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Special Article - Tools for Experiment and Theory

Normal ordering normal modes

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Abstract In a soliton sector of a quantum field theory, it is often convenient to expand the quantum fields in terms of normal modes. Normal mode creation and annihilation operators can be normal ordered, and their normal ordered products have vanishing expectation values in the one-loop soliton ground state. The Hamiltonian of the theory, however, is usually normal ordered in the basis of operators which create plane waves. In this paper we find the Wick map between the two normal orderings. For concreteness, we restrict our attention to Schrodinger picture scalar fields in 1+1 dimensions, although we expect that our results readily generalize beyond this case. We find that plane wave ordered n-point functions of fields are sums of terms which factorize into j-point functions of zero modes, breather and continuum normal modes. We find a recursion formula in j and, for products of fields at the same point, we solve the recursion formula at all j.

1 Introduction

In perturbation theory about a translation-invariant vacuum, it is customary to decompose the quantum fields into operators $a_p^\dagger$ and a_p which create and annihilate plane wave excitations. The free vacuum is annihilated by a_p and is the initial state in the perturbative expansion. This perturbation theory is simplest when the Hamiltonian is normal ordered, so that all $a_p^\dagger$ appear to the left of all a_p.

At the same leading order,[1] the ground state of a quantum soliton is given by a coherent state formed by shifting the fields by the functions corresponding to their classical solutions [3–5]. The normal modes of the quantum soliton are, at linear order, described by quantum harmonic oscillators. The one-loop ground state of the soliton sector consists of the tensor product of the ground states of these oscillators [6]. If the fields are decomposed into the normal modes of the soliton, with operators $b_k^\dagger$ and b_k corresponding to the raising and lowering operators in the corresponding quantum harmonic oscillators, then the one-loop soliton ground state is, after the shift operator noted above, the state annihilated by all of the b_k.

Again a sensible perturbation theory exists which describes the spectrum of the one soliton sector. It is similar to that of the vacuum sector, except that treating zero modes requires special care [7–9]. In particular the calculation is simplest if the Hamiltonian is normal ordered by placing all $b_k^\dagger$ to the left of b_k.

However, the Hamiltonian is usually given normal ordered in terms of plane waves, as is convenient for vacuum sector perturbation theory. Therefore the first step in soliton perturbation theory is to convert plane wave normal ordering to normal mode normal ordering. The goal of the present note is to describe how this can be done in the case of a scalar field theory in 1+1 dimensions. The fact that the theory is scalar and only in 1+1 dimensions does not appear to play a central role in our analysis, and so we expect that the approach in this paper can be trivially generalized to more complicated theories in more dimensions.

We find that the problem of converting plane wave normal ordering into normal mode normal ordering can be achieved in two steps. First, as will be described in Sect. 3, we show that plane wave normal ordered products of the form $:\phi^n(x):_a$ can be decomposed into sums of products of factors of the form $:\phi_M^j(x):_a$ where

$$\phi(x)=\sum_M \phi_M(x) \tag{1.1}$$

is a decomposition into different kinds of normal modes, such as even and odd breather modes. Next, in Sect. 4, we find that these factors can each be converted according to the Wick formula

[1] Even at weak coupling quantum corrections can affect the existence itself of the solution [1,2].

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$$
:\phi_M^j(x):_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \times \frac{j!}{2^m m!(j-2m)!} \mathcal{I}_M^m(x) : \phi_M^{j-2m}(x):_b \tag{1.2}
$$

where the contraction, except for the case of zero modes, is schematically

$$
\mathcal{I}_M(x) = \left\langle \frac{1}{2\omega_k} - \frac{1}{2\omega_p} \right\rangle \tag{1.3}
$$

with ω_k and ω_p the energy of a normal mode and plane wave respectively. Combining the decomposition (1.1) with the Wick formula (1.2) for each component, one arrives at our Wick's theorem

$$
:\phi^j(x):_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} \mathcal{I}^m(x) : \phi^{j-2m}(x):_b
$$

$$
\mathcal{I}(x) = \sum_M \mathcal{I}_M(x) \tag{1.4}
$$

where $\mathcal{I}(x)$ can be found using the identity [9]

$$
\partial_x \mathcal{I}(x) = \int \frac{dk}{2\pi} \frac{\partial_x |g_k(x)|^2}{2\omega_k} + \sum_i \frac{\partial_x |g_i(x)|^2}{2\omega_i} \tag{1.5}
$$

in terms of continuum normal modes $g_k(x)$ and breathers $g_i(x)$ together with the condition that it vanish at as $|x|$ tends to infinity.

We begin in Sect. 2 with a review of our formalism.

2 The setup

In this section we will review the one loop description of kinks developed in Refs. [10–12] using the formalism developed in Refs. [6,13,14], which has the advantage that it resolves the ambiguity noted in Ref. [15]. The key elements of our notation are summarized in Table 1.

For concreteness, we consider a theory of a real scalar field $\phi(x)$ and its canonical momentum $\pi(x)$ in 1+1 dimensions, described by a Hamiltonian

$$
H = \int dx \mathcal{H}(x), \quad \mathcal{H}(x) = \frac{1}{2} : \pi(x)\pi(x) :_a + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) :_a + \frac{M^2}{g^2} : V[g\phi(x)] :_a \tag{2.1}
$$

where M has dimensions of mass and g has dimensions of action$^{-1/2}$. The perturbative expansion will be an expansion in $g^2\hbar$ and we will set $\hbar = 1$. The plane-wave normal-ordering $::_a$ will be defined momentarily.

We assume that the potential V has degenerate minima so that the classical equations of motion admit a time-independent kink solution

$$
\phi(x,t) = f(x). \tag{2.2}
$$

In the Schrodinger picture of the quantum theory, the translation operator

$$
\mathcal{D}_f = \exp\left(-i \int dx f(x) \pi(x)\right) \tag{2.3}
$$

satisfies the identity [13]

$$
: F[\pi(x), \phi(x)] :_a \mathcal{D}_f = \mathcal{D}_f : F[\pi(x), \phi(x) + f(x)] :_a \tag{2.4}
$$

for any functional F and maps the vacuum sector to the kink sector. For example, the kink ground state may be written

$$
|K\rangle = \mathcal{D}_f \mathcal{O} |\Omega\rangle \tag{2.5}
$$

where $|\Omega\rangle$ is the free scalar vacuum state and $\mathcal{O}$ may be calculated in perturbation theory. As $|K\rangle$ is a Hamiltonian eigenstate, $\mathcal{O}|\Omega\rangle$ is an eigenstate of its similarity transform

$$
H' = \mathcal{D}_f^{-1} H \mathcal{D}_f = Q_0 + H_2 + H_I
$$

$$
H_2 = \frac{1}{2} \int dx \Big[: \pi^2(x) :_a + : (\partial_x \phi(x))^2 :_a + M^2 V''[gf(x)] : \phi^2(x) :_a \Big] \tag{2.6}
$$

where Q_0 is the classical kink mass and H_I consists of higher order terms in the g expansion. Note that $gf(x)$ is dimensionless and so contains no powers of $\hbar$ and so no powers of g. We assume that the field has been shifted so that $\phi = 0$ is a global minimum of the potential V.

Defining

$$
\tilde{V}_p = \int dx M^2 V''[gf(x)] e^{-ipx} \tag{2.7}
$$

one easily evaluates

$$
H_2 = -\frac{m^2}{4} \int \frac{dp}{2\pi} \frac{1}{\omega_p} \left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p} \right) + \frac{1}{2} \int \frac{dp}{2\pi} \frac{m^2 + 2p^2}{\omega_p} a_p^\dagger a_p + \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{\tilde{V}_{p_1+p_2}}{\sqrt{\omega_{p_1}\omega_{p_2}}} \times \left(a_{p_1}^\dagger a_{p_2}^\dagger + 2 a_{p_1}^\dagger a_{-p_2} + a_{-p_1} a_{-p_2} \right). \tag{2.8}
$$

It is a mess. The perturbative ground state, which is annihilated by a_p, is not an eigenstate. As our shift function $f(x)$ has explicit x dependence, this expression does not even have manifest translation invariance. This motivates us to search for a new basis of the operator algebra in terms of which

Table 1 Summary of notation

Operator	Description
$\phi(x),\ \pi(x)$	The real scalar field and its conjugate momentum
$a_p^\dagger,\ a_p$	Creation and annihilation operators in plane wave basis
$b_k^\dagger,\ b_k$	Creation and annihilation operators in normal mode basis
$b_{BE/BO}^\dagger,\ b_{BE/BO}$	Creation/annihilation operators for even/odd breather modes
$\phi_0,\ \pi_0$	Zero mode of $\phi(x)$ and $\pi(x)$ in normal mode basis
$::_a,\ ::_b$	Normal ordering with respect to a or b operators respectively
$S[]$	Symmetrization with respect to momenta

Indices	Description
m	Contractions
i	Breather modes
I	Bound states including both breather modes and the zero mode
M	Normal mode type: zero mode, breather or continuum mode
$B,\ BE,\ BO,\ C$	The zero mode, bound even and odd modes and continuum modes

Hamiltonian	Description
H	The original Hamiltonian
H'	H with $\phi(x)$ shifted by soliton solution $f(x)$
H_n	The ϕ^n term in H'

Symbol	Description
$f(x)$	The classical soliton solution
$\mathcal{D}_f$	Operator that translates $\phi(x)$ by the classical soliton solution
$g_B(x)$	The soliton linearized translation mode
$g_{BE,i}(x),\ g_{BO,i}(x)$	The ith even/odd breather mode
$g_k(x)$	Continuum normal mode
p	Momentum
k_i	The analog of momentum for soliton perturbations
$\omega_k,\ \omega_p$	The frequency corresponding to k or p
$\tilde{g}$	Inverse Fourier transform of g
$\mathcal{I}_M(x)$	Contraction arising from type M normal mode
$N_k,\ N_k^M$	Plane wave normal ordered product of k $a^\dagger + a$ or $a_M^\dagger + a_M$ factors
B_n^M	Normal mode normal ordered product of n $b^\dagger \pm b$ factors
$\alpha_{nm},\ a_{nm}$	Dimensionful/less coefficients for n field products with m contractions

State	Description
$\vert K\rangle,\ \vert\Omega\rangle$	Kink and vacuum sector ground states
$\mathcal{O}\vert\Omega\rangle$	Translation of $\vert K\rangle$ by $\mathcal{D}_f^{-1}$
$\mathcal{O}_1\vert\Omega\rangle$	Translation of $\vert K\rangle$ at one loop by $\mathcal{D}_f^{-1}$

H_2 is simple. More precisely, while (2.8) is complicated it is nonetheless quadratic in the operators a and $a^\dagger$ and so it can be diagonalized by a Bogoliubov transform, which we now construct.

As Q_0 is $O(g^{-2})$ and H_2 is $O(g^0)$, these are the only terms which appear at one loop. In particular the one-loop kink ground state $\mathcal{O}_1|\Omega\rangle$ is an eigenstate of H_2. To find it, one expands the fields in terms of the fixed frequency ω solutions $g(x)$ of the classical equations of motion for H_2

$$\phi(x,t) = e^{-i\omega t} g(x), \quad M^2 V''[gf(x)]g(x) = \omega^2 g(x) + g''(x). \tag{2.9}$$

This is a wave equation for a particle in a potential and its solutions are the normal modes of the field theory in the kink background. It generally has bound state and continuum solutions. We will refer to even and odd bound state solutions as

$g_{BE,i}(x)$ and $g_{BO,i}(x)$ respectively, where the index i runs over distinct solutions if there is more than one. The corresponding frequencies, defined using (2.9), will be denoted $\omega_{BE,i}$ and $\omega_{BO,i}$. There will always be an even bound state solution corresponding to the translation symmetry, which we call

$$g_B(x) = \frac{1}{\sqrt{Q_0}} f'(x). \tag{2.10}$$

As it corresponds to a symmetry, it is a zero mode $\omega_B = 0$. The other bound state solutions correspond to breather modes. Let n_e and n_o be the number of even and odd breather modes. We will name the continuum states $g_k(x)$ where k is defined by $\omega_k^2 = k^2 + m^2$ and the sign of k is fixed by demanding that asymptotically it becomes the corresponding plane wave. All of these solutions are clearly mutually orthogonal and we normalize them such that

$$\int dx g_{k_1}(x) g^*_{k_2}(x) = 2\pi\delta(k_1 - k_2), \quad \int dx |g_B(x)|^2 = \int dx |g_{BE}(x)|^2 = \int dx |g_{BO}(x)|^2 = 1. \tag{2.11}$$

We also impose

$$g(-x) = g^*(x). \tag{2.12}$$

Their inverse Fourier transforms

$$\tilde{g}(p) = \int dx g(x) e^{ipx} \tag{2.13}$$

satisfy the completeness relations [16]

$$\sum_I (-1)^I \tilde{g}_I(p) \tilde{g}_I(q) + \int \frac{dk}{2\pi} \tilde{g}_k(p) \tilde{g}_{-k}(q) = 2\pi\delta(p+q)$$
$$\times \sum_I g_I(x) g^*_I(y) + \int \frac{dk}{2\pi} g_k(x) g_{-k}(y) = \delta(x-y) \tag{2.14}$$

where I runs over all $n_e + n_o + 1$ bound state field labels $\{B, \{BE, i\}, \{BO, i\}\}$ and the symbol $(-1)^I$ is equal to -1 for odd breathers and otherwise is 1.

This entire paper will be in the Schrodinger picture, and so the field $\phi(x)$ and $\pi(x)$ will be independent of time and of any choice of Hamiltonian, and of any decomposition of the Hamiltonian into a free and imaginary part. They satisfy canonical commutation relations exactly, in the full interacting theory.[2] We define the linear combinations

$$a_p^\dagger = \int dx \left(\sqrt{\frac{\omega_p}{2}} \phi(x) - \frac{i}{\sqrt{2\omega_p}} \pi(x) \right) e^{ipx}, \quad a_{-p} = \int dx \left(\sqrt{\frac{\omega_p}{2}} \phi(x) + \frac{i}{\sqrt{2\omega_p}} \pi(x) \right) e^{ipx}$$
$$\phi_0 = \int dx \phi(x) g_B(x), \quad \pi_0 = \int dx \pi(x) g_B(x)$$
$$b^\dagger_{BE,i} = \int dx \left(\sqrt{\frac{\omega_{BE,i}}{2}} \phi(x) - \frac{i}{\sqrt{2\omega_{BE,i}}} \pi(x) \right) g^*_{BE,i}(x)$$
$$b_{BE,i} = \int dx \left(\sqrt{\frac{\omega_{BE,i}}{2}} \phi(x) + \frac{i}{\sqrt{2\omega_{BE,i}}} \pi(x) \right) g^*_{BE,i}(x)$$
$$b^\dagger_{BO,i} = \int dx \left(\sqrt{\frac{\omega_{BO,i}}{2}} \phi(x) - \frac{i}{\sqrt{2\omega_{BO,i}}} \pi(x) \right) g^*_{BO,i}(x)$$
$$b_{BO,i} = \int dx \left(-\sqrt{\frac{\omega_{BO,i}}{2}} \phi(x) - \frac{i}{\sqrt{2\omega_{BO,i}}} \pi(x) \right) g^*_{BO,i}(x). \tag{2.15}$$

From the canonical algebra satisfied by $\phi(x)$ and $\pi(x)$ one easily finds the algebra satisfied by their components

$$[a_p, a_q^\dagger] = 2\pi\delta(p-q), \quad [\phi_0, \pi_0] = i, \quad [b_{BE,i}, b^\dagger_{BE,j}] = \delta_{ij}$$
$$[b_{BO,i}, b^\dagger_{BO,j}] = \delta_{ij}, \quad [b_{k_1}, b^\dagger_{k_2}] = 2\pi\delta(k_1 - k_2) \tag{2.16}$$

with all other commutators within each decomposition vanishing.

Using the completeness relations (2.14) one may invert (2.15) to arrive at two distinct expansions of the same fields. The first is in terms of plane waves

$$\phi(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} \left(a_p^\dagger + a_{-p} \right) e^{-ipx},$$
$$\omega_p = \sqrt{m^2 + p^2}$$
$$\pi(x) = i \int \frac{dp}{2\pi} \sqrt{\frac{\omega_p}{2}} \left(a_p^\dagger - a_{-p} \right) e^{-ipx}. \tag{2.17}$$

Here the the mass m is defined by

$$m = M\sqrt{V''[0]}. \tag{2.18}$$

The other decomposition is in terms of normal modes[3]

$$\phi(x) = \sum_I \phi_I(x) + \phi_C(x),$$
$$\pi(x) = \sum_I \pi_I(x) + \pi_C(x)$$
$$\phi_B(x) = \phi_0 g_B(x),$$
$$\phi_{BE,i}(x) = \frac{1}{\sqrt{2\omega_{BE,i}}} \left(b^\dagger_{BE,i} + b_{BE,i} \right) g_{BE,i}(x)$$

[2] See for example Eq. (2.1) of Ref. [17].

[3] Note that these decompositions can be done even if H is not the Hamiltonian of the theory. Decompositions and even normal ordering relating various choices of decomposition of H into free and interacting parts were used in [17] and in this context in [6].

$$\phi_{BO,i}(x) = \frac{1}{\sqrt{2\omega_{BO,i}}}\left(b^\dagger_{BO,i} - b_{BO,i}\right) g_{BO,i}(x),$$
$$\phi_C(x) = \int \frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(b^\dagger_k + b_{-k}\right) g_k(x)$$
$$\pi_B(x) = \pi_0 g_B(x)$$
$$\pi_{BE,i}(x) = i\sqrt{\frac{\omega_{BE,i}}{2}}\left(b^\dagger_{BE,i} - b_{BE,i}\right) g_{BE,i}(x)$$
$$\pi_{BO,i}(x) = i\sqrt{\frac{\omega_{BO,i}}{2}}\left(b^\dagger_{BO,i} + b_{BO,i}\right) g_{BO,i}(x),$$
$$\pi_C(x) = i\int \frac{dk}{2\pi}\sqrt{\frac{\omega_k}{2}}\left(b^\dagger_k - b_{-k}\right) g_k(x). \tag{2.19}$$

For each g here there is a corresponding ω, defined by Eq. (2.9). In the case of the continuum modes in ϕ_C, these are equal to

$$\omega_k = \sqrt{m^2 + k^2}. \tag{2.20}$$

It may appear surprising that m, which was defined in (2.18) using the other expansion, appears here. However (2.20) is merely our choice of parametrization of $|k|$, while its sign is fixed by (2.12). The only parametrization-independent content of (2.20) is that the minimum ω_k for a continuum mode is $\omega_0 = m$ which arises because this delocalized mode has only measure zero support on the kink, and so its mass agrees with that of the lowest mass vacuum sector mode, m.

Normal ordering may be defined with respect to either decomposition. Plane wave normal ordering $::_a$ places all $a^\dagger$ to the left of each a. Normal mode normal ordering $::_b$ places all $b^\dagger$ and ϕ_0 on the left of all b and π_0.

Finally one may simplify H_2

$$H_2 = Q_1 + \frac{\pi_0^2}{2} + \sum_{i=1}^{n_e}\omega_{BE,i} b^\dagger_{BE,i} b_{BE,i} + \sum_{i=1}^{n_o}\omega_{BO,i} b^\dagger_{BO,i} b_{BO,i} + \int\frac{dk}{2\pi}\omega_k b^\dagger_k b_k \tag{2.21}$$

where Q_1 is the one-loop correction to the kink energy. One recognizes this system as the sum of a free quantum mechanical particle with position ϕ_0 and momentum π_0 plus an infinite set of quantum harmonic oscillators. The one-loop vacuum therefore is annihilated by π_0 and also by all operators b

$$\pi_0\mathcal{O}_1|\Omega\rangle = b_{BE,i}\mathcal{O}_1|\Omega\rangle = b_{BO,i}\mathcal{O}_1|\Omega\rangle = b_k\mathcal{O}_1|\Omega\rangle = 0. \tag{2.22}$$

We can now describe the entire Fock space of H_2. It consists of the vacuum, plus an arbitrary center of mass momentum proportional to π_0, where relativistic corrections appear only at higher orders in pertubation theory [10]. Also there is an infinite tower of oscillator excitations, created by $b^\dagger_k$ with mass ω_k.

Equation (2.22) implies that normal mode normal ordered operators $:A:_b$, with vanishing c-number component, have vanishing expectation values at one loop

$$\langle\Omega|\mathcal{O}_1^\dagger :A:_b \mathcal{O}_1|\Omega\rangle = 0. \tag{2.23}$$

This is one motivation for considering normal mode normal ordering. Another is that it allows an efficient computation of states and energies beyond one loop [9].

3 Factorization

3.1 Factorization

The Hamiltonian H is plane wave normal ordered and a similarity transform by $\mathcal{D}_f$ preserves the normal ordering [13]. Therefore the Hamiltonian H' is also plane wave normal ordered. However for several applications, normal mode normal ordering is most efficient. In this paper we will study how to relate the two.

The Hamiltonian H', at nth order, for $n > 2$ is

$$H_n = \frac{M^2 g^{n-2}}{n!}V^{(n)}[gf(x)] :\phi^n(x):_a \tag{3.1}$$

where $V^{(n)}$ is the nth functional derivative of the potential V with respect to its argument. To calculate the soliton spectrum and energy corrections in perturbation theory, beginning with $\mathcal{O}_1|\Omega\rangle$, it is easiest to normal mode normal order the Hamiltonian. The plane wave normal ordering is defined in terms of $a^\dagger$ and a and so, to evaluate these terms, we must use the plane wave expansion (2.17)

$$:\phi^n(x):_a = \int\frac{d^n p}{(2\pi)^n}\frac{\exp\left(-ix\sum_{i=1}^n p_i\right)}{\sqrt{2^n\omega_{p_1}\dots\omega_{p_n}}} :\prod_{i=1}^n \times\left(a^\dagger_{p_i} + a_{-p_i}\right):_a . \tag{3.2}$$

To rewrite this in terms of normal mode operators, one need only insert (2.19) into the inverse of (2.17) to obtain the Bogoliubov transformations

$$a^\dagger_p = \sum_I a^\dagger_{I,p} + a^\dagger_{C,p},$$
$$a_{-p} = \sum_I a_{I,-p} + a_{C,-p}$$
$$a^\dagger_{B,p} = \tilde{g}_B(p)\left[\sqrt{\frac{\omega_p}{2}}\phi_0 - \frac{i}{\sqrt{2\omega_p}}\pi_0\right],$$
$$a_{B,-p} = \tilde{g}_B(p)\left[\sqrt{\frac{\omega_p}{2}}\phi_0 + \frac{i}{\sqrt{2\omega_p}}\pi_0\right].$$
$$a^\dagger_{BE,i,p} = \frac{\tilde{g}_{BE,i}(p)}{2}\left(\frac{\omega_p + \omega_{BE,i}}{\sqrt{\omega_p\omega_{BE,i}}}b^\dagger_{BE,i} + \frac{\omega_p - \omega_{BE,i}}{\sqrt{\omega_p\omega_{BE,i}}}b_{BE,i}\right)$$

$$a_{BE,i,-p} = \frac{\tilde{g}_{BE,i}(p)}{2}\left(\frac{\omega_p - \omega_{BE,i}}{\sqrt{\omega_p\omega_{BE,i}}}b^\dagger_{BE,i} + \frac{\omega_p + \omega_{BE,i}}{\sqrt{\omega_p\omega_{BE,i}}}b_{BE,i}\right)$$
$$a^\dagger_{BO,i,p} = \frac{\tilde{g}_{BO,i}(p)}{2}\left(\frac{\omega_p + \omega_{BO,i}}{\sqrt{\omega_p\omega_{BO,i}}}b^\dagger_{BO,i} + \frac{-\omega_p + \omega_{BO,i}}{\sqrt{\omega_p\omega_{BO,i}}}b_{BO,i}\right)$$
$$a_{BO,i,-p} = \frac{\tilde{g}_{BO,i}(p)}{2}\left(\frac{\omega_p - \omega_{BO,i}}{\sqrt{\omega_p\omega_{BO,i}}}b^\dagger_{BO,i} + \frac{-\omega_p - \omega_{BO,i}}{\sqrt{\omega_p\omega_{BO,i}}}b_{BO,i}\right)$$
$$a^\dagger_{C,p} = \int\frac{dk}{2\pi}\frac{\tilde{g}_k(p)}{2}\left(\frac{\omega_p + \omega_k}{\sqrt{\omega_p\omega_k}}b^\dagger_k + \frac{\omega_p - \omega_k}{\sqrt{\omega_p\omega_k}}b_{-k}\right)$$
$$a_{C,-p} = \int\frac{dk}{2\pi}\frac{\tilde{g}_k(p)}{2}\left(\frac{\omega_p - \omega_k}{\sqrt{\omega_p\omega_k}}b^\dagger_k + \frac{\omega_p + \omega_k}{\sqrt{\omega_p\omega_k}}b_{-k}\right). \quad (3.3)$$

The key simplification comes from the fact that the modes from distinct oscillators commute with each other and they all commute with the zero modes. Thus, after inserting (3.3) into (3.2), one can separate the modes of each oscillator and the zero modes

$$:\prod_{i=1}^{n}\left(a^\dagger_{p_i} + a_{-p_i}\right):_a$$
$$= \sum_{\{J^M|\cup_M J^M=[1,n]\}}\prod_M\left(:\prod_{i\in J^M}\left(a^\dagger_{M,p_i} + a_{M,-p_i}\right):_a\right) \quad (3.4)$$

where M runs over $\{B,\{BE,i\},\{BO,i\},C\}$, the J^M are disjoint and their union is $[1,n]$.

For example, in the case of two point functions in the Sine-Gordon model, $n_e = n_0 = 0$ and $n = 2$. Thus M runs over the labels B and C corresponding to the translation zero mode and the continuum. J^M runs over the four subsets of $\{1,2\}$, leading to four summands

$$:\prod_{i=1}^{2}\left(a^\dagger_{p_i} + a_{-p_i}\right):_a$$
$$=:\prod_{i=1}^{2}\left(a^\dagger_{B,p_i} + a_{B,-p_i}\right):_a + :$$
$$\times\left(a^\dagger_{B,p_1} + a_{B,-p_1}\right):_a:\left(a^\dagger_{C,p_2} + a_{C,-p_2}\right):_a$$
$$+:\left(a^\dagger_{B,p_2} + a_{B,-p_2}\right):_a:\left(a^\dagger_{C,p_1} + a_{C,-p_1}\right):_a$$
$$+:\prod_{i=1}^{2}\left(a^\dagger_{C,p_i} + a_{C,-p_i}\right):_a. \quad (3.5)$$

Note that in a local Hamiltonian, normal-ordered products appear in the combination (3.2) where this product is integrated over a kernel which is symmetric with respect to permutations of the p_i. Thus only the symmetric part of the product contributes to the Hamiltonian. This depends on the subsets J^M only via their cardinalities $j_M = |J^M|$ which sum to n

$$S\left[:\prod_{i=1}^{n}\left(a^\dagger_{p_i} + a_{-p_i}\right):_a\right]$$
$$= n!\sum_{\{j_M|\sum_M j_M=n\}} S\left[\prod_M\left(\frac{1}{j_M!}:\prod_{i=1+\sum_{N=1}^{M-1}j_N}^{\sum_{N=1}^{M}j_N}\left(a^\dagger_{M,p_i} + a_{M,-p_i}\right):_a\right)\right] \quad (3.6)$$

where S symmetrizes all values of p_i. Where the letter M appears in the limits of the sum, it is understood that we have numbered the $n_o + n_e + 2$ values of M from 1 to $n_o + n_e + 2$. The ordering chosen does not matter.

For example, (3.5) becomes

$$S\left[:\prod_{i=1}^{2}\left(a^\dagger_{p_i} + a_{-p_i}\right):_a\right]$$
$$= S\left[:\prod_{i=1}^{2}\left(a^\dagger_{B,p_i} + a_{B,-p_i}\right):_a\right]$$
$$+2S\left[:\left(a^\dagger_{B,p_1} + a_{B,-p_1}\right):_a:\left(a^\dagger_{C,p_2} + a_{C,-p_2}\right):_a\right]$$
$$+S\left[:\prod_{i=1}^{2}\left(a^\dagger_{C,p_i} + a_{C,-p_i}\right):_a\right] \quad (3.7)$$

where the three terms correspond to $\{j_B = 2,\ j_C = 0\}$, $\{j_B = 1,\ j_C = 1\}$ and $\{j_B = 0,\ j_C = 2\}$. To avoid clutter, below the operator S will not be written explicitly, but we will write in the text when we symmetrize.

If we decompose $\phi(x)$ similarly to the plane wave operators

$$\phi(x) = \sum_M \phi_M(x),$$
$$\phi_M(x) = \int\frac{dp}{2\pi}\frac{1}{\sqrt{2\omega_p}}\left(a^\dagger_{M,p} + a_{M,-p}\right)e^{-ipx} \quad (3.8)$$

then we can use (3.6) to decompose

$$:\phi^n(x):_a = n!\sum_{\{j_M|\sum_M j_M=n\}}\prod_M\left(\frac{1}{j_M!}:\phi_M^{j_M}(x):_a\right). \quad (3.9)$$

Note that the symmetrization is automatic here because of the symmetric kernel of the p integration in (3.2).

The normal ordering on the right hand side of (3.4) is defined to be whatever one obtains from (3.2) when all of the different oscillators are separated. This is well-defined. But is it a normal ordering?

3.2 The problem

To simplify this question, let us restrict our attention momentarily to the case $n_e = n_o = 0$, as in the Sine-Gordon theory.

The generalization to other values is trivial. Clearly, whatever $::_a$ on the a_M means, it is linear since the factorization above can be performed separately for each summand. So consider one summand in (3.5)

$$: a^{\dagger}_{B,p_1} a^{\dagger}_{B,p_2} :_a . \tag{3.10}$$

The simplest guess would be that $::_a$ places the $a^{\dagger}_B$ on the left, and so the answer could be $a^{\dagger}_{B,p_1} a^{\dagger}_{B,p_2}$ or $a^{\dagger}_{B,p_2} a^{\dagger}_{B,p_1}$. The trouble is that these are not equal because

$$\begin{aligned}
[a^{\dagger}_{B,p_1}, a^{\dagger}_{B,p_2}] &= \left[\tilde{g}_B(p_1)\left(\sqrt{\frac{\omega_{p_1}}{2}}\phi_0 - \frac{i}{\sqrt{2\omega_{p_1}}}\pi_0\right),\right.\\
&\quad \left.\times \tilde{g}_B(p_2)\left(\sqrt{\frac{\omega_{p_2}}{2}}\phi_0 - \frac{i}{\sqrt{2\omega_{p_2}}}\pi_0\right)\right]\\
&= \frac{1}{2}\left(\sqrt{\frac{\omega_{p_1}}{\omega_{p_2}}} - \sqrt{\frac{\omega_{p_2}}{\omega_{p_1}}}\right)\tilde{g}_B(p_1)\tilde{g}_B(p_2).
\end{aligned} \tag{3.11}$$

Similarly, in the case of continuum modes

$$\begin{aligned}
[a^{\dagger}_{C,p_1}, a^{\dagger}_{C,p_2}] &= \int \frac{d^2k}{(2\pi)^2}\left[\frac{\tilde{g}_{k_1}(p_1)}{2}\right.\\
&\quad \times \left(\frac{\omega_{p_1}+\omega_{k_1}}{\sqrt{\omega_{p_1}\omega_{k_1}}} b^{\dagger}_{k_1} + \frac{\omega_{p_1}-\omega_{k_1}}{\sqrt{\omega_{p_1}\omega_{k_1}}} b_{-k_1}\right),\\
&\quad \left.\times \frac{\tilde{g}_{k_2}(p_2)}{2}\left(\frac{\omega_{p_2}+\omega_{k_2}}{\sqrt{\omega_{p_2}\omega_{k_2}}} b^{\dagger}_{k_2} + \frac{\omega_{p_2}-\omega_{k_2}}{\sqrt{\omega_{p_2}\omega_{k_2}}} b_{-k_2}\right)\right]\\
&= \frac{1}{2}\left(\sqrt{\frac{\omega_{p_1}}{\omega_{p_2}}} - \sqrt{\frac{\omega_{p_2}}{\omega_{p_1}}}\right)\int \frac{d^1k}{(2\pi)^1}\tilde{g}_{k_1}(p_1)\tilde{g}_{-k_1}(p_2).
\end{aligned} \tag{3.12}$$

Thus the action of $::_a$ on $a^{\dagger}_{M,p}$ and $a_{M,-p}$ is more complicated than simply putting all $a^{\dagger}_M$ on the left, since their order matters. This was not the case with the undecomposed plane wave oscillator modes because

$$\begin{aligned}
[a^{\dagger}_{p_1}, a^{\dagger}_{p_2}] &= [a^{\dagger}_{B,p_1}, a^{\dagger}_{B,p_2}] + [a^{\dagger}_{C,p_1}, a^{\dagger}_{C,p_2}]\\
&= \frac{1}{2}\left(\sqrt{\frac{\omega_{p_1}}{\omega_{p_2}}} - \sqrt{\frac{\omega_{p_2}}{\omega_{p_1}}}\right)\\
&\quad \times \left(\tilde{g}_B(p_1)\tilde{g}_B(p_2) + \int \frac{d^1k}{(2\pi)^1}\tilde{g}_{k_1}(p_1)\tilde{g}_{-k_1}(p_2)\right)\\
&= \frac{1}{2}\left(\sqrt{\frac{\omega_{p_1}}{\omega_{p_2}}} - \sqrt{\frac{\omega_{p_2}}{\omega_{p_1}}}\right)2\pi\delta(p_1+p_2) = 0
\end{aligned} \tag{3.13}$$

where we used the completeness relations (2.14) and the product of zero and a delta function vanishes at $p_1 = p_2$ because this is the commutator of an operator with itself.

Conclusion: One may freely interchange the undecomposed plane wave mode operators $a^{\dagger}$ and also a inside of $::_a$, for example in Eq. (3.2). However, this shuffling fixes the order of the components $a^{\dagger}_M$ and also a_M in (3.4). In particular, the ordering of the components must be the same for all M, as this ordering is that chosen for the undecomposed operators.

Note that the symmetrized commutators vanish, and so this problem does not arise in the symmetrized products relevant to the computations of products of fields at the same point, as appear for example in the Hamiltonian.

3.3 A practical convention

In the previous section we learned that we need to make a choice. We need to choose the ordering of the $a^{\dagger}_{p_i}$ and also of the a_{-p_i} in (3.2). This choice does not affect our answer but it fixes the orderings of each component in (3.4). In this section we will choose an ordering which will facilitate the computations in the next section.

Let us define the shorthand

$$N_k(p_1 \dots p_k) =: \prod_{i=1}^{k}\left(a^{\dagger}_{p_i} + a_{-p_i}\right) :_a . \tag{3.14}$$

We choose the ordering defined by

$$\begin{aligned}
&N_0 = 1, \quad N_{k+1}(p_1 \dots p_{k+1})\\
&\quad = a^{\dagger}_{p_{k+1}} N_k(p_1 \dots p_k) + N_k(p_1 \dots p_k) a_{-p_{k+1}}.
\end{aligned} \tag{3.15}$$

We remind the reader that the value of N_k does not depend on this choice of ordering, as all $a^{\dagger}$ commute with each other as do all a. However it does affect the definition of the normal ordering of the components.

Our strategy will be the following. First we will guess a formula for the normal ordering of the components

$$N^M_k(p_1 \dots p_k) =: \prod_{i=1}^{k}\left(a^{\dagger}_{M,p_i} + a_{M,-p_i}\right) :_a . \tag{3.16}$$

Then we will show that, using the factorization formula (3.4) the guess yields the correct value of N_k. Recall that our definition of $::_a$ on components is that it satisfies (3.4) and so once we have shown this, we will have verified that our guess indeed satisfies the definition and so corresponds to a valid convention.

Our guess is

$$\begin{aligned}
&N^M_0 = 1,\\
&N^M_{k+1}(p_1 \dots p_{k+1}) = a^{\dagger}_{M,p_{k+1}} N^M_k(p_1 \dots p_k)\\
&\quad + N^M_k(p_1 \dots p_k) a_{M,-p_{k+1}}.
\end{aligned} \tag{3.17}$$

The factorization formula (3.4) in the case $n_e = n_o = 0$ is

$$N_k(p_1 \dots p_k) = \sum_{J \subset [1,k]} N^B_{|J|}(p_J) N^C_{k-|J|}(p_{[1,k]\setminus J}). \tag{3.18}$$

Here we have adopted the shorthand p_S for the ordered set of all p_j with $j \in S$. The ordering is just the ascending order, since that appeared on the left hand side of the equation. We need to show that our guess (3.17) satisfies (3.18).

Our proof will be by induction. The base case, $k = 0$ is trivial as the only term in the sum is $J = \varnothing$ and so (3.18) becomes $1 = 1$. Next assume that (3.18) is satisfied for some value of k and define

$$\hat{N}_{k+1}(p_1 \dots p_{k+1}) = \sum_{J\subset[1,k+1]} N^B_{|J|}(p_J) N^C_{k+1-|J|}(p_{[1,k+1]\backslash J}) \quad (3.19)$$

where the right hand side is defined using (3.17). We need to prove that $\hat{N} = N$ to complete the induction.

Each J either does or does not contain the element $\{k+1\}$ and so we may respectively divide the sum in two parts, redefining the dummy set J in the first sum by removing $\{k+1\}$

$$\begin{aligned}\hat{N}_{k+1}(p_1 \dots p_{k+1}) &= \sum_{J\subset[1,k]} N^B_{|J|+1}(p_J, p_{k+1}) N^C_{k-|J|}(p_{[1,k]\backslash J}) \\ &\quad + \sum_{J\subset[1,k]} N^B_{|J|}(p_J) N^C_{k+1-|J|}(p_{[1,k]\backslash J}, p_{k+1}) \\ &= \sum_{J\subset[1,k]} \left(a^\dagger_{B,p_{k+1}} N^B_{|J|}(p_J, p_k)\right. \\ &\quad \left.+N^B_{|J|}(p_J, p_k) a_{B,-p_{k+1}}\right) N^C_{k-|J|}(p_{[1,k]\backslash J}) \\ &\quad + \sum_{J\subset[1,k]} N^B_{|J|}(p_J, p_k) \left(a^\dagger_{C,p_{k+1}} N^C_{k-|J|}(p_{[1,k]\backslash J})\right. \\ &\quad \left.+N^C_{k-|J|}(p_{[1,k]\backslash J}) a_{C,-p_{k+1}}\right) \\ &= a^\dagger_{p_{k+1}} N_k(p_1 \dots p_k) \\ &\quad + N_k(p_1 \dots p_k) a_{-p_{k+1}} = N_{k+1}(p_1 \dots p_{k+1})\end{aligned} \quad (3.20)$$

completing the induction.

In summary, we have shown that if we adopt the definition (3.17) for the plane wave normal ordering of component fields $a^\dagger_M$ and a_M, then the factorization formula (3.18) is satisfied and so these components N^M_k can be assembled to determine the plane wave normal ordered product N_k of the undecomposed operators. Although our proof was for the case with no breather modes $n_e = n_o = 0$, the equation (3.17) works in general and indeed the proof can be trivially generalized to show the compatibility of (3.17) and (3.4).

4 Recursion formulas

4.1 Zero modes

Define the coefficients α_{nm} by

$$N^B_n(p_1 \dots p_n) = \left(\prod_{i=1}^n \sqrt{2\omega_{p_i}} \tilde{g}_B(p_i)\right) \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} \phi_0^{n-2m} \quad (4.1)$$

where N^B_n was defined in (3.16). Then using (3.17) we can find the next product

$$N^B_{n+1}(p_1 \dots p_{n+1}) = \left(\prod_{i=1}^n \sqrt{2\omega_{p_i}} \tilde{g}_B(p_i)\right) \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} \left(a^\dagger_{B,p_{n+1}} \phi_0^{n-2m} + \phi_0^{n-2m} a_{B,-p_{n+1}}\right) \quad (4.2)$$

$$\begin{aligned}&= \frac{1}{2}\left(\prod_{i=1}^{n+1} \sqrt{2\omega_{p_i}} \tilde{g}_B(p_i)\right) \\ &\quad \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} \left[\left(\phi_0 - \frac{i}{\omega_{p_{n+1}}}\pi_0\right)\phi_0^{n-2m}\right. \\ &\quad \left.+\phi_0^{n-2m}\left(\phi_0 + \frac{i}{\omega_{p_{n+1}}}\pi_0\right)\right] \\ &= \left(\prod_{i=1}^{n+1} \sqrt{2\omega_{p_i}} \tilde{g}_B(p_i)\right) \\ &\quad \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} \left(\phi_0^{n-2m+1} - \frac{1}{2\omega_{p_{n_1}}}\phi_0^{n-2m-1}\right).\end{aligned} \quad (4.3)$$

Dividing through by the product on the left one finds

$$\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \alpha_{n+1,m} \phi_0^{n-2m+1} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} \left(\phi_0^{n-2m+1} - \frac{n-2m}{2\omega_{p_{n+1}}}\phi_0^{n-2m-1}\right). \quad (4.4)$$

Finally matching terms with the same power of ϕ_0 we arrive at the recursion relation

$$\alpha_{n+1,m} = \alpha_{nm} - \frac{n-2m+2}{2\omega_{p_{n+1}}} \alpha_{n,m-1} \quad (4.5)$$

which, together with the initial condition $\alpha_{0m} = \delta_{m,0}$ fixes all of the coefficients α.

The recursion relation (4.5) has a simple interpretation in terms of a Wick's theorem. m is the number of contractions. The $(n+1)$st operator may either not contract, leading to the first term on the right hand side, or else it may contract. If it does contract, since there are m contractions in all, the first n operators have $m-1$ contractions. Therefore the $n+1$st operator may contract with any one of the $n-2(m-1)$ uncontracted operators, yielding the factor of $n-2m+2$ in the second term. Each contraction yields a factor of $-1/(2\omega_{p_{n+1}})$. Note that this contraction factor is not symmetric with respect to a permutation of the p_i, since it depends only on the p_i with the highest value of i among the two contracted operators, which is p_{n+1}.

4.2 Solving the recursion formula

Recall that to compute the Hamiltonian we only need the symmetrized N_n. In this case the choice of ω_{p_i} is irrelevant, it is only important that no N have two ω_{p_i} with the same i. Said differently, adding an antisymmetric piece to N will not change the symmetrized N and so will not change H. We can thus shift N to be of the form

$$N_n^B(p_1\dots p_n) = \left(\prod_{i=1}^n \sqrt{2\omega_{p_i}}\tilde{g}_B(p_i)\right)\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor} a_{nm}\phi_0^{n-2m}\prod_{i=1}^m\left(\frac{-1}{2\omega_{p_i}}\right) \tag{4.6}$$

where a is a pure number which simply counts the number of ways to make m contractions. a satisfies the recursion relation

$$a_{n+1,m} = a_{nm} + (n-2m+2)a_{n,m-1}. \tag{4.7}$$

As the contractions are interchangeable, a_{nm} contains a factor of $1/m!$. This is multiplied by the number of choices for the jth contraction, which is $(n-2j+2)(n-2j+1)/2$, for each j from 1 to m. In all one finds

$$a_{nm} = \frac{1}{m!}\prod_{j=1}^m \frac{(n-2j+2)(n-2j+1)}{2} = \frac{1}{2^m}\frac{n!}{m!(n-2m)!}. \tag{4.8}$$

In the case with no breathers, the decomposition of the fields (3.9) becomes

$$:\phi^n(x):_a = \sum_{j=0}^n \frac{n!}{j!(n-j)!} :\phi_B^j(x):_a :\phi_C^{n-j}(x):_a . \tag{4.9}$$

Assembling the results above, we have evaluated the first factor in (4.9)

$$\begin{aligned}
:\phi_B^j(x):_a &= \int \frac{d^jp}{(2\pi)^j}\frac{e^{-ix\sum_i p_i}}{\sqrt{2^j\omega_{p_1}\dots\omega_{p_j}}} N_j^B(p_1\dots p_j) \\
&= \sum_{m=0}^{\lfloor \frac{j}{2}\rfloor}\frac{1}{2^m}\frac{j!}{m!(j-2m)!}\phi_0^{j-2m} \\
&\quad\times \int\frac{d^jp}{(2\pi)^j}e^{-ix\sum_i p_i}\left(\prod_{i=1}^j \tilde{g}_B(p_i)\right)\prod_{i=1}^m\left(\frac{-1}{2\omega_{p_i}}\right) \\
&= \sum_{m=0}^{\lfloor \frac{j}{2}\rfloor}\frac{j!}{2^m m!(j-2m)!} g_B^{j-2m}(x)\mathcal{I}_B^m(x)\phi_0^{j-2m} \\
&= \sum_{m=0}^{\lfloor \frac{j}{2}\rfloor}\frac{j!}{2^m m!(j-2m)!}\mathcal{I}_B^m(x)\phi_B^{j-2m}(x)
\end{aligned} \tag{4.10}$$

where we have introduced the contraction factor

$$\mathcal{I}_B(x) = g_B(x)\hat{g}_B(x),$$

$$\hat{g}_B(x) = -\int\frac{dp}{2\pi}e^{-ipx}\frac{\tilde{g}_B(p)}{2\omega_p}. \tag{4.11}$$

4.3 Odd breathers

Similarly to the plane wave ordered products $N_n(p)$ we will define the normal mode ordered products

$$B_n^{BO} =: \left(b_{BO}^\dagger - b_{BO}\right)^n :_b . \tag{4.12}$$

Our goal in this section is to learn how to expand $N_n(p)$ in terms of $B_n^{BO}(k)$.

Using the identity

$$B_n^{BO} = \sum_{k=0}^n (-1)^k\frac{n!}{k!(n-k)!}b_{BO}^{\dagger n-k}b_{BO}^k \tag{4.13}$$

one readily derives the anticommutator

$$\{b_{BO}^\dagger - b_{BO}, B_n^{BO}\} = 2B_{n+1}^{BO} - 2nB_{n-1}^{BO} \tag{4.14}$$

and the commutator

$$\left[b_{BO}^\dagger + b_{BO}, B_n^{BO}\right] = 2nB_{n-1}^{BO} \tag{4.15}$$

which will be useful momentarily.

Proceeding as for the zero mode, we define coefficients α_{nm} by

$$N_n^{BO}(p_1\dots p_n) = \left(\prod_{i=1}^n\sqrt{\frac{\omega_{p_i}}{\omega_{BO}}}\tilde{g}_{BO}(p_i)\right)\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\alpha_{nm}B_{n-2m}^{BO}. \tag{4.16}$$

Then using (3.17)

$$\begin{aligned}
N_{n+1}^{BO}(p_1\dots p_{n+1}) &= \left(\prod_{i=1}^n\sqrt{\frac{\omega_{p_i}}{\omega_{BO}}}\tilde{g}_{BO}(p_i)\right) \\
&\times\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\alpha_{nm}\left(a_{BO,p_{n+1}}^\dagger B_{n-2m}^{BO} + B_{n-2m}^{BO}a_{BO,-p_{n+1}}\right) \\
&= \frac{1}{2}\left(\prod_{i=1}^{n+1}\sqrt{\frac{\omega_{p_i}}{\omega_{BO}}}\tilde{g}_{BO}(p_i)\right) \\
&\times\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\alpha_{nm}\left(\{b_{BO}^\dagger - b_{BO}, B_{n-2m}^{BO}\} + \frac{\omega_{BO}}{\omega_{p_{n+1}}}\right. \\
&\times\left.[b_{BO}^\dagger + b_{BO}, B_{n-2m}^{BO}]\right) \\
&= \left(\prod_{i=1}^{n+1}\sqrt{\frac{\omega_{p_i}}{\omega_{BO}}}\tilde{g}_{BO}(p_i)\right) \\
&\times\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\alpha_{nm}\left(B_{n-2m+1}^{BO} + (n-2m)\right. \\
&\times\left.\left(-1+\frac{\omega_{BO}}{\omega_{p_{n+1}}}\right)B_{n-2m-1}^{BO}\right)
\end{aligned} \tag{4.17}$$

and so

$$\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} \left(B^{BO}_{n-2m+1} + (n-2m) \left(-1 + \frac{\omega_{BO}}{\omega_{p_{n+1}}} \right) B^{BO}_{n-2m-1} \right) = \sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \alpha_{n+1,m} B^{BO}_{n-2m+1}. \tag{4.18}$$

Matching coefficients we obtain the recursion relation

$$\alpha_{n+1,m} = \alpha_{nm} + (n-2m+2) \left(-1 + \frac{\omega_{BO}}{\omega_{p_{n+1}}} \right) \alpha_{n,m-1}. \tag{4.19}$$

So far we have not used symmetrization, and so our recursion relation may be applied to computing any n-point function. Again, for calculating n-point functions at the same point, as in our interaction terms, we may shift N^{BO} by an operator which vanishes when symmetrized

$$N_n^{BO}(p_1 \dots p_n) = \left(\prod_{i=1}^{n} \sqrt{\frac{\omega_{p_i}}{\omega_{BO}}} \tilde{g}_{BO}(p_i) \right) \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} a_{nm} B^{BO}_{n-2m} \prod_{i=1}^{m} \left(-1 + \frac{\omega_{BO}}{\omega_{p_i}} \right). \tag{4.20}$$

Note that the product on the right can be rewritten

$$\prod_{i=1}^{m} \left(-1 + \frac{\omega_{BO}}{\omega_{p_i}} \right) = (2\omega_{BO})^m \prod_{i=1}^{m} \left(-\frac{1}{2\omega_{BO}} + \frac{1}{2\omega_{p_i}} \right) \tag{4.21}$$

so that it resembles the contraction terms in (4.6). Proceeding as above, the recursion formula satisfied by the a_{nm} is again (4.7) and so the a_{nm} are given by (4.8).

$$\begin{aligned}
:\phi^j_{BO}(x):_a &= \int \frac{d^j p}{(2\pi)^j} \frac{e^{-ix\sum_i p_i}}{\sqrt{2^j \omega_{p_1} \dots \omega_{p_j}}} N_j^{BO}(p_1 \dots p_j) \\
&= \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{2^m} \frac{j!}{m!(j-2m)!} B^{BO}_{j-2m} \\
&\quad \times \int \frac{d^j p}{(2\pi)^j} e^{-ix\sum_i p_i} (2\omega_{BO})^{(2m-j)/2} \\
&\quad \times \left(\prod_{i=1}^{j} \tilde{g}_{BO}(p_i) \right) \prod_{i=1}^{m} \left(-\frac{1}{2\omega_{BO}} + \frac{1}{2\omega_{p_i}} \right) \\
&= \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} \mathcal{I}^m_{BO}(x) \frac{g^{j-2m}_{BO}(x) B^{BO}_{j-2m}}{(2\omega_{BO})^{(j-2m)/2}} \\
&= \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} :\phi^{j-2m}_{BO}(x):_b \mathcal{I}^m_{BO}(x)
\end{aligned} \tag{4.22}$$

where we have defined the contraction factor

$$\mathcal{I}_{BO}(x) = g_{BO}(x)\hat{g}_{BO}(x), \quad \hat{g}_{BO}(x) = \int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_{BO}(p) \times \left(-\frac{1}{2\omega_{BO}} + \frac{1}{2\omega_p} \right). \tag{4.23}$$

The contraction factor is similar to $\mathcal{I}_B(x)$ except that the contraction contains two terms $1/(2\omega_{BO})$ and $1/(2\omega_p)$ with a relative sign. These are respectively the contraction arising from the normal mode normal ordering and the plane wave normal ordering. In the case of $\mathcal{I}_B(x)$ the normal mode normal ordering was fundamentally different, as it was a rule for the placement of the canonical variables ϕ_0 and π_0 and not for the oscillator modes.

The occurrence of a difference of contractions in $\mathcal{I}_{BO}$ is reminiscent of the general contraction defined in Ref. [18].

4.4 Even breathers

The normal ordering of even breathers is identical to that of odd breathers except for a few sign differences. Defining

$$B_n^{BE} =: \left(b^\dagger_{BE} + b_{BE} \right)^n :_b \tag{4.24}$$

and using the identity

$$B_n^{BE} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} b^{\dagger n-k}_{BE} b^k_{BE} \tag{4.25}$$

one finds

$$\begin{aligned}
\{b^\dagger_{BE} + b_{BE}, B_n^{BE}\} &= 2B^{BE}_{n+1} + 2nB^{BE}_{n-1}, \\
\left[b^\dagger_{BE} - b_{BE}, B_n^{BE} \right] &= -2nB^{BE}_{n-1}.
\end{aligned} \tag{4.26}$$

Then defining

$$N_n^{BE}(p_1 \dots p_n) = \left(\prod_{i=1}^{n} \sqrt{\frac{\omega_{p_i}}{\omega_{BE}}} \tilde{g}_{BO}(p_i) \right) \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{nm} B^{BE}_{n-2m}. \tag{4.27}$$

The same computation as in the odd case yields the recursion relation

$$\alpha_{n+1,m} = \alpha_{nm} + (n-2m+2) \left(1 - \frac{\omega_{BO}}{\omega_{p_{n+1}}} \right) \alpha_{n,m-1}. \tag{4.28}$$

Comparing (4.19) and (4.28) one sees that the contractions of even and odd breathers differ by an overall sign.

In the symmetric case one may shift N^{BE} to

$$N_n^{BE}(p_1 \dots p_n) = \left(\prod_{i=1}^{n} \sqrt{\frac{\omega_{p_i}}{\omega_{BE}}} \tilde{g}_{BE}(p_i) \right) \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} a_{nm} B^{BE}_{n-2m} \prod_{i=1}^{m} \left(1 - \frac{\omega_{BE}}{\omega_{p_i}} \right). \tag{4.29}$$

where a_{bm} again satisfies (4.7) and so we conclude that

$$: \phi^j_{BE}(x) :_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} : \phi^{j-2m}_{BE}(x) :_b \mathcal{I}^m_{BE}(x) \tag{4.30}$$

where we have defined the contraction factor

$$\mathcal{I}_{BE}(x) = g_{BE}(x)\hat{g}_{BE}(x),$$
$$\hat{g}_{BE}(x) = \int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_{BE}(p) \left(\frac{1}{2\omega_{BE}} - \frac{1}{2\omega_p} \right). \tag{4.31}$$

The relative sign in the recursion relation has indeed translated into a relative sign in the contraction factor with respect to $\mathcal{I}_{BO}$. As $g_{BO}(x)$ is imaginary and $g_{BE}(x)$ is real due to our convention (2.12), the relative sign may be absorbed by taking the complex conjugate of $g(x)$ in the definition of $\mathcal{I}(x)$. We will now see in the continuum case that this definition arises quite naturally.

4.5 Continuum modes

Define

$$B^C_n(k_1 \ldots k_n) =: \prod_{i=1}^n \left(\frac{b^\dagger_{k_i} + b_{-k_i}}{\sqrt{2\omega_{k_i}}} \right) :_b . \tag{4.32}$$

Using the identity

$$B^C_n(k_1 \ldots k_n) = \sum_{J \subset [1,n]} \left(\prod_{j \in J} \frac{b^\dagger_{k_j}}{\sqrt{2\omega_{k_j}}} \right) \left(\prod_{j \in [1,n] \setminus J} \frac{b_{-k_j}}{\sqrt{2\omega_{k_j}}} \right) \tag{4.33}$$

one finds the commutator

$$\begin{aligned}
&\left[\frac{b^\dagger_{k'} - b_{-k'}}{\sqrt{2\omega_{k'}}}, B^C_n(k_1 \ldots k_n) \right] \\
&= -\frac{1}{2\omega_{k'}} \sum_{J \subset [1,n]} \left[\sum_{j' \in J} 2\pi\delta(k_{j'} + k') \prod_{j \in J \setminus j'} \right. \\
&\quad \times \frac{b^\dagger_{k_j}}{\sqrt{2\omega_{k_j}}} \prod_{j \in [1,n] \setminus J} \frac{b_{-k_j}}{\sqrt{2\omega_{k_j}}} \\
&\quad + \sum_{j' \in [1,n] \setminus J} 2\pi\delta(k_{j'} + k') \prod_{j \in J} \\
&\quad \left. \times \frac{b^\dagger_{k_j}}{\sqrt{2\omega_{k_j}}} \prod_{j \in [1,n] \setminus J \setminus j'} \frac{b_{-k_j}}{\sqrt{2\omega_{k_j}}} \right] \\
&= -\frac{2}{2\omega_{k'}} \sum_{j' \in [1,n]} \\
&\quad \times 2\pi\delta(k_{j'} + k') B^C_{n-1}(k_1 \ldots \hat{k}_{j'} \ldots k_n)
\end{aligned} \tag{4.34}$$

and similarly the anticommutator

$$\begin{aligned}
&\left\{ \frac{b^\dagger_{k'} + b_{-k'}}{\sqrt{2\omega_{k'}}}, B^C_n(k_1 \ldots k_n) \right\} \\
&= 2B^C_{n+1}(k_1 \ldots k_n, k') + \frac{2}{2\omega_{k'}} \sum_{j' \in [1,n]} \\
&\quad \times 2\pi\delta(k_{j'} + k') B^C_{n-1}(k_1 \ldots \hat{k}_{j'} \ldots k_n).
\end{aligned} \tag{4.35}$$

We will need the integrals of these identities, where the integral over k' is performed using the Dirac delta function

$$\begin{aligned}
&\left[\frac{1}{2} \int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1}) \frac{\omega_{k'}}{\omega_{p_{n+1}}} \frac{\left(b^\dagger_{k'} - b_{-k'}\right)}{\sqrt{2\omega_{k'}}}, \right. \\
&\quad \left. \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \alpha^{k_1 \ldots k_{n-2m}}_{nm} B^C_{n-2m}(k_1 \ldots k_{n-2m}) \right] \\
&= -\frac{1}{2\omega_{p_{n+1}}} \sum_{j'=1}^{n-2m} \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \tilde{g}_{-k_{j'}}(p_{n+1}) \alpha^{k_1 \ldots k_{n-2m}}_{nm} \\
&\quad \times B^C_{n-2m-1}(k_1 \ldots \hat{k}_{j'} \ldots k_{n-2m})
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
&\left\{ \frac{1}{2} \int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1}) \frac{\left(b^\dagger_{k'} + b_{-k'}\right)}{\sqrt{2\omega_{k_{n+1}}}}, \right. \\
&\quad \left. \times \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \alpha^{k_1 \ldots k_{n-2m}}_{nm} B^C_{n-2m}(k_1 \ldots k_{n-2m}) \right\} \\
&= \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1}) \alpha^{k_1 \ldots k_{n-2m}}_{nm} \\
&\quad \times B^C_{n-2m+1}(k_1 \ldots k_{n-2m}, k') \\
&\quad + \sum_{j'=1}^{n-2m} \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \tilde{g}_{-k_{j'}}(p_{n+1}) \alpha^{k_1 \ldots k_{n-2m}}_{nm} \\
&\quad \times \frac{1}{2\omega_{k_{j'}}} B^C_{n-2m-1}(k_1 \ldots \hat{k}_{j'} \ldots k_{n-2m})
\end{aligned} \tag{4.37}$$

for arbitrary matrices α_{nm}.
We will define the matrices α_{nm} by

$$\begin{aligned}
N^C_n(p_1 \ldots p_n) &= \left(\prod_{i=1}^n \sqrt{2\omega_{p_i}} \right) \\
&\quad \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \alpha^{k_1 \ldots k_{n-2m}}_{nm} B^C_{n-2m}(k_1 \ldots k_{n-2m}).
\end{aligned} \tag{4.38}$$

Then (3.17) implies

$$\begin{aligned}
N^C_{n+1}(p_1 \ldots p_{n+1}) &= \frac{1}{2} \left(\prod_{i=1}^{n+1} \sqrt{2\omega_{p_i}} \right) \\
&\quad \times \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \left(\left\{ \int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1}) \frac{\left(b^\dagger_{k'} + b_{-k'}\right)}{\sqrt{2\omega_{k_{n+1}}}}, \right. \right.
\end{aligned}$$

$$\times \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \alpha_{nm}^{k_1 \dots k_{n-2m}} B_{n-2m}^C(k_1 \dots k_{n-2m}) \Big\}$$
$$+ \Big[\int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1}) \frac{\omega_{k'}}{\omega_{p_{n+1}}} \frac{\left(b_{k'}^\dagger - b_{-k'}\right)}{\sqrt{2\omega_{k'}}},$$
$$\times \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \alpha_{nm}^{k_1 \dots k_{n-2m}} B_{n-2m}^C(k_1 \dots k_{n-2m}) \Big] \Big)$$
$$= \left(\prod_{i=1}^{n+1} \sqrt{2\omega_{p_i}} \right) \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor}$$
$$\times \Big[\int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1}) \alpha_{nm}^{k_1 \dots k_{n-2m}}$$
$$\times B_{n-2m+1}^C(k_1 \dots k_{n-2m}, k')$$
$$+ \sum_{j'=1}^{n-2m} \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \left(\frac{1}{2\omega_{k_{j'}}} - \frac{1}{2\omega_{p_{n+1}}} \right) \tilde{g}_{-k_{j'}}(p_{n+1}) \alpha_{nm}^{k_1 \dots k_{n-2m}}$$
$$\times B_{n-2m-1}^C(k_1 \dots \hat{k}_{j'} \dots k_{n-2m}) \Big]. \tag{4.39}$$

Summarizing, we find

$$\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \int \frac{d^{n-2m+1}k}{(2\pi)^{n-2m+1}} \alpha_{n+1,m}^{k_1 \dots k_{n-2m+1}} B_{n-2m+1}^C(k_1 \dots k_{n-2m+1})$$
$$= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \Big[\int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \int \frac{dk'}{2\pi} \tilde{g}_{k'}(p_{n+1})$$
$$\times \alpha_{nm}^{k_1 \dots k_{n-2m}} B_{n-2m+1}^C(k_1 \dots k_{n-2m}, k')$$
$$+ \sum_{j'=1}^{n-2m} \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}} \left(\frac{1}{2\omega_{k_{j'}}} - \frac{1}{2\omega_{p_{n+1}}} \right)$$
$$\times \tilde{g}_{-k_{j'}}(p_{n+1}) \alpha_{nm}^{k_1 \dots k_{n-2m}} B_{n-2m-1}^C(k_1 \dots \hat{k}_{j'} \dots k_{n-2m}) \Big]$$
$$= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \int \frac{d^{n-2m+1}k}{(2\pi)^{n-2m+1}} \tilde{g}_{k_{n-2m+1}}(p_{n+1})$$
$$\times \alpha_{nm}^{k_1 \dots k_{n-2m}} B_{n-2m+1}^C(k_1 \dots k_{n-2m+1})$$
$$+ \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor + 1} \sum_{j'=1}^{n-2m+2} \int \frac{d^{n-2m+2}k}{(2\pi)^{n-2m+2}} \left(\frac{1}{2\omega_{k_{j'}}} - \frac{1}{2\omega_{p_{n+1}}} \right)$$
$$\times \tilde{g}_{-k_{j'}}(p_{n+1}) \alpha_{n,m-1}^{k_1 \dots k_{n-2m+2}}$$
$$\times B_{n-2m+1}^C(k_1 \dots \hat{k}_{j'} \dots k_{n-2m+2}) \tag{4.40}$$

where $\hat{k}_{j'}$ indicates that $k_{j'}$ is omitted. Matching yields the recursion relation

$$\alpha_{n+1,m}^{k_1 \dots k_{n-2m+1}}$$
$$= \tilde{g}_{k_{n-2m+1}}(p_{n+1}) \alpha_{nm}^{k_1 \dots k_{n-2m}}$$
$$+ \int \frac{dk'}{2\pi} \tilde{g}_{-k'}(p_{n+1}) \left(\frac{1}{2\omega_{k'}} - \frac{1}{2\omega_{p_{n+1}}} \right)$$
$$\times \sum_{j'=1}^{n-2m+2} \alpha_{n,m-1}^{k_1 \dots k_{j'-1} k' k_{j'} \dots k_{n-2m+1}}. \tag{4.41}$$

Symmetrizing we may write

$$N_n^C(p_1 \dots p_n) = \left(\prod_{i=1}^n \sqrt{2\omega_{p_i}} \right) \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \int \frac{d^{n-2m}k}{(2\pi)^{n-2m}}$$
$$\times \left(\prod_{i=1}^{n-2m} \tilde{g}_{k_i}(p_i) \right) a_{nm} B_{n-2m}^C(k_1 \dots k_{n-2m})$$
$$\times \int \frac{d^m k'}{(2\pi)^m} \prod_{i=1}^m \Big(\tilde{g}_{-k_i'}(p_{n-2m+2i-1}) \tilde{g}_{k_i'}(p_{n-2m+2i})$$
$$\times \left(\frac{1}{2\omega_{k_i'}} - \frac{1}{2\omega_{p_{n-2m+2i}}} \right) \Big) \tag{4.42}$$

where again a_{nm} satisfies (4.7) and so is given by (4.8). We therefore conclude

$$: \phi_C^j(x) :_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \mathcal{I}_C^m(x) \int \frac{d^{j-2m}k}{(2\pi)^{j-2m}}$$
$$\times \left(\prod_{i=1}^{j-2m} g_{k_i}(x) \right)$$
$$\times \frac{j!}{2^m m!(j-2m)!} B_{j-2m}^C(k_1 \dots k_{j-2m})$$
$$= \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} \mathcal{I}_C^m(x) : \phi_C^{j-2m}(x) :_b$$
$$\mathcal{I}_C(x) = \int \frac{dk}{2\pi} g_{-k}(x) \hat{g}_k(x),$$
$$\hat{g}_k(x) = \int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_k(p)$$
$$\times \left(\frac{1}{2\omega_k} - \frac{1}{2\omega_p} \right). \tag{4.43}$$

While the algebra leading up to our result seemed more complicated than in the case of the bound states, our final result is essentially the same. The only difference is that $\mathcal{I}_C$ is integrated over normal modes k. However, even in the case of breathers, there will be a sum over breather modes i, and so this distinction is superficial.

4.6 Wick's Theorem

Finally we may use our decomposition (3.9) to reassemble $: \phi^j(x) :_a$ by inserting (4.10), (4.22), (4.30) and (4.43)

$$: \phi^j(x) :_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} \mathcal{I}^m(x) : \phi^{j-2m}(x) :_b,$$
$$\mathcal{I}(x) = \sum_M \mathcal{I}_M(x). \tag{4.44}$$

The contraction factor $\mathcal{I}(x)$ can be efficiently evaluated using the completeness relations (2.14)

$$\mathcal{I}(x)=g_B(x)\hat{g}_B(x)+\sum_i g_i(x)\hat{g}_i(x)+\int\frac{dk}{2\pi}g_k(x)\hat{g}_{-k}(x) \tag{4.45}$$

$$\begin{aligned}
&=-\int\frac{dq}{2\pi}e^{-iqx}\int\frac{dp}{2\pi}\frac{e^{-ipx}}{2\omega_p}\\
&\quad\times\left[\tilde{g}_B(q)\tilde{g}_B(p)+\sum_{i=1}^{n_e}\tilde{g}_{BE,i}(q)\tilde{g}_{BE,i}(p)\right.\\
&\quad-\sum_{i=1}^{n_0}\tilde{g}_{BO,i}(q)\tilde{g}_{BO,i}(p)\\
&\quad\left.+\int\frac{dk}{2\pi}\tilde{g}_k(q)\tilde{g}_{-k}(p)\right]\\
&\quad+\int\frac{dk}{2\pi}\frac{1}{2\omega_k}g_k(x)\int\frac{dp}{2\pi}e^{-ipx}\tilde{g}_{-k}(p)\\
&\quad+\sum_{i=1}^{n_e}\frac{1}{2\omega_i}g_{BE,i}(x)\int\frac{dp}{2\pi}e^{-ipx}\tilde{g}_{BE,i}(p)\\
&\quad-\sum_{i=1}^{n_o}\frac{1}{2\omega_i}g_{BO,i}(x)\int\frac{dp}{2\pi}e^{-ipx}\tilde{g}_{BO,i}(p)\\
&=-\int\frac{dq}{2\pi}e^{-iqx}\int\frac{dp}{2\pi}\frac{e^{-ipx}}{2\omega_p}2\pi\delta(p+q)+\int\frac{dk}{2\pi}\\
&\quad\times\frac{1}{2\omega_k}g_k(x)\int\frac{dp}{2\pi}e^{-ipx}\tilde{g}_{-k}(p)\\
&\quad+\sum_{i=1}^{n_e}\frac{1}{2\omega_i}g_{BE,i}(x)\int\frac{dp}{2\pi}e^{-ipx}\tilde{g}_{BE,i}(p)\\
&\quad-\sum_{i=1}^{n_o}\frac{1}{2\omega_i}g_{BO,i}(x)\int\frac{dp}{2\pi}e^{-ipx}\tilde{g}_{BO,i}(p)\\
&=\int\frac{dp}{2\pi}\left[-\frac{1}{2\omega_p}+e^{-ipx}\left(\sum_{i=1}^{n_e}\frac{1}{2\omega_i}g_{BE,i}(x)\tilde{g}_{BE,i}(p)\right.\right.\\
&\quad-\sum_{i=1}^{n_o}\frac{1}{2\omega_i}g_{BO,i}(x)\tilde{g}_{BO,i}(p)\\
&\quad\left.\left.+\int\frac{dk}{2\pi}\frac{1}{2\omega_k}g_k(x)\tilde{g}_{-k}(p)\right)\right].
\end{aligned} \tag{4.46}$$

It is finite. The first term is x-independent, and integrates to infinity. Now we take the derivative with respect to x, and so the first term vanishes

$$\partial_x\mathcal{I}(x)=\sum_{i=1}^{n_e+n_0}\frac{\partial_x|g_i(x)|^2}{2\omega_I}+\int\frac{dk}{2\pi}\frac{\partial_x|g_k(x)|^2}{2\omega_k}. \tag{4.47}$$

Notice that the relative sign between the even and odd breathers has disappeared, because $g_i(x)^2$ is positive in the even case and negative in the odd case. This, together with the boundary condition that $\mathcal{I}(x)$ vanishes as x goes to infinity, completely determines $\mathcal{I}(x)$.

5 Applications to the Sine-Gordon soliton

5.1 Normal ordering the Hamiltonian

In the Sine-Gordon theory the interaction Hamiltonian density in H' is [19]

$$\begin{aligned}
\mathcal{H}_I&=\frac{m^2}{\sqrt{\lambda}}\sin(\sqrt{\lambda}f(x))\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{(2n+1)!}:\phi^{2n+1}(x):_a\\
&\quad-\frac{m^2}{\lambda}\cos(\sqrt{\lambda}f(x))\sum_{n=2}^{\infty}\frac{(-\lambda)^n}{2n!}:\phi^{2n}(x):_a.
\end{aligned} \tag{5.1}$$

The contribution arising from bound states is

$$\begin{aligned}
\mathcal{H}_B&=-\frac{m^2}{\lambda}\cos(\sqrt{\lambda}f(x))h_e+\frac{m^2}{\sqrt{\lambda}}\sin(\sqrt{\lambda}f(x))h_o\\
h_e&=\sum_{n=2}^{\infty}\frac{(-\lambda)^n}{2n!}:\phi_B^{2n}(x):_a\\
,\quad h_o&=\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{(2n+1)!}:\phi_B^{2n+1}(x):_a.
\end{aligned} \tag{5.2}$$

Using (4.10) the plane wave normal ordering may be evaluated explicitly

$$\begin{aligned}
h_e&=\sum_{n=2}^{\infty}(-\lambda)^n\sum_{m=0}^{n}\\
&\quad\times\frac{1}{2^m m!(2n-2m)!}\mathcal{I}_B^m(x)\phi_B^{2n-2m}(x).
\end{aligned} \tag{5.3}$$

To simplify this sum, we will include the terms at $n=0$ and $n=1$, which are present in the Hamiltonian although they are not the only terms at their orders. These terms only affect the noninteracting part of the Hamiltonian, which is known to be the Poschl-Teller Hamiltonian. So we define

$$\begin{aligned}
\tilde{h}_e&=\sum_{n=0}^{\infty}(-\lambda)^n\sum_{m=0}^{n}\frac{1}{2^m m!(2n-2m)!}\mathcal{I}_B^m(x)\phi_B^{2n-2m}(x)\\
&=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\lambda)^{p+m}}{2^m m!(2p)!}\mathcal{I}_B^m(x)\phi_B^{2p}(x)\\
&=\cos\left(\sqrt{\lambda}\phi_B(x)\right)\exp\left(-\lambda\mathcal{I}_B(x)/2\right).
\end{aligned} \tag{5.4}$$

The $n=0$ and $n=1$ terms that we have added are

$$\tilde{h}_e-h_e=1-\lambda\left(\frac{\mathcal{I}_B+\phi_B^2(x)}{2}\right). \tag{5.5}$$

Similarly, including the $n=0$ term,

$$\tilde{h}_o = \sum_{n=0}^{\infty}(-\lambda)^n \sum_{m=0}^{n} \times \frac{1}{2^m m!(2n-2m+1)!}\mathcal{I}_B^m(x)\phi_B^{2n-2m+1}(x)$$
$$= \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\lambda)^{p+m}}{2^m m!(2p+1)!}\mathcal{I}_B^m(x)\phi_B^{2p+1}(x)$$
$$= \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\phi_B(x)\right)\exp\left(-\lambda\mathcal{I}_B(x)/2\right) \quad (5.6)$$

where

$$\tilde{h}_o - h_o = -\phi_B(x). \quad (5.7)$$

Substituting this back into (5.2) and letting the tilde remind the reader that the noninteracting (quadratic in ϕ and below) terms are incorrect, we find

$$\tilde{\mathcal{H}}_B = -\frac{m^2}{\lambda}\cos\left(\sqrt{\lambda}\left(\phi_B(x)+f(x)\right)\right)\exp\left(-\lambda\mathcal{I}_B(x)/2\right). \quad (5.8)$$

where

$$\tilde{\mathcal{H}}_B - \mathcal{H}_B = m^2\cos(\sqrt{\lambda}f(x))\left(-\frac{1}{\lambda}+\frac{\mathcal{I}_B+\phi_B^2(x)}{2}\right) - m^2\sin(\sqrt{\lambda}f(x))\frac{\phi_B(x)}{\sqrt{\lambda}}. \quad (5.9)$$

As the terms in (5.9) are at most $O(\lambda^0)$, and $\mathcal{H}_I$ anyway does not include all terms at this order, we will not be interested in these terms. The terms at these leading orders are given in this example in Ref. [19], and more generally are of the form (2.21). In particular they include no interactions and the tadpole vanishes in the full expression.

Equation (5.8) has a straightforward interpretation. The combination $\phi_B(x)+f(x)$ is just the $\mathcal{D}_f$ translated field, brutally truncated to the zero mode part. The prefactor and the cosine term are thus just the original Sine-Gordon action, translated and truncated. However we see that the plane wave normal ordering is now gone, indeed it was our goal to eliminate it, and instead there is an exponential of a contraction term. Thus plane wave normal ordering is equivalent to multiplication by the exponent of the bound state contraction. Of course only the bound state contraction appeared because we have truncated our Hamiltonian by only considering the bound component of the field. Our result is trivially normal mode normal ordered as it only involves the operator ϕ_0.

More generally we may expect the exponential to include the sum of the contractions of the various normal modes

$$\tilde{\mathcal{H}}_I = -\frac{m^2}{\lambda} :\cos\left(\sqrt{\lambda}\left(\phi(x)+f(x)\right)\right):_b \times\exp\left(-\lambda\mathcal{I}(x)/2\right)$$

$$\mathcal{I}(x) = \sum_M \mathcal{I}_M(x) = \mathcal{I}_B(x)+\mathcal{I}_C(x). \quad (5.10)$$

The appearance of contractions in an exponential in (5.10) is also similar to the generalized Wick's theorem postulated in Ref. [18]. It would be useful to understand this connection more precisely, as the generalized Wick's theorem may provide a simple extension of our results to more complicated and interesting models.

5.2 Application: time independent perturbation theory

We have already noted that Eq. (5.10) is only correct beyond $O(\lambda^0)$, as we did not include all terms at lower orders. Recalling that f is of order $\lambda^{-1/2}$, the cos term may be expanded as in (5.1)

$$\tilde{\mathcal{H}}_I = \left(\frac{m^2}{\sqrt{\lambda}}\sin(\sqrt{\lambda}f(x))\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{(2n+1)!} :\phi^{2n+1}(x):_b - \frac{m^2}{\lambda}\cos(\sqrt{\lambda}f(x))\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{2n!} :\phi^{2n}(x):_b\right) \times\sum_{j=0}^{\infty}\frac{(-\lambda\mathcal{I}(x))^j}{2^j j!}. \quad (5.11)$$

In particular the $O\left(\lambda^{1/2}\right)$ terms in $\mathcal{H}_I$ are

$$\mathcal{H}_3 = -\sqrt{\lambda}m^2\sin(\sqrt{\lambda}f(x))\left(\frac{:\phi^3(x):_b}{6}+\frac{:\phi(x):_b\,\mathcal{I}(x)}{2}\right). \quad (5.12)$$

This was reported already in Ref. [9], but now using Wick's theorem a calculation that originally required several days has been reduced to less than an hour.

Using (5.11) we can easily go to higher orders. For example at $O(\lambda)$ we find that $\mathcal{H}_I$ contains

$$\mathcal{H}_4 = -\lambda m^2\cos(\sqrt{\lambda}f(x))\left(\frac{:\phi^4(x):_b}{24}+\frac{\mathcal{I}(x):\phi^2(x):_b}{4}+\frac{\mathcal{I}^2(x)}{8}\right). \quad (5.13)$$

To apply this formula, first expand the soliton ground state

$$\mathcal{O}|\Omega\rangle = \sum_{n=0}^{\infty}|0\rangle_n \quad (5.14)$$

where $|0\rangle_0 = \mathcal{O}_1|\Omega\rangle$ is the one-loop state described above and each successive term comes with another power of $\lambda^{1/2}$. Then, similarly expanding the Hamiltonian the Schrodinger equation can be expanded order by order, beginning with

$$H_2|0\rangle_0 = Q_1|0\rangle_0,$$
$$H_3|0\rangle_0 = -(H_2-Q_1)|0\rangle_1,$$
$$(H_4-Q_2)|0\rangle_0 = -H_3|0\rangle_1-(H_2-Q_1)|0\rangle_2. \quad (5.15)$$

The first equation was solved in Ref. [16] and the second in [9]. However Q_2 and $|0\rangle_2$ are still unknown. We can see from the third equation that Q_2 receives three contributions, one from each term in the decomposition of $|0\rangle_0$. From (5.13) we see that the $|0\rangle_0$ contribution to the two-loop correction to the soliton mass Q_2 is

$$Q_2^{(0)} = -\frac{\lambda m^2}{8}\int dx \cos(\sqrt{\lambda} f(x))\mathcal{I}^2(x). \tag{5.16}$$

The $|0\rangle_1$ contribution $Q_2^{(1)}$ can be found by acting (5.12) on the expression for $|0\rangle_1$ in [9]. To compute the $|0\rangle_2$ contribution $Q_2^{(2)}$ one needs to use the translation invariance of the ground state as described in [9]. The full calculation appears in Ref. [20].

5.3 Another application: correlation functions

The unsymmetrized Wick's theorem can be used to calculate Schrodinger picture correlation functions of plane wave normal ordered operators, should one be interested in such objects

$$F(x_1\ldots x_n) = {}_0\langle 0| :\phi(x_1)\ldots\phi(x_n):_a |0\rangle_0. \tag{5.17}$$

We can decompose $\phi(x) = \phi_B(x) + \phi_C(x)$. As each $\phi_B(x)$ and $\phi_C(x)$ commute, this product can be factorized into a sum over all subsets S of $[1, n]$

$$F(x_1\ldots x_n) = \sum_{S\subset[1,n]} {}_0\langle 0| :\prod_{i\in S}\phi_C(x_i):_a: \prod_{i\in[1,n]\setminus S}\phi_C(x_i):_a |0\rangle_0. \tag{5.18}$$

A complete set of states in the one-soliton sector is generated by functions $\psi(\phi_0)$ times products of $b_k^\dagger$ acting on $|0\rangle_0$. Inserting the identity written in terms of such a complete set of states in between $\phi_C(x_n)$ and $\phi_B(x_1)$, one sees that all terms with $b^\dagger$ operators vanish because the right matrix element contains no other $b^\dagger$ operators. Similarly, if one considers an orthogonal basis of functions of ϕ_0, then only the trivial function 1 will contribute as for all other functions the left matrix element will vanish by orthogonality, as none of the operators on the left contain ϕ_0 or π_0 and they all commute with ϕ_0. Thus we have argued

$$F(x_1\ldots x_n) = \sum_{S\subset[1,n]} F_C(x_S)F_B(x_{[1,n]\setminus S})$$
$$F_M(x_1\ldots x_n) = {}_0\langle 0| :\phi_M(x_1)\ldots\phi_M(x_n):_a |0\rangle_0 \tag{5.19}$$

where for any set $S = i_1, .., i_n$ we have adopted the shorthand that x_S is $x_{i_1}\ldots x_{i_n}$.

These products of fields can be rewritten in terms of our symbols N as in (4.10) but with no symmetrization

$$F_M(x_1\ldots x_n) = \int \frac{d^n p}{(2\pi)^n}\frac{e^{-i\sum_i x_i p_i}}{\sqrt{2^n\omega_{p_1}\ldots\omega_{p_n}}} \times {}_0\langle 0|N_n^M(p_1\ldots p_n)|0\rangle_0. \tag{5.20}$$

As all matrix elements of normal mode normal ordered operators vanish, only the completely contracted operators $(n = 2m)$ contribute. These depend on the coefficients $\alpha_{n,n/2}$. Immediately we see that the n-point functions vanish unless n is even.

Let us begin with the bound state components. Solving the recursion relation (4.5) in a few cases we find

$$\alpha_{n,0} = 1,$$
$$\alpha_{n,1} = -\sum_{j=1}^{n-1}\frac{j}{2\omega_{p_{j+1}}},$$
$$\alpha_{4,2} = \frac{1}{4\omega_{p_2}\omega_{p_4}} + \frac{1}{2\omega_{p_3}\omega_{p_4}}. \tag{5.21}$$

The completely contracted terms

$$F_B(x_1\ldots x_n) = \int\frac{d^n p}{(2\pi)^n}e^{-i\sum_i x_i p_i}\left(\prod_{j=1}^n \tilde{g}_B(p_j)\right)\alpha_{n,n/2} \tag{5.22}$$

can be written immediately in the first few cases

$$F_B(x_1, x_2) = g_B(x_1)\hat{g}_B(x_2)$$
$$F_B(x_1, x_2, x_3, x_4) = g_B(x_1)\hat{g}_B(x_4) \times\left(\hat{g}_B(x_2)g_B(x_3) + 2g_B(x_2)\hat{g}_B(x_3)\right). \tag{5.23}$$

It is remarkably asymmetrical in the x_i. This is to be expected, as we made an arbitrary choice in our convention for the bound state normal ordering. The correlator F_B depends on that choice. Only the product F is independent.

Now let us turn our attention to the continuum modes ϕ_C. Solving the recursion relation (4.41) in the first few cases we find

$$\alpha_{n,0}^{k_1\ldots k_n} = \prod_{j=1}^n \tilde{g}_{k_j}(p_j),$$
$$\alpha_{2,1} = \int\frac{dk'}{2\pi}\tilde{g}_{-k'}(p_2)\left(\frac{1}{2\omega_{k'}} - \frac{1}{2\omega_{p_2}}\right)\tilde{g}_{k'}(p_1)$$
$$\alpha_{3,1}^{k_1} = \tilde{g}_{k_1}(p_3)\int\frac{dk'}{2\pi}\tilde{g}_{-k'}(p_2) \times\left(\frac{1}{2\omega_{k'}} - \frac{1}{2\omega_{p_2}}\right)\tilde{g}_{k'}(p_1) + \int\frac{dk'}{2\pi}\tilde{g}_{-k'}(p_3)\left(\frac{1}{2\omega_{k'}} - \frac{1}{2\omega_{p_3}}\right) \times\left(\tilde{g}_{k'}(p_1)\tilde{g}_{k_1}(p_2) + \tilde{g}_{k_1}(p_1)\tilde{g}_{k'}(p_2)\right)$$
$$\alpha_{4,2} = \int\frac{dk_2'}{2\pi}\tilde{g}_{-k_2'}(p_4)\left(\frac{1}{2\omega_{k_2'}} - \frac{1}{2\omega_{p_4}}\right)\alpha_{3,1}^{k_2'}$$
$$= \int\frac{d^2k'}{(2\pi)^2}\tilde{g}_{-k_2'}(p_4)\left(\frac{1}{2\omega_{k_2'}} - \frac{1}{2\omega_{p_4}}\right)$$

$$\times\tilde{g}_{k_2'}(p_3)\tilde{g}_{-k_1'}(p_2)\left(\frac{1}{2\omega_{k_1'}}-\frac{1}{2\omega_{p_2}}\right)\tilde{g}_{k_1'}(p_1)$$
$$+\int\frac{d^2k'}{(2\pi)^2}\tilde{g}_{-k_2'}(p_4)\left(\frac{1}{2\omega_{k_2'}}-\frac{1}{2\omega_{p_4}}\right)$$
$$\times\tilde{g}_{k_2'}(p_1)\tilde{g}_{-k_1'}(p_3)\left(\frac{1}{2\omega_{k_1'}}-\frac{1}{2\omega_{p_3}}\right)\tilde{g}_{k_1'}(p_2)$$
$$+\int\frac{d^2k'}{(2\pi)^2}\tilde{g}_{-k_2'}(p_4)\left(\frac{1}{2\omega_{k_2'}}-\frac{1}{2\omega_{p_4}}\right)$$
$$\times\tilde{g}_{k_2'}(p_2)\tilde{g}_{-k_1'}(p_3)\left(\frac{1}{2\omega_{k_1'}}-\frac{1}{2\omega_{p_3}}\right)\tilde{g}_{k_1'}(p_1). \tag{5.24}$$

The first completely contracted term

$$F_C(x_1\dots x_n)=\int\frac{d^np}{(2\pi)^n}e^{-i\sum_i x_ip_i}\alpha_{n,n/2} \tag{5.25}$$

is

$$F_C(x_1,x_2)=\int\frac{dk'}{2\pi}g_{k'}(x_1)\hat{g}_{-k'}(x_2) \tag{5.26}$$

and so, using the fact that for the empty set S, $F_M(x_S)=1$ we find

$$F(x_1,x_2)=F_B(x_1,x_2)+F_C(x_1,x_2)=g_B(x_1)\hat{g}_B(x_2)+\int\frac{dk'}{2\pi}g_{k'}(x_1)\hat{g}_{-k'}(x_2). \tag{5.27}$$

This is looks like the completeness relation (2.14) but is different because of the hats.

As we are working in the Schrodinger picture, all operators are time-independent and so these are equal time correlators. When $x_1\neq x_2$ the points are spacelike separated. Nonetheless this trivial calculation shows that the correlator does not vanish for spacelike separated x_1 and x_2. Also it is not translation invariant. Neither of these properties should be a surprise. This is not a correlation function in the vacuum, but in the 1-loop kink state $|0\rangle_1$ which is not translation invariant. It is, to one-loop, a Hamiltonian eigenstate and this condition implies that it has nonlocal correlations, just as the phase in a superconductor varies slowly over macroscopic distances or as parton wave functions are correlated in a hadron.

What about causality? Causality demands that

$${}_0\langle 0|[\phi(x_1),\phi(x_2)]|0\rangle_0=0 \tag{5.28}$$

for spacelike separated x_1 and x_2. This expression is not normal ordered and one needs to be careful when normal ordering a commutator as commutation and normal ordering do not commute

$$0=a^\dagger a-a^\dagger a=:aa^\dagger:-:a^\dagger a:$$
$$\neq 1=:1:=:(aa^\dagger-a^\dagger a):=:[a,a^\dagger]:. \tag{5.29}$$

If we start by using the equal time canonical commutation relation $[\phi(x_1),\phi(x_2)]=0$ then (5.28) is trivially satisfied and we have not succeeded in testing our correlation function. So instead rewrite (5.28) as

$${}_0\langle 0|\phi(x_1)\phi(x_2)|0\rangle_0={}_0\langle 0|\phi(x_2)\phi(x_1)|0\rangle_0. \tag{5.30}$$

Now we can normal order both sides. So long as we do not to use the commutation relations, equality before normal ordering will imply equality after normal ordering and thus we will obtain a weaker but still necessary condition for causality

$$F(x_1,x_2)=F(x_2,x_1). \tag{5.31}$$

Is this satisfied by (5.27)?

$$F(x_1,x_2)=\int\frac{d^2p}{(2\pi)^2}e^{-i(x_1p_1+x_2p_2)}\left(-\frac{\tilde{g}_B(p_1)\tilde{g}_B(p_2)}{2\omega_{p_2}}+\int\frac{dk'}{2\pi}\tilde{g}_{-k'}(p_2)\left(\frac{1}{2\omega_{k'}}-\frac{1}{2\omega_{p_2}}\right)\tilde{g}_{k'}(p_1)\right)$$
$$=\int\frac{d^2p}{(2\pi)^2}e^{-i(x_1p_1+x_2p_2)}\times\left(-\frac{2\pi\delta(p_1+p_2)}{2\omega_{p_2}}+\int\frac{dk'}{2\pi}\tilde{g}_{-k'}(p_2)\times\left(\frac{1}{2\omega_{k'}}\right)\tilde{g}_{k'}(p_1)\right)$$
$$=-\int\frac{dp}{2\pi}\frac{e^{-ip(x_1-x_2)}}{2\omega_p}+\int\frac{dk'}{2\pi}\frac{g_{-k'}(x_2)g_{k'}(x_1)}{2\omega_{k'}}. \tag{5.32}$$

We used the completeness relation to arrive at the second line. The last line has two terms. The first is symmetric under $x_1\leftrightarrow x_2$ because ω_p is an even function of p and so the odd piece of the exponential does not contribute to the integral. To see that the second term is also symmetric, it suffices to relabel the dummy variable $k'\to -k'$. Therefore the causality condition (5.31) is satisfied.

The continuum contribution to the four-point function is

$$F_C(x_1,x_2,x_3,x_4)=\int\frac{d^4p}{(2\pi)^4}e^{-i\sum_i x_ip_i}\alpha_{4,2}$$
$$=\int\frac{d^2k'}{(2\pi)^2}\Big(g_{k_1'}(x_1)\hat{g}_{-k_1'}(x_2)g_{k_2'}(x_3)+g_{k_1'}(x_2)\hat{g}_{-k_1'}(x_3)g_{k_2'}(x_1)+g_{k_1'}(x_1)\hat{g}_{-k_1'}(x_3)g_{k_2'}(x_2)\Big)\hat{g}_{-k_2'}(x_4). \tag{5.33}$$

The total four-point function

$$F(x_1,x_2,x_3,x_4)=F_C(x_1,x_2,x_3,x_4)+F_C(x_1,x_2)F_B(x_3,x_4)+F_C(x_1,x_3)F_B(x_2,x_4)+F_C(x_1,x_4)F_B(x_2,x_3)$$

$$+F_C(x_2,x_3)F_B(x_1,x_4)+F_C(x_2,x_4)F_B(x_1,x_3)$$
$$+F_C(x_3,x_4)F_B(x_1,x_2)+F_B(x_1,x_2,x_3,x_4) \tag{5.34}$$

is then given in terms of Eqs. (5.23), (5.26) and (5.33).

6 Remarks

We have found that plane wave normal ordering can be converted into normal mode normal ordering by following a simple rule, playing the role of Wick's theorem. After decomposing a product of n fields into products of j field components, where each component corresponds to a set of normal modes, the components can be decomposed by summing over all possible contractions. For each contraction one replaces the pair of field components with the difference between the inverse plane wave energy ω_p and inverse normal mode energy, suitably normalized over the spectrum. Intuitively the first term arises from eliminating the plane wave normal ordering and the second from imposing the normal mode normal ordering. Of course with no normal ordering at all, one expects divergences. However the difference between these two energies is, when suitably averaged, quite small and thus all expressions are finite given either normal ordering scheme. Once we go beyond scalar theories and 1+1 dimensions there will be other divergences which must be regularized and renormalized.

In Ref. [9] the conversion between normal orderings was the most complicated part of the perturbation theory treatment of the one soliton sector. Now that we have treated this problem at all orders, and in a much more general class of theories, we expect that it will be easier to extend that calculation to two loops or beyond. In Ref. [20], we have used the Wick's theorem here to find the two-loop ground state and mass of a scalar kink in a theory with an arbitrary potential. Previously the scalar kink mass was only found in the Sine-Gordon model [21–23], and our results agree in that case. However it is still not obvious that the solution to the zero mode problem in Ref. [9] also solves the problem at higher loops. If it does not, then it may be necessary to use other formalisms such as that of [7] and [8].

To go beyond perturbation theory, we will eventually need supersymmetry. In this context, coherent states have been constructed in Refs. [24,25]. This will require a fermionic generalization of the Wick's theorem found here. Perhaps the generalized Wick's theorem of Ref. [18] can provide an efficient derivation.

Another application of our Wick's theorem is the calculation of plane wave normal ordered correlation functions in soliton states. So far our calculations have been in the Schrodinger picture. As a result we can only evaluate equal time correlators. However we suspect that a generalization to the interaction picture would be straightforward, as interaction picture operators evolve according to the free Hamiltonian. This would allow us to compute correlators of operators at arbitrary time. We note that there are two interaction pictures, corresponding to the free Hamiltonians H and H' which describe perturbations in the zero-soliton and one-soliton sectors. For the calculation of one-soliton state correlation functions, one would use the latter as the one-loop one-soliton ground state is an eigenstate of the noninteracting part. As the one-soliton ground state is an eigenstate of H', it is even plausible that an LSZ reduction formula exists in the one-soliton sector. This would be the starting point for a treatment of scattering in this sector.

The recent discovery of spectral walls [26] caused by transitions between breather and continuum states has rekindled interest in kink scattering [27,28]. The treatment of this phenomenon has so far been largely classical. While the current methodology is most straightforwardly applied to the one kink sector, it could nonetheless allow an understanding of the role played by breathers in fully quantum scattering. In particular the scattering of a kink with a plane wave or wave packet could be treated in the one kink sector.

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References

1. G. Delfino, W. Selke, A. Squarcini, Vortex mass in the three-dimensional $O(2)$ scalar theory. Phys. Rev. Lett. **122**(5), 050602 (2019). https://doi.org/10.1103/PhysRevLett.122.050602. arXiv:1808.09276 [cond-mat.stat-mech]
2. D. Davies, Quantum solitons in any dimension: Derrick's Theorem v. AQFT (2020). arXiv:1907.10616 [hep-th]

3. K. Hepp, The classical limit for quantum mechanical correlation functions. Commun. Math. Phys. **35**, 265 (1974). https://doi.org/10.1007/BF01646348
4. J.G. Taylor, Solitons as infinite constituent bound states. Ann. Phys. **115**, 153 (1978). https://doi.org/10.1016/0003-4916(78)90179-3
5. J. Sato, T. Yumibayashi, Quantum-classical correspondence via coherent state in integrable field theory (2020), arXiv:1811.03186 [quant-ph]
6. K.E. Cahill, A. Comtet, R. Glauber, Mass formulas for static solitons. Phys. Lett. B **64**, 283–285 (1976). https://doi.org/10.1016/0370-2693(76)90202-1
7. N. Christ, T. Lee, Quantum expansion of soliton solutions. Phys. Rev. D **12**, 1606 (1975). https://doi.org/10.1103/PhysRevD.12.1606
8. J.L. Gervais, B. Sakita, Extended particles in quantum field theories. Phys. Rev. D **11**, 2943 (1975). https://doi.org/10.1103/PhysRevD.11.2943
9. J. Evslin, Constructing quantum soliton states despite zero modes (2020). arXiv:2006.02354 [hep-th]
10. R.F. Dashen, B. Hasslacher, A. Neveu, Nonperturbative methods and extended Hadron models in field theory 2. Two-dimensional models and extended Hadrons. Phys. Rev. D **10**, 4130 (1974). https://doi.org/10.1103/PhysRevD.10.4130
11. R. Rajaraman, Some nonperturbative semiclassical methods in quantum field theory: a pedagogical review. Phys. Rept. **21**, 227 (1975). https://doi.org/10.1016/0370-1573(75)90016-2
12. A. Aguirre, G. Flores-Hidalgo, A note on one-loop soliton quantum mass corrections. Mod. Phys. Lett. A **33**, 2050102 (2020). https://doi.org/10.1142/S0217732320501023. arXiv:1912.13051 [hep-th]
13. J. Evslin, Manifestly finite derivation of the quantum kink mass. JHEP **11**, 161 (2019). https://doi.org/10.1007/JHEP11(2019)161. arXiv:1908.06710 [hep-th]
14. J. Evslin, Well-defined quantum soliton masses without supersymmetry. Phys. Rev. D **101**(6), 065005 (2020). https://doi.org/10.1103/PhysRevD.101.065005. arXiv:2002.12523 [hep-th]
15. A. Rebhan, P. van Nieuwenhuizen, No saturation of the quantum Bogomolnyi bound by two-dimensional supersymmetric solitons. Nucl. Phys. B **508**, 449 (1997). https://doi.org/10.1016/S0550-3213(97)00625-1. arXiv:hep-th/9707163]
16. J. Evslin, The ground state of the sine-gordon soliton. JHEP **07**, 099 (2020). https://doi.org/10.1007/JHEP07(2020)099. arXiv:2003.11384 [hep-th]
17. S.R. Coleman, The quantum sine-gordon equation as the massive thirring model. Phys. Rev. D **11**, 2088 (1975). https://doi.org/10.1103/PhysRevD.11.2088
18. L. Diósi, Wick theorem for all orderings of canonical operators. J. Phys. A **51**(36), 365201 (2018). https://doi.org/10.1088/1751-8121/aad0a6. arXiv:1712.08811 [quant-ph]
19. H. Guo, J. Evslin, Finite derivation of the one-loop sine-Gordon soliton mass. JHEP **02**, 140 (2020). https://doi.org/10.1007/JHEP02(2020)140. arXiv:1912.08507 [hep-th]
20. J. Evslin, H. Guo, Two-loop scalar kinks (2020). arXiv:2012.04912 [hep-th]
21. R.F. Dashen, B. Hasslacher, A. Neveu, The particle spectrum in model field theories from semiclassical functional integral techniques. Phys. Rev. D **11**, 3424 (1975). https://doi.org/10.1103/PhysRevD.11.3424
22. A. Luther, Eigenvalue spectrum of interacting massive fermions in one-dimension. Phys. Rev. B **14**, 2153–2159 (1976). https://doi.org/10.1103/PhysRevB.14.2153
23. H. de Vega, Two-loop quantum corrections to the soliton mass in two-dimensional scalar field theories. Nucl. Phys. B **115**, 411–428 (1976). https://doi.org/10.1016/0550-3213(76)90497-1
24. M. Bianchi, M. Firrotta, DDF operators, open string coherent states and their scattering amplitudes. Nucl. Phys. B **952**, 114943 (2020). https://doi.org/10.1016/j.nuclphysb.2020.114943. arXiv:1902.07016 [hep-th]
25. A. Aldi, M. Firrotta, String coherent vertex operators of Neveu-Schwarz and Ramond states. Nucl. Phys. B **955**, 115050 (2020). https://doi.org/10.1016/j.nuclphysb.2020.115050. arXiv:1912.06177 [hep-th]
26. C. Adam, K. Oles, T. Romanczukiewicz, A. Wereszczynski, Spectral Walls in Soliton Collisions. Phys. Rev. Lett. **122**(24), 241601 (2019). https://doi.org/10.1103/PhysRevLett.122.241601. arXiv:1903.12100 [hep-th]
27. Y. Zhong, X.L. Du, Z.C. Jiang, Y.X. Liu, Y.Q. Wang, Collision of two kinks with inner structure. JHEP **02**, 153 (2020). https://doi.org/10.1007/JHEP02(2020)153. arXiv:1906.02920 [hep-th]
28. J.G. Campos, A. Mohammadi, Interaction between kinks and antikinks with double long-range tails (2020). arXiv:2006.01956 [hep-th]