Gravitational deflection of massive particles in Schwarzschild-de Sitter spacetime

Guansheng He^{1,a}, Xia Zhou², Zhongwen Feng², Xueling Mu³, Hui Wang^{4,5}, Weijun Li¹, Chaohong Pan^{1,b}, Wenbin Lin^{1,4,c}

¹ School of Mathematics and Physics, University of South China, Hengyang 421001, China

² Physics and Space Science College, China West Normal University, Nanchong 637009, China

³ School of information science and Engineering, Chengdu University, Chengdu 610106, China

⁴ School of Physical Science and Technology, Southwest Jiaotong University, Chengdu 610031, China

⁵ Xinjiang Astronomical Observatory, Chinese Academy of Sciences, Xinjiang 830011, China

Received: 26 June 2020 / Accepted: 18 August 2020 / Published online: 8 September 2020
© The Author(s) 2020

Abstract In this paper, the gravitational deflection of a relativistic massive neutral particle in the Schwarzschild-de Sitter spacetime is studied via the Rindler–Ishak method in the weak-field limit. When the initial velocity v_0 of the particle tends to the speed of light, the result is consistent with that obtained in the previous work for the light-bending case. Our result is reduced to the Schwarzschild deflection angle of massive particles up to the second order, if the contributions from the cosmological constant Λ are dropped. The observable correctional effects due to the deviation of v_0 from light speed on the Λ -induced contributions to the deflection angle of light are also analyzed.

1 Introduction

Since its original proposal by Einstein [1] for cosmological reasons in 1917, the cosmological constant Λ plays an important historical role in modern astronomy. Nowadays, much attention has been paid to it again with the consideration that the cosmological constant (or, more generally, dark energy) is regarded to be one of the main candidates responsible for the acceleration of the cosmic expansion (see, e.g., Refs. [2–12], and references therein). Being a special gravitational source, Λ should exhibit its effect in all of the gravitational phenomena such as the classical tests of general relativity. In 1983, Islam [13] placed constraints on the absolute value of Λ by calculating the advance of Mercury’s perihelion in the Kottler geometry [14] to get its upper limit. Meanwhile, the author gave a significant conclusion that the grav-

itational deflection of light is independent on Λ , which was later approved by many groups (see, for instance, Refs. [15–19], and references therein).

This conventional opinion had been believed for a long time until Rindler and Ishak [20] drew an opposite conclusion in 2007. They applied the invariant formula of cosine to achieve the Λ -induced contribution to the weak-field bending of light in the Schwarzschild-de Sitter (SdS) spacetime. Dramatically, their conclusion that Λ does contribute to light bending has been recently confirmed by various authors [21–45], although the Λ -induced contribution to the bending angle takes diverse ways in different approaches. In particular, Sereno [23,24] investigated the effects of Λ on the lens equation and the main observable quantities of images in the gravitational lensing by a SdS source via the method of the finite-distance corrections [40,45,46]. The presence of the leading correction to light deflection due to the coupling between the lens mass and Λ was proposed in Ref. [23], though in a different form from the one given in Ref. [20]. Arakida and Kasai [33] calculated the second-order gravitational deflection of light and the lens equation in the SdS geometry by redefining the impact parameter to absorb the cosmological constant, and were in favor of the proposal of the dependence of the null orbital equation on Λ in Ref. [27]. In order to examine the influence of Λ on the gravitomagnetic deflection, Sultana [34] extended the Rindler–Ishak method [20,21,26] to the axially symmetric geometry, and computed the effect of Λ on the equatorial bending angle of light in the Kerr-de Sitter spacetime. The Time–Transfer–Function approach [47] was also adopted to revisit the Λ -induced contributions to the light deflection caused by a SdS black hole [41]. It seems fair to mention that further work is needed with respect to the issue of whether or not Λ con-

^a e-mail: hgs@usc.edu.cn (corresponding author)

^b e-mail: pch@usc.edu.cn (corresponding author)

^c e-mail: lwb@usc.edu.cn (corresponding author)

tributes directly to the bending of light, to get a perfect agreement.

To our knowledge, the previous works have been devoted mainly to the investigation of the contributions from Λ to the gravitational lensing of light, while the effect of Λ on the gravitational bending of timelike particles has not yet been considered. On the one hand, although Islam [13] showed that Λ modifies the propagation of massive particles, it is necessary to examine the influence of the cosmological constant on the gravitational deflection of massive particles, for which two reasons are responsible. One is that it is of theoretical interest to determine the form and degree of the effect of Λ on the gravitational lensing phenomenon of massive particles, which acts as a test independent from the perihelion advance effect [13]. The other is that the decrease of the initial velocity v_0 of a test particle leads to the increase in the total deflection angle for a given lens system, which may make it more accessible to distinguish the contributions of Λ from the Schwarzschild contributions to the bending in the astronomical detection. Moreover, it provides a chance to comprehend the characteristics of the particle itself with the help of the observable correctional effect [48, 49] induced by the deviation of v_0 from light speed c on the bending angle. On the other hand, there have been some other works focusing on the perihelion shift of the orbit of a massive particle caused by the cosmological constant (see, e.g., Refs. [30, 50–57]), besides Islam's early work [13]. For example, the authors of Ref. [51] derived the analytical form for the tiny influence from Λ on the pericenter precession of the orbit motion of a point mass for arbitrary orbital eccentricity. The discrepancy among different formulae in previous studies for the contribution of Λ to the perihelion advance of celestial bodies was also discussed, and a prescription for this confusion was presented [56]. There is no doubt that the presence of the Λ -induced effect on the orbital perihelion advance, verified by all these works, provides us an excellent basis for considering the gravitational lensing of massive particles due to Λ in turn.

In present work, we adopt the Rindler–Ishak approach to calculate the gravitational deflection of a relativistic massive neutral particle up to the second post-Minkowskian (2PM) order in the SdS spacetime. We focus our discussions on the Λ -induced contributions to the deflection angle of massive particles, and analyze the correctional effects caused by the deviation of v_0 from c on the Λ -induced deflection, within the weak-field and small-angle approximation. For simplicity, the source and the observer of massive particles are assumed to be static relative to the gravitational lens.

The organization of this paper is as follows. In Sect. 2, we derive the gravitational deflection of massive particles up to the 2PM order in the SdS geometry. The discussions of the magnitudes and possible detections of the velocity effects on the gravitational deflection angle are given in Sect. 3,

followed by a summary in Sect. 4. Throughout this paper, the geometrized units where $G = c = 1$ is used, and Greek indices run over the four space-time coordinate labels 0, 1, 2, and 3. We will work in the metric convention $(-, +, +, +)$.

2 Gravitational deflection of massive particles due to a SdS black hole

Consider the gravitational deflection of a relativistic massive particle in the Schwarzschild-de Sitter geometry in the weak-field limit. The SdS metric in Boyer–Lindquist coordinates reads

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{1}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

with M being the rest mass of the lens. The cosmological constant Λ (> 0) is of the order of $10^{-52} m^{-2}$ from cosmology [3, 11, 21].

The Euler–Lagrange equation with the Lagrangian $\mathcal{L} = -\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ for a single particle is given by

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \quad (2)$$

where the dot denotes differentiation with respect to the affine parameter λ which describes the trajectory of a test particle. Since the field is spherically symmetric, we consider the case where the orbits of massive particles are restricted in the equatorial plane ($\theta = \pi/2$). Thus, we have

$$\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \frac{dt}{d\lambda} = E, \quad (3)$$

$$r^2 \frac{d\phi}{d\lambda} = L. \quad (4)$$

Here, the conserved quantities E and L may be given as $E = 1/\sqrt{1 - v_0^2}$ and $L = b v_0/\sqrt{1 - v_0^2}$ [49, 58–63], respectively, where v_0 represents the initial velocity of a massive particle, and the impact parameter b is defined by $L/E = v_0 b$, which is consistent with the definition $b \equiv L/E$ for the case of light ($v_0 = 1$) [34]. Notice that the source and the observer situated in the extensive and essentially flat region between the Schwarzschild and de Sitter geometries are still very far away from the gravitational lens, although the value of r cannot exceed the de Sitter radius ($r_{dS} = \sqrt{3/\Lambda}$).

The substitution of Eqs. (3)–(4) into the r -component of the Euler–Lagrange equation leads to

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \left(1 + \frac{L^2}{r^2}\right), \quad (5)$$

Fig. 1 The geometrical diagram for the equatorial propagation (blue) of a massive particle, adopted from Ref. [20]. Let d and δ be the orbital direction of massive particles and the radial direction where $\phi = \text{const.}$, respectively. The one-side bending angle is then defined by $\epsilon \equiv \psi - \phi$

which gives by means of a new variable $u = 1/r$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} + \frac{\Lambda}{3} - \left(1 - \frac{1}{v_0^2}\right) \frac{2Mu}{b^2} - u^2 + 2Mu^3 - \left(1 - \frac{1}{v_0^2}\right) \frac{\Lambda}{3b^2} \frac{1}{u^2}. \quad (6)$$

It can be seen that Eq. (6) is reduced to the null geodesic equation in the limit $v_0 \rightarrow 1$ [33, 34]. Moreover, with the consideration of the extremely small value of Λ , we roughly regard Λr^2 as a first-order quantity, as done in Ref. [34].

We then differentiate Eq. (6) over ϕ and get

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 + \left(1 - \frac{1}{v_0^2}\right) \left(\frac{\Lambda}{3b^2u^3} - \frac{M}{b^2}\right). \quad (7)$$

In order to obtain the trajectory of the massive particle up to the 2PM order, we employ the standard perturbation theory analysis to write the solution of Eq. (7) as

$$u = u_0 + u_1 + u_2. \quad (8)$$

Here, $u_0 = \frac{\sin \phi}{b}$ denotes the undeflected trajectory which is a straight line. u_1 and u_2 are respectively the first and second order perturbations, and the differential equations satisfied by them are given as follows:

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3Mu_0^2 + \left(1 - \frac{1}{v_0^2}\right) \left(\frac{\Lambda}{3b^2u_0^3} - \frac{M}{b^2}\right), \quad (9)$$

$$\frac{d^2u_2}{d\phi^2} + u_2 = 6Mu_0u_1 - \left(1 - \frac{1}{v_0^2}\right) \frac{\Lambda}{b^2u_0^4}u_1. \quad (10)$$

The substitution of the solutions of Eqs. (9)–(10) into Eq. (8) gives

$$u = \frac{\sin \phi}{b} + \frac{M}{2b^2} \left(1 + \frac{2}{v_0^2} + \cos 2\phi\right) + \left(1 - \frac{1}{v_0^2}\right) \frac{b\Lambda}{6} \frac{\cos^2 \phi}{\sin \phi} + \frac{3M^2}{16b^3} \left[2 \left(1 + \frac{4}{v_0^2}\right) (\pi - 2\phi) \cos \phi - \sin 3\phi\right]$$

$$- \frac{M\Lambda(1-v_0^2)}{24v_0^4 \sin^2 \phi} \left[2 - 6 \cos 2\phi + 2v_0^2 \cos \phi (\cos 3\phi - 3 \cos \phi) + 3(\cos 3\phi - \cos \phi + 4v_0^2 \cos \phi \sin^2 \phi) \ln \left(\cot \frac{\phi}{2}\right)\right], \quad (11)$$

where the second-order terms in Λ have been dropped since we focus our attentions on the first-order contributions from Λ to the gravitational deflection. Equation (11) is reduced to the weak-field null trajectory [21, 26, 34] when $v_0 = 1$.

For computing the deflection angle of massive particles, we then apply the invariant formula for the cosine of the angle (denoted by ψ) between two coordinate directions $d = d^i$ and $\delta = \delta^i$ shown in Fig. 1, which reads [20]

$$\cos \psi = \frac{g_{ij}d^i\delta^j}{\sqrt{g_{ij}d^id^j}\sqrt{g_{ij}\delta^i\delta^j}}, \quad (12)$$

where $d^i = (dr, d\phi) = (A, 1)d\phi$, $\delta^i = (dr, 0)$, and the 2-metric g_{ij} is given by Eq. (1) for the special case of $t = \text{const.}$ and $\theta = \pi/2$. Equation (12) is rewritten explicitly as

$$\cos \psi = \frac{|A|}{\sqrt{A^2 + \frac{g_{\phi\phi}}{g_{rr}}}}, \quad (13)$$

or equivalently,

$$\tan \psi = \frac{\sqrt{\frac{g_{\phi\phi}}{g_{rr}}}}{|A|} = \frac{r\sqrt{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}}{|A|}, \quad (14)$$

with $A(r, \phi) = dr/d\phi$.

As shown in Fig. 1, the one-side gravitational deflection angle at a general reception point $P(r, \phi)$ is given to be $\epsilon(\phi) = \psi(\phi) - \phi$, which gives traditionally the total two-side deflection angle $\alpha(v_0) = 2\epsilon(0)$ of a massive particle when $\phi = 0$ [20]. However, some of the terms in Eq. (11) will be divergent and become unphysical when $\phi = 0$, similar to the case in the Kerr-de Sitter spacetime [34]. In order to avoid the irrationality resulted from the divergence, we drop the terms including the factor $\frac{1}{\sin \phi}$ on the right hand side of Eq. (11), and obtain the value of the radial variable at $\phi = 0$ as follows:

$$r_b = b \left[\left(1 + \frac{1}{v_0^2}\right) \frac{M}{b} + \frac{3\pi}{8} \left(1 + \frac{4}{v_0^2}\right) \frac{M^2}{b^2} \right]^{-1} + O(M^2, M\Lambda, \Lambda) \\ = \frac{v_0^2 b^2}{(1 + v_0^2) M} \left[1 - \frac{3\pi}{8} \frac{(4 + v_0^2) M}{(1 + v_0^2) b} + \frac{9\pi^2 (4 + v_0^2)^2 M^2}{64 (1 + v_0^2)^2 b^2} + O(M^2, M\Lambda, \Lambda) \right], \quad (15)$$

which yields the explicit form of $|A|$ for the reception point $(r_b, 0)$ of the massive particle

$$\begin{aligned}
 |A| &= b \left[1 - \frac{3}{16} \left(7 + \frac{16}{v_0^2} \right) \frac{M^2}{b^2} - \left(1 - \frac{1}{v_0^2} \right) \frac{\Lambda b^2}{6} \right] \\
 &\quad \times \left[\left(1 + \frac{1}{v_0^2} \right) \frac{M}{b} + \frac{3\pi}{8} \left(1 + \frac{4}{v_0^2} \right) \frac{M^2}{b^2} \right]^{-2} \\
 &\quad + O(M^2, M\Lambda, \Lambda) \\
 &= \frac{v_0^4 b^3}{(1 + v_0^2)^2 M^2} \left\{ 1 - \frac{3\pi(4 + v_0^2)M}{4(1 + v_0^2)b} + \frac{1 - v_0^2}{6v_0^2} \Lambda b^2 \right. \\
 &\quad - \frac{3}{16} \left[7 + \frac{16}{v_0^2} - \frac{9\pi^2(4 + v_0^2)^2}{4(1 + v_0^2)^2} \right] \frac{M^2}{b^2} \\
 &\quad \left. + \frac{\pi(v_0^2 - 1)(4 + v_0^2)}{8v_0^2(1 + v_0^2)} M\Lambda b + O(M^2, M\Lambda, \Lambda) \right\}. \quad (16)
 \end{aligned}$$

In addition, by performing series expansions, we have

$$\begin{aligned}
 &\sqrt{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}} \\
 &= 1 - \frac{M}{r} - \frac{M^2}{2r^2} - \frac{M\Lambda r}{6} - \frac{\Lambda r^2}{6} + O(M^2, M\Lambda, \Lambda). \quad (17)
 \end{aligned}$$

Substituting Eqs. (15)–(17) into Eq. (14), we achieve the analytical expression of the gravitational deflection angle of a relativistic massive particle up to the 2PM order as follows:

$$\begin{aligned}
 \alpha(v_0) &= 2 \left(1 + \frac{1}{v_0^2} \right) \frac{M}{b} + \frac{3\pi}{4} \left(1 + \frac{4}{v_0^2} \right) \frac{M^2}{b^2} \\
 &\quad - \frac{v_0^2}{3(1 + v_0^2)} \frac{\Lambda b^3}{M} + \frac{v_0^2(4 + v_0^2)\pi}{8(1 + v_0^2)^2} \Lambda b^2 \\
 &\quad + O(M^2, M\Lambda, \Lambda), \quad (18)
 \end{aligned}$$

which acts as our main result. Notice that we have omitted the second-order contributions from Λ , as well as the second-order contribution including the factor $M\Lambda b$ (see Eq. (17) of Ref. [34] for details) on the right hand side of Eq. (18), since their magnitudes are much smaller than that of the fourth term in Eq. (18).

Interestingly, dropping the contributions from the cosmological constant, Eq. (18) is reduced to

$$\alpha(v_0) = 2 \left(1 + \frac{1}{v_0^2} \right) \frac{M}{b} + \frac{3\pi}{4} \left(1 + \frac{4}{v_0^2} \right) \frac{M^2}{b^2}, \quad (19)$$

which matches well with the result of the Schwarzschild deflection angle of massive particles up to the second order via other approaches [58–62, 64–71]. In the limit $v_0 \rightarrow 1$, Eq. (18) can be simplified to the weak-field Schwarzschild-

de Sitter deflection angle of light [21, 34]:

$$\alpha(1) = \frac{4M}{b} + \frac{15\pi}{4} \frac{M^2}{b^2} - \frac{\Lambda b^3}{6M} + \frac{5\pi}{32} \Lambda b^2. \quad (20)$$

3 Discussion of velocity corrections

In this section, we discuss the correctional effects induced by the deviation of the initial velocity v_0 of a relativistic massive particle from light speed on the SdS deflection angle of light. Based on Eqs. (18) and (20), we easily get

$$\Delta_M(v_0) = \left(\frac{1}{v_0^2} - 1 \right) \frac{2M}{b}, \quad (21)$$

$$\Delta_{M^2}(v_0) = \left(\frac{1}{v_0^2} - 1 \right) \frac{3\pi M^2}{b^2}, \quad (22)$$

$$\Delta_{\Lambda_1}(v_0) = \frac{1 - v_0^2}{1 + v_0^2} \frac{\Lambda b^3}{6M}, \quad (23)$$

$$\Delta_{\Lambda_2}(v_0) = -\frac{(5 - 6v_0^2 + v_0^4)\pi}{32(1 + v_0^2)^2} \Lambda b^2, \quad (24)$$

where $\Delta_M(v_0)$ and $\Delta_{M^2}(v_0)$ denote respectively the velocity corrections to the first and second order Schwarzschild contributions to the deflection angle given in Eq. (20). $\Delta_{\Lambda_1}(v_0)$ and $\Delta_{\Lambda_2}(v_0)$ represent the velocity corrections to the third and forth terms (the Λ -induced contributions) on the right-hand side of Eq. (20), respectively.

In order to evaluate their magnitudes, we take $\Lambda = 1.29 \times 10^{-52} \text{ m}^{-2}$ [21] and $M = 4.2 \times 10^6 M_\odot$ (the mass of the supermassive black hole Sgr A* [72–74] at the galactic center) as an example, with $M_\odot$ being the rest mass of the Sun. The impact parameter b is preset to be the Einstein radius $R_E (= 5.74 \times 10^{-5} \text{ kpc})$ of Sgr A* for the special case of $D_L = D_{LS}$, where $D_L (= 8.2 \text{ kpc [73–75]})$ and D_{LS} denote the angular-diameter distances between the lens and the observer on earth, and the lens and the source, respectively. The velocity corrections acting as the functions of the initial velocity v_0 of massive particles are plotted in Fig. 2. For the convenience of illustration, a particle with v_0 being larger than 0.05 is roughly regarded to be relativistic.

We know that great progress in astronomical measurement techniques has been made in the past few decades. Today's high-accuracy angular measurement in astronomic programs such as the GAIA mission [77–79], the Space Interferometry Mission (SIM) [80, 81], the Search for Terrestrial Exo-Planet (STEP) mission [82, 83], and the planned Nearby Earth Astrometric Telescope (NEAT) mission [84–86], is at the level of 1–10 μarcsec (μas) or even better. For instance, the NEAT mission aims at an unprecedented accuracy of 0.05 μas ($\sim 2.42 \times 10^{-13} \text{ rad}$). We can see from Figure 2 that for Sgr A* being the lens, there is a relatively large possi-

Fig. 2 The four velocity corrections are plotted for Sgr A* being the lens system, with $M = 4.2 \times 10^6 M_\odot$ and $b = R_E = 5.74 \times 10^{-5}$ kpc

Fig. 3 The Λ -dependent velocity corrections are plotted for the galaxy cluster Abell 2219L ($M = 4.53 \times 10^{13} M_\odot$, $b = R_E = 86.3$ kpc) serving as the lens system

bility to detect the velocity corrections to the first and second order Schwarzschild contributions to the deflection angle for almost all relativistic massive particles ($0.05 \leq v_0 \leq 0.999$), within the capability of the NEAT. The possibility to detect $\Delta_M(v_0)$ and $\Delta_{M^2}(v_0)$ becomes more and more small in the process of v_0 approaching the speed of light. Furthermore, the relativistic velocity corrections to the Λ -induced contributions to the light deflection for Sgr A*-like lens system are not possible to be detected in current techniques, since their magnitudes are much smaller than $0.05 \mu\text{as}$.

Due to the dependence of the Λ -induced deflections in Eq. (18) on the radius of the gravitational system, for comparison, we also present in Fig. 3 the Λ -dependent velocity corrections for the case of the galaxy cluster Abell 2219L [76] (a much larger system) as the lens system, with $M = 4.53 \times 10^{13} M_\odot$ and $R_E = 86.3$ kpc [21]. Figure 3 shows for a large system such as a cluster of galaxies, the magnitude of the relativistic velocity correction to the leading Λ -induced deflection of light is still larger than the NEAT's accuracy even when v_0 is extremely close to 1 ($1 - v_0 \approx 4.0 \times 10^{-8}$ for the cluster Abell 2219L), not to

speak of the case of a smaller value v_0 takes. For example, $\Delta_{A_1}(v_0)$ is about $125 \mu\text{as}$ ($\gg 0.05 \mu\text{as}$) for a massive particle with $v_0 = 0.9999$, when the lens is the cluster Abell 2219L. In addition, Fig. 3 indicates that it is also likely to detect $\Delta_{A_2}(v_0)$ for a relativistic massive particle with a proper value of v_0 ($v_0 \leq 0.998$ for Abell 2219L as the lens), through the NEAT telescope.

Correspondingly, in order to examine our main result given in Eq. (18) for the massive-particle deflection more conveniently, two numerical examples are given as follows:

- (1) For an ordinary massive neutrino with an initial velocity $v \approx 1 - 1.0 \times 10^{-6}$ [87–90] as the test particle, the total bending angle $\alpha(v_0)$ should be about 2.89 as and 19.46 as, for Sgr A* ($M = 4.2 \times 10^6 M_\odot$, $b = 5.74 \times 10^{-5}$ kpc) and the cluster Abell 2219L ($M = 4.53 \times 10^{13} M_\odot$, $b = 86.3$ kpc) acting as the gravitational lens, respectively. The primary contribution from Λ is $3.98 \times 10^{-6} \mu\text{as}$ and 1.25 as, respectively, in these two cases.
- (2) The hypervelocity celestial bodies are not rare in the universe [91–94]. For a small interstellar object with an initial velocity $v = 0.1$ (e.g., a relativistic celestial body with an Oumuamua-like size) as the test particle, the total bending angle $\alpha(v_0)$ should be about 145.84 as and 1045.93 as for Sgr A* and the cluster Abell 2219L (respectively with the same M and b as above) as the lens, respectively. The corresponding primary contribution of Λ is $7.88 \times 10^{-8} \mu\text{as}$ and 0.025 as, respectively.

4 Summary

In this paper, we have extended the Rindler–Ishak approach to investigate the gravitational deflection of a relativistic massive particle in Schwarzschild-de Sitter geometry in the weak-field limit. On the basis of this approach, it is found that the cosmological constant appears in the orbital equation of a massive particle and contributes to its deflection angle. Dropping the contributions from the cosmological constant, our deflection angle is reduced to the second-order Schwarzschild deflection angle of massive particles reported in the previous works. When the initial velocity of the massive particle is equal to light speed, our result matches well with that obtained in the previous work for the second-order Schwarzschild-de Sitter deflection angle of light. In addition, the correctional effects due to the particle's initial velocity deviating from the speed of light on the first and second order Schwarzschild contributions as well as the Λ -induced contributions to the light bending angle have also been discussed. Different from the case where the supermassive black hole at the galaxy center acts as the lens system, the correctional effects on the Λ -induced deflection of light under the scenario

where a cluster of galaxies serves as the lens are possible to be detected in current resolution.

Acknowledgements We would like to thank the referee very much for his/her useful comments and suggestions to improve the quality of this work. This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11801263, 11947018, 11947128, and 11973025) and the Natural Science Foundation of Hunan (Grant No. 2018JJ3418).

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.
Funded by SCOAP³.

References

1. H.A. Lorentz, A. Einstein, H. Minkowski, H. Weyl, *The principle of relativity* (Methuen, 1923, reprinted by Dover Publications)
2. M.S. Turner, Phys. Rep. **333**, 619 (2000)
3. P.J.E. Peebles, B. Ratra, Rev. Mod. Phys. **75**, 559 (2003)
4. T. Padmanabhan, Phys. Rep. **380**, 235 (2003)
5. S.M. Carroll, M. Hoffman, M. Trodden, Phys. Rev. D **68**, 023509 (2003)
6. Z. Guo, Y. Piao, X. Zhang, Y. Zhang, Phys. Lett. B **608**, 177 (2005)
7. P. Wu, H. Yu, Phys. Lett. B **643**, 315 (2006)
8. J.A. Frieman, M.S. Turner, D. Huterer, Annu. Rev. Astron. Astrophys. **46**, 385 (2008)
9. R.J. Yang, S.N. Zhang, Mon. Not. R. Astron. Soc. **407**, 1835 (2010)
10. M. Li, X.D. Li, S. Wang, Y. Wang, Commun. Theor. Phys. **56**, 525 (2011)
11. E. Komatsu et al., Astrophys. J. Suppl. **192**, 18 (2011)
12. V. Bonvin et al., Mon. Not. R. Astron. Soc. **465**, 4914 (2017)
13. N.J. Islam, Phys. Lett. **97A**, 239 (1983)
14. F. Kottler, Ann. Phys. **361**, 401 (1918)
15. W.H.C. Freire, V.B. Bezerra, J.A.S. Lima, Gen. Relat. Gravit. **33**, 1407 (2001)
16. K. Lake, Phys. Rev. D **65**, 087301 (2002)
17. F. Finelli, M. Galaverni, A. Gruppiso, Phys. Rev. D **75**, 043003 (2007)
18. F. Simpson, J.A. Peacock, A.F. Heavens, Mon. Not. R. Astron. Soc. **402**, 2009 (2010)
19. L.M. Butcher, Phys. Rev. D **94**, 083011 (2016)
20. W. Rindler, M. Ishak, Phys. Rev. D **76**, 043006 (2007)
21. M. Ishak, W. Rindler, J. Dossett, J. Moldenhauer, C. Allison, Mon. Not. R. Astron. Soc. **388**, 1279 (2008)
22. M. Ishak, Phys. Rev. D **78**, 103006 (2008)
23. M. Sereno, Phys. Rev. D **77**, 043004 (2008)
24. M. Sereno, Phys. Rev. Lett. **102**, 021301 (2009)

25. T. Schücker, Gen. Relat. Gravit. **41**, 67 (2009)
26. M. Ishak, W. Rindler, Gen. Relat. Gravit. **42**, 2247 (2010)
27. A. Bhadra, S. Biswas, K. Sarkar, Phys. Rev. D **82**, 063003 (2010)
28. R. Kantowski, B. Chen, X. Dai, Astrophys. J. **718**, 913 (2010)
29. M. Ishak, W. Rindler, J. Dossett, Mon. Not. R. Astron. Soc. **403**, 2152 (2010)
30. H. Miraghaei, M. Nouri-Zonoz, Gen. Relat. Gravit. **42**, 2947 (2010)
31. T. Biressa, J.A. de Freitas Pacheco, Gen. Relat. Gravit. **43**, 2649 (2011)
32. A. Bhattacharya, G.M. Garipova, E. Laserra, A. Bhadra, K.K. Nandi, J. Cosmol. Astropart. Phys. **1102**, 028 (2011)
33. H. Arakida, M. Kasai, Phys. Rev. D **85**, 023006 (2012)
34. J. Sultana, Phys. Rev. D **88**, 042003 (2013)
35. F. Hammad, Mod. Phys. Lett. A **28**, 1350181 (2013)
36. D. Lebedev, K. Lake, [arXiv:1308.4931](https://arxiv.org/abs/1308.4931)
37. D. Batic, S. Nelson, M. Nowakowski, Phys. Rev. D **91**, 104015 (2015)
38. F. Zhao, J. Tang, Phys. Rev. D **92**, 083011 (2015)
39. F. Zhao, J. Tang, F. He, Phys. Rev. D **93**, 123017 (2016)
40. A. Ishihara, Y. Suzuki, T. Ono, T. Kitamura, H. Asada, Phys. Rev. D **94**, 084015 (2016)
41. H. Arakida, Universe **2**, 5 (2016)
42. H. He, Z. Zhang, J. Cosmol. Astropart. Phys. **08**, 036 (2017)
43. Y. Lim, Q. Wang, Phys. Rev. D **95**, 024004 (2017)
44. M. Ali, S. Bhattacharya, Phys. Rev. D **97**, 024029 (2018)
45. S. Haroon, M. Jamil, K. Jusufi, K. Lin, R.B. Mann, Phys. Rev. D **99**, 044015 (2019)
46. T. Ono, A. Ishihara, H. Asada, Phys. Rev. D **96**, 104037 (2017)
47. P. Teyssandier, C. Le Poncin-Lafitte, Class. Quant. Grav. **25**, 145020 (2008)
48. O. Wucknitz, U. Sperhake, Phys. Rev. D **69**, 063001 (2004)
49. X. Liu, N. Yang, J. Jia, Class. Quant. Grav. **33**, 175014 (2016)
50. R.A. Alpher, Am. J. Phys. **35**, 771 (1967)
51. A.W. Kerr, J.C. Hauck, B. Mashhoon, Class. Quant. Grav. **20**, 2727 (2003)
52. G.V. Kraniotis, S.B. Whitehouse, Class. Quant. Grav. **20**, 4817 (2003)
53. N. Cruz, M. Olivares, J.R. Villanueva, Class. Quant. Grav. **22**, 1167 (2005)
54. W. Rindler, *Relativity: special, general, and cosmological* (Oxford University Press, New York, 2006)
55. L. Iorio, Adv. Astron. **2008**, 268647 (2008)
56. H. Arakida, Int. J. Theor. Phys. **52**, 1408 (2013)
57. S. Liang, Y. Xie, Res. Astron. Astrophys. **14**, 527 (2014)
58. A. Accioly, S. Ragusa, Class. Quant. Grav. **19**, 5429 (2002)
59. A. Accioly, S. Ragusa, Class. Quant. Grav. **20**, 4963 (2003)
60. G. Crisnejo, E. Gallo, Phys. Rev. D **97**, 124016 (2018)
61. K. Jusufi, Phys. Rev. D **98**, 064017 (2018)
62. K. Jusufi, A. Banerjee, G. Gyulchev, M. Amir, Eur. Phys. J. C **79**, 28 (2019)
63. K. Jusufi, [arXiv:1906.12186](https://arxiv.org/abs/1906.12186)
64. G. Crisnejo, E. Gallo, K. Jusufi, Phys. Rev. D **100**, 104045 (2019)
65. X. Pang, J. Jia, Class. Quant. Grav. **36**, 065012 (2019)
66. G. Crisnejo, E. Gallo, J.R. Villanueva, Phys. Rev. D **100**, 044006 (2019)
67. Z. Li, G. He, T. Zhou, Phys. Rev. D **101**, 044001 (2020)
68. Z. Li, J. Jia, Eur. Phys. J. C **80**, 157 (2020)
69. J. Jia, Eur. Phys. J. C **80**, 242 (2020)
70. G. He, W. Lin, Class. Quant. Grav. **33**, 095007 (2016)
71. G. He, W. Lin, Class. Quant. Grav. **34**, 105006 (2017)
72. S. Chatzopoulos et al., Mon. Not. R. Astron. Soc. **447**, 948 (2015)
73. J. Bland-Hawthorn, O. Gerhard, Annu. Rev. Astron. Astrophys. **54**, 529 (2016)
74. M. Parsa et al., Astrophys. J. **845**, 22 (2017)
75. P.J. McMillan, Mon. Not. R. Astron. Soc. **465**, 76 (2017)
76. S. Allen, Mon. Not. R. Astron. Soc. **296**, 392 (1998)
77. M.A.C. Perryman et al., Astron. Astrophys. **369**, 339 (2001)
78. F. Mignard et al., Earth. Moon Planets **101**, 97 (2007)
79. T. Prusti et al., Gaia Collaboration. Astron. Astrophys. **595**, A1 (2016)
80. R.A. Laskin, Proc. SPIE Int. Soc. Opt. Eng. **6268**, 626823 (2006)
81. M. Shao, B. Nemati, Publ. Astron. Soc. Pac. **121**, 41 (2009)
82. D. Chen, J. Instrum. **9**, C04040 (2014)
83. H. Li, B. Li, Y. Cao, D. Chen, L. Li, Proc. SPIE **9622**, 96221A (2015)
84. F. Malbet et al., Exp. Astron. **34**, 385 (2012)
85. F. Malbet et al., Int. J. Space Sci. En. **2**, 3 (2014)
86. F. Malbet et al., Proc. SPIE **9143**, 91432L (2014)
87. G.R. Kalbeisch, N. Baggett, E.C. Fowler, J. Alspector, Phys. Rev. Lett. **43**, 1361 (1979)
88. P. Alvarez Sanchez et al., Phys. Lett. B **716**, 401 (2012)
89. T. Adam et al., J. High Energy Phys. **10**, 093 (2012)
90. P. Adamson et al., Phys. Rev. D **92**, 052005 (2015)
91. S. Chatterjee et al., Astrophys. J. **630**, L61 (2005)
92. Z. Zheng et al., Astrophys. J. **785**, L23 (2014)
93. D. Boubert, J. Guillochon, K. Hawkins, I. Ginsburg, N.W. Evans, J. Strader, Mon. Not. R. Astron. Soc. **479**, 2789 (2018)
94. D. Boubert, D. Erkal, A. Gualandris, Mon. Not. R. Astron. Soc. **497**, 2930 (2020)