Real classical geometry with arbitrary deficit parameter(s) $\alpha(I)$ in deformed Jackiw–Teitelboim gravity

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Abstract An interesting deformation of Jackiw–Teitelboim (JT) gravity has been proposed by Witten by adding a potential term $U(\phi)$ as a self-coupling of the scalar dilaton field. During calculating the path integral over fields, a constraint comes from integration over ϕ as $R(x) + 2 = 2\alpha\delta(\vec{x} - \vec{x}')$. The resulting Euclidean metric suffered from a conical singularity at $\vec{x} = \vec{x}'$. A possible geometry is modeled locally in polar coordinates (r, φ) by $ds^2 = dr^2 + r^2 d\varphi^2$, $\varphi \cong \varphi + 2\pi - \alpha$. In this letter we show that there exists another family of “exact” geometries for arbitrary values of the α . A pair of exact solutions are found for the case of $\alpha = 0$. One represents the static patch of the AdS and the other one is the non-static patch of the AdS metric. These solutions were used to construct the Green function for the inhomogeneous model with $\alpha \neq 0$. We address a type of phase transition between different patches of the AdS in theory because of the discontinuity in the first derivative of the metric at $x = x'$. We extended the study to the exact space of metrics satisfying the constraint $R(x) + 2 = 2 \sum_{i=1}^k \alpha_i \delta^{(2)}(x - x'_i)$ as a modulus diffeomorphisms for an arbitrary set of deficit parameters $(\alpha_1, \alpha_2, \dots, \alpha_k)$. The space is the moduli space of Riemann surfaces of genus g with k conical singularities located at x'_k , denoted by $\mathcal{M}_{g,k}$.

1 Introduction

One of the most exciting branches of the current research is finding duality between an exactly solvable lower dimensional quantum system and certain types of the bulk theories with gravity. This duality was inspired by AdS/CFT where at specific regimes of the coupling, certain types of the string theories (with gravity) are mapped to strongly coupled large N CFT theories on the flat boundary of the AdS spacetime [1]. Depending on the gravitational sector in the bulk, the resultant boundary quantum theory could be differ-

ent from the CFT one, and the bulk could be a non-black-hole background. Some basic deformations of AdS/CFT are dualities between asymptotically AdS spaces and QFTs with UV fixed points. For example pure AdS is dual to zero temperature CFTs, black holes on AdS are duals to finite temperature CFTs, while the AdS soliton is dual to QFT with mass gap. The idea of making exact duality between realistic condensed matter systems (exactly solvable) and black-hole physics work was initiated with the simple Kitaev model [2] and later its dual bulk theory investigated widely in [3,4]. For certain types of exact solvable Kitaev models, the holographic bulk was investigated via gauge/gravity duality. In Ref. [5], the authors demonstrated the duality between random Hermitian matrix theory (RMT) and the corresponding $2d$ bulk theory with gravity. Gravity in $2d$ is very special and has been well studied in the past by many authors; see for example [6–15]. A reason for such a wide study was firstly its ultraviolet (UV) divergence free form along with its simplicity and secondly its viable physical interpretations in the context of the string theory. Gravity in $2d$ can be understood as a dilaton gravity as well as a natural reduction of the standard classical general relativity (GR) action from higher D gravities to $D = 2$; see e.g. [16]. Furthermore, Euclidean fully integrable forms of the theory in $2d$ were studied in [17].

JT gravity is a family of scalar field theories where the scalar field (dilaton) ϕ coupled to gravity in two dimensions is a minimal theory of gravity in $2d$ [18,19]. It is possible to remove UV divergences in two dimensional bulk theory for gravity; as a result it can be considered as a “toy” model for quantum gravity. A rigorous derivation of the pure JT gravity action can be easily done using the trick of conformal transformations in D dimensions and then taking the limit of $D \rightarrow 2$ [20].

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The Euclidean action of the pure JT gravity with a negative cosmological constant is represented by

$$S = -\frac{1}{2} \int_{\Omega} d^2x \sqrt{g} (\phi R + 2\phi) + \partial S_{bdy} \quad (1)$$

where ∂S_{bdy} stands for any boundary topological term (or the usual GHY boundary term), e.g. the Euler characteristic, R is the Ricci scalar curvature of the Euclidean metric tensor $g_{\mu\nu}$, $\mu, \nu = 0, 1$. By Ω we mean a region of the spacetime with (without) boundary. It has been demonstrated that the JT gravity action given in (1) enjoys several interesting features from symmetry up to solvability of the equations of motion (EoMs); see for example the work done by Refs. [21–24]. If one adds nontrivial couplings between the dilaton and the Abelian 1-form, the model still shows several physical properties as an exact solvable model [25]. In Ref. [26] the authors studied defects in JT gravity holographically by studying the deformation of the Schwarzian theory as the dual quantum boundary action.

Very recently a deformation of the pure JT gravity was proposed by Witten [27]. The model still has the RMT dual as a boundary gauge theory in a similar manner to the original JT gravity [28] (see [29] also for the RMT dual and the relation between critical $3d$ gravity and JT). Witten investigated a simple deformation of JT gravity by adding a potential term $U(\phi)$ as a self-coupling of the scalar dilaton field. The model reduced to pure JT and furthermore a remarkable observation was that the density of energy levels is different from the pure JT. The action for deformed JT (dJT) as proposed in Ref. [28] takes the form

$$S = -\frac{1}{2} \int_{\Omega} d^2x \sqrt{g} (\phi R + U(\phi)). \quad (2)$$

In this work we will study the above model as proposed in Ref. [28] for $U(\phi) = 2\phi + W(\phi)$, where $W(\phi)$ is

$$W(\phi) = 2 \sum_{i=1}^r \epsilon_i \exp\{-\alpha_i \phi\}, \quad \pi < \alpha < 2\pi \quad (3)$$

To write the above potential function, we assume that the potential function $U(\phi) \sim 2\phi$, $\phi \rightarrow +\infty$. It is required to get JT theory in the asymptotic limit $\phi \rightarrow \infty$. One possible constraint to satisfy the above requirement is to restrict the potential to be as follows:

$$\lim_{\phi \rightarrow +\infty} |\phi^{1+\delta} (U(\phi) - 2\phi)| < 1, \quad \delta \geq 1. \quad (4)$$

One possible simple form for such potential function is the one written in Eq. (3). A remarkable program for the quantization of the JT gravity like theories and viable higher order corrections to it widely studied in the in Refs. [30–34]. Some new exact solutions for dJT were studied recently in [35] in favor of Maldacena's duality conjecture and boundary Schwarzian theories. In Ref. [35] we showed that how pure AdS seed metric for pure JT gravity will be deformed in the

dJT. In this work we continued our study about dJT. The problem we want to address here is how this perturbative potential (3) will deform the pure JT gravity bulk geometry. The problem statement will be clearly present in Sect. 2. It will be formulated on a conformal gauge for Euclidean metric as a nonlinear PDE.

The structure of the paper is as follows. In Sect. 2 we formulate the problem of the deformed singular metrics with a single deficit parameter α . In Sect. 3, we formulate weak problem via integral balance technique. In addition, we construct Green function as a solution for the nonlinear PDE formulated in Sect. 4. Moreover, in Sect. 5, we introduce the phase transition from AdS to AdS in dJT. In Sect. 6 black-hole solutions studied with more than one deficit parameters. In Sect. 7 a brief discussion given about time-dependent geometries with singularity. We finally conclude our results in the last section.

2 Problem statement

The Euclidean action for dJT theory Eq. (2) with potential function (3) for $\epsilon_i = 0$ coincides to the pure JT gravity. The aim is to compute the Euclidean path integral (EPI) for first order $\mathcal{O}(\epsilon)$ perturbatively. Following the assumptions made by Witten, it is possible to take the bulk action of order ϵ for a typical exponential dilatonic potential given by $U(\phi) = 2\epsilon e^{-\alpha\phi}$ in the following form:

$$I = I_{JT} - \epsilon \int \sqrt{g} d^2x e^{-\alpha\phi} \quad (5)$$

There are higher order terms of $\mathcal{O}(\epsilon^2)$, $n \geq 2$ will be considered if one is interested to see the effects of higher orders. For simplicity we kept the term of order one. The EPI is explicitly written in a perturbative form as follows:

$$\begin{aligned} \text{EPI} = & \int D\phi Dg \exp\{-I_{JT}\} + \epsilon \int D\phi Dg \exp\{-I_{JT}\} \\ & \times \int d^2x_1 \sqrt{g(x_1)} e^{-\alpha\phi} + \mathcal{O}(\epsilon^2) \end{aligned} \quad (6)$$

If we only consider the pure JT gravity, the EPI reduces to the partition function of the JT gravity. The first order correction needs an evaluation of an integral in the following form (after a normalization to the volume):

$$\begin{aligned} & \int D\phi Dg \exp\left\{\frac{1}{2} \int d^2x_2 \sqrt{g(x_2)} (R + 2)\phi(x_2)\right\} \\ & \times \int d^2x_1 \sqrt{g(x_1)} e^{-\alpha\phi(x_1)} \end{aligned} \quad (7)$$

The trick to calculate the following integral is to write $\phi(x_2) = \int d^2x_1 \sqrt{g(x_1)} \phi(x_1) \delta(x_1 - x_2)$ where the Dirac

delta function is defined as

$$\int \sqrt{g(x_1)} d^2 x_1 \delta(x_1 - x_2) = 1. \quad (8)$$

Using this field representation, we have

$$\int d^2 x_1 \sqrt{g(x_1)} \int D\phi Dg \exp \left\{ \frac{1}{2} \int d^2 x_2 \sqrt{g(x_2)} \phi(x_2) \times (R(g(x_2)) + 2 - 2\alpha \delta(x_2 - x_1)) \right\} \quad (9)$$

We interchange the integration orders, firstly by taking the integral over field ϕ ,

$$\int Dg \delta \left(\frac{1}{2} (R(g(x_2)) + 2 - 2\alpha \delta(x_2 - x_1)) \right) \quad (10)$$

The above delta integral can be reduced to a simpler form via the following functional delta function formula:

$$\int Dg \delta(f(g)) = \int Dg \sum_{i=1} \frac{\delta f(g)}{\delta g} \Big|_{g=g_i} \delta(g - g_i); \quad (11)$$

the functional in our case is $f(g) = \frac{1}{2} (R(g(x_2)) + 2 - 2\alpha \delta(x_2 - x_1))$ and the root g_i lies on the hypersurface given by $R(g_i(x_2)) + 2 - 2\alpha \delta(x_2 - x_1)$ in the functional space. Furthermore $\frac{\delta f(g)}{\delta g} \Big|_{g=g_i} = \frac{\delta R}{\delta g} \Big|_{g=g_i} = R^{\mu\nu} \Big|_{g=g_i} \delta g_{\mu\nu}$. Using the above consideration one can complete the partition function steps adequately. The constraint appearing here defines a specific geometry for the metric $g_{\mu\nu}(x_2)$. If we can solve the following partial differential equation (PDE) for $\alpha \neq 0$, we can find the specific geometry for the hypersurface. This is the main task of our paper and we formulate the problem as follows.

Problem 1 Find all exact two dimensional geometries satisfying the constraint equation:

$$R(x) + 2 = 2\alpha \delta(x - x'), \quad \alpha \neq 0. \quad (12)$$

The resulting geometries suffer from a conical singularity at $x = x'$.

In Witten's paper, it has been claimed that the geometry "can be modeled locally" by a conical flat geometry,

$$ds^2 = dr^2 + r^2 d\varphi^2, \quad \varphi \cong \varphi + 2\pi - \alpha, \quad (13)$$

and later it has been claimed that "there is no real classical geometry" for $\alpha > 2\pi$. In this paper we show that there is a classical geometry for any arbitrary value of the α . The problem of finding a solution to this problem reduces to constructing a proper Green function for a non-linear operator. Before we solve the above problem and find an "exact" non-trivial geometry for it, in the next section we will give a physical interpretation to the deficit parameter α via the method of integral balance.

3 The integral balance method and the meaning of α

The method of integral balance is to formulate the weak problem for a given general PDE [36]. Indeed, we know that there is a naive connection between an integral balance and the associated differential operator equation. It is worth mentioning here that the method of integral balance is more fundamental and can only be inverted into a PDE form, when the field functions are sufficiently smooth. Because we need the explicit form of the PDE for our problem formulated in the previous section, i.e., (12), we adopt an Euclidean two dimensional geometry in the following null coordinates:

$$ds^2 = e^{\psi(u,v)} du dv, \quad u = z + t, \quad v = z - t; \quad (14)$$

here t is the Euclidean time. An explicit expression for the Ricci scalar is $R = -4e^{-\psi} \partial_u \partial_v \psi$. Because we have to satisfy the normalization condition given in Eq. (8), we multiply both sides of Eq. (12) by the factor $\sqrt{g} = \frac{1}{2} e^{\psi(u,v)}$, and, by dropping the factor 2, we obtain the following nonlinear PDE:

$$\underbrace{\frac{e^{\psi}}{2}}_{\sqrt{g}} - \partial_u \partial_v \psi = \alpha \underbrace{\frac{e^{\psi}}{2}}_{\sqrt{g}} \delta(u - u') \delta(v - v'). \quad (15)$$

Now the Problem simplifies to finding a non-trivial solution for the non-linear inhomogeneous PDE Eq. (15). Actually the solution is nothing but the Green function for any "arbitrary value of the deficit parameter α ". We use the term of the Green equation and consequently the non-trivial solution for the above PDE is basically a Green function defined in the following form:

$$\hat{O}G(x|x') = \frac{\alpha}{2} \exp\{G(x|x')\} \delta(u - u') \delta(v - v') \quad (16)$$

where the non-linear differential operator $\hat{O}$ is defined as

$$\hat{O}[\dots] \equiv \frac{e^{[\dots]}}{2} - \partial_u \partial_v [\dots] \quad (17)$$

and $\psi \equiv G(x|x')$, $x \equiv (u, v)$. Although the operator is not Hermitian (we will check it later) or linear, we are very lucky to have at least two exact solutions for it. One exact solution for the operator is found in Ref. [35]. That solution corresponds to the pure AdS metric written in the null coordinates. The other solution as we will show in the next section will represent another AdS solution but in the non-static patch (cosmological patch). Both solutions are exact solutions and will be used effectively to find the Green function in the next section. Solving the above PDE is our plan in the next section.

Let us see whether the operator $\hat{O}$ is self-adjoint on the domain Ω_2 as a compact version of the real domain Ω_1 . We know that the usual Euclidean AdS (half plane) coordinates (hyperbolic geometry) are living in the following domain:

$$\Omega_1 = \{z, t \in (0, \infty) \times (-\infty, +\infty)\}; \quad (18)$$

the above non-compact domain will be mapped to another non-compact null domain,

$$\Omega_2 = \{u, v \in (u_-, u_+) \times (v_-, v_+)\}. \quad (19)$$

Basically $|u_{\pm}|, |v_{\pm}| \rightarrow \infty$ but we kept it as some types of "conformal boundaries" for the new mapping domain which actually live in a very far away region. We know that the operator is a self-adjoint operator (or equivalently, a Hermitian operator) if and only if, for a pair of the functions, it satisfies the following integral:

$$\begin{aligned} \int_{\Omega} \sqrt{g} du dv \Phi_1^*(u, v) \hat{O} \Phi_2(u, v) \\ = \int_{\Omega} \sqrt{g} du dv \left(\hat{O} \Phi_1(u, v) \right)^* \Phi_2(u, v). \end{aligned} \quad (20)$$

We simply use the usual definition of the self-adjoint operator on a Hilbert space in the finite-dimensional space, the only difference exists in the integral measure: instead of the flat space we use the covariant volume element. It is illustrative that the partial derivative term in Eq. (17) looks just like a kinetic energy operator, consequently it is a Hermitian term. Meanwhile the first exponential part does not satisfy Eq. (20) conditions. As a result the operator is non-Hermitian (not self-adjoint). But at the end of day, still the operator enjoys some linearity at the level of solutions. That means for a pair of exact solutions for the operator, i.e., the kernel pair functions $\psi_{1,2}(u, v)$,

$$\hat{O} \psi_{1,2}(u, v) = 0, \quad (21)$$

one can show that the solutions remain linear independent even when the operator is nonlinear by itself. In the language of PDEs, the partial Wronskian of the functions is non-zero. The partial Wronskian for solutions, $W_u(\psi_1, \psi_2)$, is

$$W_u(\psi_1, \psi_2) = \psi_1 \partial_u \psi_2 - \psi_2 \partial_u \psi_1. \quad (22)$$

We will prove it later in the next section (4). In this section we only focus on studying a special discrete regime of the above PDE, i.e., the integral balance technique. We integrate the PDE (15) from the lower to the upper boundaries, and we obtain

$$\begin{aligned} \underbrace{\frac{1}{2} \int_{v_-}^{v_+} dv \int_{u_-}^{u_+} e^{\psi} du - \int_{v_-}^{v_+} dv \int_{u_-}^{u_+} du \partial_u \partial_v \psi}_{\text{Entirely space volume}} \\ = \underbrace{\alpha \int_{v_-}^{v_+} dv \int_{u_-}^{u_+} \frac{1}{2} \delta(u - u') \delta(v - v') e^{\psi} du}_{1}. \end{aligned} \quad (23)$$

We assume that $u_- < u' < u_+, v_- < v' < v_+$, although the case of $\leq$ also satisfies our arguments, but it is too risky to work with an integral of the distribution functions over domains where the initial or final points are singularities of the integrand. As we understood ψ is just the Green function,

and it always remains continuous and symmetric under the exchange of $x \rightarrow x'$ and with a discontinuity at the point $x = x'$. Using the mean value theorem, the first term of Eq. (23) is the total volume of the spacetime manifold, i.e., V_{tot} , the right hand side is just simply α (remembering the normalization condition (8)). The second term can be integrated by parts carefully and finally we obtain

$$\begin{aligned} \alpha = V_{tot} - \psi(u_+, v_+) + \psi(u_+, v_-) \\ + \psi(u_-, v_+) - \psi(u_-, v_-). \end{aligned} \quad (24)$$

Equation (24) is considered as a weak problem version of the original (12). In particular, it is suitable to integrate numerically and find the metric profile function ψ . As regards the physical meaning of the α is encoded in (24): the deficit parameter α is the particular volume of the spacetime by excluding the upper $\psi(u_+, v_+)$ and the lower $\psi(u_-, v_-)$ values of the metric function and including the corner sides. We now demonstrate the construction of a weak solution $\psi(u, v)$ that is a continuously differentiable function over the whole plane. The only exception is for discontinuities along a curve $u = \gamma(v)$. Because the solution is smooth on both sides of γ , it is easy to show that it satisfies the PDE (15). We write the weak solution in the following form:

$$\psi(u_-, u_+ | v) = \int_{u_-}^{u_+} du \psi(\zeta, v) d\zeta. \quad (25)$$

By plugging it into (15) and by integration we obtain

$$\begin{aligned} \frac{1}{2} \exp \left\{ \int_{u_-}^{u_+} du \psi(\zeta, v) d\zeta \right\} \\ - \partial_v (\psi(u_+, v) - \psi(u_-, v)) = \alpha. \end{aligned} \quad (26)$$

We need to compute γ . For this purpose we write the weak formulation (26) in the form

$$\begin{aligned} \frac{1}{2} \exp \left\{ \int_{u_-}^{\gamma(v)} \psi(\zeta, v) d\zeta + \int_{\gamma(v)}^{u_+} \psi(\zeta, v) d\zeta \right\} \\ - \partial_v (\psi(u_+, v) - \psi(u_-, v)) = \alpha. \end{aligned} \quad (27)$$

Differentiating the above integrals with respect to γ and using the PDE itself and after performing the integration we can show that the curve γ propagates at uniform speed. The speed is given by

$$\frac{dv}{d\gamma} = \frac{\partial_v \psi(u_+, v) - \partial_v \psi(u_-, v)}{\partial_v \psi(u_-, u_+ | v)}. \quad (28)$$

We have the average of the propagation speeds on the left and right ends. After this interpretation, we go back to solving the PDE (15) and finding the metric functions which are associated with it.

4 Constructing Green function

The metric function $\psi(x, x')$ defines Green functions for the non-linear differential operator $\mathcal{O}$. Explicitly we observe that there are a pair of exact solutions for the homogeneous case, $\mathcal{O}\psi_{1,2} = 0$ given as follows:

$$\psi_1(x) = 2 \log(2|v + u|^{-1}), \quad \psi_2(x) = 2 \log(2|v - u|^{-1}). \quad (29)$$

These solutions are found using a direct ansatz for the solutions $\psi \sim |u \pm v|^n$. After substituting into the field equation (21), we find $n = -2$ for both cases, $\psi_{1,2}$. The proportionality constant can be fitted after a further investigation of the homogeneous PDE. In the above solutions (whose are not unique solutions because of the non-linearity), the first ψ_1 is suitable for AdS boundary regimes $z \rightarrow 0$, while the second one works for non-static cosmological AdS spacetime and is suitable for regions with $z \rightarrow \infty$ (spatial boundary regions). Using the above pair of "exact" modes we can show that the following symmetric preposition is a suggestion for the Green function:

$$G(x|x') = \begin{cases} C\psi_1(u, v)\psi_2(u', v') & \text{if } u < u', v < v', \\ C\psi_1(u', v')\psi_2(u, v) & \text{if } u > u', v > v'. \end{cases} \quad (30)$$

We kept the expression for the Green function in terms of any pair of exact modes, including the hypothetical solutions we presented in Eqs. (29). To fix the parameter C , one needs to integrate both sides of the inhomogeneous PDE (16). in a domain close to the singularity point, $x = x'$, we assume that the Green function remains continuous on the singularity, it provides the following limiting integral constraint via the mean value theorem in calculus:

$$\lim_{\epsilon_a \rightarrow 0} \int_{v' - \epsilon_2}^{v' + \epsilon_2} dv \int_{u' - \epsilon_1}^{u' + \epsilon_1} du \sqrt{g} = 0. \quad (31)$$

The above integral denotes shrinking of the volume enclosed by the singularity point $x = x'$. The other terms in the integration process can be calculated easily, after a simple usage of the Leibniz formula we obtain

$$\begin{aligned} & \psi_1(u', v') \left(\psi_2(u' + \epsilon_1, v' + \epsilon_2) - \psi_2(u' - \epsilon_1, v' + \epsilon_2) \right) \\ & - \psi_2(u', v') \left(\psi_1(u' + \epsilon_1, v' - \epsilon_2) \right. \\ & \left. - \psi_1(u' - \epsilon_1, v' - \epsilon_2) \right) = -C^{-1}. \end{aligned} \quad (32)$$

We apply the Taylor series by assuming that $|\epsilon_a| \ll |x'|$, as follows:

$$\begin{aligned} & \psi_a(u' \pm \epsilon_1, v' \pm \epsilon_2) \\ & = \psi_a(u', v') \pm \epsilon_1 \partial_u \psi_a|_{u', v'} \pm \epsilon_2 \partial_v \psi_a|_{u', v'} + \dots; \end{aligned} \quad (33)$$

we simplify Eq. (32) as follows:

$$C = -\frac{\alpha'}{2W_u(\psi_1, \psi_2)|_{u', v'}} \quad (34)$$

and we urge to redefine the deficit angle $\alpha' \equiv \frac{\alpha}{\epsilon_1}$. By inserting C into the multi-part Green function (59) we finally obtain the metric function associated with Eq. (12),

$$\psi(x, x') = -\frac{\alpha'}{2W_u(\psi_1, \psi_2)|_{u', v'}} \psi_1(u_<, v_<) \psi_2(u_>, v_>) \quad (35)$$

here $x_<, x_>$ refers to x, x' . A non-trivial fully classical metric with deficit parameter can be written in the following explicit form:

$$ds^2 = \exp \left\{ -\frac{\alpha'}{2W_u(\psi_1, \psi_2)|_{u', v'}} \psi_1(u_<, v_<) \psi_2(u_>, v_>) \right\} du dv. \quad (36)$$

Using the pair of the trial solutions (29) one can show that

$$W_u(\psi_1, \psi_2) = \psi_1 \exp \left\{ \frac{\psi_2}{2} \right\} + \psi_2 \exp \left\{ \frac{\psi_1}{2} \right\}, \quad (37)$$

and we end with a smooth (but discontinuous at the singularity point) multi-part metric,

$$\begin{aligned} ds^2 = du dv & \times \begin{cases} \exp \left\{ -\left(\frac{\alpha(v'-u')}{2\epsilon_1 \log(\frac{v'-u'}{v'+u'})} \right) \log(2|v+u|^{-1}) \log(2|v'-u'|^{-1}) \right\} & \text{if } u < u', v < v', \\ \exp \left\{ -\left(\frac{\alpha(v'-u')}{2\epsilon_1 \log(\frac{v'-u'}{v'+u'})} \right) \log(2|v-u|^{-1}) \log(2|v'+u'|^{-1}) \right\} & \text{if } u > u', v > v'. \end{cases} \end{aligned} \quad (38)$$

The metric is continuous at the boundary $x = x'$,

$$g_{\mu\nu}^>|_{x=x'} = g_{\mu\nu}^<|_{x=x'}. \quad (39)$$

However, there is a discontinuity in the first derivative coming from the Green function,

$$\partial_u g_{\mu\nu}^>|_{x=x'} \neq \partial_u g_{\mu\nu}^<|_{x=x'}. \quad (40)$$

This can be interpreted as a discontinuity in the affine connection for the spacetime. The reason is that the non-vanishing components for the Christoffel symbol are given either by $\partial_u \psi$ or $\partial_v \psi$. Note that the Green functions have a discontinuity (or a jump) for both $\partial_{u,v}$. The discontinuity which is located at $\partial_u G(x|x')$ is proportional to (more precisely equals) the Christoffel symbol $\Gamma_{\mu\nu}^t$. Any discontinuity in Γ will be transferred directly to the geodesics of the test particle. Basically we guess the trajectory of a test particle undergoes a critical process. A more interesting interpretation for the discontinuity will be presented in the next section, where we will address a phase transition between metrics which represent the trial solutions ψ_1, ψ_2 .

5 More about the metrics associated to ψ_1, ψ_2

The trial solutions given in Eqs. (29) define two geometries with very interesting features,

$$ds_1^2 = \frac{4dudv}{(u+v)^2}, \quad (41)$$

$$ds_2^2 = \frac{4dudv}{(v-u)^2}. \quad (42)$$

The Ricci scalar is $R = -2$ for both metrics except for the asymptotic regions $u \rightarrow \pm v$ (conformal boundaries). If one writes the metrics in the standard Poincaré coordinates, it represents a half plane metric for $z > 0$. We mention here that any other solution except for AdS is a quotient AdS/Γ , where Γ denotes a discrete subgroup of $\text{SL}(2, \mathcal{R})$ and as a result it is not a unique solution. It is instructive to rewrite the above metrics in the usual Poincaré coordinates (not the Euclidean) where $u = z + i\tau, v = z - i\tau$; one immediately find that the metrics corresponding to the ψ_1 and ψ_2 both represent AdS_2 ,

$$ds_1^2 = \frac{dz^2 + d\tau^2}{z^2}, \quad \text{AdS}, \quad (43)$$

$$ds_2^2 = -\frac{dz^2 + d\tau^2}{\tau^2}, \quad \text{AdS}. \quad (44)$$

The first can be interpreted as AdS_2 in the static path including the AdS boundary, the second one, after a signature change, represents a non-static patch of the AdS (if such patch existed at all). For the first metric, the dual system is supposed to lie on the conformal boundary $z = 0$. Technically the conformal boundary is located in an infinitely far away region. The signature change from first to the second metric shows that the trial solutions belong to different metrics. That implies that, although the PDE for these functions is nonlinear, a somehow "non-linear" independence of the solutions still remains valid. Although one can easily show that the AdS metric with wrong signature (signature changed) can be transformed to the AdS metric (both solutions are obtained as exact solutions for the homogeneous case with $\alpha = 0$), if we let the metric coordinates (z, τ) undergo a complex conformal transformation,

$$(z \rightarrow i\tau, \tau \rightarrow z) \implies (z \rightarrow iz, \tau \rightarrow \tau) \quad (45)$$

$$ds_2^2 = -\frac{dz^2 + d\tau^2}{z^2}. \quad (46)$$

The new metric is considered as the standard AdS which undergoes a signature change. Note that dJT gravity is a diffeomorphism invariance as well as conformal invariance theory, i.e., any signature change of the metric $g_{\mu\nu} \rightarrow -g_{\mu\nu}$ does not change the action. The reason is that under such a transformation, the Ricci scalar remains unchanged because $R = g_{\mu\nu} R^{\mu\nu} \rightarrow R$. But the signature change probably

makes a difference for the dilaton profile. Under such a transformation, the dilaton probably will change.

The signature change from static AdS to the non-static patch AdS, or equivalently from $ds_1^2 \rightarrow ds_2^2$, can be understood by studying the trajectories of a test particle in the background of both metrics. For this purpose we have to write down the set of equations of motion (EoM) for the trajectory. For simplicity we just consider a photon path, basically with a suitable parametrization. We find the trajectories $(\tau(\zeta), z(\zeta))$ by minimization of the following string-like actions:

$$S_1 = \pm \int \frac{d\zeta}{z} \sqrt{\dot{z}^2 + \dot{\tau}^2}, \quad \text{for } ds_1^2, \quad (47)$$

$$S_2 = \pm \int \frac{d\zeta}{\tau} \sqrt{\dot{z}^2 + \dot{\tau}^2}, \quad \text{for } ds_2^2. \quad (48)$$

For both action functionals there are a pair of conserved charges, reading

$$E_1 = \frac{\dot{\tau}}{z\sqrt{\dot{z}^2 + \dot{\tau}^2}}, \quad E_2 = \frac{\dot{z}}{\tau\sqrt{\dot{z}^2 + \dot{\tau}^2}}. \quad (49)$$

Here E_1 has the meaning of an energy while E_2 defines a conserved translational momentum along the z coordinate. One can easily show that the above first integrals are the unique EoM for the trajectories, the second EoM being trivially satisfied. By using the conserved charges (49), we can show that the trajectory of the test particle reduces to the points existing on the Hamilton surface (first integral),

$$\tau^2 - \left(\frac{E_1}{E_2}\right) z^2 = b; \quad (50)$$

here b is constant. Depending on the sign of this parameter, we have the following cases:

- If $b > 0$, $\frac{E_1}{E_2} < 0$, then the trajectory is either elliptic (for $E_1 \neq E_2$) or circular (for $E_1 = E_2$). The trajectory corresponds to non-static patch of the AdS.
- If $b \geq 0$, $\frac{E_1}{E_2} > 0$, then the trajectory is either hyperbolic (for $E_1 \neq E_2$) or a line (for $E_1 = E_2, b = 0$); it is simply the typical trajectory in static patch of the AdS.
- For $b < 0$, one can show that the situation remains the same as the above cases if one performs the coordinate transformations as $\tau \rightarrow z\sqrt{|\frac{E_1}{E_2}|}$.

The signature change can be understood as a change in the photon trajectory from static region in the AdS to non-static (cosmological) patch adequately. If the conserved charges have different signatures, then the trajectory is closed, it corresponds to AdS with a possibility to have minimal surfaces in hyperbolic spaces with negative curvature (that is, AdS). For the case when the conserved charges have different signs, the particle trajectory falls into a non-static patch (or equivalently an open dS universe), there is no minimal surface that

corresponds to the non-static patch of the AdS. We can understand the transition from AdS to AdS as a phase transition. It corresponds to the Green function we obtained. Indeed as we have seen, the derivative of the Green function has a jump discontinuity at $x = x'$. This type of discontinuity is interpreted as a discontinuity in the first derivative of the metric or more precisely the Christoffel symbols. Let us explain it in a more concrete way: The Green function as the response function of the inhomogeneous Witten equation is dual to the two point function of a boundary operator $\hat{O}$. In our case it is related to the probability of measuring a field $\phi(x)$ when the source is at x' . A discontinuity in the metric function implies a jump in the free energy of the system. It is known that the free energy can be written in terms of the thermal Green functions for a bulk/boundary theory. Remember that the Green function is a two point correlation function and the expectation value of the operators can be expressed in terms of the partition function using the path integrals. We postulate that the following discontinuity in the metric for a non-zero deficit parameter α addresses the phase transition from static to cosmological patch of the AdS in dJT at least in a formal form. For a better understanding of the phase transition one should compare the free energy for the bulk theory in dJT for different patches of the AdS. We guess that a more careful calculation will support the argument which we stated here.

6 Black hole solutions

The Green function metric derived in the previous section and its interesting phase transition scenario can be recast into a spherically symmetric Euclidean metric in the Schwarzschild coordinates $x^\mu = (t, r)$. As we learned from study of the exact solutions in the null coordinates, to define the metric for the geometry we need to specify only one gauge function, which here could be a function of (r) as follows:

$$ds^2 = A(r)dt^2 + \frac{dr^2}{A(r)}. \quad (51)$$

Any other representation of the metric with two arbitrary functions, for example $A(r)$, $B(r)$, can be recast into the form of the above case after a suitable reparametrization of the radial (spatial) coordinate r . For the case of a time-dependent metric it is very difficult to reduce the metric to the null coordinate form or eliminate one of the metric functions. A reason is that one cannot deduce a simple form of Birkhoff's theorem for JT gravity. It is very hard to prove that JT gravity possesses only static and asymptotically flat solutions in the absence of any other matter field contents. We do not study the validity of such a fundamental theorem in JT gravity as also we are not sure of such a proof in more general cases of the UV free two dimensional theories for quantum gravity.

If we limit our study to the case where the metric remains time independent, it is easy to show that for the non-singular case, i.e., when $\alpha = 0$, the classical field equations provide the class of exact solutions for the metric found in [28]. Furthermore for a general class of the deformation potentials $U(\phi)$, considering the metric function given by

$$A(r) = \int_{r_h}^r U(r') dr' \quad (52)$$

in the above suggested solution, we assume that there exists a black-hole solution with a null hyperbolic surface (horizon) which is located at $r = r_h$ and $A(r_h) = 0$. The value of the dilaton field $\phi(r_h) = \phi_h$ is kept finite and positive. One can show that the near horizon geometry of the metric is a thermal region with the temperature given by

$$T = \frac{U(\phi_h)}{4\pi}. \quad (53)$$

If one tries to satisfy the JT gravity asymptotically bound by the potential function $U(\phi)$ defined in Eq. (8), one can show that the asymptotic form for the metric in the limit $\phi \rightarrow \infty$ is

$$A(r) = r^2 - b + \mathcal{O}(r^{-\delta}). \quad (54)$$

Using simple arguments, one can show that the first law of thermodynamics still holds if one identifies the parameter b as a quasi-energy $E = \frac{b}{2}$ defined via the Gibbons–Hawking–York surface.

Our aim in this section is to find an exact solution for the Witten equation (12) with a single deficit angle with metric form given by (51). A remarkable observation is that the metric keeps the Dirac function the same as in the flat Euclidean metric. That is because the metric is uni-modular, i.e., $\det g_{\mu\nu} = 1$, the normalization condition for the Dirac delta function being

$$\begin{aligned} & \int \sqrt{g(x_1)} d^2 x_1 \delta(x_1 - x_2) \\ &= \int_{M^2} dr dt \delta(r - r') \delta(t - t') = 1. \end{aligned} \quad (55)$$

Using the spherical symmetry of the metric one can write the Dirac delta function only in a r dependent form,

$$\delta(x - x') = \delta(r - r'), \quad (56)$$

where we implied the time independence of the metric and spherical symmetry. Using the metric, we can compute the Ricci scalar as $R = -A''$, and the equation for the modular space reduces to the following linear ODE:

$$A'' = 2(1 - \alpha \delta(r - r')), \quad \alpha \neq 0, \quad (57)$$

for the single deficit parameter α , and a single conical singularity located at $r = r'$. An exact solution for the metric can

be obtained using simple integration techniques,

$$A(r) = r^2 + c_1 + rc_2 - 2\alpha(r - r')\theta(r - r'). \quad (58)$$

Here $\theta(x)$ is the Heaviside theta (step function),

$$\theta(r - r') = \begin{cases} 0 & \text{if } r < r', \\ 1 & \text{if } r > r'. \end{cases} \quad (59)$$

One can directly show that the above metric function solves the ODE given in Eq. (57), because $(x - x')\delta(x - x') = 0$, and as a result $2\delta(x - x') + (x - x')\frac{d\delta(x - x')}{dx} \equiv \delta(x - x')$. The metric is explicitly written as

$$ds^2 = \begin{cases} (r^2 + c_1 + rc_2)dt^2 + \frac{dr^2}{r^2 + c_1 + rc_2} & \text{if } r < r', \\ (r^2 + c_1 + rc_2 - 2\alpha(r - r'))dt^2 + \frac{dr^2}{r^2 + c_1 + rc_2 - 2\alpha(r - r')} & \text{if } r > r'. \end{cases} \quad (60)$$

The metric suffered from a first jumping singularity at $r = r'$, where the metric radial derivative $\partial_r g_{\mu\nu}|_{r=r'-0} \neq \partial_r g_{\mu\nu}|_{r=r'+0}$. It can be interpreted as a discontinuity in the connections. For the exterior region $r > r'$, we observe that, if we apply the following transformation to the metric within the interior region $r < r'$, the metrics $ds^2_{r > r'}$ will be mapped to the one on the exterior region:

$$c_1 \rightarrow c_1 + 2\alpha r', \quad c_2 \rightarrow c_2 - 2\alpha. \quad (61)$$

Consequently we deduce that the exterior metric can be mapped analytically to the interior metric as follows:

$$ds^2(r < r') \Big[_{c_1 \rightarrow c_1 + 2\alpha r', \quad c_2 \rightarrow c_2 - 2\alpha} \Big] \rightarrow ds^2(r > r'). \quad (62)$$

The singularity sphere with equation $r = r'$ can be identified only with a unique metric for inner or outer regions.

The metric function has the following regular form for both regions but with different values of the metric parameters (c_1, c_2) :

$$g_{tt} = g_{rr}^{-1} = r^2 + c_1 + rc_2. \quad (63)$$

The metric can be considered as a black hole with horizon radius r_h ,

$$r_h = \frac{1}{2}(-c_2 + \sqrt{c_2^2 - 4c_1}), \quad c_1 < 0, c_2 > 0. \quad (64)$$

Note that one can study the near horizon geometry; we have

$$A(r) = \sqrt{c_2^2 - 4c_1}(r - r_h) + \mathcal{O}((r - r_h)^2). \quad (65)$$

We define a new coordinate $z = r - r_h$, and using this coordinate the near horizon geometry is represented as follows:

$$ds^2 \approx \frac{4}{\sqrt{c_2^2 - 4c_1}} \left(dz^2 + \left(\frac{c_2^2 - 4c_1}{4} \right) z^2 dt^2 \right). \quad (66)$$

The metric remains smooth in the vicinity of the $z = 0$, if and only if

$$t \cong t + \frac{4\pi}{\sqrt{c_2^2 - 4c_1}}. \quad (67)$$

The temperature then is defined as $T = \frac{\sqrt{c_2^2 - 4c_1}}{4\pi}$.

For the case with more than one deficit parameter, when $\alpha_I \equiv (\alpha_1, \alpha_2, \dots, \alpha_k)$ and using the metric form as we used in the case with a single deficit parameter α , Witten's equation for the deformed hyperbolic geometry changes to the following form:

$$A'' = 2 \left(1 - \sum_{i=1}^k \alpha_i \delta^{(2)}(x - x'_i) \right). \quad (68)$$

We are lucky to have a linear ODE, using the superposition principle, exact solution for a set of the deficit parameters and singularities can be obtained using the general solution which we obtained for the singular parameter i.e., Eq. (58), as $A_k(r) = \sum_{i=1}^k A_i$ where

$$A_i = \frac{r^2}{k} + c_i^{(1)} + rc_i^{(2)} - 2\alpha_i(r - r'_i)\theta(r - r'_i). \quad (69)$$

The general metric solution for $\mathcal{M}_{g,k}$ is

$$A_k(r) = r^2 + \sum_{i=1}^k \left(c_i^{(1)} + rc_i^{(2)} + 2\alpha_i(r'_i - r)\theta(r - r'_i) \right). \quad (70)$$

It can be understood as a superposition of the single mode solutions, as one of the rare cases where the Einstein gravity still respects the superposition principle. The metric is explicitly written as

$$ds_k^2 = \begin{cases} (r^2 + \sum_{i=1}^k (c_i^{(1)} + rc_i^{(2)}))dt^2 + \frac{dr^2}{r^2 + \sum_{i=1}^k (c_i^{(1)} + rc_i^{(2)})} & \text{if } r < r', \\ (r^2 + \sum_{i=1}^k (c_i^{(1)} + rc_i^{(2)} + 2\alpha_i(r'_i - r))dt^2 + \frac{dr^2}{r^2 + \sum_{i=1}^k (c_i^{(1)} + rc_i^{(2)} + 2\alpha_i(r'_i - r))} & \text{if } r > r'. \end{cases} \quad (71)$$

The second root of the equation $g_{tt}(r) = 0$ is negative and completely will be removed from further considerations.

In the above metric, the set of the integration parameters

$$c_i^{(1)} \rightarrow c_i^{(1)} + 2\alpha r', \quad c_i^{(2)} \rightarrow c_i^{(2)} - 2\alpha \quad (72)$$

under the following transformations the metrics for interior and exterior regions remains unaltered:

$$ds_k^2(r < r') \left[c_i^{(1)} \rightarrow c_i^{(1)} + 2\alpha_i r', \quad c_i^{(2)} \rightarrow c_i^{(2)} - 2\alpha_i \right] \longrightarrow ds_k^2(r > r'). \quad (73)$$

The near horizon geometry and temperature can be recovered as we obtained in the case with a single deficit parameter only if we replace c_1, c_2 as follows:

$$c_1 \rightarrow \sum_{i=1}^k c_i^{(1)}, \quad c_2 \rightarrow \sum_{i=1}^k c_i^{(2)}. \quad (74)$$

The temperature for the case with many finite deficit parameters can be written as the following expression:

$$T = \frac{\sum_{i,j=1}^k c_i^{(2)} c_j^{(2)} - 4 \sum_{i=1}^k c_i^{(1)}}{4\pi} \quad (75)$$

where we assumed that $\sum_{i=1}^k c_i^{(1)} < 0$.

7 On singular manifold with time-dependent metrics

In the previous section we investigated exact black-hole solutions when the metric was considered as time independent, i.e., when $A = A(r)$. Under this assumption we integrate a linear ODE with Dirac delta source term and the analysis could be extended to the situation when there is more than one deficit parameter α_i but it still remains finite. In this section we want to see, as we relax the constraint of time independence and allow the metric to be time dependent, how the singular surfaces will be deformed adequately. Firstly we note that although in two dimensions it is possible to reduce the metric to a conformal flat Euclidean metric in general, when the metric is considered time dependent,

$$ds^2 = g_{tt}(t, r) dt^2 + 2g_{tr}(t, r) dt dr + g_{rr}(t, r) dr^2, \quad (76)$$

the metric cannot not be written as a conformal flat form. Let us see what is going to happen if one factorizes the above arbitrary time dependent metric,

$$ds^2 = g_{tt}(t, r) \left(dt + A_r(t, r) dr \right)^2 + \left(g_{rr}(t, r) - \frac{g_{tr}(t, r)^2}{g_{tt}(t, r)} \right) dr^2 \quad (77)$$

where the non-static vector potential (or angular velocity in the terminology of the lower dimensional black holes) is defined as $A_r(t, r) \equiv \frac{g_{tr}(t, r)}{g_{tt}(t, r)}$. If we need a conformal flat form of the above metric, we are required to find a set of new coordinates T, R such that

$$dT(t, r) = dt + A_r(t, r) dr, \quad (78)$$

$$dR(t, r) = \sqrt{g_{rr}(t, r) - \frac{g_{tr}(t, r)^2}{g_{tt}(t, r)}} dr. \quad (79)$$

The first equation is not a regular Pfaffian form. Let us try to make it Pfaffian using an auxiliary function $\mu(t, r)$, where

$$\frac{\partial \mu(t, r)}{\partial r} = \frac{\partial (\mu(t, r) A_r(t, r))}{\partial t}. \quad (80)$$

The aim is to find at least one auxiliary function $\mu(t, r)$ to make (78) a Pfaffian form and to make it possible to integrate (79). After a carefully investigation, we see that the following cases are possible:

- $\mu(t, r) = \mu(t)$: Following this constraint we obtain $\mu(t) A_r(t, r) = C(r)$ and finally it suggests that $\mu(t) = \frac{C(r)}{A_r(t, r)}$. The only possibility is when $A_r(t, r) = A_1(r) A_2(t)$. With this choice of the vector potential, we can show that

$$\frac{g_{tr}(t, r)}{g_{tt}(t, r)} = \frac{C(r)}{A_1(r) A_2(t)} = \frac{\tilde{C}(r)}{A_2(t)}, \quad (81)$$

under these constraints, the first (78) converts to a Pfaffian form,

$$dT(t, r) = \mu(t) dt + C(r) dr = d\Phi(t, r) \quad (82)$$

where the potential function is

$$\Phi(t, r) = \tilde{C}(r) \int \frac{dt}{A_2(t)} + \int dr C(r), \quad (83)$$

then the second integral gives

$$dR(t, r) = dr \sqrt{g_{rr}(t, r) - \left(\frac{\tilde{C}(r)}{A_2(t)} \right)^2 g_{tt}(t, r)}. \quad (84)$$

The above differential form can be integrated only if $g_{rr}(t, r) - \left(\frac{\tilde{C}(r)}{A_2(t)} \right)^2 g_{tt}(t, r) = B(r)$. Using the above set of the new coordinates T, R , the metric reduces to a conformal flat form after defining a set of the null coordinates U, V adequately (see for example [35]).

- $\mu(t, r) = \mu(r)$: In this situation, the factor $\mu(r)$ can be obtained:

$$\mu(r) = \exp \frac{\partial}{\partial t} \int dr A_r(t, r). \quad (85)$$

It becomes meaningful if and only if $A_r(t, r) \propto t$. Under this condition we obtain $\mu(r) = \exp \int dr A_r(r)$. This case also provides a set of appropriate null coordinates.

But as we checked, in general one cannot reduce any two dimensional metric to a flat conformal form.

In dJT, the singular surfaces with time-dependent metric are completely different from the static ones. Let us simply consider a very restricted form of such metrics when the metric is given by a single gauge $A(t, r) = g_{tt} = g_{rr}^{-1}$, $g_{tr} = 0$. Even if we consider the case with one deficit parameter α , Witten's equation for the singular metrics reduces to a nonlinear second order PDE, with a source term $\delta(r - r')\delta(t - t')$. The model can be considered as a nonlinear wave equation in an inhomogeneous medium. There will not be an easy exact solution for the metric function $A(t, r)$, which means one cannot construct (easily) an analytic Green function for such a wave equation in such an inhomogeneous medium. A further study can be done if we provide an appropriate set of the initial conditions for the Green function and integrate numerically the Green function very carefully in the vicinity of the singularity point. Otherwise, we have to find the Green function perturbatively, order by order if any dimensionless perturbation parameter can be found in the model.

8 Conclusion

In this work, we investigated deformed geometries for deformed JT gravity recently proposed by Witten [28]. As claimed by Witten, there should not be any type of classical geometry for deficit angle $\alpha > 2\pi$. We found a class of metric solutions in the null coordinate as a non-trivial solution for the nonlinear PDE formulated as Witten's problem. The metric was expressed as the Green function for the operator. There is a discontinuity in the metric derivative and that implies a type of phase transition between the AdS and AdS metrics. Although Witten's equation for the deformed hypersurface with $\alpha \neq 0$ is nonlinear, we demonstrated that some linear independence still remains in theory. As an attempt to prove the difference between two independent exact solutions to the homogeneous equation, we studied the null geodesics of two metrics. It has been shown that the trajectory of the test particle coincides with the classical trajectories in two different patches of the AdS. Although we did not compute the free energy, a qualitative discussion provides more evidence for a coexistence phase of both patches of the AdS in dJT. It will be very interesting to compare such a formal phase transition with the realistic description which has been investigated in detail in Ref. [28]. The thermodynamical phase transition proposed for dJT in the above reference is based on the study of the free energy of a given exact dilatonic black hole obtained in a suitable scale invariant (gauge fixed) Euclidean form for the metric. In our study a pair of patches for the AdS spacetime surprisingly appeared in the model as exact solutions (with a signature change) to the

homogeneous Green equation. The phase transition from a static patch of the AdS to a non-static one which we proposed in our work can be interpreted as a simple phase transition from an Euclidean AdS to another Euclidean AdS along with a signature change. We have

$$\text{AdS} \Rightarrow \text{Signature-changed AdS}. \quad (86)$$

Because a signature change in the Euclidean (or even Lorentzian) metrics usually occurs when we pass a horizon (or get close to the singularity), the phenomena here look a bit odd and we have to study them in more detail in our forthcoming work. One possible description could be as follows: small quantum fluctuations in the dJT could make a cloud surrounding the AdS black hole. Although it cannot make a horizon, it is possible to make a transitive horizon with a very short life time. Consequently with the metric in the outer region of the cloud described by the usual Euclidean AdS, while one passes the temporal cloudy horizon the metric signature eventually changes. We mention here that the phase transition is described in a purely classical sense using the classical trajectories. We expanded our study by considering black-hole solutions in theory with a single and finite number of deficit parameters α . In the case of a single parameter we found a class of exact static, spherically symmetric two dimensional black-hole solutions with a non-zero temperature and a well behaved horizon. We show that the metric for the exterior region $r > r'$ beyond the singularity point $r = r'$ can be mapped smoothly to the interior metric within the region $r < r'$ if one applies a class of simple algebraic transformations between the set of integration constants c_1, c_2 . When we have more than one deficit parameter, we showed that the equation for the singular surfaces remains linear. Using the superposition principle we construct the finite k metric from k copies of the single singular metric. By constraining the parameters of the metric we show that the horizon remained thermal with a similar temperature for the single hole metric.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is theoretical study and no experimental data has been listed.]

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