding low-energy effective action should have boundary terms as well. In this case, $\partial S_{\rm eff}(\psi)$ represents the reduction of the $(p-1)$-brane boundary action along the circle transverse to the brane, i.e., $\partial M^{(D)} = \partial M^{(p)} \times M^{(D-p)}$ where the $(p-1)$-brane boundary action is in the subspace $\partial M^{(p)}$ and $M^{(D-p)} = S^{(1)} \times M^{(D-p-1)}$, and $\partial S_{\rm eff}(\psi')$ represents T-duality transformation of the reduction of the p-brane boundary action along the circle tangent to the brane, i.e., $\partial M^{(D)} = \partial M^{(p+1)} \times$

$M^{(D-p-1)}$ where the p-brane boundary action is in the subspace $\partial M^{(p+1)}$ and $\partial M^{(p+1)} = S^{(1)} \times \partial M^{(p)}$.

Similarly, the extension of the S-duality constraint (2) to the spacetime with boundary has two parts. One part is exactly as in (2) in which the bulk action $\mathbf{S}_{\rm eff}$ in the string frame has no total derivative term, however, the total derivative terms **TD** resulting from transforming the string frame action to the Einstein frame are transferred to the boundary using the Stokes's theorem. We call them ∂**TD**. In the second part, one should combine them with the boundary action to be written in the S-duality invariant form, i.e.,

$$\begin{aligned}&\partial\mathbf{TD}(n, G, \phi, B, C^{(0)}, C^{(2)}, C^{(4)})\\&\quad+\partial\mathbf{S}_{\rm eff}(n, G, \phi, B, C^{(0)}, C^{(2)}, C^{(4)})\\&= \partial\mathbf{S}_{\rm eff}(G^E, \tau, \bar{\tau}, \mathcal{H}, C^{(4)}) + \mathbf{td}\end{aligned}$$

where **td** represents some total derivative terms, however, since the boundary of boundary is zero, they are zero by using the Stokes's theorem. The sum of the bulk constraint (2) and the above boundary constraint means the total bulk and boundary actions are invariant under the S-duality, i.e.,

$$\begin{aligned}\mathbf{S}_{\rm eff} + \partial\mathbf{S}_{\rm eff} &= \mathbf{S}_{\rm eff}(G^E, \tau, \bar{\tau}, \mathcal{H}, C^{(4)})\\&\quad+\partial\mathbf{S}_{\rm eff}(G^E, \tau, \bar{\tau}, \mathcal{H}, C^{(4)}) \qquad (5)\end{aligned}$$

up to some total derivative terms in the boundary $\partial M^{(D)}$ which are zero by the Stokes's theorem. The above S-duality constraint has been used in [29] on the couplings that the T-duality produces at the leading order of α'. This constraint removes the extra couplings in the boundary that the T-duality produces.

The S-duality constraint (5) for D_3-brane/O_3-plane is such that the combination of world-volume action and its boundary terms should be written in an S-duality invariant form up to some total derivative terms in the world-volume boundary $\partial M^{(4)}$ which are zero by the Stokes's theorem..

In this paper, we are going to apply the T-duality constraint (4) and the S-duality constraint (5) on the effective actions of O-plane when spacetime has boundary. We are interested in NS–NS couplings of O-planes of type II superstring theory. At the leading order of α' there is no boundary term and the bulk action which is given by DBI action is invariant under T-duality and S-duality (see e.g., [13]). The first corrections to the DBI action is at order α'^2. At this order, the T-duality transformations are given only by the Buscher rules because the first corrections to the effective action of type II superstring theory are at order α'^3. To study the S-duality at order α'^2, one needs to take into account R–R fields as well in which we are not interested in this paper. However, it has been observed in [32] that it is impossible to combine couplings in the Einstein frame involving odd number of dilatons and zero B-field with corresponding R–R couplings to be written in an S-duality invariant form. Hence the S-duality constraint on the NS–NS couplings is such that the O_3-plane couplings with zero B-field which involve odd number of dilatons must be zero. The T-duality constraint as well as this S-duality constraint may fix the NS–NS couplings in the bulk and boundary actions of O-planes.

The outline of the paper is as follows: In Sect. 2.1, we first impose gauge symmetry to show that there are 48 independent bulk couplings. In Sect. 2.2, we impose T-duality to fix the 48 couplings up to an overall factor, and up to some total derivative terms in the base space which are transferred to the boundary by using the Stokes's theorem. In Sect. 2.3, we show that the bulk couplings that are fixed by the gauge symmetry and the T-duality, are consistent with S-duality up to some total derivative terms which are transferred to the boundary by using the Stokes's theorem. In Sect. 3.1, we first impose gauge symmetry to show that there are 78 independent boundary couplings. In Sect. 3.2, we show that the T-duality can not fix all parameters. In fact we find, apart from the boundary couplings that are needed to make the total derivative terms in the bulk to be invariant under the T-duality, there are 17 boundary multiplets that are T-duality invariant. In Sect. 3.3, we impose the S-duality constraint on the T-duality invariant couplings. We find, apart from the boundary couplings that are needed to make the total derivative terms in the bulk to be invariant under the T-duality and S-duality, there are also three other boundary multiplets that are invariant under the T-duality and S-duality. In Sect. 4, we briefly discuss our results.

2 Bulk couplings

The NS–NS couplings in the O_p-plane bulk action at order α'^2 have been found in [25,26] by the T-duality method. The total derivative terms in the base space, i.e., the TD on the right-hand side of (1), are needed for the calculations of the boundary action in (3). So we reproduce the bulk couplings here again to find the corresponding total derivative terms in the base space. To this end, we need first to find minimum number of independent and gauge invariant terms at order α'^2 and then reduce them on a circle to apply the T-duality constraint (1). So let us find how many independent gauge invariant couplings are in the bulk.

2.1 Minimal gauge invariant couplings in the bulk

In this subsection we would like to find all independent and gauge invariant couplings on the O_p-plane bulk action involving the NS–NS fields at order α'^2 in the string frame, i.e.,

$$\mathbf{S}_p = -\frac{T_p\pi^2\alpha'^2}{48}\int_{M^{(p+1)}} d^{p+1}\sigma\, e^{-\phi}\sqrt{-\tilde{g}}\,\mathcal{L}_p \qquad (6)$$

where the 10-dimensional spacetime is written as $M^{(10)} = M^{(p+1)} \times M^{(9-p)}$ and the O_p-plane is along the subspace $M^{(p+1)}$. In above equation, $\widetilde{g}$ is determinant of the pull-back metric

$$\widetilde{g}_{ab} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} G_{\mu\nu}. \tag{7}$$

The O_p-plane is specified in the spacetime by vectors $X^\mu(\sigma^a)$, T_p is tension of O_p-plane and $\mathcal{L}_p$ is the Lagrangian we are after which includes all independent couplings.

As it has been argued in [25], since we are interested in O_p-plane as a probe, it does not have back reaction on the spacetime. As a result, the massless closed string fields must satisfy the bulk equations of motion at order α'^0, i.e.,

$$\begin{aligned}
0 &= R + 4\nabla_\mu\nabla^\mu\phi - 4\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + \cdots\\
0 &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_\mu^{\rho\sigma}H_{\nu\rho\sigma} + \cdots\\
0 &= \nabla_\mu\nabla^\mu\phi - 2\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + \cdots\\
0 &= \nabla^\rho H_{\mu\nu\rho} - 2\nabla^\rho\phi H_{\mu\nu\rho}
\end{aligned} \tag{8}$$

where dots represent terms involving the R–R fields in which we are not interested in this paper. In the third line we use subtraction of the first equation and the contraction of second equation with metric $G^{\mu\nu}$. To impose these equations, we remove R, $R_{\mu\nu}$, $\nabla_\mu\nabla^\mu\phi$ and $\nabla^\mu H_{\mu\nu\rho}$ and their derivatives from the Lagrangian $\mathcal{L}_p$. As a result, one can rewrite the terms in the world-volume theory which have contraction of two transverse indices, e.g., $\nabla_i\nabla^i\Phi$, $R_{i\mu}{}^i{}_\nu$, or $\nabla^i H_{i\mu\nu}$ in terms of contraction of two world-volume indices, e.g., $\nabla_a\nabla^a\Phi$, $R_{a\mu}{}^a{}_\nu$, or $\nabla^a H_{a\mu\nu}$. This indicates that the former couplings are not independent. Moreover, the O_p-plane effective action has no open string couplings, no couplings that have odd number of transverse indices on metric and dilaton and their corresponding derivatives, and no couplings that have even number of transverse indices on B-field and its corresponding derivatives [33]. This orientifold projection makes the construction of the O-plane effective action to be much more easier than the construction of the D-brane action at a given order of α'.

The couplings involving the Riemann curvature and its derivatives and the couplings involving derivatives of $H = dB$ satisfy the following Bianchi identities

$$\begin{aligned}
&R_{\mu[\nu\alpha\beta]} = 0\\
&\nabla_{[\mu}R_{\nu\alpha]\beta\gamma} = 0\\
&dH = 0.
\end{aligned} \tag{9}$$

Moreover, the couplings involving the commutator of two covariant derivatives of a tensor are not independent of the couplings involving the contraction of this tensor with the Riemann curvature, i.e.,

$$[\nabla, \nabla]\mathcal{O} = R\mathcal{O}. \tag{10}$$

This indicates that if one considers all couplings involving the Riemann curvature, then only one ordering of covariant derivatives is needed to be considered as independent coupling.[1]

To find all independent and gauge invariant couplings at order α'^2, we first consider all even-parity contractions of $\widetilde{G}, \perp, H, \nabla H, \nabla\nabla H, \nabla\Phi, \nabla\nabla\Phi, \nabla\nabla\nabla\Phi, \nabla\nabla\nabla\nabla\Phi, R, \nabla R, \nabla\nabla R$ at four-derivative order, where the first fundamental form $\widetilde{G}^{\mu\nu}$ and the tensor $\perp^{\mu\nu}$ project the spacetime tensors along the O-plane and orthogonal to the O-plane, respectively. We then remove the terms which are projected out by the orientifold projection and by the equations of motion. We call the remaining terms, with coefficients $a'_1, a'_2, \cdots$, the Lagrangian L_p. Not all terms in this Lagrangian, however, are independent. Some of them are related by total derivative terms and by Bianchi identities (9) and (10). To remove the redundancy corresponding to the total derivative terms, we add to L_p all total derivative terms at order α'^2 with arbitrary coefficients. To this end we first write all even-parity contractions of $\widetilde{G}, \perp, H, \nabla H, \nabla\Phi, \nabla\nabla\Phi, \nabla\nabla\nabla\Phi, R, \nabla R$ at three-derivative order with one free world-volume index. Then we remove the terms which are projected out by the orientifold projection and by the equations of motion as we have done for L_p. We call the remaining terms, with arbitrary coefficients, the vector I_a. The total derivative terms are then

$$\alpha'^2\int d^{p+1}\sigma\,\sqrt{-\widetilde{g}}J = \alpha'^2\int d^{p+1}\sigma\,\sqrt{-\widetilde{g}}\,\widetilde{g}^{ab}\nabla_a(e^{-\phi}I_b) \tag{11}$$

where $\widetilde{g}^{ab}$ is inverse of the pull-back metric.

Adding the total derivative terms with arbitrary coefficients to L_p, one finds the same Lagrangian but with different parameters $a_1, a_2, \ldots$. We call the new Lagrangian $\mathcal{L}_p$. Hence

$$\Delta - J = 0 \tag{12}$$

where $\Delta = \mathcal{L}_p - L_p$ is the same as L_p but with coefficients $\delta a_1, \delta a_2, \ldots$ where $\delta a_i = a_i - a'_i$. Solving the above equation, one finds some linear relations between only $\delta a_1, \delta a_2, \ldots$ which indicate how the couplings are related among themselves by the total derivative terms. The above equation also gives some relation between the coefficients of the total derivative terms and $\delta a_1, \delta a_2, \ldots$ in which we are not interested.

However, to accurately solve the equation (12) one should write it in terms of independent couplings, i.e., one has to consider the terms in Δ and in total derivatives which are not related to each other by the Bianchi identities (9). To

[1] We have used the package "xAct" [34] for performing the calculations in this paper.

impose the Bianchi identities in gauge invariant form, one may contract the left-hand side of each Bianchi identity with field strengths of dilaton, B-field and metric to produce terms at order α'^2. The coefficients of these terms are arbitrary. Adding these terms to the equation (12), then one can solve the equation to find the linear relations between only $\delta a_1, \delta a_2, \ldots$. Alternatively, to impose the Bianchi identities in non-gauge invariant form, one may rewrite the terms in (12) in the local frame in which the first derivative of metric is zero, and rewrite the terms in (12) which have derivatives of H in terms of B-field, i.e., $H = dB$. In this way, the Bianchi identities satisfy automatically [35]. In fact, writing the couplings in terms of potential rather than field strength, there would be no Bianchi identity at all. We find that this latter approach is easier to impose the Bianchi identities by computer. Moreover, in this approach one does not need to introduce a large number of arbitrary parameters to include the Bianchi identities to the equation (12).

Using the above steps, one can rewrite the different terms on the left-hand side of (12) in terms of independent but non-gauge invariant couplings. Some combinations of the parameters appear as coefficients of the independent couplings. The solution to the equation (12) which corresponds to setting all these coefficients to zero, then has two parts. One part is 48 relations between only δa_i's, and the other part is some relations between the coefficients of the total derivative terms and δa_i's in which we are not interested. The number of relations in the first part gives the number of independent couplings in $\mathcal{L}_p$. In a particular scheme, one may set some of the coefficients in L_p to zero, however, after replacing the non-zero terms in (12), the number of relations between only δa_i's should not be changed, i.e., there must be always 48 relations. We set the coefficients of the couplings in which each term has more than two derivatives, to zero. After setting this coefficients to zero, there are still 48 relations between δa_i's. This means we are allowed to remove these terms. We choose some other coefficients to zero such that the remaining coefficients satisfy the 48 relations $\delta a_i = 0$. In this way one can find the minimum number of gauge invariant couplings. One particular choice for the 48 gauge invariant couplings is the following:

$$\begin{aligned}
\mathcal{L}_p = {} & a_1\ H_a{}^{cj} H^{abi} H_b{}^d{}_j H_{cdi} + a_2\ H_a{}^c{}_i H^{abi} H_b{}^{dj} H_{cdj} \\
& + a_3\ H_{ab}{}^j H^{abi} H_{cdj} H^{cd}{}_i \\
& + a_4\ H_{abi} H^{abi} H_{cdj} H^{cdj} + a_5\ H_a{}^{cj} H^{abi} H_{bc}{}^k H_{ijk} \\
& + a_6\ H_{ab}{}^j H^{abi} H_i{}^{kl} H_{jkl} \\
& + a_7\ H_{abi} H^{abi} H_{jkl} H^{jkl} + a_8\ H_i{}^{lm} H^{ijk} H_{jl}{}^n H_{kmn} \\
& + a_9\ H_{ij}{}^l H^{ijk} H_k{}^{mn} H_{lmn} \\
& + a_{10}\ H_{ijk} H^{ijk} H_{lmn} H^{lmn} + a_{11}\ H^{abi} H^{cd}{}_i R_{abcd} \\
& + a_{12}\ H^{abi} H_i{}^{jk} R_{abjk} \\
& + a_{13}\ H_{ijk} H^{ijk} R^{ab}{}_{ab} + a_{14}\ R_{abcd} R^{abcd} \\
& + a_{15}\ R_{abij} R^{abij} + a_{16}\ R_{aibj} R^{aibj} \\
& + a_{17}\ H_{ij}{}^l H^{ijk} R^a{}_{kal} + a_{18}\ H_a{}^c{}_i H^{abi} R_b{}^d{}_{cd} \\
& + a_{19}\ R^{ab}{}_a{}^c R_b{}^d{}_{cd} + a_{20}\ H_a{}^{cj} H^{abi} R_{bicj} \\
& + a_{21}\ R^{ai}{}_a{}^j R^b{}_{ibj} + a_{22}\ H_{abi} H^{abi} R^{cd}{}_{cd} \\
& + a_{23}\ R^{ab}{}_{ab} R^{cd}{}_{cd} + a_{24}\ H_{ab}{}^j H^{abi} R^c{}_{icj} \\
& + a_{25}\ R_{ijkl} R^{ijkl} + a_{26}\ H_i{}^{lm} H^{ijk} R_{jklm} \\
& + a_{27}\ \nabla_a H_{bci} \nabla^a H^{bci} + a_{28}\ \nabla_a H_{ijk} \nabla^a H^{ijk} \\
& + a_{29}\ H_{bci} H^{bci} \nabla_a \phi \nabla^a \phi \\
& + a_{30}\ H_{ijk} H^{ijk} \nabla_a \phi \nabla^a \phi + a_{31}\ H_{bci} H^{bci} \nabla^a \nabla_a \phi \\
& + a_{32}\ H_{ijk} H^{ijk} \nabla^a \nabla_a \phi \\
& + a_{33}\ R^{bc}{}_{bc} \nabla^a \nabla_a \phi + a_{34}\ H_a{}^{ci} H_{bci} \nabla^a \nabla^b \phi \\
& + a_{35}\ R_a{}^c{}_{bc} \nabla^a \nabla^b \phi + \ a_{36}\ \nabla^a \nabla^b \phi \nabla_b \nabla_a \phi \\
& + a_{37}\ H_{bci} \nabla^a \phi \nabla^b H_a{}^{ci} + a_{38}\ H_a{}^{ci} H_{bci} \nabla^a \phi \nabla^b \phi \\
& + a_{39}\ R_a{}^c{}_{bc} \nabla^a \phi \nabla^b \phi \\
& + a_{40}\ \nabla^a \phi \nabla_b \nabla_a \phi \nabla^b \phi + a_{41}\ \nabla^a \nabla_a \phi \nabla^b \nabla_b \phi \\
& + a_{42}\ \nabla^a H_a{}^{bi} \nabla^c H_{bci} + a_{43}\ \nabla_i H_{abc} \nabla^i H^{abc} \\
& + a_{44}\ H_{abj} H^{ab}{}_i \nabla^i \nabla^j \phi \\
& + a_{45}\ H_i{}^{kl} H_{jkl} \nabla^i \nabla^j \phi + a_{46}\ R^a{}_{iaj} \nabla^i \nabla^j \phi \\
& + a_{47}\ \nabla^i H^{ajk} \nabla_j H_{aik} + a_{48}\ \nabla^i \nabla^j \phi \nabla_j \nabla_i \phi
\end{aligned} \tag{13}$$

where $a_1, \ldots, a_{48}$ are 48 arbitrary p-independent coefficients that should be fixed by the duality constraint. In writing the above couplings we have used the fact that the first fundamental form $\widetilde{G}^{\mu\nu}$ for O-plane has non-zero components only for world-volume indices, and tensor $\perp^{\mu\nu}$ has non-zero components only for transverse indices. For example, the last term above in terms of 10-dimensional indices is

$$\nabla^i \nabla^j \phi \nabla_j \nabla_i \phi = \perp^{\mu\nu} \perp^{\rho\sigma}\ \nabla_\mu \nabla_\rho \phi \nabla_\sigma \nabla_\nu \phi. \tag{14}$$

Similarly for all other terms in (13).

Since the above string-frame couplings involve NS–NS fields which transform into each others under T-duality, the T-duality constraint should produce relations between all the 48 coefficients in (13). On the other hand, the S-duality relates the above couplings to the couplings involving R–R fields in which we are not interested in this paper. As we argued in the Sect. 1, the S-duality on the NS–NS couplings constrains the Einstein frame couplings involving zero B-field and odd number of dilaton to be zero. In the next subsection, we impose the T-duality constraint to the above couplings to find relations between the coefficients. In fact, as we will see, this constraint fixes all coefficients up to an overall factor. It also produces some total derivative terms in the base space which are needed for studying the T-duality of the boundary action. In the subsequent subsection we impose the S-duality constraint on the resulting coefficients. Since all parameters are already fixed by the T-duality constraint, the S-duality satisfies automatically up to some total derivative terms that should be included in the study of S-duality of the boundary action.

2.2 T-duality constraint in the bulk

In this subsection we are going to impose the T-duality constraint (1) on the gauge invariant couplings (13) to fix their parameters. To find the reduction $S_{\rm eff}(\psi)$ we need to dimensionally reduce $\mathrm{O}_{(p-1)}$-plane bulk action along the circle orthogonal to the O-plane (transverse reduction), and to find $S_{\rm eff}(\psi')$ we need to dimensionally reduce O_p-plane action along the circle tangent to the O-plane (world-volume reduction) and then transform it under the T-duality. The reduction of the spacetime fields $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ and their derivatives which appear in (13), are independent of orientation of the O-plane. However, the reduction of the first fundamental form $\widetilde{G}^{\mu\nu}$ and the tensor $\perp^{\mu\nu}$ which also appear in the couplings (13), do depend on the orientation of the O-plane.

When one of the spatial dimensions is circle with coordinate y, i.e., $M^{(10)} = S^{(1)} \times M^{(9)}$, the reduction of metric $G_{\mu\nu}$ and $B_{\mu\nu}$ are [36]

$$G_{\mu\nu} = \begin{pmatrix} g_{\alpha\beta} + e^{\varphi} g_\alpha g_\beta & e^{\varphi} g_\alpha \\ e^{\varphi} g_\beta & e^{\varphi} \end{pmatrix};$$
$$B_{\mu\nu} = \begin{pmatrix} \bar{b}_{\alpha\beta} - \frac{1}{2} g_\alpha b_\beta + \frac{1}{2} g_\beta b_\alpha & b_\alpha \\ -b_\beta & 0 \end{pmatrix}. \qquad (15)$$

Inverse of this metric is

$$G^{\mu\nu} = \begin{pmatrix} g^{\alpha\beta} & -g^\alpha \\ -g^\beta & e^{-\varphi} + \ \ g_\sigma g^\sigma \end{pmatrix}. \qquad (16)$$

Using these reductions, it is straightforward to calculate the reduction of the spacetime tensors $R_{\mu\nu\rho\sigma}$, $\nabla_\mu H_{\nu\rho\sigma}$, $H_{\mu\nu\rho}$, $\nabla_\mu\phi$, and $\nabla_\mu\nabla_\nu\phi$ which appear in the couplings (13). For example the reduction of $\nabla_\mu\nabla_\nu\phi$ when both indices are in the 9-dimensional base space is

$$\begin{aligned}\nabla_\mu\nabla_\nu\phi =& \frac{1}{2} e^{\varphi} g^{\beta\alpha} g_\nu \nabla_\alpha\phi\nabla_\beta g_\mu + \frac{1}{2} e^{\varphi} g^{\beta\alpha} g_\mu \nabla_\alpha\phi\nabla_\beta g_\nu \\ &+ \frac{1}{2} e^{\varphi} g^{\beta\alpha} g_\mu g_\nu \nabla_\alpha\phi\nabla_\beta\varphi \\ &- \frac{1}{2} e^{\varphi} g^{\alpha\beta} g_\nu \nabla_\alpha\phi\nabla_\mu g_\beta - \frac{1}{2} e^{\varphi} g^{\alpha\beta} g_\mu \nabla_\alpha\phi\nabla_\nu g_\beta \\ &+ \nabla_\nu\nabla_\mu\phi. \end{aligned} \qquad (17)$$

One can find the expression for the reduction of all other tensors in [25].

When O_p-plane is along the y-direction, i.e., $M^{(10)} = M^{(p+1)} \times M^{(9-p)}$ and $M^{(p+1)} = S^{(1)} \times M^{(p)}$, the reduction of pull-back metric $\tilde{g}_{ab}$ and its inverse are

$$\tilde{g}_{ab} = \begin{pmatrix} g_{\tilde{a}\tilde{b}} + e^{\varphi} g_{\tilde{a}} g_{\tilde{b}} & e^{\varphi} g_{\tilde{a}} \\ e^{\varphi} g_{\tilde{b}} & e^{\varphi} \end{pmatrix};$$
$$\tilde{g}^{ab} = \begin{pmatrix} g^{\tilde{a}\tilde{b}} & -g^{\tilde{a}} \\ -g^{\tilde{b}} & e^{-\varphi} + g_{\tilde{c}} g^{\tilde{c}} \end{pmatrix} \qquad (18)$$

where the indices $\tilde{a}, \tilde{b}$ are world-volume indices that do not include the world-volume index y, i.e., they are belong to $M^{(p)}$. In above equation we have used the static gauge and assumed the O-plane is at the origin, i.e.,

$$X^a = \sigma^a; \quad X^i = 0 \qquad (19)$$

where the world-volume index a belong to $M^{(p+1)}$ and the transverse index i belong to $M^{(9-p)}$. The reduction of the first fundamental form $\tilde{G}^{\mu\nu} = \frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\tilde{g}^{ab}$ and $\perp^{\mu\nu} = G^{\mu\nu} - \tilde{G}^{\mu\nu}$ in this case have the following non-zero components:

$$\tilde{G}^{ab} = \begin{pmatrix} g^{\tilde{a}\tilde{b}} & -g^{\tilde{a}} \\ -g^{\tilde{b}} & e^{-\varphi} + g_{\tilde{c}} g^{\tilde{c}} \end{pmatrix}; \quad \perp^{ij} = g^{ij} \qquad (20)$$

where we have used the fact that $g^{\tilde{a}i}$ and vector g_i are projected out by the orientifold projection.

When $\mathrm{O}_{(p-1)}$-plane is orthogonal to the y-direction, i.e., $M^{(10)} = M^{(p)} \times M^{(10-p)}$ and $M^{(10-p)} = S^{(1)} \times M^{(9-p)}$, $g^{\tilde{a}i}$ and the vector $g_{\tilde{a}}$ are projected out by the orientifold projection. Then the reduction of the pull-back metric becomes $\tilde{g}_{ab} = g_{\tilde{a}\tilde{b}}$, and the non-zero components of the first fundamental form and $\perp^{\mu\nu}$ are

$$\tilde{G}^{\tilde{a}\tilde{b}} = g^{\tilde{a}\tilde{b}}; \quad \perp^{\tilde{i}\tilde{j}} = \begin{pmatrix} g^{ij} & -g^i \\ -g^j & e^{-\varphi} + g_k g^k \end{pmatrix} \qquad (21)$$

where the indices $\tilde{i}, \tilde{j}$ are transverse indices that include the transverse index y, i.e., they are belong to $M^{(10-p)}$. Note that the determinate of the pull-back metric is gauge invariant in both cases, i.e., when O_p-plane is along the y-direction it is $\sqrt{-\tilde{g}} = e^{\varphi/2}\sqrt{-g}$, and when $\mathrm{O}_{(p-1)}$-plane is orthogonal to the y-direction it is $\sqrt{-\tilde{g}} = \sqrt{-g}$.

Using the above reductions, one can calculate reduction of each gauge invariant coupling in (13) when O-plane is along or orthogonal to the circle. Since the 10-dimensional couplings are gauge invariant, one expects the dimensional reduction of the couplings to be gauge invariant under various 9-dimensional gauge transformations. In particular they should be invariant under the $U(1) \times U(1)$ gauge transformations corresponding to the two vectors g_μ, b_μ. This observation has been used in [22] to simplify greatly the complexity of the calculations at six derivatives order. Using this trick, one should keep the terms in the reductions of various tensors which are invariant under the $U(1) \times U(1)$ gauge transformations. The gauge invariant terms in the reduction of the spacetime tensors $R_{\mu\nu\rho\sigma}$, $\nabla_\mu H_{\nu\rho\sigma}$, $H_{\mu\nu\rho}$, $\nabla_\mu\phi$, and $\nabla_\mu\nabla_\nu\phi$ are

$$
\begin{aligned}
\nabla_\mu\nabla_\nu\phi &= \nabla_\mu\nabla_\nu\phi\\
\nabla_\mu\nabla_y\phi &= \frac{1}{2}e^{\varphi}V^\alpha{}_\mu\nabla_\alpha\phi\\
\nabla_y\nabla_y\phi &= \frac{1}{2}e^{\varphi}\nabla^\alpha\phi\nabla_\alpha\varphi\\
\nabla_\mu\phi &= \nabla_\mu\phi\\
\nabla_y\phi &= 0\\
\nabla_y H_{\nu\rho y} &= -\frac{1}{2}e^{\varphi}V_\rho{}^\alpha W_{\nu\alpha}+\frac{1}{2}e^{\varphi}V_\nu{}^\alpha W_{\rho\alpha}+\frac{1}{2}e^{\varphi}\bar{H}_{\nu\rho\alpha}\nabla^\alpha\varphi\\
\nabla_y H_{\nu\rho\sigma} &= \frac{1}{2}\Big(e^{\varphi}V_\rho{}^\alpha\bar{H}_{\nu\sigma\alpha}-e^{\varphi}V_\sigma{}^\alpha\bar{H}_{\nu\rho\alpha}-e^{\varphi}V_\nu{}^\alpha\bar{H}_{\rho\sigma\alpha}\\
&\quad -W_{\rho\sigma}\nabla_\nu\varphi+W_{\nu\sigma}\nabla_\rho\varphi-W_{\nu\rho}\nabla_\sigma\varphi\Big)\\
\nabla_\mu H_{\nu\rho y} &= -\frac{1}{2}e^{\varphi}V_\mu{}^\alpha\bar{H}_{\nu\rho\alpha}+\nabla_\mu W_{\nu\rho}-\frac{1}{2}W_{\nu\rho}\nabla_\mu\varphi\\
\nabla_\mu H_{\nu\rho\sigma} &= \frac{1}{2}V_{\mu\sigma}W_{\nu\rho}-\frac{1}{2}V_{\mu\rho}W_{\nu\sigma}+\frac{1}{2}V_{\mu\nu}W_{\rho\sigma}+\nabla_\mu\bar{H}_{\nu\rho\sigma}\\
H_{\nu\rho\sigma} &= \bar{H}_{\nu\rho\sigma}\\
H_{\mu\nu\mathbf{y}} &= W_{\mu\nu}\\
R_{\mu\nu\rho\sigma} &= \hat{R}_{\mu\nu\rho\sigma}+\frac{1}{4}e^{\varphi}V_{\mu\sigma}V_{\nu\rho}-\frac{1}{4}e^{\varphi}V_{\mu\rho}V_{\nu\sigma}\\
&\quad -\frac{1}{2}e^{\varphi}V_{\mu\nu}V_{\rho\sigma}\\
R_{\mu\nu\rho y} &= \frac{1}{4}e^{\varphi}V_{\nu\rho}\nabla_\mu\varphi-\frac{1}{4}e^{\varphi}V_{\mu\rho}\nabla_\nu\varphi\\
&\quad -\frac{1}{2}e^{\varphi}\nabla_\rho V_{\mu\nu}-\frac{1}{2}e^{\varphi}V_{\mu\nu}\nabla_\rho\varphi\\
R_{\mu y\nu y} &= \frac{1}{4}e^{2\varphi}V_\mu{}^\rho V_{\nu\rho}-\frac{1}{4}e^{\varphi}\nabla_\mu\varphi\nabla_\nu\varphi-\frac{1}{2}e^{\varphi}\nabla_\nu\nabla_\mu\varphi
\end{aligned}
\tag{22}
$$

where $\bar{H}_{\mu\nu\rho} \equiv 3\partial_{[\mu}\bar{b}_{\nu\rho]} - \frac{3}{2}g_{[\mu}W_{\nu\rho]} - \frac{3}{2}b_{[\mu}V_{\nu\rho]}$, $V_{\mu\nu} = \partial_\mu g_\nu - \partial_\nu g_\mu$ and $W_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ are 9-dimensional field strengths. The base space field strengths $\bar{H}$, V and W satisfy the following Bianchi identities:

$$
\begin{aligned}
d\bar{H} &= -\frac{3}{2}V\wedge W\\
dV &= 0\\
dW &= 0.
\end{aligned}
\tag{23}
$$

Note that the field strengths $\bar{H}$, W have odd parity. Similarly, the gauge invariant components of the tensors in (20) are

$$
\tilde{G}^{ab} = \begin{pmatrix} g^{\tilde{a}\tilde{b}} & 0\\ 0 & e^{-\varphi}\end{pmatrix};\quad \perp^{ij} = g^{ij}
\tag{24}
$$

and the gauge invariant components of the tensors in (21) are

$$
\tilde{G}^{\tilde{a}\tilde{b}} = g^{\tilde{a}\tilde{b}};\quad \perp^{\tilde{i}\tilde{j}} = \begin{pmatrix} g^{ij} & 0\\ 0 & e^{-\varphi}\end{pmatrix}.
\tag{25}
$$

Note that the yy component of the metric needed for producing scalars from the tensors on the right-hand side of reductions (22) is $e^{-\varphi}$. Then one can easily observe that in the couplings in the base space the tensors V, W always appear as $e^{\varphi/2}V$, $e^{-\varphi/2}W$.

Using the above gauge invariant parts of the reductions, one can calculate reduction of various terms in (13) along or orthogonal to the circle. For example, the reduction of last term in (13) when O_p-plane is along the circle is

$$
\perp^{\mu\nu}\perp^{\rho\sigma}\ \nabla_\mu\nabla_\rho\phi\nabla_\sigma\nabla_\nu\phi = \nabla^i\nabla^j\phi\nabla_j\nabla_i\phi.
\tag{26}
$$

The reduction of this term when $O_{(p-1)}$-plane is orthogonal to the circle is

$$
\begin{aligned}
&\perp^{\mu\nu}\perp^{\rho\sigma}\ \nabla_\mu\nabla_\rho\phi\nabla_\sigma\nabla_\nu\phi\\
&= \nabla^i\nabla^j\phi\nabla_j\nabla_i\phi+\frac{1}{4}(\nabla^\alpha\phi\nabla_\alpha\varphi)^2.
\end{aligned}
\tag{27}
$$

Similarly one can calculate reduction of all other 10-dimensional covariant terms in (13). It is important to note that if one keeps all gauge invariant and non-gauge invariant terms in the reduction of tensors, one would find the same result for the reduction of 10-dimension covariant couplings.

After reduction, one has to impose the orientifold projection which means the O-plane couplings in the base space do not have couplings with odd number of transverse indices on metric, $\nabla\phi$, $\nabla\varphi$, and their corresponding derivatives, and do not have couplings with even number of transverse indices on $\bar{H}$ and its derivatives. When O-plane is along (orthogonal) the circle, the reduction of O-plane couplings do not have odd number of transverse indices on V (W) and its derivatives, and do not have couplings with even number of transverse indices on W (V) and its derivatives.

After applying the above O-plane conditions, one has to also impose T-duality transformations which are the Buscher rules [19,20]. The base space metric and $\bar{H}$ are invariant under T-duality and the other fields transform as

$$
\phi \to \phi - \frac{1}{2}\varphi,\quad \varphi \to -\varphi,\quad V_{\mu\nu} \longleftrightarrow W_{\mu\nu}.
\tag{28}
$$

Using the above transformations, then one can calculate the left-hand side of T-duality constraint (1). Note that the overall factor of $e^{-\phi}\sqrt{-\tilde{g}}$ in O_p-plane action (6) transforms under the T-duality in the world-volume reduction, to $e^{-\phi}\sqrt{-g}$ which is the same as the transverse reduction of the corresponding term in $O_{(p-1)}$-plane. So the T-duality constraint (1) is only on the couplings in the Lagrangian (13).

To construct the total derivative terms in the constraint (1), we use the observation made in [22] that the T-duality constraint (1) for flat base space and for curved base space produces identical constraint on the coefficients of the gauge invariant couplings in the bosonic effective action at orders α', α'^2. This is as expected because the effective action should be independent of the details of the geometry of the base space. Hence, we continue our calculations on the T-duality constraint (1) for flat base space. To construct the most general total derivative terms for the right-hand side of (1), we consider all even-parity contractions of the base space tensors $\widetilde{G}$, $\perp$, $\partial\phi$, $\partial\partial\phi$, $\partial\partial\partial\phi$, $\partial\varphi$, $\partial\partial\varphi$, $\partial\partial\partial\varphi$, $\bar{H}$, $\partial\bar{H}$, $e^{\varphi/2}V$, $e^{\varphi/2}\partial V$, $e^{-\varphi/2}W$ and $e^{-\varphi/2}\partial W$ at three derivatives with one free world-volume index. Then we remove the terms which are projected out by the orientifold projection. We call the remaining terms, with arbitrary coefficients, the vector $\mathcal{I}_{\tilde{a}}$. The most general gauge invariant even-parity total derivative terms in the base space is then

$$TD = -\frac{T_{p-1}\pi^2\alpha'^2}{48}\int d^p\sigma\,\sqrt{-g}\,g^{\tilde{a}\tilde{b}}\partial_{\tilde{a}}(e^{-\phi}\mathcal{I}_{\tilde{b}}) \tag{29}$$

where $\mathcal{I}_{\tilde{a}}$ is

$$\begin{aligned}
\mathcal{I}_{\tilde{d}} = {} & u_4\, V^{\tilde{a}i}\bar{H}_{\tilde{d}\tilde{b}i}W_{\tilde{a}}{}^{\tilde{b}} + u_1\, V_{\tilde{d}}{}^{i}\bar{H}_{\tilde{a}\tilde{b}i}W^{\tilde{a}\tilde{b}} + u_2\, V^{\tilde{a}i}\bar{H}_{\tilde{a}\tilde{b}i}W_{\tilde{d}}{}^{\tilde{b}}\\
&+u_3\, V^{\tilde{a}i}\bar{H}_{daj}W_i{}^j + u_5\, V_{\tilde{d}}{}^i\bar{H}_{ijk}W^{jk}\\
&+u_6\, e^{\varphi}V_{\tilde{d}}{}^i\partial_{\tilde{a}}V^{\tilde{a}}{}_i + u_7\, e^{\varphi}V^{\tilde{a}i}\partial_{\tilde{a}}V_{di} + u_8\,\partial_{\tilde{a}}\partial^{\tilde{a}}\partial_{\tilde{d}}\varphi\\
&+u_9\,\partial_{\tilde{a}}\partial^{\tilde{a}}\partial_{\tilde{d}}\phi + u_{10}\, e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^i\partial^{\tilde{a}}\varphi\\
&+u_{12}\,\bar{H}_{\tilde{a}\tilde{b}i}\bar{H}_{\tilde{d}}{}^{bi}\partial^{\tilde{a}}\varphi + u_{11}\, e^{-\varphi}W_{\tilde{a}\tilde{b}}W_{\tilde{d}}{}^{\tilde{b}}\partial^{\tilde{a}}\varphi\\
&+u_{13}\, e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^i\partial^{\tilde{a}}\phi + u_{15}\,\bar{H}_{\tilde{a}\tilde{b}i}\bar{H}_{\tilde{d}}{}^{bi}\partial^{\tilde{a}}\phi\\
&+u_{14}\, e^{-\varphi}W_{\tilde{a}\tilde{b}}W_{\tilde{d}}{}^{\tilde{b}}\partial^{\tilde{a}}\phi + u_{18}\,\bar{H}_{\tilde{d}}{}^{\tilde{a}i}\partial_{\tilde{b}}\bar{H}_{\tilde{a}}{}^{\tilde{b}}{}_i\\
&+u_{19}\,\bar{H}^{\tilde{a}\tilde{b}i}\partial_{\tilde{b}}\bar{H}_{dai} + u_{16}\, e^{-\varphi}W_{\tilde{d}}{}^{\tilde{a}}\partial_{\tilde{b}}W_{\tilde{a}}{}^{\tilde{b}}\\
&+u_{17}\, e^{-\varphi}W^{\tilde{a}\tilde{b}}\partial_{\tilde{b}}W_{\tilde{d}\tilde{a}} + u_{20}\, e^{\varphi}V^{\tilde{a}i}\partial_{\tilde{d}}V_{\tilde{a}i}\\
&+u_{23}\,\bar{H}^{\tilde{a}\tilde{b}i}\partial_{\tilde{d}}\bar{H}_{\tilde{a}\tilde{b}i} + u_{24}\,\bar{H}^{ijk}\partial_{\tilde{d}}\bar{H}_{ijk}\\
&+u_{21}\, e^{-\varphi}W^{\tilde{a}\tilde{b}}\partial_{\tilde{d}}W_{\tilde{a}\tilde{b}} + u_{22}\, e^{-\varphi}W^{ij}\partial_{\tilde{d}}W_{ij}\\
&+u_{25}\, e^{\varphi}V_{\tilde{a}i}V^{\tilde{a}i}\partial_{\tilde{d}}\varphi + u_{28}\,\bar{H}_{\tilde{a}\tilde{b}i}\bar{H}^{\tilde{a}\tilde{b}i}\partial_{\tilde{d}}\varphi\\
&+u_{29}\,\bar{H}_{ijk}\bar{H}^{ijk}\partial_{\tilde{d}}\varphi + u_{26}\, e^{-\varphi}W_{\tilde{a}\tilde{b}}W^{\tilde{a}\tilde{b}}\partial_{\tilde{d}}\varphi\\
&+u_{27}\, e^{-\varphi}W_{ij}W^{ij}\partial_{\tilde{d}}\varphi + u_{30}\,\partial_{\tilde{a}}\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi\\
&+u_{31}\,\partial_{\tilde{a}}\partial^{\tilde{a}}\phi\partial_{\tilde{d}}\varphi + u_{32}\,\partial_{\tilde{a}}\varphi\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi\\
&+u_{33}\,\partial_{\tilde{a}}\phi\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi + u_{34}\,\partial_{\tilde{a}}\phi\partial^{\tilde{a}}\phi\partial_{\tilde{d}}\varphi\\
&+u_{35}\, e^{\varphi}V_{\tilde{a}i}V^{\tilde{a}i}\partial_{\tilde{d}}\phi + u_{38}\,\bar{H}_{\tilde{a}\tilde{b}i}\bar{H}^{\tilde{a}\tilde{b}i}\partial_{\tilde{d}}\phi\\
&+u_{39}\,\bar{H}_{ijk}\bar{H}^{ijk}\partial_{\tilde{d}}\phi + u_{36}\, e^{-\varphi}W_{\tilde{a}\tilde{b}}W^{\tilde{a}\tilde{b}}\partial_{\tilde{d}}\phi\\
&+u_{37}\, e^{-\varphi}W_{ij}W^{ij}\partial_{\tilde{d}}\phi + u_{40}\,\partial_{\tilde{a}}\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\phi\\
&+u_{41}\,\partial_{\tilde{a}}\partial^{\tilde{a}}\phi\partial_{\tilde{d}}\phi + u_{42}\,\partial_{\tilde{a}}\varphi\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\phi + u_{43}\,\partial_{\tilde{a}}\phi\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\phi\\
&+u_{44}\,\partial_{\tilde{a}}\phi\partial^{\tilde{a}}\phi\partial_{\tilde{d}}\phi + u_{45}\,\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\partial_{\tilde{a}}\varphi\\
&+u_{46}\,\partial^{\tilde{a}}\phi\partial_{\tilde{d}}\partial_{\tilde{a}}\varphi + u_{47}\,\partial^{\tilde{a}}\varphi\partial_{\tilde{d}}\partial_{\tilde{a}}\phi + u_{48}\,\partial^{\tilde{a}}\phi\partial_{\tilde{d}}\partial_{\tilde{a}}\phi\\
&+u_{49}\,\partial_{\tilde{d}}\partial_{\tilde{a}}\partial^{\tilde{a}}\varphi + u_{50}\,\partial_{\tilde{d}}\partial_{\tilde{a}}\partial^{\tilde{a}}\phi + u_{51}\, e^{\varphi}V^{\tilde{a}i}\partial_i V_{\tilde{d}\tilde{a}}\\
&+u_{53}\,\bar{H}^{\tilde{a}\tilde{b}i}\partial_i\bar{H}_{\tilde{d}\tilde{a}\tilde{b}} + u_{52}\, e^{-\varphi}W_{\tilde{d}}{}^{\tilde{a}}\partial_i W_{\tilde{a}}{}^i\\
&+u_{54}\,\partial_{\tilde{d}}\varphi\partial_i\partial^i\varphi + u_{55}\,\partial_{\tilde{d}}\phi\partial_i\partial^i\varphi + u_{56}\,\partial_{\tilde{d}}\varphi\partial_i\partial^i\phi\\
&+u_{57}\,\partial_{\tilde{d}}\phi\partial_i\partial^i\phi + u_{58}\,\partial_i\partial^i\partial_{\tilde{d}}\varphi\\
&+u_{59}\,\partial_i\partial^i\partial_{\tilde{d}}\phi + u_{60}\, e^{\varphi}V_{\tilde{d}}{}^i\partial_j V_i{}^j\\
&+u_{62}\,\bar{H}_{\tilde{d}}{}^{\tilde{a}i}\partial_j\bar{H}_{\tilde{a}i}{}^j + u_{61}\, e^{-\varphi}W^{ij}\partial_j W_{\tilde{d}i}\\
&+u_{63}\,\bar{H}^{ijk}\partial_k\bar{H}_{\tilde{d}ij}.
\end{aligned}$$

The parameters $u_1,\ldots,u_{63}$ are yet some arbitrary coefficients. Note that the base space is $M^{(9)} = M^{(p)}\times M^{(9-p)}$, the indices $\tilde{a},\tilde{b}$ belong to $M^{(p)}$ and the indices i, j belong to $M^{(9-p)}$. Replacing the above total derivative terms to the right-hand side of the T-duality constraint (1), one should then write the couplings in the form of independent structures by imposing the Bianchi identities (23) in the base space. Here again we write the field strengths $\bar{H}, V, W$ in terms of potentials $\bar{b}_{\mu\nu}, g_\mu, b_\mu$ to satisfy the Bianchi identities automatically.

Writing the couplings in the T-duality constraint (1) in terms of independent and non-gauge invariant structures, then one makes the coefficients of the independent structures which include the parameters of the gauge invariant Lagrangian (13) and the above total derivative terms, to be zero. These linear equations produce the following 47 relations between the 48 parameters of the Lagrangian (13):

$$\begin{aligned}
&a_{23}\to 0, a_{22}\to 0, a_{19}\to 12a_{28}, a_{18}\\
&\quad\to -6a_{28}, a_{14}\to -6a_{28}, a_{11}\to 6a_{28},\\
&a_{44}\to 9a_{28}, a_4\to 0, a_3\to -\frac{3}{2}a_{28}, a_2\to 0, a_1\\
&\quad\to -\frac{3}{4}a_{28}, a_{24}\to 9a_{28}, a_{20}\to 0,\\
&a_5\to a_{28}, a_{21}\to -12a_{28}, a_{16}\to 0, a_{12}\\
&\quad\to -6a_{28}, a_{15}\to 6a_{28}, a_{13}\to 0, a_7\to 0,\\
&a_6\to \frac{3}{2}a_{28}, a_{17}\to -3a_{28}, a_{26}\to 0, a_{25}\\
&\quad\to 0, a_{10}\to 0, a_9\to 0, a_8\to -\frac{1}{4}a_{28},\\
&a_{42}\to 0, a_{27}\to -3a_{28}, a_{37}\to 0, a_{47}\\
&\quad\to 0, a_{46}\to -24a_{28}, a_{40}\to 0, a_{41}\to 0,\\
&a_{36}\to 12a_{28}, a_{35}\to 24a_{28}, a_{34}\to -6a_{28}, a_{43}\\
&\quad\to 2a_{28}, a_{48}\to -12a_{28}, a_{45}\to -3a_{28},\\
&a_{29}\to 0, a_{39}\to 0, a_{38}\to 0, a_{30}\\
&\quad\to 0, a_{33}\to 0, a_{31}\to 0, a_{32}\to 0.
\end{aligned}\tag{30}$$

Which fix the bulk effective action up to one overall factor a_{28}, i.e.,

$$\begin{aligned}
\mathcal{L}_p = a_{28}\Big[&-\frac{3}{4}H_a{}^{cj}H^{abi}H_b{}^d{}_jH_{cdi}\\
&-\frac{3}{2}H_{ab}{}^jH^{abi}H_{cdj}H^{cd}{}_i + H_a{}^{cj}H^{abi}H_{bc}{}^kH_{ijk}\\
&+\frac{3}{2}H_{ab}{}^jH^{abi}H_i{}^{kl}H_{jkl}\\
&-\frac{1}{4}H_i{}^{lm}H^{ijk}H_{jl}{}^nH_{kmn} + 6H^{abi}H^{cd}{}_iR_{abcd}\\
&-6H^{abi}H_i{}^{jk}R_{abjk}\\
&-6R_{abcd}R^{abcd} + 6R_{abij}R^{abij} - 6H_a{}^{ci}H_{bci}\mathcal{R}^{ab}\\
&+12\mathcal{R}_{ab}\mathcal{R}^{ab}\\
&+9H_{abj}H^{ab}{}_i\mathcal{R}^{ij} - 3H_i{}^{kl}H_{jkl}\mathcal{R}^{ij} - 12\mathcal{R}_{ij}\mathcal{R}^{ij}\\
&+\nabla_aH_{ijk}\nabla^aH^{ijk} - 3\nabla_cH_{abi}\nabla^cH^{abi} + 2\nabla_iH_{abc}\nabla^iH^{abc}\Big]
\end{aligned}\tag{31}$$

where $\mathcal{R}_{\mu\nu} = \widetilde{G}^{\rho\sigma}R_{\rho\mu\sigma\nu} + \nabla_\mu\nabla_\nu\phi$. This is exactly the action that has been found in [26]. For $a_{28} = -\frac{1}{6}$, it is consistent with S-matrix element of two vertex operators [28]. In finding the above action we have imposed the equations of motion in finding the independent gauge invariant couplings in (13). If one does not impose the equations of motion to find the independent couplings, then the T-duality would produce the above T-duality invariant multiple with coefficient a_{28} and 10 other T-duality invariant multiples which includes terms like $\nabla_i\nabla^i\Phi$, or $R_{i\mu}{}^i{}_\nu$.

The linear equations also fix the following relations between the parameters of the total derivative terms and a_{28}:

$$
\begin{aligned}
&u_{10}\to 6a_{28},\ u_{13}\to 0,\ u_{14}\to -u_{11},\ u_{16}\\
&\quad\to 6a_{28},\ u_{17}\to 0,\ u_{18}\to -u_{12},\ u_2\to 0,\\
&u_{20}\to 6a_{28},\ u_{21}\to 0,\ u_{22}\to 18a_{28}+2u_1,\ u_{24}\\
&\quad\to 6a_{28},\ u_{27}\to 9a_{28}+u_1,\ u_{29}\to \frac{1}{2}u_{23},\\
&u_3\to -12a_{28}-2u_1,\ u_{32}\to 3a_{28}+\frac{1}{2}u_{25},\ u_{34}\\
&\quad\to 6a_{28},\ u_{35}\to 0,\ u_{36}\to 0,\ u_{37}\to 0,\\
&u_{38}\to 0,\ u_{39}\to 0,\ u_4\to\ 0,\ u_{40}\to -\frac{15}{2}a_{28},\ u_{41}\\
&\quad\to \frac{3}{2}a_{28},\ u_{42}\to -6a_{28},\ u_{44}\to -3a_{28},\\
&u_{45}\to 6a_{28},\ u_{46}\to -u_{43},\ u_{47}\to 0,\ u_{48}\\
&\quad\to 0,\ u_{49}\to 0,\ u_5\to 3a_{28}+3u_{28},\ u_{50}\to 0,\\
&u_{51}\to 0,\ u_{52}\to 0,\ u_{53}\to 0,\ u_{54}\to 0,\ u_{55}\\
&\quad\to -u_{43},\ u_{56}\to 0,\ u_{57}\to 0,\ u_{58}\to u_{43},\\
&u_{59}\to 0,\ u_6\to -6a_{28},\ u_{60}\to 6a_{28},\ u_{61}\\
&\quad\to u_{43},\ u_{62}\to -u_{43},\ u_{63}\to 0,\ u_{64}\to -u_{11},\\
&u_{65}\to -u_{12},\ u_{66}\to u_{15},\ u_{67}\\
&\quad\to u_{19},\ u_{68}\to u_{26},\ u_{69}\\
&\quad\to -9a_{28}-u_1,\ u_7\to 6a_{28}-u_{26},\\
&u_{70}\to u_{30},\ u_{71}\to 0,\ u_{72}\to 0,\ u_{73}\to 0,\ u_{74}\\
&\quad\to 0,\ u_{75}\to 0,\ u_{76}\to 0,\ u_{77}\to 0,\\
&u_{78}\to 2u_{31},\ u_{79}\to 2u_{33},\ u_8\\
&\quad\to -u_{30},\ u_{80}\to -u_{15},\ u_{81}\to -u_{19},\ u_{82}\to -3u_{28},\\
&u_9\to -12a_{28}.
\end{aligned}\tag{32}
$$

The parameters which are not fix in terms of a_{28} are cancelled when one replaces the above relations on the right-hand side of (29) and imposed the Bianchi identities. So the unfixed parameters represent the redundancy of the couplings in (29). Hence one can set them to zero. The fixed parameters then produce the following vector

$$
\begin{aligned}
\mathcal{I}_{\tilde d} = a_{28}\Big[&2V^{\tilde a i}W_{\tilde d}{}^{\tilde b}\bar H_{\tilde a\tilde b i}-\frac{1}{2}V_{\tilde d}{}^iW^{jk}\bar H_{ijk}+e^{\varphi}V_{\tilde d}{}^i\partial_{\tilde a}V^{\tilde a}{}_i\\
&-e^{\varphi}V^{\tilde a i}\partial_{\tilde a}V_{\tilde d i}+2e^{\varphi}V_{\tilde a i}V_{\tilde d}{}^i\partial^{\tilde a}\varphi\\
&-e^{-\varphi}W_{\tilde a\tilde b}W_{\tilde d}{}^{\tilde b}\partial^{\tilde a}\varphi-e^{\varphi}V_{\tilde a i}V_{\tilde d}{}^i\partial^{\tilde a}\phi-e^{-\varphi}W_{\tilde a\tilde b}W_{\tilde d}{}^{\tilde b}\partial^{\tilde a}\phi\\
&-e^{-\varphi}W_{\tilde d}{}^{\tilde a}\partial_{\tilde b}W_{\tilde a}{}^{\tilde b}-3\bar H^{\tilde a\tilde b i}\partial_{\tilde b}\bar H_{\tilde d\tilde a i}\\
&-\frac{1}{2}e^{-\varphi}W^{\tilde a\tilde b}\partial_{\tilde d}W_{\tilde a\tilde b}-\frac{3}{2}\bar H^{\tilde a\tilde b i}\partial_{\tilde d}\bar H_{\tilde a\tilde b i}\\
&-e^{\varphi}V_{\tilde a i}V^{\tilde a i}\partial_{\tilde d}\varphi+\frac{5}{4}e^{-\varphi}W_{\tilde a\tilde b}W^{\tilde a\tilde b}\partial_{\tilde d}\varphi-\frac{1}{4}e^{-\varphi}W_{ij}W^{ij}\partial_{\tilde d}\varphi\\
&+\partial_{\tilde a}\partial^{\tilde a}\varphi\partial_{\tilde d}\varphi+\frac{1}{2}\partial_{\tilde a}\varphi\partial^{\tilde a}\varphi\partial_{\tilde d}\varphi-\partial_{\tilde a}\phi\partial^{\tilde a}\varphi\partial_{\tilde d}\varphi\\
&-\partial^{\tilde a}\varphi\partial_{\tilde d}\partial_{\tilde a}\varphi+\frac{3}{2}\bar H^{\tilde a\tilde b i}\partial_i\bar H_{\tilde d\tilde a\tilde b}\Big].
\end{aligned}\tag{33}
$$

Note that only for simplicity we have assumed the base space is flat. If it is not flat, then the partial derivatives in above equation would be covariant derivative. In fact we have performed the calculations for curved base space and find the same Lagrangian (31) and the same total derivative as above in which the partial derivatives are replaced by covariant derivatives.

Now we assume the subspace $M^{(p)}$ in the base space $M^{(9)}=M^{(p)}\times M^{(9-p)}$ has boundary $\partial M^{(p)}$, i.e., $\partial M^{(10)}=S^{(1)}\times\partial M^{(9)}$ and $\partial M^{(9)}=\partial M^{(p)}\times M^{(9-p)}$. The Stokes's theorem in this subspace is (see Appendix)

$$
\begin{aligned}
&\int_{M^{(p)}}d^p\sigma\sqrt{-g}\,g^{\tilde a\tilde b}\partial_{\tilde a}(e^{-\phi}\mathcal{I}_{\tilde b})\\
&=\int_{\partial M^{(p)}}d^{p-1}\tau\, e^{-\phi}\sqrt{|\bar g|}\,g^{\tilde a\tilde b}n_{\tilde a}\mathcal{I}_{\tilde b}
\end{aligned}\tag{34}
$$

where $n^{\tilde a}$ is the normal vector to the boundary $\partial M^{(p)}$ which is outward-pointing (inward-pointing) if the boundary is space-like (timelike), and the boundary in the static gauge is specified by the functions $\sigma^{\tilde a}=\sigma^{\tilde a}(\tau^{\bar a})$. In the square root on the right-hand side $\bar g$ is determinant of the induced metric, i.e.,

$$
\bar g_{\bar a\bar b}=\frac{\partial\sigma^{\tilde a}}{\partial\tau^{\bar a}}\frac{\partial\sigma^{\tilde b}}{\partial\tau^{\bar b}}g_{\tilde a\tilde b}.\tag{35}
$$

The coordinates of the boundary $\partial M^{(p)}$ are $\tau^0,\tau^1,\ldots,\tau^{p-2}$. Using the above Stokes's theorem, one finds that the contribution of the total derivative terms in the boundary is

$$
\begin{aligned}
\partial TD=-\frac{T_{p-1}\pi^2\alpha'^2a_{28}}{48}\int_{\partial M^{(p)}}&d^{p-1}\tau\, e^{-\phi}\sqrt{|\bar g|}\,n^{\tilde d}\Big[2V^{\tilde a i}W_{\tilde d}{}^{\tilde b}\bar H_{\tilde a\tilde b i}\\
&-\frac{1}{2}V_{\tilde d}{}^iW^{jk}\bar H_{ijk}+e^{\varphi}V_{\tilde d}{}^i\partial_{\tilde a}V^{\tilde a}{}_i\\
&-e^{\varphi}V^{\tilde a i}\partial_{\tilde a}V_{\tilde d i}+2e^{\varphi}V_{\tilde a i}V_{\tilde d}{}^i\partial^{\tilde a}\varphi-e^{-\varphi}W_{\tilde a\tilde b}W_{\tilde d}{}^{\tilde b}\partial^{\tilde a}\varphi\\
&-e^{\varphi}V_{\tilde a i}V_{\tilde d}{}^i\partial^{\tilde a}\phi e^{-\varphi}W_{\tilde a\tilde b}W_{\tilde d}{}^{\tilde b}\partial^{\tilde a}\phi\\
&-e^{-\varphi}W_{\tilde d}{}^{\tilde a}\partial_{\tilde b}W_{\tilde a}{}^{\tilde b}-3\bar H^{\tilde a\tilde b i}\partial_{\tilde b}\bar H_{\tilde d\tilde a i}-\frac{1}{2}e^{-\varphi}W^{\tilde a\tilde b}\partial_{\tilde d}W_{\tilde a\tilde b}\\
&-\frac{3}{2}\bar H^{\tilde a\tilde b i}\partial_{\tilde d}\bar H_{\tilde a\tilde b i}-e^{\varphi}V_{\tilde a i}V^{\tilde a i}\partial_{\tilde d}\varphi\\
&+\frac{5}{4}e^{-\varphi}W_{\tilde a\tilde b}W^{\tilde a\tilde b}\partial_{\tilde d}\varphi-\frac{1}{4}e^{-\varphi}W_{ij}W^{ij}\partial_{\tilde d}\varphi+\partial_{\tilde a}\partial^{\tilde a}\varphi\partial_{\tilde d}\varphi\\
&+\frac{1}{2}\partial_{\tilde a}\varphi\partial^{\tilde a}\varphi\partial_{\tilde d}\varphi-\partial_{\tilde a}\phi\partial^{\tilde a}\varphi\partial_{\tilde d}\varphi\\
&-\partial^{\tilde a}\varphi\partial_{\tilde d}\partial_{\tilde a}\varphi+\frac{3}{2}\bar H^{\tilde a\tilde b i}\partial_i\bar H_{\tilde d\tilde a\tilde b}\Big].
\end{aligned}\tag{36}
$$

In each term the tensors $n_\alpha, V_{\alpha\beta}, W_{\alpha\beta},\cdots$ in the base space $M^{(9)}$ are contracted with projections $\tilde{G}^{\alpha\beta}=\frac{\partial X^\alpha}{\partial\sigma^{\tilde{a}}}\frac{\partial X^\beta}{\partial\sigma^{\tilde{b}}}g^{\tilde{a}\tilde{b}}$ and $\perp^{\alpha\beta}=G^{\alpha\beta}-\tilde{G}^{\alpha\beta}$ at the boundary. Note that in the static gauge $X^{\tilde{a}}=\sigma^{\tilde{a}}$ and $X^i=0$. The above boundary terms are zero if the subspace $M^{(p)}$ has no boundary. However, if it has boundary $\partial M^{(p)}$, then the above terms should be included in the T-duality constraint of the boundary action to have full T-duality in the bulk and boundary. We will consider the above terms in the T-duality of the boundary action in Sect. 3.

2.3 S-duality constraint in the bulk

We have seen that the coefficients of the gauge invariant couplings in (13) are all fixed up to an overall factor by imposing the T-duality constraint. Hence the resulting couplings should be consistent with S-duality for the case of O_3-plane up to some total derivative terms. To fully have an S-duality invariant action for O_3-plane one should include appropriate R–R couplings in which we are not interested in this paper. However, the S-duality has also constraint on the couplings involving only metric and dilaton. In the Einstein frame, i.e., $G_{\mu\nu}=e^{\phi/2}G^{(E)}_{\mu\nu}$, there must be no such couplings involving odd number of dilatons because they can not be combined with appropriate R–R scalar couplings to make S-duality invariant [32]. Note that the couplings involving B-field and odd number of dilaton can be combined with appropriate R–R couplings to be written in S-duality invariant form. Hence, we are going to check that, up to some total derivative terms, in the Einstein frame there should be no couplings involving metric and odd number of dilaton. The total derivative terms should be transferred to the boundary using the Stokes's theorem.

The overall factor $e^{-\phi}\sqrt{-\tilde{g}}$ in the string frame action (6) transforms to the following factor in the Einstein frame:

$$e^{\frac{p-3}{4}\phi}\sqrt{-\tilde{g}^E} \tag{37}$$

which is invariant under the S-duality for $p=3$. Hence the Lagrangian (31) should be consistent with the S-duality separately. The string-frame Lagrangian (31) transforms to the following Lagrangian in the Einstein frame:

$$\begin{aligned}\mathcal{L}^E_p=a_{28}e^{-\phi}\Big[&-6R_{abcd}R^{abcd}+6R_{abij}R^{abij}\\&+12R^{ab}{}_a{}^cR_b{}^d{}_{cd}-12R^{ai}{}_a{}^jR^b{}_{ibj}-6R^{bc}{}_{bc}\nabla_a\nabla^a\phi\\&+6R^{bi}{}_{bi}\nabla_a\nabla^a\phi+\left(9-\frac{3}{2}p\right)R^{bc}{}_{bc}\nabla_a\phi\nabla^a\phi\\&+\frac{3}{2}(-4+p)R^{bi}{}_{bi}\nabla_a\phi\nabla^a\phi\\&+3(-5+p)\nabla_a\nabla^a\phi\nabla_b\nabla^b\phi+\frac{3}{8}(67\\&-30p+3p^2)\nabla_a\phi\nabla^a\phi\nabla_b\nabla^b\phi\\&+\frac{3}{2}(-11+p)R_a{}^c{}_{bc}\nabla^a\phi\nabla^b\phi+\frac{3}{32}(-64+57p\\&-14p^2+p^3)\nabla_a\phi\nabla^a\phi\nabla_b\phi\nabla^b\phi\\&-\frac{3}{8}(47-16p+p^2)\nabla^a\phi\nabla_b\nabla_a\phi\nabla^b\phi\\&-6(p-7)R_a{}^c{}_{bc}\nabla^b\nabla^a\phi\\&+\frac{3}{4}(27-12p+p^2)\nabla_b\nabla_a\phi\nabla^b\nabla^a\phi\\&-\frac{3}{2}(p-3)\nabla_a\nabla^a\phi\nabla_i\nabla^i\phi+6(p-3)R^a{}_{iaj}\nabla^j\nabla^i\phi\\&-\frac{3}{8}(12-7p+p^2)\nabla_a\phi\nabla^a\phi\nabla_i\nabla^i\phi\\&-\frac{3}{4}(p-3)^2\nabla_j\nabla_i\phi\nabla^j\nabla^i\phi+\cdots\Big]\end{aligned} \tag{38}$$

where we have imposed the O-plane condition that $\nabla_i\phi=0$. In above Lagrangian dots represent the couplings including H and its derivatives. For $p=3$, it becomes

$$\begin{aligned}\mathcal{L}^E_3=a_{28}e^{-\phi}\Big[&-6R_{abcd}R^{abcd}+6R_{abij}R^{abij}\\&+12R^{ab}{}_a{}^cR_b{}^d{}_{cd}-12R^{ai}{}_a{}^jR^b{}_{ibj}-6R^{bc}{}_{bc}\nabla_a\nabla^a\phi\\&+6R^{bi}{}_{bi}\nabla_a\nabla^a\phi+\frac{9}{2}R^{bc}{}_{bc}\nabla_a\phi\nabla^a\phi\\&-\frac{3}{2}R^{bi}{}_{bi}\nabla_a\phi\nabla^a\phi-6\nabla_a\nabla^a\phi\nabla_b\nabla^b\phi\\&+\frac{3}{2}\nabla_a\phi\nabla^a\phi\nabla_b\nabla^b\phi-12R_a{}^c{}_{bc}\nabla^a\phi\nabla^b\phi\\&+\frac{3}{4}\nabla_a\phi\nabla^a\phi\nabla_b\phi\nabla^b\phi\\&-3\nabla^a\phi\nabla_b\nabla_a\phi\nabla^b\phi+24R_a{}^c{}_{bc}\nabla^b\nabla^a\phi+\cdots\Big].\end{aligned} \tag{39}$$

To study the S-duality of these terms, one should include the R–R couplings in which we are not interested in this paper. To make the overall factor $e^{-\phi}$ to be invariant under the S-duality, one should include loop and non-perturbative effects [28]. The terms in the bracket which have odd number of dilaton must be zero up to some total derivative terms. Since we have already imposed the equations of motion in the string frame, we have to impose the equations of motion in the Einstein frame as well. The equations of motion are

$$R^{\mu\nu}-\frac{1}{2}\partial^\mu\phi\partial^\nu\phi+\cdots=0,\quad\text{and}\quad\nabla_\mu\partial^\mu\phi=0 \tag{40}$$

where dots represent terms that involve H. Using the above Einstein frame equations of motion, one can rewrite the Lagrangian (39) as

$$\begin{aligned}\mathcal{L}^E_3=a_{28}e^{-\phi}\Big[&-6R_{abcd}R^{abcd}+6R_{abij}R^{abij}\\&+12R^{ab}{}_a{}^cR_b{}^d{}_{cd}-12R^{ai}{}_a{}^jR^b{}_{ibj}+6R^{bc}{}_{bc}\nabla_a\phi\nabla^a\phi\\&-6\nabla_a\nabla^a\phi\nabla_b\nabla^b\phi-12R_a{}^c{}_{bc}\nabla^a\phi\nabla^b\phi\end{aligned}$$

$$
\begin{aligned}
&-12R^{bd}{}_{bd}\nabla_a\phi\nabla^a\phi-\alpha\nabla_a\nabla_b\nabla^b\phi\nabla^a\phi\\
&-\alpha\nabla_a\nabla^a\phi\nabla_b\nabla^b\phi-2(6+\alpha)\nabla^a\phi\nabla_b\nabla^b\nabla_a\phi\\
&+24R_a{}^d{}_{bd}\nabla^a\phi\nabla^b\phi\\
&+\left(\frac{9}{2}+\alpha\right)\nabla_a\phi\nabla^a\phi\nabla_b\phi\nabla^b\phi\\
&-2(6+\alpha)\nabla_b\nabla_a\phi\nabla^b\nabla^a\phi+\dots\Big]\\
&+a_{28}\nabla^d\Big[e^{-\phi}\Big(24R_d{}^b{}_{ab}\nabla^a\phi\\
&-12R^{ab}{}_{ab}\nabla_d\phi+\alpha\nabla_a\nabla^a\phi\nabla_d\phi\\
&+\left(\frac{9}{2}+\alpha\right)\nabla_a\phi\nabla^a\phi\nabla_d\phi+2(6+\alpha)\nabla^a\phi\nabla_d\nabla_a\phi\Big)\Big]
\end{aligned}
\tag{41}
$$

where α is an arbitrary parameter. The terms in the first bracket are consistent with the S-duality for any value for the parameter α. The terms in the second bracket are not consistent with the S-duality because they have couplings with odd number of dilaton. However, using the Stokes's theorem they becomes zero if spacetime has no boundary and they are transferred to the boundary if the spacetime has boundary.

Now we assume the subspace $M^{(4)}$ in the spacetime $M^{(10)}=M^{(4)}\times M^{(6)}$ has boundary $\partial M^{(4)}$, i.e., $\partial M^{(10)}=\partial M^{(4)}\times M^{(6)}$. The Stokes's theorem in the world-volume of O_3-plane in the Einstein frame is (see Appendix)

$$
\int_{M^{(4)}} d^4\sigma\sqrt{-\tilde{g}^E}\,\nabla_a V^a=\int_{\partial M^{(4)}} d^3\tau\,\sqrt{|\hat{g}^E|}\,n^E_a V^a \tag{42}
$$

where n^E_a is the normal vector to the boundary $\partial M^{(4)}$ in the Einstein frame and the boundary is specified by the functions $\sigma^a=\sigma^a(\tau^{\hat{a}})$. The coordinates of the boundary are $\tau^0,\tau^1,\tau^2,\tau^3$. In the square root on the right-hand side $\hat{g}^E_{\hat{a}\hat{b}}$ is the induced metric in the coordinates $\tau^{\hat{a}}$, i.e., $\hat{g}^E_{\hat{a}\hat{b}}=\frac{\partial\sigma^a}{\partial\tau^{\hat{a}}}\frac{\partial\sigma^b}{\partial\tau^{\hat{b}}}\tilde{g}^E_{ab}$. Then the total derivative terms in the last line of (41) produce the following boundary terms:

$$
\begin{aligned}
\partial\mathbf{TD}=-\frac{T_3\pi^2\alpha'^2a_{28}}{48}\int_{\partial M^{(4)}} d^3\tau\, e^{-\phi}\sqrt{|\hat{g}^E|}\,n^E_c\Big[2(6\\
+\alpha)\nabla^a\phi\nabla^c\nabla_a\phi+\alpha\nabla_a\nabla^a\phi\nabla^c\phi\\
+24R^{cb}{}_{ab}\nabla^a\phi-12R^{ab}{}_{ab}\nabla^c\phi\\
+\left(\frac{9}{2}+\alpha\right)\nabla_a\phi\nabla^a\phi\nabla^c\phi\Big].
\end{aligned}
\tag{43}
$$

They involve the projection tensor $\tilde{G}^{\mu\nu}$ evaluated at the boundary of O_3-plane. Here again the overall dilaton factor can be extended to an S-duality invariant form by including the loop and non-perturbative effects [28]. While the terms in the first line can be extended to an S-duality invariant form by including the R–R couplings, the terms in the last line have odd number of dilaton which can not be extended to the S-duality invariant form. They should be cancelled with the appropriate terms in the boundary action to be consistent with the S-duality. The consistency of the boundary action with the S-duality may then fix the parameter α. We are going to consider the boundary action in the next section.

3 Boundary couplings

When spacetime has boundary, the O_p-planes in this manifold may end on the boundary. For example, if one writes the spacetime as $M^{(10)}=M^{(p+1)}\times M^{(9-p)}$ where the O_p-plane is along the subspace $M^{(p+1)}$ and this subspace has boundary $\partial M^{(p+1)}$, then the effective action of O_p-plane at specific order of α' has world-volume couplings on the bulk of the O_p-plane, i.e., in $M^{(p+1)}$, as well as boundary couplings on the boundary of the O_p-plane, i.e., in $\partial M^{(p+1)}$. We have seen in the previous section that the invariance under gauge transformations and under T-duality transformations constructs the bulk action at order α'^2. The T-duality constraint however is not fully satisfied. It produces some total derivative terms in the 9-dimensional base space $M^{(9)}=M^{(p)}\times M^{(9-p)}$ which is not zero. When base space has boundary, i.e., $\partial M^{(9)}=\partial M^{(p)}\times M^{(9-p)}$, they produce some couplings in the boundary $\partial M^{(p)}$ which are proportional to the unit vector $n^{\tilde{a}}$ orthogonal to the boundary, i.e., (36). They should be included in the T-duality of the boundary action. Similarly, writing the spacetime as $M^{(10)}=M^{(4)}\times M^{(6)}$ with boundary $\partial M^{(10)}=\partial M^{(4)}\times M^{(6)}$, we have seen that the bulk O_3-plane couplings that the T-duality produces are consistent with the S-duality provided that the boundary terms (43) are included in the S-duality of the boundary action. In this section we are going to study string duality of the boundary action.

To impose the T-duality constraint (3) on the boundary action, we need first to find minimum number of independent and gauge invariant couplings at three derivative order on the boundary and then reduce them on the circle to apply the T-duality constraint (1). So let us find how many independent gauge invariant couplings are in the boundary.

3.1 Minimal gauge invariant couplings in the boundary

In this subsection we would like to find all independent and gauge invariant couplings on the boundary of O_p-plane involving NS–NS fields at order α'^2 in the string frame. Inspired by the boundary couplings (43) in the Einstein frame for $p=3$ case, one realizes that the effective boundary action in the string frame should be as

$$
\partial\mathbf{S}_p=-\frac{T_p\pi^2\alpha'^2}{48}\int_{\partial M^{(p+1)}} d^p\tau\, e^{-\phi}\sqrt{|\hat{g}|}\,\partial\mathcal{L}_p \tag{44}
$$

where $\hat{g}$ is the determinant of the induced metric on the boundary of O$_p$-plane, i.e.,

$$\hat{g}_{\hat{a}\hat{b}} = \frac{\partial\sigma^a}{\partial\tau^{\hat{a}}}\frac{\partial\sigma^b}{\partial\tau^{\hat{b}}}\widetilde{g}_{ab}. \tag{45}$$

The boundary of O$_p$-plane is specified by the vectors $\sigma^a(\tau^{\hat{a}})$ where $\tau^0, \tau^1, \dots \tau^{p-1}$ are coordinates of the boundary, and $\partial\mathcal{L}_p$ in (44) is the boundary Lagrangian at three-derivative order which includes all couplings involving the projection tensors $\tilde{G}^{\mu\nu}$ and $\perp^{\mu\nu}$ evaluated at the boundary of O$_p$-plane.

Since the boundary of spacetime has a unite normal vector n^μ, the boundary Lagrangian $\partial\mathcal{L}_p$ should include this vector as well as the tensors $K_{\mu\nu}$, $H_{\mu\nu\rho}$, $R_{\mu\nu\rho\sigma}$, $\nabla_\mu\phi$ and their derivatives. They should be contracted with the projection tensors $\tilde{G}^{\mu\nu}$ and $\perp^{\mu\nu}$. The extrinsic curvature of boundary, i.e., $K_{\mu\nu}$, is defined as $K_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \nabla_{(\alpha}n_{\beta)}$ where the projection tensor $P^{\mu\nu}$ is $P^{\mu\nu} = G^{\mu\nu} - n^\mu n^\nu$. Using the fact that n^μ is unit vector orthogonal to the boundary, i.e.,

$$n^\mu = (\nabla_\alpha f \nabla^\alpha f)^{-1/2}\nabla^\mu f \tag{46}$$

where boundary is specified by the function f to be a constant f^*, one can rewrite $K_{\mu\nu}$ as

$$K_{\mu\nu} = \nabla_\mu n_\nu - n_\mu a_\mu \tag{47}$$

where $a_\nu = n^\rho\nabla_\rho n_\nu$ is acceleration. It satisfies the relation $n^\mu a_\mu = 0$. Note that the extrinsic curvature is symmetric and satisfies $n^\mu K_{\mu\nu} = 0$ which can easily be seen by writing it in terms of function f. Using this symmetry and $n^\mu n_\mu = 1$, one finds the most general couplings have the structures KH^2, $nH\nabla H$, KR, $n\nabla R$, $n(\nabla\phi)^3$, $K(\nabla\phi)^2$, $n\nabla\nabla\nabla\phi$, $K\nabla\nabla\phi$, $nH^2\nabla\phi$, $n\nabla\phi\nabla\nabla\phi$, $\nabla K\nabla\phi$, $\nabla\nabla K$, K^3, $K\nabla Kn$, $n^2\nabla K\nabla\phi$, $n^2\nabla\nabla K$, $nK^2\nabla\phi$, n^2KH^2, n^2KR, $n^2K(\nabla\phi)^2$, $n^2K\nabla\nabla\phi$, $n^3\nabla R$, $n^3H\nabla H$, $n^3R\nabla\phi$, $n^3H^2\nabla\phi$, $n^3\nabla\nabla\nabla\phi$, $n^3\nabla\nabla\phi\nabla\phi$, $n^3(\nabla\phi)^3$, n^4kH^2, $n^5H\nabla H$, n^4KR, $n^5\nabla R$, $n^5(\nabla\phi)^3$, $n^4K(\nabla\phi)^2$, $n^5\nabla\nabla\nabla\phi$, $n^4K\nabla\nabla\phi$, $n^5H^2\nabla\phi$, $n^5\nabla\phi\nabla\nabla\phi$, $n^4\nabla K\nabla\phi$, $n^4\nabla\nabla K$, n^2K^3, $K\nabla Kn^3$, $n^3K^2\nabla\phi$. One should impose the equations of motion (8) and the orientifold projections for the bulk fields as in the bulk action. The orientifold projection for the boundary fields requires to remove the following boundary terms:

$$\begin{aligned}
&K_{bi} = 0, \quad \nabla_b K_{ai} = 0, \quad \nabla_i K_{ab} = 0,\\
&\nabla_j K_{ik} = 0, \quad \nabla_a\nabla_b K_{ci} = 0\\
&\nabla_a\nabla_j K_{ik} = 0, \quad \nabla_j\nabla_l K_{ai} = 0, \quad n_i = 0.
\end{aligned} \tag{48}$$

After imposing the equations of motion and the orientifold projection, one finds that the corresponding Lagrangian has 108 couplings. We call this Lagrangian, with coefficients $b'_1, b'_2, \cdots, b'_{108}$, the boundary Lagrangian ∂L_p. Not all terms in this Lagrangian, however, are independent. Some of them are related by total derivative terms and by the Bianchi identities (9) and (10). The unit vector also satisfies the relation

$$n_{[\mu}\nabla_\nu n_{\rho]} = 0 \tag{49}$$

which can easily be seen by writing it in terms of function f using (46).

To remove the redundancy corresponding to the total derivative terms, we add to ∂L_p all total derivative terms at order α'^2 with arbitrary coefficients. In this case, however, the total derivative terms in the boundary have different structure than the total derivative terms in the bulk. According to the Stokes's theorem, the total derivative terms in the boundary which have the following structure are zero (see Appendix):

$$\begin{aligned}
&\alpha'^2\int_{\partial M^{(p+1)}} d^p\tau\sqrt{|\hat{g}|}\mathcal{J}\\
&\quad = \alpha'^2\int_{\partial M^{(p+1)}} d^p\tau\sqrt{|\hat{g}|}n_a\nabla_b(e^{-\phi}\mathcal{F}^{ab}) = 0
\end{aligned} \tag{50}$$

where $\mathcal{F}^{ab}$ is an arbitrary antisymmetric tensor constructed from n, K, ∇K, H^2, $\nabla\phi$, $\nabla\nabla\phi$ at two-derivative order, i.e.,

$$\begin{aligned}
\mathcal{F}_{de} = {} & z_1\left(H_{abi}H_e{}^{bi}n^a n_d - H_{abi}H_d{}^{bi}n^a n_e\right)\\
&+z_2\left(K^b{}_b K_{ea}n^a n_d - K^b{}_b K_{da}n^a n_e\right)\\
&+z_3\left(K_{ab}K_e{}^b n^a n_d - K_{ab}K_d{}^b n^a n_e\right)\\
&+z_4\left(K_{ea}K^i{}_i n^a n_d - K_{da}K^i{}_i n^a n_e\right)\\
&+z_5\left(K_{bc}K_{ea}n^a n^b n^c n_d - K_{bc}K_{da}n^a n^b n^c n_e\right)\\
&+z_6\left(n^a n_e R_d{}^b{}_{ab} - n^a n_d R_e{}^b{}_{ab}\right)\\
&+z_7\left(n_e\nabla_a K_d{}^a - n_d\nabla_a K_e{}^a\right)\\
&+z_8\left(K_{eb}n^a n^b n_d\nabla_a\phi - K_{db}n^a n^b n_e\nabla_a\phi\right)\\
&+z_9\left(K_{ea}n_d\nabla^a\phi - K_{da}n_e\nabla^a\phi\right)\\
&+z_{10}\left(n^a n^b n_e\nabla_b K_{da} - n^a n^b n_d\nabla_b K_{ea}\right)\\
&+z_{11}\left(n^a n^b n_e\nabla_d K_{ab} - n^a n^b n_d\nabla_e K_{ab}\right)\\
&+z_{12}\left(n_e\nabla_d K^a{}_a - n_d\nabla_e K^a{}_a\right)\\
&+z_{13}\left(n^a\nabla_d K_{ea} - n^a\nabla_e K_{da}\right)\\
&+z_{14}\left(n_e\nabla_d K^i{}_i - n_d\nabla_e K^i{}_i\right)\\
&+z_{15}\left(K_{ea}n^a\nabla_d\phi - K_{da}n^a\nabla_e\phi\right)\\
&+z_{16}\left(K^a{}_a n_e\nabla_d\phi - K^a{}_a n_d\nabla_e\phi\right)\\
&+z_{17}\left(K^i{}_i n_e\nabla_d\phi - K^i{}_i n_d\nabla_e\phi\right)\\
&+z_{18}\left(K_{ab}n^a n^b n_e\nabla_d\phi - K_{ab}n^a n^b n_d\nabla_e\phi\right)\\
&+z_{19}\left(n^a n_e\nabla_a\phi\nabla_d\phi - n^a n_d\nabla_a\phi\nabla_e\phi\right)
\end{aligned}$$

$$+z_{20}\left(n^a n_e \nabla_d \nabla_a \phi - n^a n_d \nabla_e \nabla_a \phi\right)$$
$$+z_{21}\left(n_e \nabla_i K_d{}^i - n_d \nabla_i K_e{}^i\right) \tag{51}$$

where $z_1, \ldots, z_{21}$ are arbitrary parameters. It is important to note that these total derivative terms connect the boundary terms which have different number of the unit vector. For example, consider $\mathcal{F}_{de} = n_e \nabla_i K_d{}^i - n_d \nabla_i K_e{}^i$. Then the boundary integral of the following term is zero

$$n^e \nabla^d\Big(e^{-\phi}\Big(n_e \nabla_i K_d{}^i - n_d \nabla_i K_e{}^i\Big)$$
$$= e^{-\phi}\Big[-n^d n^e \nabla_e \nabla_i K_d{}^i + \nabla_e \nabla_i K^{ei} - n^d \nabla_e n^e \nabla_i K_d{}^i$$
$$+n^d n^e \nabla_d \phi \nabla_i K_e{}^i - \nabla^e \phi \nabla_i K_e{}^i\Big] \tag{52}$$

where we have used the fact that n^μ is unite vector orthogonal to the boundary and $n^\mu K_{\mu\nu} = 0$. The right-hand side then gives a relation between one n and three n's.

Adding the total derivative terms with arbitrary coefficients to ∂L_p, one finds the same Lagrangian but with different parameters $b_1, b_2, \cdots$. We call the new Lagrangian $\partial \mathcal{L}_p$. Hence

$$\Delta - \mathcal{J} = 0 \tag{53}$$

where $\Delta = \partial\mathcal{L}_p - \partial L_p$ is the same as ∂L_p but with coefficients $\delta b_1, \delta b_2, \ldots$ where $\delta b_i = b_i - b_i'$. Solving the above equation, one finds some linear relations between only $\delta b_1, \delta b_2, \ldots$ which indicate how the couplings are related among themselves by the total derivative terms. The above equation also gives some relation between the coefficients of the total derivative terms and $\delta b_1, \delta b_2, \ldots$ in which we are not interested.

To impose in (53) the Bianchi identities (9), (10) we go to the local frame, and to impose the identities corresponding to the unit vector we write n^μ in terms of function f using (46). Then one finds 78 independent couplings. In this case, there is no scheme in which the couplings involve only terms with one and two derivatives, e.g., there is no scheme which has no ∇R. One particular choice for the 78 gauge invariant boundary couplings is the following:

$$\begin{aligned}
\partial\mathcal{L}_p = {} & b_1\, H_{bci} H^{bci} K^a{}_a + b_2\, H_{ijk} H^{ijk} K^a{}_a \\
& + b_3\, H_a{}^{ci} H_{bci} K^{ab} + b_4\, K_a{}^c K^{ab} K_{bc} \\
& + b_5\, K^a{}_a K_{bc} K^{bc} \\
& + b_6\, K^a{}_a K^b{}_b K^c{}_c + b_7\, H_{abj} H^{abj} K^i{}_i \\
& + b_8\, H_{jkl} H^{jkl} K^i{}_i + b_9\, K^a{}_a K^b{}_b K^i{}_i \\
& + b_{10}\, H_{abj} H^{ab}{}_i K^{ij} \\
& + b_{11}\, H_i{}^{kl} H_{jkl} K^{ij} + b_{12}\, K_i{}^k K^{ij} K_{jk} \\
& + b_{13}\, K^a{}_a K^i{}_i K^j{}_j \\
& + b_{15}\, K^i{}_i K^j{}_j K^k{}_k + b_{16}\, H_a{}^{di} H_{bdi} K^c{}_c n^a n^b \\
& + b_{17}\, H_{ac}{}^i H_{bdi} K^{cd} n^a n^b + b_{18}\, H_a{}^{cj} H_{bcj} K^i{}_i n^a n^b \\
& + b_{19}\, H_a{}^c{}_i H_{bcj} K^{ij} n^a n^b + b_{20}\, K^{cd} n^a n^b R_{acbd} \\
& + b_{21}\, K^{ab} R_a{}^c{}_{bc} + b_{22}\, K^i{}_i n^a n^b R_a{}^c{}_{bc} \\
& + b_{23}\, K^c{}_c n^a n^b R_a{}^d{}_{bd} + b_{24}\, K^{ij} n^a n^b R_{aibj} \\
& + b_{25}\, K^i{}_i R^{ab}{}_{ab} + b_{26}\, K^{ij} R^a{}_{iaj} + b_{27}\, K^a{}_a R^{bc}{}_{bc} \\
& + b_{29}\, \nabla_a \nabla^a K^i{}_i + b_{28}\, n^a n^b \nabla_b \nabla_a K^i{}_i \\
& + b_{30}\, H^{bci} n^a \nabla_a H_{bci} + b_{31}\, K^i{}_i n^a \nabla_a K^b{}_b \\
& + b_{34}\, K^{ij} n^a \nabla_a K_{ij} + b_{32}\, K^b{}_b n^a \nabla_a K^i{}_i \\
& + b_{33}\, K^i{}_i n^a \nabla_a K^j{}_j + b_{35}\, n^a \nabla_a R^{bc}{}_{bc} \\
& + b_{37}\, H_{bci} H^{bci} n^a \nabla_a \phi + b_{38}\, H_{ijk} H^{ijk} n^a \nabla_a \phi \\
& + b_{39}\, K^b{}_b K^c{}_c n^a \nabla_a \phi + b_{40}\, K^b{}_b K^i{}_i n^a \nabla_a \phi \\
& + b_{41}\, K^i{}_i K^j{}_j n^a \nabla_a \phi + b_{42}\, n^a R^{bc}{}_{bc} \nabla_a \phi \\
& + b_{43}\, n^a \nabla_a \nabla_b \nabla^b \phi + b_{71}\, \nabla_a \nabla_i K^{ai} \\
& + b_{45}\, \nabla_a K^b{}_b \nabla^a \phi + b_{46}\, \nabla_a K^i{}_i \nabla^a \phi \\
& + b_{47}\, K^b{}_b \nabla_a \phi \nabla^a \phi + b_{48}\, K^i{}_i \nabla_a \phi \nabla^a \phi \\
& + b_{51}\, K^i{}_i n^a \nabla_b K_a{}^b + b_{52}\, K^c{}_c n^a n^b \nabla_a \phi \nabla_b \phi \\
& + b_{53}\, K^i{}_i n^a n^b \nabla_a \phi \nabla_b \phi \\
& + b_{54}\, \nabla_b \nabla_a K^{ab} + b_{55}\, n^a n^b \nabla_b \nabla_a K^c{}_c + b_{56}\, K^{ab} \nabla_b \nabla_a \phi \\
& + b_{57}\, K^c{}_c n^a n^b \nabla_b \nabla_a \phi \\
& + b_{58}\, K^i{}_i n^a n^b \nabla_b \nabla_a \phi + b_{59}\, \nabla_b \nabla^b K^a{}_a \\
& + b_{60}\, n^a n^b \nabla_b \nabla_c K_a{}^c + b_{44}\, n^a n^b \nabla_b \nabla_i K_a{}^i \\
& + b_{61}\, H_a{}^{ci} H_{bci} n^a \nabla^b \phi + b_{62}\, n^a R_a{}^c{}_{bc} \nabla^b \phi \\
& + b_{63}\, n^a \nabla_a \phi \nabla_b \phi \nabla^b \phi + b_{64}\, n^a \nabla_b \nabla_a \phi \nabla^b \phi \\
& - b_{49}\, H^{bci} n^a \nabla_c H_{abi} + b_{68}\, H_a{}^{bi} n^a \nabla_c H_b{}^c{}_i \\
& + b_{50}\, H_a{}^{di} n^a n^b n^c \nabla_c H_{bdi} + b_{65}\, n^a n^b \nabla_a \phi \nabla_c K_b{}^c \\
& + b_{36}\, n^a n^b n^c \nabla_c R_a{}^d{}_{bd} + b_{66}\, n^a n^b n^c \nabla_a \phi \nabla_b \phi \nabla_c \phi \\
& + b_{67}\, n^a n^b n^c \nabla_a \phi \nabla_c \nabla_b \phi \\
& + b_{69}\, n^a n^b \nabla_b K_{ac} \nabla^c \phi + b_{70}\, n^a n^b n^c n^d \nabla_d \nabla_c K_{ab} \\
& + b_{73}\, K^b{}_b n^a \nabla_i K_a{}^i \\
& + b_{74}\, n^a n^b \nabla_a \phi \nabla_i K_b{}^i + b_{78}\, \nabla_i \nabla^i K^a{}_a \\
& + b_{75}\, \nabla_j \nabla_i K^{ij} + b_{76}\, K^{ij} \nabla_j \nabla_i \phi \\
& + b_{77}\, \nabla_j \nabla^j K^i{}_i + b_{72}\, H^{ijk} n^a \nabla_k H_{aij} + b_{14}\, K^i{}_i K_{jk} K^{jk}
\end{aligned} \tag{54}$$

where $b_1, \ldots, b_{78}$ are 78 arbitrary p-independent coefficients that may be fixed by the duality constraints.

3.2 T-duality constraint in the boundary

In this subsection we are going to impose the T-duality constraint (3) on the gauge invariant couplings (54) to fix their parameters. To find $\partial S_{\rm eff}(\psi)$ we need to dimensionally reduce $\mathrm{O}_{(p-1)}$-plane boundary action along the circle orthogonal to the O-plane (transverse reduction), i.e., $\partial M^{(10)} = \partial M^{(p)} \times M^{(10-p)}$ and $M^{(10-p)} = S^{(1)} \times M^{(9-p)}$, and to find $\partial S_{\rm eff}(\psi')$ we need to dimensionally reduce O_p-plane boundary action along the circle tangent to the O-plane (world-volume reduction), i.e., $\partial M^{(10)} = \partial M^{(p+1)} \times M^{(9-p)}$ and $\partial M^{(p+1)} = S^{(1)} \times \partial M^{(p)}$.

To find the T-duality of the pull-back metric (45) in the boundary, we assume the killing direction in the boundary space to be y. It is implicitly assumed in the T-duality prescription that everything should be independent of the killing coordinate y, hence, the boundary should be specified as $\sigma^a(\tau^{\hat{a}}) = (y, \sigma^{\tilde{a}}(\tau^{\bar{a}}))$. Then one can show that when O_p-plane is along the y-direction the reduction of $e^{-\phi}\sqrt{|\hat{g}|} = e^{-\phi+\varphi/2}\sqrt{|\bar{g}|}$ where $\bar{g}$ is determinant of the induced metric (35), and when $O_{(p-1)}$-plane is orthogonal to the y-direction the reduction of $e^{-\phi}\sqrt{|\hat{g}|} = e^{-\phi}\sqrt{|\bar{g}|}$. The former transforms under the T-duality transformation (28) to the latter. Hence, to find the T-duality constraints on the boundary action (44), one should consider only the T-duality constraint on $\partial\mathcal{L}_p$ in (54).

The reductions of projection tensors $\widetilde{G}$, $\perp$ and the spacetime tensors $\nabla\phi$, H, R and their covariant derivative are exactly as we have found in the Sect. 2.2, so we need to find reduction of the boundary tensor $K_{\mu\nu}$ and its covariant derivatives, and the reduction of the unite vector n^μ which appear in (54). Using the fact that everything should be independent of the killing coordinate y, one finds $n^y \sim \partial^y f = 0$ and n^μ when μ is not the y-index, is the unite vector orthogonal to the boundary in the base space. The reduction of the extrinsic curvature and its derivatives again have both $U(1)\times U(1)$ gauge invariant part and non-gauge invariant part. The non-gauge invariant part will be cancelled in any covariant couplings. Hence, we need to keep only the gauge invariant part of the reduction of the extrinsic curvature and its covariant derivatives. Writing each tensor in terms of metric and n^μ, and using the reductions (15) and (16), one finds the gauge invariant part of the reductions when the base space is flat, are

$$
\begin{aligned}
K_{\mu\nu} &= \hat{K}_{\mu\nu}\\
K_{\mu y} &= \frac{1}{2}e^{\varphi}n^\alpha V_{\alpha\mu} \equiv \frac{1}{2}e^{\varphi}\mathcal{V}_\mu\\
K_{yy} &= \frac{1}{2}e^{\varphi}n^\alpha\partial_\alpha\varphi \equiv \frac{1}{2}e^{\varphi}\Theta\\
\nabla_\rho K_{\mu\nu} &= e^{\varphi}\Big(-\frac{1}{4}V_{\nu\rho}\mathcal{V}_\mu - \frac{1}{4}V_{\mu\rho}\mathcal{V}_\nu + e^{-\varphi}\partial_\rho\hat{K}_{\mu\nu}\Big)\\
\hat{\nabla}_\rho K_{\nu y} &= e^{\varphi}\Big(-\frac{1}{2}V_{\rho\alpha}\hat{K}_\nu{}^\alpha - \frac{1}{4}V_{\nu\rho}\Theta + \frac{1}{2}\partial_\rho\mathcal{V}_\nu + \frac{1}{4}\mathcal{V}_\nu\partial_\rho\varphi\Big)\\
\nabla_y K_{\mu\nu} &= e^{\varphi}\Big(-\frac{1}{2}V_{\nu\alpha}\hat{K}^\alpha{}_\mu - \frac{1}{2}V_{\mu\alpha}\hat{K}^\alpha{}_\nu - \frac{1}{4}\mathcal{V}_\nu\partial_\mu\varphi\\
&\quad -\frac{1}{4}\mathcal{V}_\mu\partial_\nu\varphi\Big)\\
\nabla_y K_{\mu y} &= \frac{1}{2}e^{\varphi}\Big(-\frac{1}{2}e^{\varphi}V_{\mu\alpha}\mathcal{V}^\alpha + \hat{K}_\mu{}^\alpha\partial_\alpha\varphi - \frac{1}{2}\Theta\partial_\mu\varphi\Big)\\
\nabla_\rho K_{yy} &= \frac{1}{2}e^{\varphi}\Big(-e^{\varphi}V_{\rho\beta}\mathcal{V}^\beta + \partial_\rho\Theta\Big)\\
\nabla_y K_{yy} &= \frac{1}{2}e^{\varphi}\Big(e^{\varphi}\mathcal{V}^\beta\partial_\beta\varphi\Big)\\
\nabla_\sigma\nabla_\rho K_{\mu\nu} &= e^{\varphi}\Big(\frac{1}{4}V_{\mu\sigma}V_{\rho\alpha}\hat{K}^\alpha{}_\nu + \frac{1}{4}V_{\mu\alpha}V_{\rho\sigma}\hat{K}^\alpha{}_\nu\\
&\quad +\frac{1}{4}V_{\nu\sigma}V_{\rho\alpha}\hat{K}_\mu{}^\alpha + \frac{1}{4}V_{\nu\alpha}V_{\rho\sigma}\hat{K}_\mu{}^\alpha\\
&\quad +\frac{1}{8}V_{\mu\sigma}V_{\nu\rho}\Theta + \frac{1}{8}V_{\mu\rho}V_{\nu\sigma}\Theta + \frac{1}{8}V_{\rho\sigma}\mathcal{V}_\nu\partial_\mu\varphi\\
&\quad +\frac{1}{8}V_{\rho\sigma}\mathcal{V}_\mu\partial_\nu\varphi - \frac{1}{4}V_{\nu\sigma}\partial_\rho\mathcal{V}_\mu\\
&\quad -\frac{1}{4}V_{\mu\sigma}\partial_\rho\mathcal{V}_\nu - \frac{1}{8}V_{\nu\sigma}\mathcal{V}_\mu\partial_\rho\varphi - \frac{1}{8}V_{\mu\sigma}\mathcal{V}_\nu\partial_\rho\varphi\\
&\quad -\frac{1}{4}\mathcal{V}_\nu\partial_\sigma V_{\mu\rho} - \frac{1}{4}\mathcal{V}_\mu\partial_\sigma V_{\nu\rho}\\
&\quad -\frac{1}{4}V_{\nu\rho}\partial_\sigma\mathcal{V}_\mu - \frac{1}{4}V_{\mu\rho}\partial_\sigma\mathcal{V}_\nu - \frac{1}{4}V_{\nu\rho}\mathcal{V}_\mu\partial_\sigma\varphi\\
&\quad -\frac{1}{4}V_{\mu\rho}\mathcal{V}_\nu\partial_\sigma\varphi + e^{-\varphi}\partial_\sigma\partial_\rho\hat{K}_{\mu\nu}\Big)\\
\nabla_y\nabla_\rho K_{\mu\nu} &= e^{\varphi}\Big(\frac{1}{8}e^{\varphi}V_{\mu\rho}V_\nu{}^\alpha\mathcal{V}_\alpha + \frac{1}{8}e^{\varphi}V_\mu{}^\alpha V_{\nu\rho}\mathcal{V}_\alpha - \frac{1}{2}V_\rho{}^\alpha\partial_\alpha\hat{K}_{\mu\nu}\\
&\quad +\frac{1}{4}V_{\rho\alpha}\hat{K}^\alpha{}_\nu\partial_\mu\varphi\\
&\quad +\frac{1}{8}V_{\nu\rho}\Theta\partial_\mu\varphi + \frac{1}{4}V_{\rho\alpha}\hat{K}_\mu{}^\alpha\partial_\nu\varphi + \frac{1}{8}V_{\mu\rho}\Theta\partial_\nu\varphi - \frac{1}{2}V_\mu{}^\alpha\partial_\rho\hat{K}_{\alpha\nu}\\
&\quad -\frac{1}{2}V_\nu{}^\alpha\partial_\rho\hat{K}_{\mu\alpha} - \frac{1}{4}\partial_\nu\varphi\partial_\rho\mathcal{V}_\mu - \frac{1}{4}\partial_\mu\varphi\partial_\rho\mathcal{V}_\nu + \frac{1}{4}V_{\mu\alpha}\hat{K}^\alpha{}_\nu\partial_\rho\varphi\\
&\quad +\frac{1}{4}V_{\nu\alpha}\hat{K}_\mu{}^\alpha\partial_\rho\varphi\Big)\\
\nabla_\sigma\nabla_y K_{\mu\nu} &= e^{\varphi}\Big(\frac{1}{8}e^{\varphi}V_{\mu\sigma}V_\nu{}^\alpha\mathcal{V}_\alpha + \frac{1}{8}e^{\varphi}V_\mu{}^\alpha V_{\nu\sigma}\mathcal{V}_\alpha\\
&\quad +\frac{1}{8}e^{\varphi}V_\nu{}^\alpha V_{\sigma\alpha}\mathcal{V}_\mu + \frac{1}{8}e^{\varphi}V_\mu{}^\alpha V_{\sigma\alpha}\mathcal{V}_\nu\\
&\quad -\frac{1}{2}V_\sigma{}^\alpha\partial_\alpha\hat{K}_{\mu\nu} - \frac{1}{4}V_{\mu\sigma}\hat{K}_{\alpha\nu}\partial^\alpha\varphi - \frac{1}{4}V_{\nu\sigma}\hat{K}_{\mu\alpha}\partial^\alpha\varphi + \frac{1}{8}V_{\nu\sigma}\Theta\partial_\mu\varphi\\
&\quad +\frac{1}{8}V_{\mu\sigma}\Theta\partial_\nu\varphi - \frac{1}{2}\hat{K}^\alpha{}_\nu\partial_\sigma V_{\mu\alpha} - \frac{1}{2}\hat{K}_\mu{}^\alpha\partial_\sigma V_{\nu\alpha} - \frac{1}{2}V_\mu{}^\alpha\partial_\sigma\hat{K}_{\alpha\nu}\\
&\quad -\frac{1}{2}V_\nu{}^\alpha\partial_\sigma\hat{K}_{\mu\alpha} - \frac{1}{4}\partial_\nu\varphi\partial_\sigma\mathcal{V}_\mu - \frac{1}{4}\partial_\mu\varphi\partial_\sigma\mathcal{V}_\nu - \frac{1}{4}V_{\mu\alpha}\hat{K}^\alpha{}_\nu\partial_\sigma\varphi\\
&\quad -\frac{1}{4}V_{\nu\alpha}\hat{K}_\mu{}^\alpha\partial_\sigma\varphi - \frac{1}{8}\mathcal{V}_\nu\partial_\mu\varphi\partial_\sigma\varphi - \frac{1}{8}\mathcal{V}_\mu\partial_\nu\varphi\partial_\sigma\varphi - \frac{1}{4}\mathcal{V}_\nu\partial_\sigma\partial_\mu\varphi\\
&\quad -\frac{1}{4}\mathcal{V}_\mu\partial_\sigma\partial_\nu\varphi\Big)\\
\nabla_\sigma\nabla_\rho K_{\mu y} &= e^{\varphi}\Big(\frac{1}{4}e^{\varphi}V_{\mu\sigma}V_\rho{}^\alpha\mathcal{V}_\alpha + \frac{1}{8}e^{\varphi}V_\mu{}^\alpha V_{\rho\sigma}\mathcal{V}_\alpha\\
&\quad +\frac{1}{8}e^{\varphi}V_{\mu\rho}V_\sigma{}^\alpha\mathcal{V}_\alpha\\
&\quad -\frac{1}{8}e^{\varphi}V_\rho{}^\alpha V_{\sigma\alpha}\mathcal{V}_\mu - \frac{1}{4}V_{\rho\sigma}\hat{K}_{\mu\alpha}\partial^\alpha\varphi + \frac{1}{8}V_{\rho\sigma}\theta\partial_\mu\varphi - \frac{1}{2}V_\sigma{}^\alpha\partial_\rho\hat{K}_{\mu\alpha}\\
&\quad -\frac{1}{4}V_{\mu\sigma}\partial_\rho\theta + \frac{1}{2}\partial_\rho\partial_\sigma\mathcal{V}_\mu - \frac{1}{4}\theta\partial_\sigma V_{\mu\rho}\\
&\quad -\frac{1}{2}\hat{K}_\mu{}^\alpha\partial_\sigma V_{\rho\alpha} - \frac{1}{2}V_\rho{}^\alpha\partial_\sigma\hat{K}_{\mu\alpha}\\
&\quad -\frac{1}{4}V_{\mu\rho}\partial_\sigma\theta + \frac{1}{4}\partial_\rho\varphi\partial_\sigma\mathcal{V}_\mu - \frac{1}{4}V_{\rho\alpha}\hat{K}_\mu{}^\alpha\partial_\sigma\varphi - \frac{1}{8}V_{\mu\rho}\theta\partial_\sigma\varphi\\
&\quad +\frac{1}{4}\partial_\rho\mathcal{V}_\mu\partial_\sigma\varphi + \frac{1}{8}\mathcal{V}_\mu\partial_\rho\varphi\partial_\sigma\varphi + \frac{1}{4}\mathcal{V}_\mu\partial_\sigma\partial_\rho\varphi\Big)\\
\nabla_\mu\nabla_\nu K_{yy} &= e^{\varphi}\Big(\frac{1}{4}e^{\varphi}V_{\mu\beta}V_{\nu\alpha}\hat{K}^{\alpha\beta} + \frac{1}{4}e^{\varphi}V_{\mu\alpha}V_{\nu\beta}\hat{K}^{\alpha\beta}\\
&\quad -\frac{1}{4}e^{\varphi}V_\mu{}^\alpha V_{\nu\alpha}\Theta + \frac{1}{4}e^{\varphi}V_{\mu\nu}\mathcal{V}_\alpha\partial^\alpha\varphi\\
&\quad -\frac{1}{2}e^{\varphi}\mathcal{V}^\beta\partial_\mu V_{\nu\beta} - \frac{1}{2}e^{\varphi}V_\nu{}^\beta\partial_\mu\mathcal{V}_\beta - \frac{1}{2}e^{\varphi}V_\nu{}^\alpha\mathcal{V}_\alpha\partial_\mu\varphi\\
&\quad -\frac{1}{2}e^{\varphi}V_\mu{}^\beta\partial_\nu\mathcal{V}_\beta\\
&\quad -\frac{1}{4}e^{\varphi}V_\mu{}^\alpha\mathcal{V}_\alpha\partial_\nu\varphi + \frac{1}{2}\partial_\nu\partial_\mu\Theta\Big)\\
\nabla_\mu\nabla_y K_{\nu y} &= e^{\varphi}\Big(\frac{1}{4}e^{\varphi}V_{\mu\beta}V_{\nu\alpha}\hat{K}^{\alpha\beta} - \frac{1}{2}e^{\varphi}V_{\alpha\beta}V_\mu{}^\beta\hat{K}_\nu{}^\alpha\\
&\quad +\frac{1}{8}e^{\varphi}V_\mu{}^\alpha V_{\nu\alpha}\Theta\\
&\quad +\frac{1}{4}e^{\varphi}V_{\mu\nu}\mathcal{V}_\alpha\partial^\alpha\varphi - \frac{1}{4}e^{\varphi}V_\mu{}^\beta\partial_\beta\mathcal{V}_\nu
\end{aligned}
$$

$$-\frac{1}{4}e^{\varphi}\mathcal{V}^{\beta}\partial_\mu V_{\nu\beta}+\frac{1}{2}\partial^{\alpha}\varphi\partial_\mu\hat{K}_{\nu\alpha}$$
$$-\frac{1}{4}e^{\varphi}V_\nu{}^{\beta}\partial_\mu\mathcal{V}_\beta-\frac{1}{4}e^{\varphi}V_\nu{}^{\alpha}\mathcal{V}_\alpha\partial_\mu\varphi+\frac{1}{2}\hat{K}_\nu{}^{\alpha}\partial_\mu\partial_\alpha\varphi$$
$$+\frac{1}{8}e^{\varphi}V_\mu{}^{\alpha}\mathcal{V}_\alpha\partial_\nu\varphi-\frac{1}{4}\partial_\mu\Theta\partial_\nu\varphi-\frac{1}{4}\Theta\partial_\nu\partial_\mu\varphi\Big)$$
$$\nabla_y\nabla_\mu K_{\nu y}=e^{\varphi}\Big(\frac{1}{4}e^{\varphi}V_{\mu\beta}V_{\nu\alpha}\hat{K}^{\alpha\beta}-\frac{1}{4}e^{\varphi}V_{\alpha\beta}V_\mu{}^{\beta}\hat{K}_\nu{}^{\alpha}$$
$$+\frac{1}{8}e^{\varphi}V_{\mu\nu}\mathcal{V}_\alpha\partial^\alpha\varphi$$
$$-\frac{1}{4}e^{\varphi}V_\mu{}^{\beta}\partial_\beta\mathcal{V}_\nu+\frac{1}{2}\partial^\alpha\varphi\partial_\mu\hat{K}_{\nu\alpha}-\frac{1}{4}e^{\varphi}V_\nu{}^{\beta}\partial_\mu\mathcal{V}_\beta$$
$$-\frac{1}{4}\hat{K}_{\nu\alpha}\partial^\alpha\varphi\partial_\mu\varphi$$
$$+\frac{1}{4}e^{\varphi}V_\mu{}^{\alpha}\mathcal{V}_\alpha\partial_\nu\varphi-\frac{1}{4}\partial_\mu\Theta\partial_\nu\varphi+\frac{1}{8}\Theta\partial_\mu\varphi\partial_\nu\varphi\Big)$$
$$\nabla_y\nabla_y K_{\mu\nu}=e^{\varphi}\Big(\frac{1}{2}e^{\varphi}V_{\mu\alpha}V_{\nu\beta}\hat{K}^{\alpha\beta}-\frac{1}{4}e^{\varphi}V_{\alpha\beta}V_\mu{}^{\beta}\hat{K}^{\alpha}{}_\nu$$
$$-\frac{1}{4}e^{\varphi}V_{\alpha\beta}V_\nu{}^{\beta}\hat{K}_\mu{}^{\alpha}$$
$$+\frac{1}{2}\partial_\alpha\hat{K}_{\mu\nu}\partial^\alpha\varphi+\frac{1}{4}e^{\varphi}V_\nu{}^{\alpha}\mathcal{V}_\alpha\partial_\mu\varphi-\frac{1}{4}\hat{K}_{\alpha\nu}\partial^\alpha\varphi\partial_\mu\varphi$$
$$+\frac{1}{4}e^{\varphi}V_\mu{}^{\alpha}\mathcal{V}_\alpha\partial_\nu\varphi-\frac{1}{4}\hat{K}_{\mu\alpha}\partial^\alpha\varphi\partial_\nu\varphi+\frac{1}{4}\Theta\partial_\mu\varphi\partial_\nu\varphi\Big)$$
$$\nabla_\mu\nabla_y K_{yy}=e^{2\varphi}\Big(\frac{1}{2}e^{\varphi}V_{\alpha\beta}V_\mu{}^{\alpha}\mathcal{V}^\beta-\frac{1}{4}V_{\mu\beta}\hat{K}_\alpha{}^{\beta}\partial^\alpha\varphi$$
$$-\frac{1}{4}V_{\mu\beta}\hat{K}^{\beta}{}_\alpha\partial^\alpha\varphi+\frac{1}{4}V_{\mu\alpha}\Theta\partial^\alpha\varphi$$
$$-\frac{1}{4}V_\mu{}^{\beta}\partial_\beta\Theta+\frac{1}{2}\partial^\beta\varphi\partial_\mu\mathcal{V}_\beta+\frac{1}{4}\mathcal{V}_\alpha\partial^\alpha\varphi\partial_\mu\varphi+\frac{1}{2}\mathcal{V}^\alpha\partial_\mu\partial_\alpha\varphi\Big)$$
$$\nabla_y\nabla_\mu K_{yy}=e^{2\varphi}\Big(\frac{1}{4}e^{\varphi}V_{\alpha\beta}V_\mu{}^{\alpha}\mathcal{V}^\beta-\frac{1}{4}V_{\mu\beta}\hat{K}_\alpha{}^{\beta}\partial^\alpha\varphi$$
$$-\frac{1}{4}V_{\mu\beta}\hat{K}^{\beta}{}_\alpha\partial^\alpha\varphi+\frac{1}{4}V_{\mu\alpha}\Theta\partial^\alpha\varphi$$
$$-\frac{1}{4}V_\mu{}^{\beta}\partial_\beta\Theta+\frac{1}{2}\partial^\beta\varphi\partial_\mu\mathcal{V}_\beta\Big)$$
$$\nabla_y\nabla_y K_{\mu y}=e^{2\varphi}\Big(\frac{1}{8}e^{\varphi}V_{\alpha\beta}V_\mu{}^{\alpha}\mathcal{V}^\beta-\frac{1}{2}V_{\mu\beta}\hat{K}_\alpha{}^{\beta}\partial^\alpha\varphi$$
$$-\frac{1}{2}V_{\alpha\beta}\hat{K}^{\beta}{}_\mu\partial^\alpha\varphi+\frac{1}{4}\partial_\beta\mathcal{V}_\mu\partial^\beta\varphi$$
$$-\frac{3}{8}\mathcal{V}_\alpha\partial^\alpha\varphi\partial_\mu\varphi\Big)$$
$$\nabla_y\nabla_y K_{yy}=e^{2\varphi}\Big(-\frac{1}{2}e^{\varphi}V_{\alpha\beta}\mathcal{V}^\beta\partial^\alpha\varphi-\frac{1}{4}\Theta\partial_\alpha\varphi\partial^\alpha\varphi$$
$$+\frac{1}{2}\hat{K}_{\alpha\beta}\partial^\alpha\varphi\partial^\beta\varphi+\frac{1}{4}\partial_\beta\Theta\partial^\beta\varphi\Big) \qquad (55)$$

where $\hat{K}_{\mu\nu}=\partial_\mu n_\nu-n_\mu n^\alpha\partial_\alpha n_\nu$ is the 9-dimensional extrinsic curvature of the boundary in the base space. The indices on the right-hand side are contracted with base space metric $g^{\alpha\beta}$. Using the above gauge invariant parts of the reductions and reductions (22), (24), (25), one can calculate reduction of various terms in (54) along or orthogonal to the circle. For example, the reduction of $K^a{}_a$ and $K^i{}_i$ when O_p-plane is along the circle are

$$\widetilde{G}^{\mu\nu}K_{\mu\nu}=\hat{K}^{\tilde{a}}{}_{\tilde{a}}+\frac{1}{2}n^\alpha\nabla_\alpha\varphi;\quad \perp^{\mu\nu}K_{\mu\nu}=\hat{K}^i{}_i. \qquad (56)$$

The reduction of these terms when $O_{(p-1)}$-plane is orthogonal to the circle are

$$\widetilde{G}^{\mu\nu}K_{\mu\nu}=\hat{K}^{\tilde{a}}{}_{\tilde{a}};\quad \perp^{\mu\nu}K_{\mu\nu}=\hat{K}^i{}_i+\frac{1}{2}n^\alpha\nabla_\alpha\varphi. \qquad (57)$$

Similarly one can calculate reduction of all 10-dimensional covariant terms in (54). It is important to note that if one keeps all gauge invariant and non-gauge invariant terms in the reduction of tensors (55), one would find the same result for the reduction of 10-dimension covariant couplings.

Using the above reductions, then one can calculate $\partial S_{\rm eff}(\psi)$ which represents the reduction of the $\mathrm{O}_{(p-1)}$-plane boundary action along the circle transverse to the O-plane, and $\partial S_{\rm eff}(\psi')$ which represents the T-duality transformation of the reduction of the O_p-plane boundary action along the circle tangent to the O-plane. The boundary term ∂TD in (3) is also given in (36).

We are free to add to the right-hand side of the constraint (3) the following total derivative terms in the boundary in the base space which are zero according to the Stokes's theorem (see Appendix):

$$\int_{\partial M^{(p)}}d^{p-1}\tau\sqrt{|\bar{g}|}n_{\tilde{a}}\partial_{\tilde{b}}(e^{-\phi}\mathcal{F}^{\tilde{a}\tilde{b}})=0 \qquad (58)$$

where $\mathcal{F}^{\tilde{a}\tilde{b}}$ is an arbitrary antisymmetric tensor constructed from the base space fields n, ∂n, $\partial\partial n$, W^2, V^2, R, $\partial\phi$, $\partial\partial\phi$ and $\bar{H}^2$ at two-derivative order,[2] i.e.,

$$\mathcal{F}_{\tilde{d}\tilde{e}}=y_5\left(e^{\varphi}V_{\tilde{a}i}V_{\tilde{e}}{}^in^{\tilde{a}}n_{\tilde{d}}-e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^in^{\tilde{a}}n_{\tilde{e}}\right)$$
$$+y_{16}\left(e^{-\varphi}n^{\tilde{a}}n_{\tilde{e}}W_{\tilde{a}\tilde{b}}W_{\tilde{d}}{}^{\tilde{b}}-e^{-\varphi}n^{\tilde{a}}n_{\tilde{d}}W_{\tilde{a}\tilde{b}}W_{\tilde{e}}{}^{\tilde{b}}\right)$$
$$+y_{15}\left(n^{\tilde{a}}n_{\tilde{e}}\bar{H}_{abi}\bar{H}_{\tilde{d}}{}^{bi}-n^{\tilde{a}}n_{\tilde{d}}\bar{H}_{\tilde{a}\tilde{b}i}\bar{H}_{\tilde{e}}{}^{\tilde{b}i}\right)$$
$$+y_3\left(n_{\tilde{e}}\partial_{\tilde{a}}\partial^{\tilde{a}}n_{\tilde{d}}-n_{\tilde{d}}\partial_{\tilde{a}}\partial^{\tilde{a}}n_{\tilde{e}}\right)$$
$$+y_2\left(n_{\tilde{e}}\partial_{\tilde{a}}\partial_{\tilde{d}}n^{\tilde{a}}-n_{\tilde{d}}\partial_{\tilde{a}}\partial_{\tilde{e}}n^{\tilde{a}}\right)$$
$$+y_1\left(n^{\tilde{a}}\partial_{\tilde{a}}\partial_{\tilde{d}}n_{\tilde{e}}-n^{\tilde{a}}\partial_{\tilde{a}}\partial_{\tilde{e}}n_{\tilde{d}}\right)$$
$$+y_{45}\left(n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial^{\tilde{a}}n_{\tilde{d}}-n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial^{\tilde{a}}n_{\tilde{e}}\right)$$
$$+y_{44}\left(n_{\tilde{e}}\partial_{\tilde{a}}\phi\partial^{\tilde{a}}n_{\tilde{d}}-n_{\tilde{d}}\partial_{\tilde{a}}\phi\partial^{\tilde{a}}n_{\tilde{e}}\right)$$
$$+y_{14}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{b}}n^{\tilde{b}}-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{b}}n^{\tilde{b}}\right)$$
$$+y_{13}\left(n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{b}}\varphi-n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{b}}\varphi\right)$$
$$+y_{12}\left(n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{b}}\phi-n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{b}}\phi\right)$$
$$+y_{11}\left(n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{e}}\partial_{\tilde{b}}\partial_{\tilde{a}}n_{\tilde{d}}-n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{d}}\partial_{\tilde{b}}\partial_{\tilde{a}}n_{\tilde{e}}\right)$$
$$+y_{10}\left(n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{e}}\partial_{\tilde{b}}\partial_{\tilde{d}}n_{\tilde{a}}-n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{d}}\partial_{\tilde{b}}\partial_{\tilde{e}}n_{\tilde{a}}\right)$$
$$+y_9\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{b}}\partial^{\tilde{b}}n_{\tilde{d}}-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{b}}\partial^{\tilde{b}}n_{\tilde{e}}\right)$$
$$+y_{42}\left(\partial^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{d}}n_{\tilde{a}}-\partial^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{e}}n_{\tilde{a}}\right)$$

[2] Note that the antisymmetric tensor $\mathcal{F}_{\tilde{a}\tilde{b}}$ should be constructed from the base space fields n, $\mathcal{V}$, $\partial\mathcal{V}$, Θ, $\partial\Theta$, $\hat{K}$, $\partial\hat{K}$, ... However, since $\partial\mathcal{V}$ and $\partial\Theta$ include ∂n, one can consider only n, ∂n, $\partial\partial n$, ...

$$
\begin{aligned}
&+y_{41}\left(n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}n^{\tilde{a}}-n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}n^{\tilde{a}}\right)\\
&+y_{40}\left(n_{\tilde{e}}\partial_{\tilde{a}}\phi\partial_{\tilde{d}}n^{\tilde{a}}-n_{\tilde{d}}\partial_{\tilde{a}}\phi\partial_{\tilde{e}}n^{\tilde{a}}\right)\\
&+y_{8}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{b}}\partial_{\tilde{d}}n^{\tilde{b}}-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{b}}\partial_{\tilde{e}}n^{\tilde{b}}\right)\\
&+y_{39}\left(\partial_{\tilde{a}}n^{\tilde{a}}\partial_{\tilde{d}}n_{\tilde{e}}-\partial_{\tilde{a}}n^{\tilde{a}}\partial_{\tilde{e}}n_{\tilde{d}}\right)\\
&+y_{38}\left(n^{\tilde{a}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}n_{\tilde{e}}-n^{\tilde{a}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}n_{\tilde{d}}\right)\\
&+y_{37}\left(n^{\tilde{a}}\partial_{\tilde{a}}\phi\partial_{\tilde{d}}n_{\tilde{e}}-n^{\tilde{a}}\partial_{\tilde{a}}\phi\partial_{\tilde{e}}n_{\tilde{d}}\right)\\
&+y_{36}\left(n_{\tilde{e}}\partial_{\tilde{a}}n^{\tilde{a}}\partial_{\tilde{d}}\varphi-n_{\tilde{d}}\partial_{\tilde{a}}n^{\tilde{a}}\partial_{\tilde{e}}\varphi\right)\\
&+y_{35}\left(n^{\tilde{a}}\partial_{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{d}}\varphi-n^{\tilde{a}}\partial_{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{e}}\varphi\right)\\
&+y_{34}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}\varphi\right)\\
&+y_{33}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\phi\partial_{\tilde{d}}\varphi-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\phi\partial_{\tilde{e}}\varphi\right)\\
&+y_{31}\left(n_{\tilde{e}}\partial_{\tilde{a}}n^{\tilde{a}}\partial_{\tilde{d}}\phi-n_{\tilde{d}}\partial_{\tilde{a}}n^{\tilde{a}}\partial_{\tilde{e}}\phi\right)\\
&+y_{30}\left(n^{\tilde{a}}\partial_{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{d}}\phi-n^{\tilde{a}}\partial_{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{e}}\phi\right)\\
&+y_{29}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}\phi-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}\phi\right)\\
&+y_{28}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\phi\partial_{\tilde{d}}\phi-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\phi\partial_{\tilde{e}}\phi\right)\\
&+y_{32}\left(\partial_{\tilde{d}}\phi\partial_{\tilde{e}}\varphi-\partial_{\tilde{d}}\varphi\partial_{\tilde{e}}\phi\right)\\
&+y_{26}\left(n_{\tilde{e}}\partial_{\tilde{d}}\partial_{\tilde{a}}n^{\tilde{a}}-n_{\tilde{d}}\partial_{\tilde{e}}\partial_{\tilde{a}}n^{\tilde{a}}\right)\\
&+y_{25}\left(n^{\tilde{a}}\partial_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{e}}-n^{\tilde{a}}\partial_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{d}}\right)\\
&+y_{24}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{d}}\partial_{\tilde{a}}\varphi-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{e}}\partial_{\tilde{a}}\varphi\right)\\
&+y_{23}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{d}}\partial_{\tilde{a}}\phi-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{e}}\partial_{\tilde{a}}\phi\right)\\
&+y_{6}\left(n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{e}}\partial_{\tilde{d}}\partial_{\tilde{b}}n_{\tilde{a}}-n^{\tilde{a}}n^{\tilde{b}}n_{\tilde{d}}\partial_{\tilde{e}}\partial_{\tilde{b}}n_{\tilde{a}}\right)\\
&+y_{22}\left(n^{\tilde{a}}\partial_{\tilde{d}}\partial_{\tilde{e}}n_{\tilde{a}}-n^{\tilde{a}}\partial_{\tilde{e}}\partial_{\tilde{d}}n_{\tilde{a}}\right)\\
&+y_{27}\left(n_{\tilde{e}}\partial_{\tilde{d}}\partial_{i}n^{i}-n_{\tilde{d}}\partial_{\tilde{e}}\partial_{j}n^{j}\right)\\
&+y_{43}\left(n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}n_{\tilde{d}}\partial_{i}n^{i}-n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}n_{\tilde{e}}\partial_{i}n^{i}\right)\\
&+y_{19}\left(\partial_{\tilde{d}}n_{\tilde{e}}\partial_{i}n^{i}-\partial_{\tilde{e}}n_{\tilde{d}}\partial_{i}n^{i}\right)\\
&+y_{18}\left(n_{\tilde{e}}\partial_{\tilde{d}}\varphi\partial_{i}n^{i}-n_{\tilde{d}}\partial_{\tilde{e}}\varphi\partial_{i}n^{i}\right)\\
&+y_{7}\left(n_{\tilde{e}}\partial_{\tilde{d}}\phi\partial_{i}n^{i}-n_{\tilde{d}}\partial_{\tilde{e}}\phi\partial_{i}n^{i}\right)\\
&+y_{21}\left(n_{\tilde{e}}\partial_{i}\partial^{i}n_{\tilde{d}}-n_{\tilde{d}}\partial_{i}\partial^{i}n_{\tilde{e}}\right)\\
&+y_{20}\left(n_{\tilde{e}}\partial_{i}\partial_{\tilde{d}}n^{i}-n_{\tilde{d}}\partial_{j}\partial_{\tilde{e}}n^{j}\right)
\end{aligned}\tag{59}
$$

where $y_1, y_2, \ldots$ are arbitrary parameters. After imposing the orientifold projection on (58), we add it to the right-hand side of (3). Finally, one should write the couplings in the form of independent structures by imposing the Bianchi identities (23) in the base space. Here again we write the field strengths $\bar{H}$, V, W in terms of potentials $\bar{b}_{\mu\nu}$, g_μ, b_μ to satisfy the Bianchi identities automatically. Using the relation (46), we also write the base space unit vector n^μ in terms of function f to impose its corresponding identities.

Writing all terms in the T-duality constraint (3) in terms of independent and non-gauge invariant structures, then one makes the coefficients of the independent structures which include the parameters of the gauge invariant Lagrangian (54), a_{28} and the arbitrary parameters in total derivative terms (58), to be zero. Unlike the bulk case, not all parameters are fixed in terms of an overall factor. The linear equations in this case produce the following boundary Lagrangian which has a_{28} and 17 other parameters:

$$
\begin{aligned}
\partial\mathcal{L}_p = {}& b_{50}\Big[H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}-2K_a{}^cK^{ab}K_{bc}-2K_i{}^kK^{ij}K_{jk}\\
&-2K^{cd}n^an^bR_{acbd}-2K^{ij}n^an^bR_{aibj}\Big]\\
&+b_6\Big[K^a{}_aK^b{}_bK^c{}_c-3K^a{}_aK^b{}_bK^i{}_i+3K^a{}_aK^i{}_iK^j{}_j\\
&-K^i{}_iK^j{}_jK^k{}_k\Big]\\
&+b_{19}\Big[-\frac{4}{3}K_a{}^cK^{ab}K_{bc}+\frac{4}{3}K_i{}^kK^{ij}K_{jk}\\
&-H_{ac}{}^iH_{bdi}K^{cd}n^an^b+H_a{}^c{}_iH_{bcj}K^{ij}n^an^b\Big]\\
&+b_{39}\Big[-K^a{}_aK^b{}_bK^i{}_i+2K^a{}_aK^i{}_iK^j{}_j-K^i{}_iK^j{}_jK^k{}_k\\
&+K^b{}_bK^c{}_cn^a\nabla_a\phi-2K^b{}_bK^i{}_in^a\nabla_a\phi+K^i{}_iK^j{}_jn^a\nabla_a\phi\Big]\\
&+b_{23}\Big[K^a{}_aK_{bc}K^{bc}+K^c{}_cn^an^b\mathcal{R}_{ab}-K^i{}_in^an^b\mathcal{R}_{ab}\\
&+K^i{}_in^a\nabla_bK_a{}^b\Big]\\
&+b_{55}\Big[n^an^b\nabla_b\nabla_aK^c{}_c-n^an^b\nabla_b\nabla_aK^i{}_i\Big]\\
&+b_{46}\Big[-\nabla_a\nabla^aK^i{}_i-\nabla_aK^b{}_b\nabla^a\phi+\nabla_aK^i{}_i\nabla^a\phi+\nabla_b\nabla^bK^a{}_a\Big]\\
&+b_{52}\Big[K^a{}_aK^i{}_iK^j{}_j-\frac{2}{3}K^i{}_iK^j{}_jK^k{}_k-2K^b{}_bK^i{}_in^a\nabla_a\phi\\
&+K^i{}_iK^j{}_jn^a\nabla_a\phi+K^c{}_cn^an^b\nabla_a\phi\nabla_b\phi\\
&-\frac{1}{3}n^an^bn^c\nabla_a\phi\nabla_b\phi\nabla_c\phi\Big]\\
&+b_{53}\Big[\frac{1}{3}K^i{}_iK^j{}_jK^k{}_k-K^i{}_iK^j{}_jn^a\nabla_a\phi+K^i{}_in^an^b\nabla_a\phi\nabla_b\phi\\
&-\frac{1}{3}n^an^bn^c\nabla_a\phi\nabla_b\phi\nabla_c\phi\Big]\\
&+b_{34}\Big[-K_i{}^kK^{ij}K_{jk}-K^{ij}n^an^bR_{aibj}+K^{ij}n^a\nabla_aK_{ij}\\
&+n^an^b\nabla_b\nabla_iK_a{}^i-n^an^b\nabla_bK_{ac}\nabla^c\phi\Big]\\
&+b_{32}\Big[-K^{ab}\mathcal{R}_{ab}+K^i{}_in^an^b\mathcal{R}_{ab}\\
&+K^i{}_in^a\nabla_aK^b{}_b+K^b{}_bn^a\nabla_aK^i{}_i-K^i{}_in^a\nabla_aK^j{}_j\\
&-\frac{1}{2}n^a\nabla_a\mathcal{R}^b{}_b+\frac{1}{2}n^a\nabla_a\nabla_b\nabla^b\phi-K^i{}_in^a\nabla_bK_a{}^b\\
&+\nabla_b\nabla_aK^{ab}+n^an^b\nabla_b\nabla_aK^i{}_i
\end{aligned}
$$

$$
\begin{aligned}
&+K^{ab}\nabla_b\nabla_a\phi - K^c{}_c n^a n^b \nabla_b\nabla_a\phi\\
&-\nabla_b\nabla^b K^a{}_a - n^a n^b \nabla_b\nabla_c K_a{}^c\\
&+n^a n^b n^c \nabla_c \mathcal{R}_{ab} - n^a n^b n^c \nabla_c\nabla_b\nabla_a\phi - n^a n^b \nabla_b K_{ac}\nabla^c\phi\Big]\\
&+b_{70}\Big[n^a n^b n^c n^d \nabla_d\nabla_c K_{ab}\Big]\\
&+b_{16}\Big[2K^a{}_a K_{bc}K^{bc} - 2K^i{}_i K_{jk}K^{jk} + H_a{}^{di}H_{bdi}K^c{}_c n^a n^b\\
&-H_a{}^{cj}H_{bcj}K^i{}_i n^a n^b + 2K^i{}_i n^a \nabla_b K_a{}^b - 2K^b{}_b n^a \nabla_i K_a{}^i\Big]\\
&+b_{61}\Big[2K^{ab}\mathcal{R}_{ab} + 2K^{ij}\mathcal{R}_{ij} + \frac{1}{2}H^{bci}n^a\nabla_a H_{bci}\\
&+\frac{1}{2}\nabla_a\nabla^a K^i{}_i - \frac{1}{2}\nabla_b\nabla^b K^a{}_a\\
&+H_a{}^{ci}H_{bci}n^a\nabla^b\phi + H^{bci}n^a\nabla_c H_{abi} + H_a{}^{bi}n^a\nabla_c H_b{}^c{}_i\\
&+\frac{1}{2}\nabla_i\nabla^i K^a{}_a - \frac{1}{2}\nabla_j\nabla^j K^i{}_i\\
&-H_a{}^{ci}H_{bci}K^{ab} - \frac{1}{2}H^{ijk}n^a\nabla_k H_{aij}\Big]\\
&+b_{64}\Big[-2K_i{}^k K^{ij}K_{jk} - 2K^{ij}n^a n^b R_{aibj} + K^{ij}\mathcal{R}_{ij}\\
&+\frac{1}{4}H^{bci}n^a\nabla_a H_{bci} - \frac{1}{2}n^a\nabla_a\mathcal{R}^b{}_b\\
&+\frac{1}{4}\nabla_a\nabla^a K^i{}_i - \frac{1}{2}n^a\nabla_a\nabla_b\nabla^b\phi - \frac{1}{4}\nabla_b\nabla^b K^a{}_a\\
&+n^a\nabla_b\nabla_a\phi\nabla^b\phi - K^b{}_b n^a\nabla_i K_a{}^i\\
&+n^a n^b\nabla_a\phi\nabla_i K_b{}^i + \frac{1}{4}\nabla_i\nabla^i K^a{}_a - \frac{1}{4}\nabla_j\nabla^j K^i{}_i\\
&-\frac{1}{4}H^{ijk}n^a\nabla_k H_{aij}\Big]\\
&+b_{65}\Big[K^a{}_a K_{bc}K^{bc} - 2K_a{}^c K^{ab}K_{bc} - 2K_i{}^k K^{ij}K_{jk}\\
&-2K^{cd}n^a n^b R_{acbd} - 2K^{ij}n^a n^b R_{aibj}\\
&+K^{ab}\mathcal{R}_{ab} + K^{ij}\mathcal{R}_{ij} + \frac{1}{4}H^{bci}n^a\nabla_a H_{bci}\\
&+\frac{1}{4}\nabla_a\nabla^a(K^i{}_i - K^b{}_b) + n^a n^b\nabla_a\phi\nabla_c K_b{}^c\\
&-K^b{}_b n^a\nabla_i K_a{}^i + n^a n^b\nabla_a\phi\nabla_i K_b{}^i\\
&+\frac{1}{4}\nabla_i\nabla^i K^a{}_a - \frac{1}{4}\nabla_j\nabla^j K^i{}_i - \frac{1}{4}H^{ijk}n^a\nabla_k H_{aij}\Big]\\
&+b_{67}\Big[-2K_a{}^c K^{ab}K_{bc} - 2K_i{}^k K^{ij}K_{jk} - 2K^{cd}n^a n^b R_{acbd}\\
&-2K^{ij}n^a n^b R_{aibj} + K^{ij}\mathcal{R}_{ij}\\
&+\frac{1}{4}H^{bci}n^a\nabla_a H_{bci} - \frac{1}{2}n^a\nabla_a\mathcal{R}^b{}_b + \frac{1}{4}\nabla_a\nabla^a K^i{}_i\\
&+\frac{1}{2}n^a\nabla_a\nabla_b\nabla^b\phi + K^{ab}\nabla_b\nabla_a\phi\\
&-K^c{}_c n^a n^b\nabla_b\nabla_a\phi - \frac{1}{4}\nabla_b\nabla^b K^a{}_a + n^a\mathcal{R}_{ab}\nabla^b\phi\\
&-n^a\nabla_b\nabla_a\phi\nabla^b\phi + n^a n^b n^c\nabla_c\mathcal{R}_{ab}\\
&+n^a n^b n^c\nabla_a\phi\nabla_c\nabla_b\phi - n^a n^b n^c\nabla_c\nabla_b\nabla_a\phi\\
&-K^b{}_b n^a\nabla_i K_a{}^i + n^a n^b\nabla_a\phi\nabla_i K_b{}^i\\
&+\frac{1}{4}\nabla_i\nabla^i K^a{}_a - \frac{1}{4}\nabla_j\nabla^j K^i{}_i - \frac{1}{4}H^{ijk}n^a\nabla_k H_{aij}\Big]\\
&+a_{28}\Big[9H_{abj}H^{ab}{}_i K^{ij} - 6H_a{}^{ci}H_{bci}K^{ab}\\
&-3H_i{}^{kl}H_{jkl}K^{ij} + 48K_i{}^k K^{ij}K_{jk} + 48K^{ij}n^a n^b R_{aibj}\\
&+12K^{ab}\mathcal{R}_{ab} - 36K^{ij}\mathcal{R}_{ij} - 3H^{bci}n^a\nabla_a H_{bci}\\
&-3\nabla_a\nabla^a K^i{}_i + 3\nabla_b\nabla^b K^a{}_a\\
&+24K^b{}_b n^a\nabla_i K_a{}^i - 24n^a n^b\nabla_a\phi\nabla_i K_b{}^i\\
&-3\nabla_i\nabla^i K^a{}_a + 3\nabla_j\nabla^j K^i{}_i + 3H^{ijk}n^a\nabla_k H_{aij}\Big]
\end{aligned}
\tag{60}
$$

where $\mathcal{R}_{\mu\nu} = \widetilde{G}^{\rho\sigma}R_{\rho\mu\sigma\nu} + \nabla_\mu\nabla_\nu\phi$. The above boundary Lagrangian is invariant under the T-duality for the above 18 parameters, i.e., it satisfies the T-duality constraint (3) for the following base space boundary total derivative terms:

$$
\begin{aligned}
\mathcal{F}_{\tilde{d}\tilde{e}} = b_{65}\Big(&\frac{1}{2}e^{\varphi}V_{\tilde{a}i}V_{\tilde{e}}{}^i n^{\tilde{a}}n_{\tilde{d}} - \frac{1}{2}e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^i n^{\tilde{a}}n_{\tilde{e}}\\
&+\frac{1}{2}e^{-\varphi}n^{\tilde{a}}n_{\tilde{e}}W_{\tilde{a}\tilde{b}}W_{\tilde{d}}{}^{\tilde{b}} - \frac{1}{2}e^{-\varphi}n^{\tilde{a}}n_{\tilde{d}}W_{\tilde{a}\tilde{b}}W_{\tilde{e}}{}^{\tilde{b}}\Big)\\
&+a_{28}\Big(-12e^{\varphi}V_{\tilde{a}i}V_{\tilde{e}}{}^i n^{\tilde{a}}n_{\tilde{d}} + 12e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^i n^{\tilde{a}}n_{\tilde{e}}\\
&+6n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi - 6n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}\varphi\Big)\\
&+b_{64}\Big(\frac{1}{2}e^{\varphi}V_{\tilde{a}i}V_{\tilde{e}}{}^i n^{\tilde{a}}n_{\tilde{d}} - \frac{1}{2}e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^i n^{\tilde{a}}n_{\tilde{e}}\\
&-\frac{1}{4}n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi + \frac{1}{4}n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}\varphi\Big)\\
&+b_{67}\Big(\frac{1}{2}e^{\varphi}V_{\tilde{a}i}V_{\tilde{e}}{}^i n^{\tilde{a}}n_{\tilde{d}} - \frac{1}{2}e^{\varphi}V_{\tilde{a}i}V_{\tilde{d}}{}^i n^{\tilde{a}}n_{\tilde{e}}\\
&-\frac{1}{4}n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{a}}\varphi\partial_{\tilde{d}}\varphi + \frac{1}{2}n^{\tilde{a}}n_{\tilde{e}}\partial_{\tilde{d}}\partial_{\tilde{a}}\varphi\\
&+\frac{1}{4}n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{a}}\varphi\partial_{\tilde{e}}\varphi - \frac{1}{2}n^{\tilde{a}}n_{\tilde{d}}\partial_{\tilde{e}}\partial_{\tilde{a}}\varphi\Big).
\end{aligned}
\tag{61}
$$

The other multiplets in (60) are invariant under the T-duality without the total derivative terms in the boundary of base space.

The Lagrangian (60) is not consistent with the S-duality for all 18 parameters. To have boundary couplings that their combinations with the bulk couplings (31) satisfy both the T-duality constraint (4) and the S-duality constraint (5), one has to consider a particular relations for the parameters. In the next subsection we are going to find these relations.

3.3 S-duality constraint in the boundary

As we mentioned before, the S-duality has a non-trivial constraint on the NS–NS couplings. In the Einstein frame, apart from the overall dilaton factor, there must be only even number of dilaton when B-field is zero. This constraint reduces the number of parameters in the T-duality invariant boundary Lagrangian (60).

The overall factor $e^{-\phi}\sqrt{|\hat{g}|}$ in the string frame action (44) transforms to the following factor in the Einstein frame $G_{\mu\nu} = e^{\phi/2}G^E_{\mu\nu}$ for the O_3-plane:

$$
e^{-\frac{1}{4}\phi}\sqrt{|\hat{g}^E|}. \tag{62}
$$

On the other hand, transformation of the string frame Lagrangian $\mathcal{L}_p$ to the Einstein frame Lagrangian has an overall dilaton factor $e^{-\frac{3}{4}\phi}$ for the gravity and dilaton couplings, i.e., $\mathcal{L}_p(G,\phi)=e^{-\frac{3}{4}\phi}\mathcal{L}_p^E(G^E,\phi)$. The couplings involving B-field, however, has another dilaton factor which is needed for making them to be invariant under the S-duality. Hence, the string frame boundary action (44) transforms to the following Einstein frame Lagrangian for O_3-plane:

$$\partial\mathbf{S}_3=-\frac{T_3\pi^2\alpha'^2}{48}\int_{\partial M^{(4)}}d^3\tau\, e^{-\phi}\sqrt{|\hat{g}^E|}\,\partial\mathcal{L}_3^E. \tag{63}$$

To make the overall dilaton factor $e^{-\phi}$ to be invariant under the S-duality, one should include loop and non-perturbative effects [28]. It is straightforward to find the Lagrangian $\mathcal{L}_3^E$ in the Einstein frame. One write each string frame term in (60) in terms of metric and its derivatives, and then replaces $G_{\mu\nu}=e^{\phi/2}G^E_{\mu\nu}$. We then add to (63) the residual boundary terms in the bulk action, i.e., the couplings (43). We are also free to add the following total derivative terms in the Einstein frame (see Appendix):

$$\int_{\partial M^{(4)}}d^3\tau\sqrt{|\hat{g}^E|}n_a^E\partial_b[e^{-\phi}(\mathcal{F}^E)^{ab}]=0 \tag{64}$$

where $\mathcal{F}^{ab}$ is an arbitrary antisymmetric tensor constructed from the boundary fields $n^E,\nabla n^E,\nabla\nabla n^E,R,\nabla R,\nabla\phi,\nabla\nabla\phi$ at two-derivative order, i.e.,

$$\begin{aligned}
\mathcal{F}^E_{de}=\,&x_2\left(n^an_eR_d{}^b{}_{ab}-n^an_dR_e{}^b{}_{ab}\right)\\
&+x_3\left(n_e\nabla_a\nabla^an_d-n_d\nabla_a\nabla^an_e\right)\\
&+x_4\left(n_e\nabla_a\phi\nabla^an_d-n_d\nabla_a\phi\nabla^an_e\right)\\
&+x_5\left(n^an_e\nabla_an_d\nabla_bn^b-n^an_d\nabla_an_e\nabla_bn^b\right)\\
&+x_6\left(n^an^bn_e\nabla_an_d\nabla_b\phi-n^an^bn_d\nabla_an_e\nabla_b\phi\right)\\
&+x_7\left(n^an^bn_e\nabla_b\nabla_an_d-n^an^bn_d\nabla_b\nabla_an_e\right)\\
&+x_8\left(n^an_e\nabla_an_b\nabla^bn_d-n^an_d\nabla_an_b\nabla^bn_e\right)\\
&+x_9\left(n^an_e\nabla_bn_a\nabla^bn_d-n^an_d\nabla_bn_a\nabla^bn_e\right)\\
&+x_{10}\left(n^an^bn^cn_e\nabla_an_d\nabla_cn_b-n^an^bn^cn_d\nabla_an_e\nabla_cn_b\right)\\
&+x_{11}\left(\nabla^an_e\nabla_dn_a-\nabla^an_d\nabla_en_a\right)\\
&+x_{12}\left(n^an_e\nabla_bn^b\nabla_dn_a-n^an_d\nabla_bn^b\nabla_en_a\right)\\
&+x_{13}\left(n^an^bn_e\nabla_b\phi\nabla_dn_a-n^an^bn_d\nabla_b\phi\nabla_en_a\right)\\
&+x_{14}\left(n^an^bn^cn_e\nabla_cn_b\nabla_dn_a-n^an^bn^cn_d\nabla_cn_b\nabla_en_a\right)\\
&+x_{15}\left(n_e\nabla_a\phi\nabla_dn^a-n_d\nabla_a\phi\nabla_en^a\right)\\
&+x_{16}\left(n^an^b\nabla_an_e\nabla_dn_b-n^an^b\nabla_an_d\nabla_en_b\right)\\
&+x_{17}\left(n^an_e\nabla_an_b\nabla_dn^b-n^an_d\nabla_an_b\nabla_en^b\right)\\
&+x_{18}\left(n^an_e\nabla_bn_a\nabla_dn^b-n^an_d\nabla_bn_a\nabla_en^b\right)\\
&+x_{19}\left(\nabla_an^a\nabla_dn_e-\nabla_an^a\nabla_en_d\right)\\
&+x_{20}\left(n^a\nabla_a\phi\nabla_dn_e-n^a\nabla_a\phi\nabla_en_d\right)\\
&+x_{21}\left(n^an^b\nabla_bn_a\nabla_dn_e-n^an^b\nabla_bn_a\nabla_en_d\right)\\
&+x_{22}\left(n_e\nabla_an^a\nabla_d\phi-n_d\nabla_an^a\nabla_e\phi\right)\\
&+x_{23}\left(n^a\nabla_an_e\nabla_d\phi-n^a\nabla_an_d\nabla_e\phi\right)\\
&+x_{24}\left(n^an_e\nabla_a\phi\nabla_d\phi-n^an_d\nabla_a\phi\nabla_e\phi\right)\\
&+x_{25}\left(n^an^bn_e\nabla_bn_a\nabla_d\phi-n^an^bn_d\nabla_bn_a\nabla_e\phi\right)\\
&+x_{26}\left(n^a\nabla_d\phi\nabla_en_a-n^a\nabla_dn_a\nabla_e\phi\right)\\
&+x_{27}\left(n_e\nabla_d\nabla_an^a-n_d\nabla_e\nabla_an^a\right)\\
&+x_{28}\left(n^a\nabla_d\nabla_an_e-n^a\nabla_e\nabla_an_d\right)\\
&+x_{29}\left(n^an_e\nabla_d\nabla_a\phi-n^an_d\nabla_e\nabla_a\phi\right)\\
&+x_{30}\left(n^an^bn_e\nabla_d\nabla_bn_a-n^an^bn_d\nabla_e\nabla_bn_a\right)\\
&+x_{31}\left(n_e\nabla_d\nabla_in^i-n_d\nabla_e\nabla_in^i\right)\\
&+x_{32}\left(n_e\nabla_d\nabla_in^i-n_d\nabla_e\nabla_in^i\right)\\
&+x_{33}\left(n^an_e\nabla_an_d\nabla_in^i-n^an_d\nabla_an_e\nabla_in^i\right)\\
&+x_{34}\left(n^an_e\nabla_dn_a\nabla_in^i-n^an_d\nabla_en_a\nabla_in^i\right)\\
&+x_{35}\left(\nabla_dn_e\nabla_in^i-\nabla_en_d\nabla_in^i\right)\\
&+x_{36}\left(n_e\nabla_d\phi\nabla_in^i-n_d\nabla_e\phi\nabla_in^i\right)
\end{aligned}\tag{65}$$

where $x_2,x_2,\ldots$ are arbitrary parameters. In the above relations n^μ is the Einstein frame unite vector. After using the Einstein frame equations of motion (40), imposing orientifold projection, various Bianchi identities, and identities corresponding to the unit vector n^μ, we impose the condition that there must no odd number of dilation when B-field is zero. This fixes the parameter in the bulk total derivative (43) to be

$$\alpha=-4. \tag{66}$$

The S-duality constraint also produces the following 4 multiplets in the string frame which are T-dual and S-dual invariant:

$$\begin{aligned}
\partial\mathcal{L}_p=a_{28}\Big[&-6H_a{}^{ci}H_{bci}K^{ab}-32K_a{}^cK^{ab}K_{bc}\\
&+\frac{40}{3}K^a{}_aK_{bc}K^{bc}-\frac{136}{27}K^a{}_aK^b{}_bK^c{}_c\\
&+\frac{52}{9}K^a{}_aK^b{}_bK^i{}_i+9H_{abj}H^{ab}{}_iK^{ij}-3H_i{}^{kl}H_{jkl}K^{ij}\\
&-16K_i{}^kK^{ij}K_{jk}+\frac{32}{9}K^a{}_aK^i{}_iK^j{}_j
\end{aligned}$$

$$+12K^i{}_iK_{jk}K^{jk}-\frac{116}{27}K^i{}_iK^j{}_jK^k{}_k-6H_a{}^{di}H_{bdi}K^c{}_cn^an^b$$
$$+12H_{ac}{}^iH_{bdi}K^{cd}n^an^b$$
$$+6H_a{}^{cj}H_{bcj}K^i{}_in^an^b-12H_a{}^c{}_iH_{bcj}K^{ij}n^an^b$$
$$-48K^{cd}n^an^bR_{acbd}+\frac{56}{3}K^{ab}\mathcal{R}_{ab}$$
$$+\frac{40}{3}K^c{}_cn^an^b\mathcal{R}_{ab}-8K^i{}_in^an^b\mathcal{R}_{ab}-24K^{ij}\mathcal{R}_{ij}$$
$$+\frac{16}{3}K^i{}_in^a\nabla_aK^b{}_b+\frac{16}{3}K^b{}_bn^a\nabla_aK^i{}_i$$
$$-\frac{16}{3}K^i{}_in^a\nabla_aK^j{}_j-\frac{8}{3}n^a\nabla_a\mathcal{R}^b{}_b+\frac{28}{3}K^b{}_bK^c{}_cn^a\nabla_a\phi$$
$$-\frac{56}{3}K^b{}_bK^i{}_in^a\nabla_a\phi+\frac{28}{3}K^i{}_iK^j{}_jn^a\nabla_a\phi$$
$$-16\nabla_a\nabla^aK^i{}_i+\frac{8}{3}n^a\nabla_a\nabla_b\nabla^b\phi$$
$$-16\nabla_aK^b{}_b\nabla^a\phi+16\nabla_aK^i{}_i\nabla^a\phi$$
$$-4K^i{}_in^a\nabla_bK_a{}^b+\frac{16}{3}\nabla_b\nabla_aK^{ab}$$
$$-\frac{32}{3}n^an^b\nabla_b\nabla_aK^c{}_c+16n^an^b\nabla_b\nabla_aK^i{}_i$$
$$-\frac{16}{3}K^c{}_cn^an^b\nabla_b\nabla_a\phi+\frac{32}{3}\nabla_b\nabla^bK^a{}_a$$
$$-\frac{16}{3}n^an^b\nabla_b\nabla_cK_a{}^c+12H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}$$
$$+12n^an^b\nabla_a\phi\nabla_cK_b{}^c+\frac{16}{3}n^an^bn^c\nabla_c\mathcal{R}_{ab}$$
$$-\frac{16}{3}n^an^bn^c\nabla_c\nabla_b\nabla_a\phi-\frac{16}{3}n^an^b\nabla_bK_{ac}\nabla^c\phi$$
$$+24K^b{}_bn^a\nabla_iK_a{}^i-12n^an^b\nabla_a\phi\nabla_iK_b{}^i$$
$$+\frac{16}{3}K^{ab}\nabla_b\nabla_a\phi\Big]$$
$$+b_{53}\Big[-\frac{1}{4}H_a{}^{ci}H_{bci}K^{ab}-\frac{1}{3}K^a{}_aK_{bc}K^{bc}$$
$$-\frac{13}{81}K^a{}_aK^b{}_bK^c{}_c+\frac{5}{54}K^a{}_aK^b{}_bK^i{}_i$$
$$+\frac{8}{27}K^a{}_aK^i{}_iK^j{}_j-\frac{1}{6}K^i{}_iK_{jk}K^{jk}$$
$$+\frac{17}{162}K^i{}_iK^j{}_jK^k{}_k+\frac{1}{12}H_a{}^{di}H_{bdi}K^c{}_cn^an^b$$
$$-\frac{1}{12}H_a{}^{cj}H_{bcj}K^i{}_in^an^b+\frac{7}{18}K^b{}_bK^c{}_cn^a\nabla_a\phi$$
$$-\frac{7}{9}K^b{}_bK^i{}_in^a\nabla_a\phi-\frac{11}{18}K^i{}_iK^j{}_jn^a\nabla_a\phi$$
$$+\frac{1}{6}K^i{}_in^a\nabla_bK_a{}^b+K^i{}_in^an^b\nabla_a\phi\nabla_b\phi$$
$$+\frac{1}{4}H_a{}^{ci}H_{bci}n^a\nabla^b\phi+\frac{1}{4}H^{bci}n^a\nabla_cH_{abi}$$
$$+\frac{1}{4}H_a{}^{bi}n^a\nabla_cH_b{}^c{}_i+\frac{1}{2}H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}$$
$$-\frac{1}{2}n^an^b\nabla_a\phi\nabla_cK_b{}^c$$
$$-\frac{1}{3}n^an^bn^c\nabla_a\phi\nabla_b\phi\nabla_c\phi+\frac{1}{3}K^b{}_bn^a\nabla_iK_a{}^i$$
$$-\frac{1}{2}n^an^b\nabla_a\phi\nabla_iK_b{}^i\Big]$$
$$+b_{52}\Big[\frac{1}{8}H_a{}^{ci}H_{bci}K^{ab}+\frac{1}{6}K^a{}_aK_{bc}K^{bc}$$
$$+\frac{37}{162}K^a{}_aK^b{}_bK^c{}_c+\frac{19}{108}K^a{}_aK^b{}_bK^i{}_i$$
$$-\frac{1}{27}K^a{}_aK^i{}_iK^j{}_j+\frac{1}{12}K^i{}_iK_{jk}K^{jk}$$
$$-\frac{11}{324}K^i{}_iK^j{}_jK^k{}_k-\frac{1}{24}H_a{}^{di}H_{bdi}K^c{}_cn^an^b$$
$$+\frac{1}{24}H_a{}^{cj}H_{bcj}K^i{}_in^an^b-\frac{31}{36}K^b{}_bK^c{}_cn^a\nabla_a\phi$$
$$-\frac{5}{18}K^b{}_bK^i{}_in^a\nabla_a\phi+\frac{5}{36}K^i{}_iK^j{}_jn^a\nabla_a\phi$$
$$-\frac{1}{12}K^i{}_in^a\nabla_bK_a{}^b+K^c{}_cn^an^b\nabla_a\phi\nabla_b\phi$$
$$-\frac{1}{8}H_a{}^{ci}H_{bci}n^a\nabla^b\phi-\frac{1}{8}H^{bci}n^a\nabla_cH_{abi}$$
$$-\frac{1}{8}H_a{}^{bi}n^a\nabla_cH_b{}^c{}_i-\frac{1}{4}H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}$$
$$+\frac{1}{4}n^an^b\nabla_a\phi\nabla_cK_b{}^c$$
$$-\frac{1}{3}n^an^bn^c\nabla_a\phi\nabla_b\phi\nabla_c\phi-\frac{1}{6}K^b{}_bn^a\nabla_iK_a{}^i$$
$$+\frac{1}{4}n^an^b\nabla_a\phi\nabla_iK_b{}^i\Big]$$
$$+b_{67}\Big[-\frac{11}{18}K^a{}_aK_{bc}K^{bc}-\frac{1}{54}K^a{}_aK^b{}_bK^c{}_c$$
$$+\frac{1}{36}K^a{}_aK^b{}_bK^i{}_i+\frac{1}{12}K^i{}_iK_{jk}K^{jk}$$
$$-\frac{1}{108}K^i{}_iK^j{}_jK^k{}_k-\frac{1}{24}H_a{}^{di}H_{bdi}K^c{}_cn^an^b$$
$$+\frac{1}{24}H_a{}^{cj}H_{bcj}K^i{}_in^an^b-\frac{8}{9}K^{ab}\mathcal{R}_{ab}$$
$$+\frac{2}{9}K^c{}_cn^an^b\mathcal{R}_{ab}-\frac{1}{3}K^i{}_in^an^b\mathcal{R}_{ab}-\frac{1}{9}K^i{}_in^a\nabla_aK^b{}_b$$
$$-\frac{1}{9}K^b{}_bn^a\nabla_aK^i{}_i$$
$$+\frac{1}{9}K^i{}_in^a\nabla_aK^j{}_j-\frac{4}{9}n^a\nabla_a\mathcal{R}^b{}_b+\frac{1}{36}K^b{}_bK^c{}_cn^a\nabla_a\phi$$
$$-\frac{1}{18}K^b{}_bK^i{}_in^a\nabla_a\phi$$
$$+\frac{1}{36}K^i{}_iK^j{}_jn^a\nabla_a\phi+\frac{1}{3}\nabla_a\nabla^aK^i{}_i+\frac{4}{9}n^a\nabla_a\nabla_b\nabla^b\phi$$
$$+\frac{1}{3}\nabla_aK^b{}_b\nabla^a\phi$$
$$-\frac{1}{3}\nabla_aK^i{}_i\nabla^a\phi+\frac{1}{4}K^i{}_in^a\nabla_bK_a{}^b-\frac{1}{9}\nabla_b\nabla_aK^{ab}$$
$$+\frac{2}{9}n^an^b\nabla_b\nabla_aK^c{}_c$$
$$+\frac{8}{9}K^{ab}\nabla_b\nabla_a\phi-\frac{8}{9}K^c{}_cn^an^b\nabla_b\nabla_a\phi-\frac{2}{9}\nabla_b\nabla^bK^a{}_a$$
$$+\frac{1}{9}n^an^b\nabla_b\nabla_cK_a{}^c$$
$$-\frac{1}{8}H_a{}^{ci}H_{bci}n^a\nabla^b\phi+n^a\mathcal{R}_{ab}\nabla^b\phi-n^a\nabla_b\nabla_a\phi\nabla^b\phi$$
$$-\frac{1}{8}H^{bci}n^a\nabla_cH_{abi}$$
$$-\frac{1}{8}H_a{}^{bi}n^a\nabla_cH_b{}^c{}_i-\frac{1}{4}H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}$$
$$-\frac{3}{4}n^an^b\nabla_a\phi\nabla_cK_b{}^c+\frac{8}{9}n^an^bn^c\nabla_c\mathcal{R}_{ab}$$
$$+n^an^bn^c\nabla_a\phi\nabla_c\nabla_b\phi-\frac{8}{9}n^an^bn^c\nabla_c\nabla_b\nabla_a\phi$$

$$+\frac{1}{9}n^a n^b \nabla_b K_{ac}\nabla^c\phi - \frac{1}{6}K^b{}_b n^a \nabla_i K_a{}^i$$
$$+\frac{1}{4}n^a n^b \nabla_a\phi\nabla_i K_b{}^i + \frac{1}{8}H_a{}^{ci}H_{bci}K^{ab}$$
$$-\frac{1}{3}n^a n^b \nabla_b\nabla_a K^i{}_i\Big]. \tag{67}$$

The form of the couplings are not unique. If one chooses another scheme for the independent couplings in (54), then the form of the above four multiplets would be changed. In other words, by using the total derivative terms and various Bianchi identities and the identities corresponding to the unite vector n^μ, one can rewrite the above couplings in various other forms. However, there is always four multiplets.

4 Discussion

In this paper we have shown that imposing the gauge symmetry on the world-volume couplings of O$_p$-plane in type II superstring theories, one finds at least 48 independent NS–NS couplings with arbitrary coefficients. We then reduce the theory on a circle to impose the T-duality on these couplings. We find that the T-duality constraint fixes all 48 parameters in terms of one overall factor. The T-duality, however, is not fully satisfied because one finds some total derivative terms in the base space if the O$_p$-plane is extended to the boundary. We have also shown that the bulk couplings that the gauge symmetry and the T-duality fix are consistent with the S-duality, again, up to some total derivative terms. Using the Stokes's theorem, one realizes that the presence of the residual total derivative terms dictates that there must be some couplings on the boundary of O$_p$-plane as well.

We have shown that imposing the gauge symmetry on the couplings in the boundary of O$_p$-plane, one finds at least 78 independent NS–NS couplings with arbitrary coefficients. We then impose the T-duality on these couplings and add the residual total derivative terms from the T-duality of bulk couplings. The T-duality then fixes the 78 parameters in terms of the overall factor of the bulk couplings and 17 other parameters. We then impose the S-duality constraint on the remaining couplings and add the residual total derivative terms from the S-duality of the bulk couplings. The S-duality finally fixes the boundary couplings up to 3 parameters and up to the overall factor of the bulk couplings. The final result for the bulk and boundary couplings in the string frame are

$$\mathbf{S}_p + \partial\mathbf{S}_p = -\frac{T_p\pi^2\alpha'^2}{48}\Bigg[\int_{M^{(p+1)}} d^{p+1}\sigma\, e^{-\phi}\sqrt{-\tilde{g}}\,\mathcal{L}_p + \int_{\partial M^{(p+1)}} d^p\tau\, e^{-\phi}\sqrt{|\hat{g}|}\,\partial\mathcal{L}_p\Bigg] \tag{68}$$

where $\mathcal{L}_p$ is

$$\mathcal{L}_p = a_{28}\Big[-\frac{3}{4}H_a{}^{cj}H^{abi}H_b{}^d{}_j H_{cdi}$$
$$-\frac{3}{2}H_{ab}{}^j H^{abi}H_{cdj}H^{cd}{}_i + H_a{}^{cj}H^{abi}H_{bc}{}^k H_{ijk}$$
$$+\frac{3}{2}H_{ab}{}^j H^{abi}H_i{}^{kl}H_{jkl} - \frac{1}{4}H_i{}^{lm}H^{ijk}H_{jl}{}^n H_{kmn}$$
$$+6H^{abi}H^{cd}{}_i R_{abcd}$$
$$-6H^{abi}H_i{}^{jk}R_{abjk} - 6R_{abcd}R^{abcd} + 6R_{abij}R^{abij}$$
$$-6H_a{}^{ci}H_{bci}\mathcal{R}^{ab} + 12\mathcal{R}_{ab}\mathcal{R}^{ab} + 9H_{abj}H^{ab}{}_i\mathcal{R}^{ij}$$
$$-3H_i{}^{kl}H_{jkl}\mathcal{R}^{ij} - 12\mathcal{R}_{ij}\mathcal{R}^{ij}$$
$$+\nabla_a H_{ijk}\nabla^a H^{ijk} - 3\nabla_c H_{abi}\nabla^c H^{abi}$$
$$+2\nabla_i H_{abc}\nabla^i H^{abc}\Big]. \tag{69}$$

The boundary Lagrangian $\partial\mathcal{L}_p$ is given in (67). The bulk Lagrangian $\mathcal{L}_p$ is consistent with linear T-duality without using total derivative terms, however, the boundary Lagrangian is not. Since one is free to add total derivative terms to the boundary, one can write (67) in other forms as well. The form of the boundary Lagrangian which is consistent with the linear T-duality is

$$\partial\mathcal{L}_p = a_{28}\Big[16\mathcal{D}_a\nabla^a\bar{K} - \frac{136}{27}\bar{K}^3 - \frac{28}{3}\bar{K}^2\mathcal{K}$$
$$-6H_a{}^{ci}H_{bci}K^{ab} + 8\bar{K}K_{ab}K^{ab} + 12\mathcal{K}K_{ab}K^{ab}$$
$$-32K_a{}^c K^{ab}K_{bc} + 9H_{abj}H^{ab}{}_i K^{ij} - 3H_i{}^{kl}H_{jkl}K^{ij}$$
$$-24\bar{K}K_{ij}K^{ij} - 12\mathcal{K}K_{ij}K^{ij}$$
$$-16K_i{}^k K^{ij}K_{jk} - 6H_a{}^{ci}H_{bci}\bar{K}n^a n^b$$
$$+12H_{ac}{}^i H_{bdi}K^{cd}n^a n^b$$
$$-12H_a{}^c{}_i H_{bcj}K^{ij}n^a n^b$$
$$-48K^{cd}n^a n^b R_{acbd} + 24K^{ab}\mathcal{R}_{ab} + 8\bar{K}n^a n^b\mathcal{R}_{ab}$$
$$-24K^{ij}\mathcal{R}_{ij} - \frac{16}{3}\bar{K}n^a\nabla_a\bar{K}$$
$$-16n^a n^b\nabla_b\nabla_a\bar{K} + 12H_a{}^{di}n^a n^b n^c\nabla_c H_{bdi}\Big]$$
$$+b_{52}\Big[\frac{37}{162}\bar{K}^3 + \frac{31}{36}\bar{K}^2\mathcal{K} + \bar{K}\mathcal{K}^2 + \frac{1}{3}\mathcal{K}^3$$
$$+\frac{1}{8}H_a{}^{ci}H_{bci}K^{ab} + \frac{1}{6}\bar{K}K_{ab}K^{ab} + \frac{1}{4}\mathcal{K}K_{ab}K^{ab}$$
$$+\frac{1}{6}\bar{K}K_{ij}K^{ij} + \frac{1}{4}\mathcal{K}K_{ij}K^{ij} - \frac{1}{24}H_a{}^{ci}H_{bci}\bar{K}n^a n^b$$
$$-\frac{1}{8}H^{bci}n^a\nabla_c H_{abi} - \frac{1}{8}H_a{}^{bi}n^a\mathcal{D}_c H_b{}^c{}_i$$
$$-\frac{1}{4}H_a{}^{di}n^a n^b n^c\nabla_c H_{bdi}\Big]$$
$$+b_{53}\Big[-\frac{13}{81}\bar{K}^3 - \frac{7}{18}\bar{K}^2\mathcal{K} + \frac{1}{3}\mathcal{K}^3 - \frac{1}{4}H_a{}^{ci}H_{bci}K^{ab}$$
$$-\frac{1}{3}\bar{K}K_{ab}K^{ab} - \frac{1}{2}\mathcal{K}K_{ab}K^{ab}$$
$$-\frac{1}{3}\bar{K}K_{ij}K^{ij} - \frac{1}{2}\mathcal{K}K_{ij}K^{ij} + \frac{1}{12}H_a{}^{ci}H_{bci}\bar{K}n^a n^b$$
$$+\frac{1}{4}H^{bci}n^a\nabla_c H_{abi}$$

$$
\begin{aligned}
&+\frac{1}{4}H_a{}^{bi}n^a\mathcal{D}_cH_b{}^c{}_i+\frac{1}{2}H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}\Big]\\
&+b_{67}\Big[-\frac{1}{54}\bar{K}^3-\frac{1}{36}\bar{K}^2\mathcal{K}-\frac{1}{2}\bar{K}K_{ab}K^{ab}\\
&-\frac{3}{4}\mathcal{K}K_{ab}K^{ab}+\frac{1}{6}\bar{K}K_{ij}K^{ij}+\frac{1}{4}\mathcal{K}K_{ij}K^{ij}\\
&-\frac{2}{3}\bar{K}n^an^b\mathcal{R}_{ab}-\mathcal{K}n^an^b\mathcal{R}_{ab}-\frac{2}{9}\bar{K}n^a\nabla_a\bar{K}\\
&-\frac{1}{3}\mathcal{K}n^a\nabla_a\bar{K}+\frac{1}{8}H_a{}^{ci}H_{bci}K^{ab}\\
&-\frac{1}{24}H_a{}^{ci}H_{bci}\bar{K}n^an^b-\frac{1}{8}H^{bci}n^a\nabla_cH_{abi}\\
&-\frac{1}{8}H_a{}^{bi}n^a\mathcal{D}_cH_b{}^c{}_i-\frac{1}{4}H_a{}^{di}n^an^bn^c\nabla_cH_{bdi}\Big]
\end{aligned}
\tag{70}
$$

where $\mathcal{D}_a \equiv \nabla_a - \nabla_a\Phi$ is dilaton-derivative and $\mathcal{R}_{\mu\nu} \equiv \widetilde{G}^{\rho\sigma}R_{\rho\mu\sigma\nu} + \nabla_\mu\nabla_\nu\phi$ is dilaton-Riemann which are consistent with the linear T-duality [27]. We have also defined $\bar{K} \equiv K^a{}_a - K^i{}_i$ and $\mathcal{K} \equiv K^i{}_i - n^a\nabla_a\phi$ which are also invariant under the linear T-duality. Note that while there is derivative of Riemann curvature in (67), this term has been cancelled in the above form of the boundary couplings by using the boundary total derivative terms. The gauge invariant action (68) is fully invariant under T-duality and is consistent with S-duality up to some terms in the boundary of boundary which are zero. There are four parameters in the above action, i.e., $a_{28}, b_{52}, b_{53}, b_{67}$. For $a_{28} = -\frac{1}{6}$, the bulk couplings are consistent with the PR_2-level or disk-level S-matrix element of two NS–NS vertex operator [28,37].

At the leading order of α', the bulk action is given by the DBI action and there is no boundary couplings. However, the action (68) indicates that at order α'^2, there are more couplings in the boundary than in the bulk. One may expect that this is a general feature of boundary couplings at the higher orders of derivative. Moreover, a general feature of higher derivative couplings is that they are depend on the scheme [38]. The metric couplings in the bosonic string theory in a particular scheme at order α' is given by the Gauss-Bonnet couplings. The corresponding boundary couplings have been found in [39]. It is known how to include the B-field and dilaton to the bulk couplings [21,38], however, it is not known how to include these fields in the boundary. It would be interesting to use the gauge symmetry and T-duality constraint to find the boundary action in the bosonic string theory at order α' which includes the metric, B-field and dilaton, as in (68).

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Appendix: Stokes's theorem

In this appendix we use the Stokes's theorem to find the formulas (34) and (50) that we have used in this paper (see Appendix E in [40] for more details). For an D-dimensional spacetime manifold M with boundary ∂M, the Stokes's theorem is the following:

$$\int_M d\omega^{(D-1)} = \int_{\partial M}\omega^{(D-1)} \tag{71}$$

where $\omega^{(D-1)}$ is an arbitrary $(D-1)$-form. If one chooses it as $\omega^{(D-1)} = *A^{(1)}$ where $A^{(1)}$ is a one-form, and uses $x^0, \ldots, x^{D-1}$ as the spacetime coordinates, then in terms of x-components, one has

$$
\begin{aligned}
\omega_{\mu_1\cdots\mu_{D-1}} &= A^\nu\epsilon_{\mu_1\cdots\mu_{D-1}\nu}\\
(d\omega)_{\lambda\mu_1\cdots\mu_{D-1}} &= \nabla_{[\lambda}A^\nu\epsilon_{\mu_1\cdots\mu_{D-1}]\nu}
\end{aligned}
\tag{72}
$$

where $\epsilon^{(D)}$ is the volume-form of the spacetime manifold M. On the other hand, since $d\omega^{(D-1)}$ is an D-form, it can be written as $d\omega^{(D-1)} = h\epsilon^{(D)} = *h$ where h is a 0-form. Using the fact that $** = 1$, one finds the function h is $h = **h = *d\omega^{(D-1)} = \epsilon^{\lambda\mu_1\cdots\mu_{D-1}}\nabla_{[\lambda}A^\nu\epsilon_{\mu_1\cdots\mu_{D-1}]\nu}$. Using the contraction of two volume-forms, one finds $h = \nabla_\nu A^\nu$. Using the fact that the volume-form in terms of x-coordinates is $\epsilon^{(D)} = \sqrt{|G|}d^Dx$, one can write the integrand on the left-hand side of (71) as

$$d\omega^{(D-1)} = \nabla_\nu A^\nu\sqrt{|G|}d^Dx. \tag{73}$$

On the right-hand side of (71), one can write $\omega^{(D-1)}$ in terms of the volume-form of the boundary space, i.e., $\omega^{(D-1)} = g\hat{\epsilon}^{(D-1)} = \hat{*}g$. Using the fact that $\hat{*}\hat{*} = 1$, one can write $g = \hat{*}\hat{*}g = \hat{*}\omega^{(D-1)}$. Then using the relation between x-components of volume-forms $\epsilon^{(D)}$ and $\hat{\epsilon}^{(D-1)}$, i.e., $\hat{\epsilon}^{\mu_1\cdots\mu_{D-1}} = n_\lambda\epsilon^{\lambda\mu_1\cdots\mu_{D-1}}$ where n^μ is unite vector orthogonal to the boundary, one finds the 0-form g to be $g = n_\lambda A^\lambda$. On the other hand, using the fact that the boundary volume-form is $\hat{\epsilon}^{(D-1)} = \sqrt{|\gamma|}d^{D-1}y$ where γ is determinate of induced metric on the boundary and the boundary with coordinates $y^0, \ldots, y^{D-2}$ is specified by the functions $x^\mu = x^\mu(y)$, one can write the integrand on the right-hand side of (71) as

$$\omega^{(D-1)} = n_\nu A^\nu \sqrt{|\gamma|} d^{D-1} y. \tag{74}$$

Replacing (73) and (74) in (71), one finds the Stokes's theorem in terms of x-components is

$$\int_M \nabla_\nu A^\nu \sqrt{|G|} d^D x = \int_{\partial M} n_\nu A^\nu \sqrt{|\gamma|} d^{D-1} y. \tag{75}$$

This is the formula that we have used in (34).

For the boundary ∂M, the Stokes's theorem is the following:

$$\int_{\partial M} d\Omega^{(D-2)} = \int_{\partial\partial M} \Omega^{(D-2)} \tag{76}$$

where $\Omega^{(D-2)}$ is an arbitrary $(D-2)$-form. Since boundary of boundary is zero, i.e., $\partial\partial M = 0$, the right-hand side this time is zero.

If one chooses $\Omega^{(D-2)} = *F^{(2)}$ where $F^{(2)}$ is a two-form, then in terms of x-components, one has

$$\begin{aligned} \Omega_{\mu_1\cdots\mu_{D-2}} &= F^{\alpha\beta}\epsilon_{\mu_1\cdots\mu_{D-2}\alpha\beta} \\ (d\Omega)_{\lambda\mu_1\cdots\mu_{D-2}} &= \nabla_{[\lambda} F^{\alpha\beta}\epsilon_{\mu_1\cdots\mu_{D-2}]\alpha\beta}. \end{aligned} \tag{77}$$

Since $d\Omega^{(D-2)}$ is an $(D-1)$-form, it can be written as $d\Omega^{(D-2)} = k\hat{\epsilon}^{(D-1)} = \hat{*}k$ where k is a 0-form. Using the fact that $\hat{*}\hat{*} = 1$, one finds the function k is $k = \hat{*}\hat{*}k = \hat{*}d\Omega^{(D-2)} = \hat{\epsilon}^{\lambda\mu_1\cdots\mu_{D-2}}\nabla_{[\lambda}F^{\alpha\beta}\epsilon_{\mu_1\cdots\mu_{D-1}]\alpha\beta}$. Then using the relation between x-components of volume-forms $\epsilon^{(D)}$ and $\hat{\epsilon}^{(D-1)}$, i.e., $\hat{\epsilon}^{\mu_1\cdots\mu_{D-1}} = n_\lambda\epsilon^{\lambda\mu_1\cdots\mu_{D-1}}$, and using the contraction of two volume-forms, one finds $k = n_\alpha\nabla_\beta F^{\alpha\beta}$. On the other hand, using the relation for the boundary volume-form $\hat{\epsilon}^{(D-1)} = \sqrt{|\gamma|}d^{D-1}y$, one can write the Stokes's theorem in the boundary as

$$\int_{\partial M} n_\alpha \nabla_\beta F^{\alpha\beta}\sqrt{|\gamma|} d^{D-1} y = 0 \tag{78}$$

where $F^{\alpha\beta}$ is an arbitrary antisymmetric tensor. This is the formula that we have used in (50).

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