Note on scalar–graviton and scalar–photon–graviton amplitudes

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Abstract In this short note, we propose an algorithm based on the expansions of amplitudes, the dimensional reduction technique and the approach by differential operators, to calculate the tree level scalar–graviton amplitudes with two massive scalars and the tree level scalar–photon–graviton amplitudes with two massive scalars and one photon. While applying the unitarity method, these amplitudes are necessary inputs for the calculation of post-Newtonian and post-Minkowskian expansions in general relativity for two massive charged objects interacting with gravity and the electromagnetic field.

1 Introduction

The progress in the study of scattering amplitudes in the past decade has revealed deep physical insights in quantum field theory and inspired efficient methods for practical calculation. Recently, an effort has been made to connect the amplitudes program to the physics of gravitational waves, which were discovered at LIGO/Virgo [1,2]. More explicitly, the modern tools for calculating scattering amplitudes at loop level provide a powerful new way to evaluate post-Newtonian and post-Minkowskian expansions in classical general relativity, giving rise to effective two-body Hamiltonians [3–16].

In this interesting new direction, an essential step is to calculate the scattering of two massive objects interacting with gravitons, at n -loop order. To achieve the goal, the tree level amplitudes with two massive scalars and $n+1$ gravitons are required, due to the so-called unitarity method. Among several methods proposed to calculate these tree amplitudes, one remarkable approach based on the CHY formalism was suggested by Naculich [17]. The advantage of this method is that the CHY formalism is valid in arbitrary space-time

dimensions [18–22]; thus the obtained results are more suitable for the dimensional regularization at intermediate steps. More recently, another more efficient way has made use of the CHY formalism and its extension to a double cover [23–28]; it was proposed by Bjerrum-Bohr, Cristofoli, Damgaard and Gomez [16]. Using their method, one can evaluate the desired amplitudes recursively, thus avoiding the treatments of contour integrals in the CHY formalism.

One can also consider the classical system including two charged massive objects, such as two charged black holes, interacting through both gravity and the electromagnetic field. Via an idea similar to that described above, one can seek the effective two-body Hamiltonian by calculating the scattering of two massive particles that interact with gravitons and photons, at loop level. Then the unitarity method motivated us to consider the tree level amplitudes whose external states include two massive scalars, gravitons, and photons, as shown in Fig. 1.

In this short note, we propose an algorithm based on the expansions of amplitudes [29–41]; we use the differential operators constructed by Cheung et al. [42–44], or equivalently the dimensional reduction technique [17,22], to calculate the tree level scalar–graviton (SG) amplitudes with two massive scalars and the tree level scalar–photon–graviton (SPG) amplitudes with two massive scalars and one photon. The first step of the algorithm is to compute the gravity (GR) amplitude with two massive gravitons by expanding it to the bi-adjoint scalar (BAS) amplitudes. The methods for calculating basis and coefficients in the expansion will be discussed in Sect. 2. The SG amplitude with two massive scalars can be generated from the GR amplitude by converting two massive gravitons to scalars, via dimensional reduction manipulation, or applying the differential operators. The SPG amplitude with two massive scalars and one photon can be obtained by transmuting the GR amplitude to the single trace Einstein–Yang–Mills (EYM) amplitude with two massive and one massless gluons, then converting two massive gluons to scalars, and identifying the remaining

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Fig. 1 Unitarity method for the scattering of two massive particles interact with gravitons and photons, at loop level. The straight lines represent the charged massive objects, the wavy lines stand for photons, and the double-wavy lines stand for gravitons

massless gluon as a photon. Our approach is also regardless of the space-time dimensions, and can be easily implemented in MATHEMATICA. As will be explained, the SPG amplitudes with more than one photons is hard to compute by the method developed in this paper. We leave this problem to future work.

The remainder of this note is organized as follows. In Sect. 2, we introduce our algorithm in detail. In Sect. 3, we compute the SG examples via this algorithm and give the results, which coincide with those obtained in [16]. In Sect. 4, the SPG examples are considered. The full expressions of the 5-point SG and SPG amplitudes are exhibited in Appendix A.

2 The algorithm

Our algorithm evaluates tree level GR amplitudes with two massive gravitons via the expansion obtained in [31], then converts gravitons to massive scalars and massless photons through the dimensional reduction procedure [17,22], or by applying differential operators [42–44], to obtain the desired scalar–graviton (SG) or scalar–photon–graviton (SPG) amplitudes. In Sect. 2.1, we will discuss the calculation of the GR amplitudes, including the techniques for computing coefficients in the expansion, and the BAS amplitudes which serve as a basis. In Sect. 2.2, we will study how to transmute gravitons to scalars or photons.

2.1 Calculating GR amplitude

When all external legs are massless, the tree level GR amplitude in D space-time dimensions can be expanded in terms of BAS amplitudes in D dimensions, in the following double copy formula [31]:

$$\mathcal{A}_{\text{GR}}(1, \dots, n) = \sum_{\sigma} \sum_{\sigma'} C^{\epsilon}(\sigma) \mathcal{A}_{\text{BAS}}(1\sigma n | 1\sigma' n) C^{\tilde{\epsilon}}(\sigma'), \quad (1)$$

where $\sigma, \sigma' \in S_{n-2}$ are permutations of $n - 2$ elements in $\{2, \dots, n - 1\}$. Each basis $\mathcal{A}_{\text{BAS}}(1\sigma n | 1\sigma' n)$ carries two color-orderings $(1\sigma n)$ and $(1\sigma' n)$. We have chosen $(1 \dots n)$ to denote legs with a color-ordering, and $(1, \dots, n)$ to denote legs without any color-ordering. This convention will be used for all amplitudes throughout this paper. Coefficients $C^{\epsilon}(\sigma)$ and $C^{\tilde{\epsilon}}(\sigma')$ are just Bern–Carrasco–Johansson (BCJ) numerators for Yang–Mills amplitudes [45–48]. The superscripts ϵ and $\tilde{\epsilon}$ indicate the dependence on polarization vectors in two sets $\{\epsilon_i\}$ and $\{\tilde{\epsilon}_i\}$, respectively. The GR amplitude carries two independent sets of polarization vectors as understood in a generalized version of the gravity theory, i.e., Einstein gravity couples to a dilaton and 2-form. It can be reduced to the amplitude for pure Einstein gravity, by setting all $\tilde{\epsilon}_i = \epsilon_i$. For our purpose, we want to seek the SP and SPG amplitudes including gravitons for pure Einstein gravity. However, at intermediate steps, we will keep $\{\epsilon_i\}$ and $\{\tilde{\epsilon}_i\}$ to be different, to make manifest the double copy structure. This structure ensures that the dimensional reduction procedure, or differential operators, act only on one piece, while keeping another one un-altered. After finishing these manipulations, we will turn all $\tilde{\epsilon}_i$ to ϵ_i , to exclude the contributions from the dilaton and 2-form.

Now we use the CHY formula to explain that such an expansion is also correct when two external legs are massive (both GR and BAS amplitudes containing two massive legs). In the CHY formula, the GR and BAS amplitudes for massless external legs arise as the contour integrals [18–22]

$$\mathcal{A}_{\text{GR}}^{\epsilon, \tilde{\epsilon}}(1, \dots, n) = \int d\mu_n \mathbf{Pf}' \Psi_n^{\epsilon} \mathbf{Pf}' \Psi_n^{\tilde{\epsilon}},$$

$$\mathcal{A}_{\text{BAS}}(1\sigma n | 1\sigma' n) = \int d\mu_n PT_n(1\sigma n) PT_n(1\sigma' n), \quad (2)$$

with the universal measure $d\mu_n$ given by

$$d\mu_n \equiv \frac{(\prod_{j=1, j \neq p, q, r}^n dz_j) |pqr|^2}{\prod_{i=1, i \neq p, q, r}^n \mathcal{E}_i(z)}, \quad (3)$$

where $|pqr| \equiv z_{pq} z_{qr} z_{rp}$, with $z_{ij} \equiv z_i - z_j$. The poles of the contour are determined by the so-called scattering equations $\mathcal{E}_i(z)$, which are given by

$$\mathcal{E}_i(z) \equiv \sum_{j \neq i} \frac{2k_i \cdot k_j}{z_{ij}} = 0. \quad (4)$$

To define the reduced Pfaffian $\mathbf{Pf}' \Psi_n^{\epsilon}$, the three matrices A , B and C are introduced as follows:

$$A_{ij} = \begin{cases} \frac{k_i \cdot k_j + \Delta_{ij}}{z_{ij}} & i \neq j, \\ 0 & i = j, \end{cases}$$

$$B_{ij} = \begin{cases} \frac{\epsilon_i \cdot \epsilon_j}{z_{ij}} & i \neq j, \\ 0 & i = j, \end{cases}$$

$$C_{ij} = \begin{cases} \frac{k_i \cdot \epsilon_j + \eta_{ij}}{z_{ij}} & i \neq j, \\ -\sum_{l=1, l \neq j}^n C_{li} & i = j. \end{cases} \quad (5)$$

The $2n \times 2n$ antisymmetric matrix Ψ_n^ϵ can be constructed from the matrices A , B and C , in the following form:

$$\Psi_n^\epsilon = \begin{pmatrix} A & C \\ -C^T & B \end{pmatrix}. \quad (6)$$

The reduced Pfaffian of Ψ_n^ϵ is defined as $\mathbf{Pf}'\Psi_n^\epsilon = \frac{(-)^{i+j}}{z_{ij}}$ $\mathbf{Pf}(\Psi_n^\epsilon)_{ij}^{ij}$, where the notation $(\Psi_n^\epsilon)_{ij}^{ij}$ means the i th and j th rows and columns in the matrix Ψ have been removed (with $1 \leq i < j \leq n$), and $\mathbf{Pf}(\Psi_n^\epsilon)_{ij}^{ij}$ stands for the Pfaffian of the antisymmetric matrix $(\Psi_n^\epsilon)_{ij}^{ij}$. Analogous notation holds for $\mathbf{Pf}'\Psi_n^{\tilde{\epsilon}}$. The Parke–Taylor factor for the ordering $(1\sigma n)$ is given by

$$PT_n(1\sigma n) = \frac{1}{z_{1\sigma_2} z_{\sigma_2\sigma_3} \cdots z_{\sigma_{n-1}n} z_{n1}}. \quad (7)$$

From the CHY point of view, expanding the GR amplitude to BAS ones can be understood as the expansions

$$\mathbf{Pf}'\Psi_n^\epsilon = \sum_{\sigma} C^\epsilon(\sigma) PT_n(1\sigma n),$$

$$\mathbf{Pf}'\Psi_n^{\tilde{\epsilon}} = \sum_{\sigma'} C^{\tilde{\epsilon}}(\sigma') PT_n(1\sigma' n). \quad (8)$$

When two external legs are massive, the Parke–Taylor factors $PT_n(1\sigma n)$ and $PT_n(1\sigma' n)$ will not be altered [16, 17, 49]. On the other hand, if one chooses the removed rows and columns for the reduced Pfaffians $\mathbf{Pf}'\Psi_n^\epsilon$ and $\mathbf{Pf}'\Psi_n^{\tilde{\epsilon}}$ to be two massive legs, two reduced Pfaffians are also unmodified [16, 17, 49]. Thus one can conclude that the expansions in (8) still hold. When two external legs are massive, the measure part $d\mu_n$ will be altered due to the modification for the scattering equations [16, 17, 49]. But it is still universal if the massive legs have the same nodes, i.e., if the i th and j th legs are massive gravitons for the GR amplitude, where i and j also denote massive scalars for the BAS amplitudes. Combining this fact with the expansions in (8), we arrive at the conclusion that the expansion in (1) is valid for amplitudes with two massive legs, if the massive legs have the same nodes.

Thus, one can use the expansion in (1) to compute the GR amplitude. To do so, one needs to calculate the BAS amplitudes which serve as the basis and the coefficients $C^\epsilon(\sigma)$ and $C^{\tilde{\epsilon}}(\sigma')$ in the expansion.

To calculate BAS amplitudes with two massive legs, we employ the method proposed by Cachazo, He and Yuan in

[20]. For a BAS amplitude whose double color-orderings are given, this method provides the corresponding Feynman diagrams and the overall sign directly from the color-orderings. We find this method to be effective, since the Feynman diagrams for scalar theory carry no gauge redundancy. Furthermore, the double color-orderings reduce the number of diagrams greatly, since only configurations satisfy two orderings simultaneously are allowed. To see how the double color-orderings restrict the number of diagrams, let us consider the 5-point example $\mathcal{A}_{\text{BAS}}(12345|14235)$. In Fig. 2, the first diagram satisfies both two color-orderings (12345) and (14235), while the second one satisfies the ordering (12345) but not (14235). Thus, the first diagram is allowed by the double color-orderings (12345|14235), while the second one is not. It is easy to see that other diagrams are also forbidden by the ordering (14235); thus the first diagram in Fig. 2 is the only diagram that contributes to the amplitude $\mathcal{A}_{\text{BAS}}(12345|14235)$. Therefore, in this example, the number of Feynman diagrams is markedly reduced by two color-orderings.

The Feynman diagrams for a given BAS amplitude can be obtained via a systematic diagrammatic rule. For the above example, one can draw a disk diagram as follows.

- Draw points on the boundary of the disk according to the first ordering (12345).
- Draw a loop of line segments connecting the points according to the second ordering (14235).

The obtained disk diagram is shown in Fig. 3. From the diagram, one can see that two orderings share the boundaries $\{1, 5\}$ and $\{2, 3\}$. These co-boundaries indicate channels $\frac{1}{s_{15}}$ and $\frac{1}{s_{23}}$; therefore we have the first Feynman diagram in Fig. 2. Then the BAS amplitude $\mathcal{A}_{\text{BAS}}(12345|14235)$ can be computed as

$$\mathcal{A}_{\text{BAS}}(12345|14235) = \frac{1}{s_{23}} \frac{1}{s_{15}}, \quad (9)$$

up to an overall sign. In this paper the kinematic variables $s_{ij\dots k}$ are defined as

$$s_{ij\dots k} = (k_i + k_j + \dots + k_k)^2 - (k_i^2 + k_j^2 + \dots + k_k^2). \quad (10)$$

The advantage of this definition is that the propagators expressed by $\frac{1}{s_{ij\dots k}}$ are valid for both massless amplitudes and amplitudes with two massive external legs, in which we are interested, as can be seen through the following discussion. The typical Feynman diagram for BAS amplitudes with two massive scalars is shown in Fig. 4. Legs 1 and n are assumed to be massive, with $P_1^2 = P_n^2 = m^2$. To distinguish them from massless particles, we have introduced P_1 and P_n to denote two massive momenta. From Fig. 4, one can observe

Fig. 2 Two 5-point diagrams **Fig. 3** Diagram for $\mathcal{A}_{\text{BAS}}(1, 2, 3, 4, 5 | 1, 4, 2, 3, 5)$ **Fig. 4** Typical diagram for BAS amplitudes with two massive scalars, the bold line representing the massive particle

that each massive virtual particle provides the propagator in the form

$$\frac{1}{(P_1 + k_2 + k_3 + \dots)^2 - m^2} = \frac{1}{(P_1 + k_2 + k_3 + \dots)^2 - P_1^2} = \frac{1}{s_{123\dots}}. \quad (11)$$

Thus, the formula $\frac{1}{s_{ij\dots k}}$ is the correct expression for any propagator in the case of two massive scalars. Notice that when two massive legs belong to α where α is a subset of external legs $\{1, \dots, n\}$, we must use $s_{\{1, \dots, n\} \setminus \alpha}$ rather than s_α , since the second one cannot reproduce the correct propagator corresponding to the channel α . For the other cases, it is easy to prove that $s_\alpha = s_{\{1, \dots, n\} \setminus \alpha}$.

As another example, let us consider the BAS amplitude $\mathcal{A}_{\text{BAS}}(12345 | 12435)$. The corresponding disk diagram is

shown in Fig. 5, and one can see that two orderings have co-boundaries $\{3, 4\}$ and $\{5, 1, 2\}$. The co-boundary $\{3, 4\}$ indicates the channel $\frac{1}{s_{34}}$. The co-boundary $\{5, 1, 2\}$ indicates the channel $\frac{1}{s_{512}}$, as well as sub-channels $\frac{1}{s_{12}}$ and $\frac{1}{s_{51}}$ ($\frac{1}{s_{52}}$ is forbidden by both two orderings). Notice that the channel $\frac{1}{s_{512}}$ is equivalent to $\frac{1}{s_{34}}$. Using the above decomposition, one can calculate $\mathcal{A}_{\text{BAS}}(12345 | 12435)$ as

$$\mathcal{A}_{\text{BAS}}(12345 | 12435) = \frac{1}{s_{34}} \left(\frac{1}{s_{12}} + \frac{1}{s_{234}} \right), \quad (12)$$

up to an overall sign.

The overall sign can be fixed by the following rule.

- Each polygon with an odd number of vertices contributes a plus sign if its orientation is the same as that of the disk and a minus sign if opposite.
- Each polygon with an even number of vertices always contributes a minus sign.
- Each intersection point contributes a minus sign.

We can apply this rule to the previous examples. In Fig. 3, the polygons are three triangles, namely $51A$, $A4B$ and $B23$, which contribute $+$, $-$, $+$, respectively, while two intersection points A and B contribute two $-$. In Fig. 5, the polygons are $512A$ and $A43$, which contribute two $-$, while the intersection point A contributes $-$. Then we arrive at the full results:

$$\begin{aligned} \mathcal{A}_{\text{BAS}}(12345 | 14235) &= -\frac{1}{s_{23}} \frac{1}{s_{234}}, \\ \mathcal{A}_{\text{BAS}}(12345 | 12435) &= -\frac{1}{s_{34}} \left(\frac{1}{s_{12}} + \frac{1}{s_{234}} \right). \end{aligned} \quad (13)$$

The BCJ numerators $C^\epsilon(\sigma)$ and $\tilde{C}^\epsilon(\sigma')$ in (1) can be obtained by the rule provided in [31, 38, 39]. To explain this rule, we chose a reference ordering $n < j_2 < \dots < j_n$, with n being fixed at the lowest position. We denote the reference ordering as $\mathcal{R}$, and denote the color-ordering $(1\sigma n)$ as $1 < \sigma_2 < \dots < \sigma_{n-1} < n$. Then the so-called ordered splittings for the ordering $(1\sigma n)$ can be constructed via the following procedure.

- At the first step, we construct all possible ordered subsets $\alpha_0 = \{1, \alpha_2^0, \dots, \alpha_{|0|-1}^0, n\}$, which satisfy two con-

Fig. 5 Diagram for $\mathcal{A}_{\text{BAS}}(1, 2, 3, 4, 5|1, 2, 4, 3, 5)$

ditions, (1) $\alpha_0 \subset \{1, \dots, n\}$, (2) $\alpha_2^0 \prec \alpha_3^0 \prec \dots \prec \alpha_{|0|-1}^0$, respecting to the color-ordering $1 \prec \sigma_2 \prec \dots \prec \sigma_{n-1} \prec n$. Here $|i|$ stands for the length of the set α_i . We call each ordered subset α_0 a root.

- For each root α_0 , we eliminate its elements in $\{1, \dots, n\}$ and $\mathcal{R}$, resulting in a reduced set $\{1, \dots, n\} \setminus \alpha_0$, and a reduced reference ordering $\mathcal{R} \setminus \alpha_0$. Suppose R_1 is the lowest element in the reduced reference ordering $\mathcal{R} \setminus \alpha_0$; we construct all possible ordered subsets α_1 as $\alpha_1 = \{\alpha_1^1, \alpha_2^1, \dots, \alpha_{|1|-1}^1, R_1\}$, with $\alpha_1^1 \prec \alpha_2^1 \prec \dots \prec \alpha_{|1|-1}^1 \prec R_1$, with regard to the color-ordering.
- By iterating the second step, one can construct $\alpha_2, \alpha_3, \dots$, until $\alpha_0 \cup \alpha_1 \cup \dots \cup \alpha_r = \{1, \dots, n\}$.

Each ordered splitting is given as an ordered set $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$, where the ordered sets α_i serve as elements. For a given ordered splitting, the root α_0 corresponds to the kinematic factor

$$(-)^{|\alpha_0|} (\epsilon_1 \cdot f_{\alpha_2^0} \cdot f_{\alpha_3^0} \cdots f_{\alpha_{|0|-1}^0} \cdot \epsilon_n), \quad (14)$$

while the other ordered sets α_i with $i \neq 0$ correspond to

$$\epsilon_{R_i} \cdot f_{\alpha_{|i|-1}^i} \cdots f_{\alpha_2^i} \cdot f_{\alpha_1^i} \cdot Z_{\alpha_1^i}. \quad (15)$$

In the above factors, the tensor $f_i^{\mu\nu}$ is defined by

$$f_i^{\mu\nu} \equiv k_i^\mu \epsilon_i^\nu - \epsilon_i^\mu k_i^\nu. \quad (16)$$

The combined momentum $Z_{\alpha_1^i}$ is the sum of momenta of external legs satisfying two conditions: (1) legs at the LHS of α_1^i in the color-ordering, (2) legs belonging to α_j at the LHS of α_i in the ordered splitting, i.e., $j < i$. The coefficient $C^\epsilon(\sigma)$ is the sum of contributions from all correct ordered splittings. An analogous algorithm holds for evaluating $C^{\tilde{\epsilon}}(\sigma')$.

To illustrate the procedure more clearly, let us consider the 4-point BCJ numerator $C^\epsilon(32)$ for the color-ordering $1 \prec 3 \prec 2 \prec 4$. The reference ordering is chosen to be $4 \prec$

$3 \prec 2 \prec 1$. The roots α_0 have the following candidates: $\{1, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 3, 2, 4\}$. The ordered set $\{1, 2, 3, 4\}$ violates the color-ordering $3 \prec 2$ and therefore can be excluded. For the root $\alpha_0 = \{1, 4\}$, the lowest element in the reduced reference ordering $3 \prec 2$ is 3; then one can construct $\alpha_1 = \{3\}$ or $\alpha_1 = \{2, 3\}$. However, the ordered set $\{2, 3\}$ violates the color-ordering $3 \prec 2$ and therefore is forbidden. Thus, we obtain the ordered splitting $\{\{1, 4\}, \{3\}, \{2\}\}$ for the root $\{1, 4\}$. Similarly, one can get $\{\{1, 2, 4\}, \{3\}\}$, $\{\{1, 3, 4\}, \{2\}\}$ and $\{\{1, 3, 2, 4\}\}$ for the other roots. After giving kinematic factors for each of these splittings, the BCJ numerator $C^\epsilon(32)$ is found to be

$$(\epsilon_1 \cdot \epsilon_4)(\epsilon_3 \cdot Z_3)(\epsilon_2 \cdot Z_2) - (\epsilon_1 \cdot f_2 \cdot \epsilon_4)(\epsilon_3 \cdot Z_3) - (\epsilon_1 \cdot f_3 \cdot \epsilon_4)(\epsilon_2 \cdot Z_2) + (\epsilon_1 \cdot f_3 \cdot f_2 \cdot \epsilon_4). \quad (17)$$

The coefficients $C^\epsilon(\sigma)$ and $C^{\tilde{\epsilon}}(\sigma')$, together with the BAS amplitudes $\mathcal{A}_{\text{BAS}}(1\sigma n|1\sigma' n)$ with two massive scalars, provide the GR amplitude $\mathcal{A}_{\text{GR}}^{\epsilon, \tilde{\epsilon}}(1, \dots, n)$ with two massive gravitons, via the expansion in (1).

2.2 Converting gravitons to scalars or photons

After obtaining the GR amplitude with two massive gravitons, one can convert gravitons to scalars or photons through the dimensional reduction procedure, or by applying differential operators to the GR amplitude, to get the desired SG and SPG amplitudes.

To get the SG amplitude with two massive scalars in D dimensions, an effective way is to consider the GR amplitude with two massive gravitons in $D+1$ dimensions. Roughly speaking, this method is to choose the polarization vectors of two massive gravitons to be in the extra dimension, while all other Lorentz vectors lie in D dimensions [17, 22]. More explicitly, one can set the momenta and polarization vectors of the external legs for the GR amplitude to be

$$\begin{aligned} P_1^\mu &= (P_1^0, P_1^1, \dots, P_1^{D-1}|0), \quad \epsilon_1^\mu = (0, 0, \dots, 0|1), \\ \tilde{\epsilon}_1^\mu &= (0, 0, \dots, 0|1), \\ P_n^\mu &= (P_n^0, P_n^1, \dots, P_n^{D-1}|0), \quad \epsilon_n^\mu = (0, 0, \dots, 0|1), \\ \tilde{\epsilon}_n^\mu &= (0, 0, \dots, 0|1), \\ k_a^\mu &= (k_a^0, k_a^1, \dots, k_a^{D-1}|0), \\ \epsilon_a^\mu &= (\epsilon_a^0, \epsilon_a^1, \dots, \epsilon_a^{D-1}|0), \\ \tilde{\epsilon}_a^\mu &= (\tilde{\epsilon}_a^0, \tilde{\epsilon}_a^1, \dots, \tilde{\epsilon}_a^{D-1}|0), \end{aligned} \quad (18)$$

where P_1 and P_n are massive momenta satisfying $P_1^2 = P_n^2 = m^2$, k_a are massless momenta with $a \in \{2, \dots, n-1\}$. For each vector, the components at the LHS of $|$ lie in D dimensions, while the component at the RHS of $|$ lies in the extra dimension. Under the above choices, two massive gravitons behave as two massive scalars in D dimensions, thus the goal is achieved. This procedure is called dimensional reduction. An equivalent approach is to apply differ-

Fig. 6 Converting two massive gluons to scalars when the number of gluons is 3

ential operators:

$$\mathcal{T}^\epsilon[1n] \equiv \frac{\partial}{\partial(\epsilon_1 \cdot \epsilon_n)}, \quad \mathcal{T}^{\tilde{\epsilon}}[1n] \equiv \frac{\partial}{\partial(\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_n)} \quad (19)$$

to the GR amplitude, as proposed in [42] and proved in [43, 44]. To exclude contributions from the dilaton and 2-form, one needs to turn all $\tilde{\epsilon}_i$ to ϵ_i at the final step.

To obtain the SPG amplitude with two massive scalars and one massless photon, we first apply the operator

$$\mathcal{T}^{\tilde{\epsilon}}[1bn] \equiv \left(\frac{\partial}{\partial(P_1 \cdot \tilde{\epsilon}_b)} - \frac{\partial}{\partial(P_n \cdot \tilde{\epsilon}_b)} \right) \mathcal{T}^{\tilde{\epsilon}}[1n] \quad (20)$$

to the GR amplitude, to generate the single trace EYM amplitude $\mathcal{A}_{\text{EYM}}^{\epsilon, \tilde{\epsilon}}(1bn; \{1, \dots, n\} \setminus \{1, b, n\})$, with two massive gluons 1, n and one massless gluon b , as well as $n - 3$ gravitons in the set $\{1, \dots, n\} \setminus \{1, b, n\}$. Then we use the operator $\mathcal{T}^\epsilon[1n]$ to convert the massive gluons 1, n to scalars. After this manipulation, the remaining gluon b can be identified as a photon. The reason is that, after converting two massive gluons to scalars, the vector boson b attached to the vertex including two scalars and b , bears the same structure as the photon–scalar vertex, as shown in Fig. 6 (The color-ordered partial amplitude does not contain any group-theoretic factor). When there is only one external gluon, the self-interaction of gluons does not occur. Thus, from the angle of scattering amplitudes, a gluon cannot be distinguished from a photon before adding the coupling constants. Consequently, we arrive at the SPG amplitude whose external particles are two scalars, one photon and $n - 3$ gravitons¹.

For the SPG amplitude with two massive scalars and one photon, it is hard to find a dimensional reduction manipulation which is equivalent to applying the differential operators, since the insertion operator

$$\mathcal{T}_{1bn}^{\tilde{\epsilon}} \equiv \frac{\partial}{\partial(P_1 \cdot \tilde{\epsilon}_b)} - \frac{\partial}{\partial(P_n \cdot \tilde{\epsilon}_b)} \quad (21)$$

cannot be interpreted directly by dimensional reduction.

It is worth to emphasize that the dimensional reduction technique and the method of applying differential operators,

¹ One may worry about the color-ordering ($1bn$) from the single trace EYM amplitude, but for three elements any way of changing the ordering leads to nothing but an overall sign, thus this color-ordering has no physical effect.

Fig. 7 Converting two massive gluons to scalars when the number of gluons is larger than 3

which are originally proposed for massless amplitudes, are also valid for amplitudes with two massive legs, as explained in [50].

Let us explain why it is hard to calculate the SPG amplitude with more than one photons by the method mentioned above. Suppose we naively apply this method, i.e., perform the trace operator $\mathcal{T}^{\tilde{\epsilon}}[1bc \dots n]$ to the GR amplitude to create the single trace EYM amplitude with more gluons; the self-interactions of gluons cannot then be removed by converting two gluons to scalars, as shown in Fig. 7. This fact indicates that the remaining external gluons can never be identified as photons. But one can ask if there are other ways to turn gravitons to photons. As is well known, the photon–graviton amplitude can be generated from the GR amplitude by dimensional reduction [22], or by applying the operator [43]

$$\mathcal{T}_{X_{2m}}^{\tilde{\epsilon}} \equiv \sum_{\rho \in \text{pair}} \prod_{i_k, j_k \in \rho} \mathcal{T}^{\tilde{\epsilon}}[i_k, j_k]. \quad (22)$$

Here the set

$$\rho = \{(i_1, j_1), \dots, (i_m, j_m)\} \quad (23)$$

is a partition of the length- $2m$ set including two massive scalars and all photons into pairs, with $i_1 < i_2 < \dots < i_m$ and $i_t < j_t, \forall t$. The summation is over all partitions ρ . For the case under consideration in this paper, it is hard to use the above manipulation to achieve the desired amplitude with correct coupling constants. On using the above method, the contributions from vertices describing that two photons interact with one graviton will occur, as shown in Fig. 8, a problem arises. For two types of vertices which include photons, as shown in Fig. 9, the coupling constants are different. Then different diagrams with the same external legs can carry different coupling constants, for instance two diagrams in Fig. 10. Since our method only concerns external states of amplitudes, and the coupling constants are omitted, how to separate different pieces with different coupling constants from a full amplitude becomes a difficult question. This is why we do

Fig. 8 Photons interact with gravitons

Fig. 9 Two vertices which contain photons

Fig. 10 Two diagrams with the same external legs and different coupling constants

not employ the dimensional reduction or the operator $\mathcal{T}_{X_{2m}}^{\tilde{\epsilon}}$ to convert gravitons to photons.²

Before ending this section, we point out there are alternative choices of expansions. For example, one can calculate the EYM amplitude with two massive and one massless gluons by expanding it to BAS amplitudes, then convert two massive gluons to scalars, to get the SPG amplitude with two massive scalars, one photon and $n - 3$ gravitons. The algorithm for evaluating coefficients for the EYM amplitude can be generated from the algorithm for computing coefficients for the GR amplitude via the differential operator $\mathcal{T}^{\tilde{\epsilon}}[1bn]$, as can be found in [31]. Thus, the two methods are totally equivalent, since the differential operators only act on coefficients rather than the basis in the expanded formula of the GR amplitude (1).

² In practical, if the number of photons is odd, there is another problem for the dimensional reduction method, or equivalently applying $\mathcal{T}_{X_{2m}}^{\tilde{\epsilon}}$. When the number of photons is odd, at least one polarization vector ϵ_b^μ lies in the extra dimension and will contract with a vector the lies in D dimensions, thus the result will be 0.

3 Scalar-graviton examples

In this section, we use the method introduced in the previous section, to calculate SG amplitudes with two massive scalars and $n - 2$ gravitons. Before going to examples, let us do a little simplification of the algorithm. The dimensional reduction manipulation and the differential operators only act on two coefficients in the expanded GR amplitude in (1). The effect of them is turning all $\epsilon_1 \cdot \epsilon_n$ and $\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_n$ to 1, while annihilating all other terms that do not contain both $\epsilon_1 \cdot \epsilon_n$ and $\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_n$. Thus, one only needs to consider ordered splittings with the root $\{1, n\}$, and turn the corresponding terms in $C^\epsilon(\sigma)$ and $C^{\tilde{\epsilon}}(\sigma')$ to $C^\epsilon(\sigma)$ and $C^{\tilde{\epsilon}}(\sigma')$, by setting $\epsilon_1 \cdot \epsilon_n = 1, \tilde{\epsilon}_1 \cdot \tilde{\epsilon}_n = 1$, respectively. Then we arrive at

$$\begin{aligned} \tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, \dots, (n-1)_h, \mathbf{n}_\varphi) \\ = \sum_{\sigma} \sum_{\sigma'} C^\epsilon(\sigma) \mathcal{A}_{\text{BAS}}(\mathbf{1}\sigma\mathbf{n}|\mathbf{1}\sigma'\mathbf{n}) C^{\tilde{\epsilon}}(\sigma'). \end{aligned} \quad (24)$$

In this and the next sections, we use a bold number to denote massive external particles. The notation $\tilde{\mathcal{A}}_{\text{SG}}$ stands for the SG amplitude, which includes contributions from the dilaton and 2-form. The last step is turning all $\tilde{\epsilon}_i$ to ϵ_i , to obtain the result

$$\begin{aligned} \mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, \dots, (n-1)_h, \mathbf{n}_\varphi) \\ = \sum_{\sigma} \sum_{\sigma'} C^\epsilon(\sigma) \mathcal{A}_{\text{BAS}}(\mathbf{1}\sigma\mathbf{n}|\mathbf{1}\sigma'\mathbf{n}) C^\epsilon(\sigma'). \end{aligned} \quad (25)$$

3.1 4-Point amplitude $\mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, \mathbf{4}_\varphi)$

The simplest example is the 4-point amplitude $\mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, \mathbf{4}_\varphi)$ with two massive scalars $\mathbf{1}_\varphi, \mathbf{4}_\varphi$, and two massless gravitons $2_h, 3_h$. This amplitude has been calculated in [16]. Our method leads to the equivalent expression.

According to the expansion in (25), we need to calculate the 4-point BAS amplitudes $\mathcal{A}_{\text{BAS}}(\mathbf{1}\sigma\mathbf{4}|\mathbf{1}\sigma'\mathbf{4})$ and the coefficients $C^\epsilon(\sigma)$ and $C^{\tilde{\epsilon}}(\sigma')$. Using the diagrammatic technique introduced in Sect. 2, the 4-point BAS amplitudes with two massive scalars $\mathbf{1}$ and $\mathbf{4}$ fixed at two ends in the color-orderings can be calculated as

$$\begin{aligned} \mathcal{A}_{\text{BAS}}(\mathbf{1}234|\mathbf{1}234) &= -\frac{1}{s_{12}} - \frac{1}{s_{23}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{1}234|\mathbf{1}324) &= \mathcal{A}_{\text{BAS}}(\mathbf{1}324|\mathbf{1}234) = \frac{1}{s_{23}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{1}324|\mathbf{1}324) &= -\frac{1}{s_{13}} - \frac{1}{s_{23}}. \end{aligned} \quad (26)$$

The coefficients can be computed by finding all correct ordered splittings, as discussed in Sect. 2. The expressions for the coefficients are given by

$$\begin{aligned} C^\epsilon(23) &= (\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1, \\ C^\epsilon(32) &= (\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13}), \end{aligned} \quad (27)$$

with the reference ordering $4 < 3 < 2 < 1$. Substituting these ingredients into the expansion (24), we get the desired SG amplitude expressed as

$$\begin{aligned} \mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, \mathbf{4}_\varphi) &= -\left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1\right)^2 \left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right) \\ &\quad + \frac{2\left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1\right)\left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13})\right)}{s_{23}} \\ &\quad - \left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13})\right)^2 \left(\frac{1}{s_{13}} + \frac{1}{s_{23}}\right). \end{aligned} \quad (28)$$

It is straightforward to verify the equivalence between this expression and the formula obtained in [16].

3.2 5-Point amplitude $\mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, 4_h, \mathbf{5}_\varphi)$

Then we consider the 5-point example $\mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, 4_h, \mathbf{5}_\varphi)$, with two massive scalars $\mathbf{1}_\varphi, \mathbf{5}_\varphi$, and three massless gravitons $2_h, 3_h, 4_h$. The 5-point BAS amplitudes with two massive legs $\mathbf{1}, \mathbf{5}$ and color-orderings (12345|1 σ 5) can be computed as

$$\begin{aligned} \mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{12345}) &= \frac{1}{s_{234}s_{34}} + \frac{1}{s_{12}s_{34}} \\ &\quad + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{23}s_{234}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{12435}) &= -\frac{1}{s_{34}s_{234}} - \frac{1}{s_{12}s_{34}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{13245}) &= -\frac{1}{s_{23}s_{45}} - \frac{1}{s_{23}s_{234}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{13425}) &= -\frac{1}{s_{34}s_{234}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{14235}) &= -\frac{1}{s_{23}s_{234}}, \\ \mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{14325}) &= \frac{1}{s_{234}s_{34}} + \frac{1}{s_{234}s_{23}}. \end{aligned} \quad (29)$$

Other 5-point BAS amplitudes can be obtained from them by changing nodes, for example, $\mathcal{A}_{\text{BAS}}(\mathbf{13245}|\mathbf{12435})$ can be generated from $\mathcal{A}_{\text{BAS}}(\mathbf{12345}|\mathbf{13425})$ via the replacement $2 \rightarrow 3, 3 \rightarrow 2$. The coefficients $C^\epsilon(\sigma)$ can be calculated as

$$\begin{aligned} C^\epsilon(234) &= (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) \\ &\quad + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_1) \\ &\quad + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_1) \\ &\quad + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{12}) \\ &\quad + (\epsilon_4 \cdot f_3 \cdot f_2 \cdot P_1), \\ C^\epsilon(243) &= (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14})(\epsilon_2 \cdot P_1) \\ &\quad + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{124}), \\ C^\epsilon(324) &= (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13}) \\ &\quad + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_1) \end{aligned}$$

$$\begin{aligned} &\quad + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_{13}) \\ &\quad + (\epsilon_4 \cdot f_2 \cdot f_3 \cdot P_1), \\ C^\epsilon(342) &= (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{134}) \\ &\quad + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_{134}), \\ C^\epsilon(423) &= (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14})(\epsilon_2 \cdot P_{14}) \\ &\quad + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_{14}), \\ C^\epsilon(432) &= (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14})(\epsilon_2 \cdot P_{134}), \end{aligned} \quad (30)$$

with the reference ordering $5 < 4 < 3 < 2 < 1$. Substituting (29) and (30) into (25), we arrive at the SG amplitude with two massive scalars, as exhibited in Appendix A.

Higher point SG amplitudes with two massive scalars follow by the same method. Although the lengths of the expressions for higher points amplitudes are large, the calculation can be easily realized in MATHEMATICA.

4 Scalar-photon-graviton examples

In this section we consider SPG amplitudes with two massive scalars, one massless photon and $n - 3$ massless gravitons. As discussed in Sect. 2, such amplitudes can be obtained by applying $\mathcal{T}^\epsilon[1n]$ and $\tilde{\mathcal{T}}^\epsilon[1bn] \equiv \tilde{\mathcal{I}}_{1bn}^\epsilon \mathcal{T}^\epsilon[1n]$ to the GR amplitude, then turning all $\tilde{\epsilon}_i$ to ϵ_i . Performing $\mathcal{T}^\epsilon[1n]$ and $\tilde{\mathcal{T}}^\epsilon[1n]$ to the GR amplitude gives rise to the SG amplitude $\tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, \dots, (n-1)_h, \mathbf{n}_\varphi)$ discussed in the previous section, thus a more efficient way is to apply the insertion operator $\tilde{\mathcal{I}}_{1bn}^\epsilon$ to $\tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, \dots, (n-1)_h, \mathbf{n}_\varphi)$, and then turn all $\tilde{\epsilon}_i$ to ϵ_i .

4.1 4-point amplitude $\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, \mathbf{4}_\varphi)$

Our first example is the 4-Point amplitude $\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, \mathbf{4}_\varphi)$, with two massive scalars $\mathbf{1}_\varphi, \mathbf{4}_\varphi$, one massless photon 2_p , and one massless graviton 3_h . As discussed above, we will apply the operator

$$\tilde{\mathcal{I}}_{124}^\epsilon \equiv \frac{\partial}{\partial(\tilde{\epsilon}_2 \cdot P_1)} - \frac{\partial}{\partial(\tilde{\epsilon}_2 \cdot k_4)}, \quad (31)$$

to the amplitude $\tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, \mathbf{4}_\varphi)$. Using the BAS amplitudes in (26) and the coefficients in (27) (coefficients $C^{\tilde{\epsilon}}(\sigma)$ can be generated from $C^\epsilon(\sigma)$ by turning all ϵ_i to $\tilde{\epsilon}_i$), we have

$$\begin{aligned} \tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, \mathbf{4}_\varphi) &= -\left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1\right) \left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right) \\ &\quad \left((\tilde{\epsilon}_3 \cdot P_1)(\tilde{\epsilon}_2 \cdot P_1) + \tilde{\epsilon}_3 \cdot \tilde{f}_2 \cdot P_1\right) \\ &\quad + \frac{\left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1\right) \left((\tilde{\epsilon}_3 \cdot P_1)(\tilde{\epsilon}_2 \cdot P_{13})\right)}{s_{23}} \end{aligned}$$

$$\begin{aligned}
& + \frac{((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13}))((\tilde{\epsilon}_3 \cdot P_1)(\tilde{\epsilon}_2 \cdot P_1) + \tilde{\epsilon}_3 \cdot \tilde{f}_2 \cdot P_1)}{s_{23}} \\
& - \left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13}) \right) \left(\frac{1}{s_{13}} + \frac{1}{s_{23}} \right) ((\tilde{\epsilon}_3 \cdot P_1)(\tilde{\epsilon}_2 \cdot P_{13})). \quad (32)
\end{aligned}$$

After applying $\tilde{\mathcal{I}}_{124}^{\tilde{\epsilon}}$, and turning all $\tilde{\epsilon}_i$ to ϵ_i , we get the result

$$\begin{aligned}
\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, \mathbf{4}_\varphi) = & - \left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1 \right) \left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) ((\epsilon_3 \cdot P_1) + \epsilon_3 \cdot k_2) \\
& + \frac{((\epsilon_3 \cdot P_1) + \epsilon_3 \cdot k_2)((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13})) + ((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + \epsilon_3 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_1)}{s_{23}} \\
& - \left((\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13}) \right) \left(\frac{1}{s_{13}} + \frac{1}{s_{23}} \right) (\epsilon_3 \cdot P_1). \quad (33)
\end{aligned}$$

4.2 5-Point amplitude $\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, 4_h, \mathbf{5}_\varphi)$

The next example is the 5-point amplitude $\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, 4_h, \mathbf{5}_\varphi)$, with two massive scalars $\mathbf{1}_\varphi, \mathbf{5}_\varphi$, one massless photon 2_p , and two massless gravitons $3_h, 4_h$. The computation of this amplitude follows in a similar way. Substituting BAS amplitudes in (29) and the coefficients in (30) into (24), one can get the SG amplitude $\tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, 4_h, \mathbf{5}_\varphi)$. Then the amplitude $\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, 4_h, \mathbf{5}_\varphi)$ can be obtained by applying the insertion operator $\tilde{\mathcal{I}}_{125}^{\tilde{\epsilon}}$ to $\tilde{\mathcal{A}}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, 4_h, \mathbf{5}_\varphi)$ and turning all $\tilde{\epsilon}_i$ to ϵ_i . The full result of $\mathcal{A}_{\text{SPG}}(\mathbf{1}_\varphi, 2_p, 3_h, 4_h, \mathbf{5}_\varphi)$ is shown in Appendix A. Higher point SPG amplitudes with two massive scalars and one photon can be calculated by the same method.

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Appendix A: Full results for 5-point examples

In this appendix, we show the full results of two 5-point examples.

The SG amplitude $\mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, 4_h, \mathbf{5}_\varphi)$ is given as follows:

$$\mathcal{A}_{\text{SG}}(\mathbf{1}_\varphi, 2_h, 3_h, 4_h, \mathbf{5}_\varphi) = \mathbf{A}_1 + \mathbf{A}_2, \quad (\text{A1})$$

where

$$\begin{aligned}
\mathbf{A}_1 = & ((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) \\
& + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_1) \\
& + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_1) \\
& + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{12}) \\
& + (\epsilon_4 \cdot f_3 \cdot f_2 \cdot P_1)) \left(\frac{1}{s_{234}s_{34}} + \frac{1}{s_{12}s_{34}} \right. \\
& \left. + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{23}s_{234}} \right) \\
& - 2((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_1) \\
& + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_1) + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{12}) \\
& + (\epsilon_4 \cdot f_3 \cdot f_2 \cdot P_1)) \left(\frac{1}{s_{34}s_{234}} + \frac{1}{s_{12}s_{34}} \right) \\
& ((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14})(\epsilon_2 \cdot P_1) \\
& + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{124})) \\
& - 2((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_1) \\
& + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_1) + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{12}) \\
& + (\epsilon_4 \cdot f_3 \cdot f_2 \cdot P_1)) \left(\frac{1}{s_{23}s_{45}} \right. \\
& \left. + \frac{1}{s_{23}s_{234}} \right) ((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{13}) \\
& + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_1) \\
& + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_{13}) \\
& + (\epsilon_4 \cdot f_2 \cdot f_3 \cdot P_1)) \\
& - 2((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_1) + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_1) \\
& + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_1) + (\epsilon_4 \cdot f_2 \cdot P_1)(\epsilon_3 \cdot P_{12}) \\
& + (\epsilon_4 \cdot f_3 \cdot f_2 \cdot P_1)) \frac{1}{s_{34}s_{234}} ((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{134})
\end{aligned}$$

$$\begin{aligned}
& \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14})(\epsilon_2 \cdot P_{14}) + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot f_2 \cdot P_{14}) \right) \\
& - \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{134}) \right. \\
& \left. + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_{134}) \right) \frac{1}{s_{24}s_{234}} \\
& \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14}) + (\epsilon_4 \cdot P_1)(\epsilon_3 \cdot k_2) \right) \\
& - \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1) + (\epsilon_4 \cdot f_3 \cdot P_1) \right) \left(\frac{1}{s_{34}s_{25}} \right. \\
& \left. + \frac{1}{s_{34}s_{234}} \right) \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14})(\epsilon_2 \cdot P_{134}) \right) \\
& - \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_1)(\epsilon_2 \cdot P_{134}) + (\epsilon_4 \cdot f_3 \cdot P_1)(\epsilon_2 \cdot P_{134}) \right) \\
& \left(\frac{1}{s_{34}s_{25}} + \frac{1}{s_{34}s_{234}} \right) \left((\epsilon_4 \cdot P_1)(\epsilon_3 \cdot P_{14}) \right). \quad (A7)
\end{aligned}$$

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