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The weak decay B_c to $Z(3930)$ and $X(4160)$ by Bethe–Salpeter method

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Abstract Considering $Z(3930)$ and $X(4160)$ as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, the semileptonic and nonleptonic of B_c decays to $Z(3930)$ and $X(4160)$ are studied by the improved Bethe–Salpeter (B–S) Method. The form factors of decay are calculated through the overlap integrals of the meson wave functions in the whole accessible kinematical range. The influence of relativistic corrections are considered in the exclusive decays. Branching ratios of B_c weak decays to $Z(3930)$ and $X(4160)$ are predicted. Some of the branching ratios are: $Br(B_c^+ \to Z(3930)e^+\nu_e) = (3.03^{+0.09}_{-0.16}) \times 10^{-4}$ and $Br(B_c^+ \to X(4160)e^+\nu_e) = (3.55^{+0.83}_{-0.35}) \times 10^{-6}$. These results may provide useful information to discover $Z(3930)$ and $X(4160)$ and the necessary information for the phenomenological study of B_c physics.

1 Introduction

During the past decade years, more and more charmonium and charmonium-like states were discovered experimentally. Such as the $X(3915)$ was reported by Belle Collaboration in $\gamma\gamma \to \omega J/\psi$ process [1]. $Z(3930)$ was observed in the process $\gamma\gamma \to D\bar{D}$ by Belle Collaboration in 2006, the corresponding mass and width were $M = 3929 \pm 5 \pm 2$ MeV and $\Gamma = 29 \pm 10 \pm 2$ MeV, respectively [2]. In 2010, BABAR Collaboration also observed the $Z(3930)$ in $\gamma\gamma$ production of the $D\bar{D}$ system, with the mass and width being $M = 3926.7 \pm 2.7 \pm 1.1$ MeV and $\Gamma = 21.3 \pm 3.8 \pm 3.6$ MeV, respectively [3]. Now Particle Data Group(PDG) give lists the mass and width of $Z(3930)$ as $M = 3927.2 \pm 2.6$ MeV and $\Gamma = 24 \pm 6$ MeV [4]. And the properties of $Z(3930)$ are consistent with the expectations for the $\chi_{c2}(2P)$ state [5–7]. Then Belle Collaboration reported a new charmonium-like state $X(4160)$ from the processes $e^+e^- \to J/\psi D^*\bar{D}^*$, which has the mass and width $M = (4156^{+25}_{-20} \pm 15)$ MeV and $\Gamma = (139^{+111}_{-61} \pm 21)$ MeV, respectively [8].

The quark structures were still not fully understood in these charmonium-like states which are called XYZ states, thus people studied the properties of XYZ states by different methods [9–22]. In this work, we only consider two of them: $Z(3930)$ and $X(4160)$. The structures of $Z(3930)$ and $X(4160)$ were already studied by some theoretical methods. Reference [9] studied B_c semileptonic decay to $Z(3930)$ and $X(4160)$ which were assumed as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states. According to study the vector-vector interaction within the framework of the hidden gauge formalism, Ref. [10] found that three resonances $Y(3940)$, $Z(3930)$ and $X(4160)$ which can be assigned to the states with $J^{PC} = 0^{++}$, 2^{++} and 2^{++}, respectively. Taking $Z(3915)$ and $Z(3930)$ as $\chi'_{c0}(2P)$ and $\chi'_{c2}(2P)$, respectively. Reference [11] investigated the $X(3915)$ and $Z(3930)$ decays into $J/\psi\omega$. Reference [12] studied the strong decay of $Z(3930)$ which was considered as $\chi'_{c2}(2P)$. Reference [13] studied the mass spectra of the hidden-charm tetraquark states in the framework of QCD sum rules, and they got the $X(4160)$ may be classified as either the scalar or tensor $qc\bar{q}\bar{c}$ tetraquark state. Using the NRQCD factorization approach, Ref. [18] calculated the branching fractions of $\Upsilon(nS) \to J/\psi + X$ with $X = X(3940)$ or $X = X(4160)$. In Ref. [19], they also explored the properties and strong decays of $X(3940)$ and $X(4160)$ as the $\eta_c(3S)$ and $\eta_c(4S)$, respectively. Reference [20] calculated the strong decay of $X(4160)$ which was assumed as $\chi_{c0}(3P)$, $\chi_{c1}(3P)$, $\eta_{c2}(2D)$ or $\eta_c(4S)$ by the $^3P_0$ model. Reference [23] studied the strong decays of $X(3940)$ and $X(4160)$ as the $\eta_c(3S)$ and $\eta_c(4S)$ with the $^3P_0$ model, and the results showed that $\eta_c(4S)$ was not

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good candidate of $X(4160)$. According to the mass spectra and the properties of $Z(3930)$ and $X(4160)$ in Refs. [9–13], $Z(3930)$ and $X(4160)$ have the possibility to be $\chi_{c2}(2P)$ and $\chi_{c2}(3P)(J^{PC}=2^{++})$, respectively.

The interpretations of $Z(3930)$ and $X(4160)$ are not the major work in this paper, we only consider $Z(3930)$ and $X(4160)$ as charmonium states with the possible quantum numbers, then study their production in B_c decays. We will consider $Z(3930)$ and $X(4160)$ as P-wave charmonium states $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$, respectively. Then we focus on the productions of $Z(3930)$ and $X(4160)$ in exclusive weak decays of B_c meson by the improved the Bethe–Salpeter (B–S) Method. On the one hand, the $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ have larger relativistic correction than that of $\chi_{c2}(1P)$, so a relativistic model is needed in a careful study; on the other hand, this study can improve the knowledge of B_c meson, which is an ideal particle to study the weak decays, since it decays weakly only. The properties of B_c meson have been studied by different relativistic constituent quark models [24–32], such as the covariant light-front quark model [33,34], the perturbative QCD factorization approach [35] and so on. We also have discussed the properties of B_c meson by the improved B–S method, include B_c decays to P-wave mesons, the rare weak decays and rare radiative decays of B_c, the nonleptonic charmless decays of B_c, and so on [36–42]. In previous work, we only studied B_c decays to $\chi_{c2}(1P)$ state [36], because when the final states are $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, the corresponding branching ratios are very small, and there were only limited data of B_c available. Now the large hadron collider (LHC) will produce as many as 5×10^{10} B_c events per year [43,44]. The huge amount of B_c events will provide us a chance to study B_c decay to $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, and some channels also provide an opportunities to discover new particles in B_c decays.

The paper is organized as follows. In Sect. 2, we give the formulations of the exclusive semileptonic and nonleptonic decays. We show the hadronic weak-current matrix elements in Sect. 3. The wave functions of initial and final mesons are given in Sect. 4. The corresponding results and conclusions are presented in Sect. 5. Finally in the Appendix, we present the instantaneous Bethe–Salpeter equation.

2 The formulations of semileptonic decays and nonleptonic decays of B_c

In this section we present the formulations of semileptonic decays and nonleptonic decays of B_c meson to $Z(3930)$ and $X(4160)$ which are considered as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, respectively.

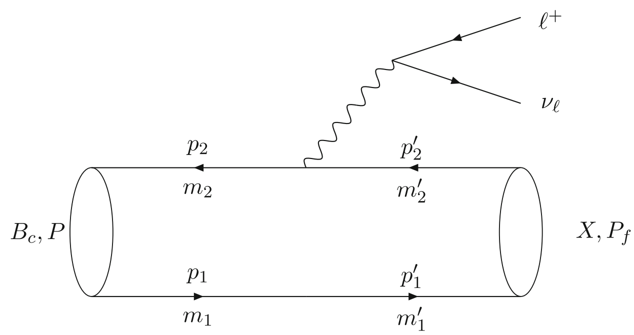

Fig. 1 Feynman diagram of the semileptonic decay $B_c \to X\ell^+\nu_\ell$, where X denotes $Z(3930)$ or $X(4160)$

2.1 Semileptonic decays of B_c

The Feynman diagram of B_c semileptonic decay to $Z(3930)$ or $X(4160)$ is shown in Fig. 1. The corresponding amplitude for the decay can be written as

$$T=\frac{G_F}{\sqrt{2}}V_{bc}\bar{u}_{\nu_\ell}\gamma_\mu(1-\gamma_5)v_\ell\langle X(P_f,\varepsilon)|J^\mu|B_c(P)\rangle, \tag{1}$$

where V_{bc} is the CKM matrix element, G_F is the Fermi constant, $J^\mu=V^\mu-A^\mu$ is the charged weak current, P and P_f are the momenta of the initial meson B_c and the final state, respectively. ε is the polarization tensor for final meson. The leptonic part $\bar{u}_{\nu_\ell}\gamma_\mu(1-\gamma_5)v_\ell$ is model independent and easy to calculate. The hadronic part $\langle X(P_f,\varepsilon)|J^\mu|B_c(P)\rangle$ can be written as,

$$\begin{aligned}\langle X(P_f,\varepsilon)|A^\mu|B_c(P)\rangle&=k(M+M_f)\varepsilon^{\mu\alpha}\frac{P_\alpha}{M}\\&+\varepsilon_{\alpha\beta}\frac{P^\alpha P^\beta}{M^2}(c_1P^\mu+c_2P_f^\mu),\\\langle X(P_f,\varepsilon)|V^\mu|B_c(P)\rangle&=\frac{2h}{M+M_f}i\varepsilon_{\alpha\beta}\frac{P^\alpha}{M}\epsilon^{\mu\beta\rho\sigma}P_\rho P_{f\sigma},\end{aligned} \tag{2}$$

where k, c_1, c_2, h are the Lorentz invariant form factors, M is the mass of B_c, M_f is the mass of the charmonium in the final state.

In the case without considering polarization, we have the squared decay-amplitude with the polarizations in final states being summed:

$$\Sigma_{s_\nu,s_l,S_X}|T|^2=\frac{G_F^2}{2}|V_{bc}|^2l_{\mu\nu}h^{\mu\nu}, \tag{3}$$

where $l_{\mu\nu}$ is the leptonic tensor:

$$l_{\mu\nu}=\Sigma_{s_\nu,s_l}\bar{v}_l(p_l)\gamma_\mu(1-\gamma_5)u_{\nu_l}(p_\nu)\bar{u}_{\nu_l}(p_\nu)\gamma_\nu(1-\gamma_5)v_l(p_l),$$

and the hadronic tensor relating to the weak-current in Eq. (1) is

$$h^{\mu\nu}\equiv\Sigma_{S_X}\langle B_c(P)|J^\mu|X(P_f)\rangle\langle X(P_f)|J^\nu|B_c(P)\rangle$$

$$
\begin{aligned}
&= -\alpha g^{\mu\nu} + \beta_{++}(P+P_f)^\mu (P+P_f)^\nu \\
&\quad +\beta_{+-}(P+P_f)^\mu (P-P_f)^\nu \\
&\quad +\beta_{-+}(P-P_f)^\mu (P+P_f)^\nu \\
&\quad +\beta_{--}(P-P_f)^\mu (P-P_f)^\nu \\
&\quad +i\gamma\epsilon^{\mu\nu\rho\sigma}(P+P_f)_\rho (P-P_f)_\sigma, \qquad (4)
\end{aligned}
$$

where the functions α, β_{++}, β_{+-}, β_{-+}, β_{--}, γ are related to the form factors.

The total decay width Γ can be written as:

$$
\begin{aligned}
\Gamma &= \frac{1}{2M(2\pi)^9}\int \frac{d^3\vec{P}_f}{2E_f}\frac{d^3\vec{p}_l}{2E_l} \\
&\quad \times \frac{d^3\vec{p}_\nu}{2E_\nu}(2\pi)^4\delta^4(P-P_f-p_l-p_\nu)\Sigma_{s_\nu,s_l,s_X}|T|^2, \qquad (5)
\end{aligned}
$$

where E_f, E_l and E_ν are the energies of the charmonium, the charged lepton and the neutrino respectively. If we define $x \equiv E_l/M$, $\;y \equiv (P-P_f)^2/M^2$, the differential width of the decay can be reduced to:

$$
\begin{aligned}
\frac{d^2\Gamma}{dxdy} &= |V_{bc}|^2\frac{G_F^2 M^5}{64\pi^3}\Bigg\{\frac{2\alpha}{M^2}\Big(y-\frac{m_l^2}{M^2}\Big) \\
&\quad +\beta_{++}\Bigg[4\Bigg(2x\Big(1-\frac{M_f^2}{M^2}+y\Big)-4x^2-y\Bigg) \\
&\quad +\frac{m_l^2}{M^2}\Bigg(8x+4\frac{M_f^2}{M^2}-3y-\frac{m_l^2}{M^2}\Bigg)\Bigg] \\
&\quad +(\beta_{+-}+\beta_{-+})\frac{m_l^2}{M^2}\Bigg(2-4x+y-2\frac{M_f^2}{M^2}+\frac{m_l^2}{M^2}\Bigg) \\
&\quad +\beta_{--}\frac{m_l^2}{M^2}\Bigg(y-\frac{m_l^2}{M^2}\Bigg) \\
&\quad -\Bigg[2\gamma y\Bigg(1-\frac{M_f^2}{M^2}-4x+y+\frac{M_l^2}{M^2}\Bigg) \\
&\quad +2\gamma\frac{M_l^2}{M^2}\Bigg(1-\frac{M_f^2}{M^2}\Bigg)\Bigg]\Bigg\}. \qquad (6)
\end{aligned}
$$

The total width of the decay is just an integration of the differential width i.e. $\Gamma = \int dx \int dy \frac{d^2\Gamma}{dxdy}$.

2.2 Nonleptonic decays of B_c

For the nonleptonic decay $B_c \to X + M_2$ in Fig. 2, the relevant effective Hamiltonian H_{eff} is [45,46]:

$$
H_{eff} = \frac{G_F}{\sqrt{2}}\left\{V_{bc}[c_1(\mu)O_1^{bc} + c_2(\mu)O_2^{bc}] + h.c.\right\}, \qquad (7)
$$

where $c_i(\mu)$ are the scale-dependent Wilson coefficients. O_i are the operators responsible for the decays constructed by four quark fields and have the structure as follows:

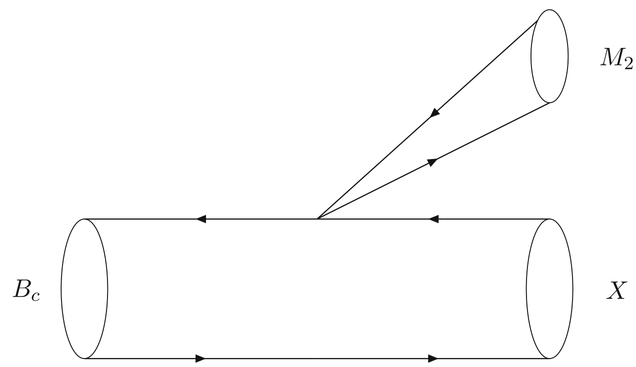

Fig. 2 Feynman diagram of the nonleptonic decay $B_c \to XM_2$, X denote $Z(3930)$ or $X(4160)$, M_2 denote a light meson: π, K, ρ, or K^*

$$
\begin{aligned}
O_1^{bc} &= [V_{ud}(\bar{d}_\alpha u_\alpha)_{V-A} + V_{us}(\bar{s}_\alpha u_\alpha)_{V-A}](\bar{c}_\beta b_\beta)_{V-A}, \\
O_2^{bc} &= [V_{ud}(\bar{d}_\alpha u_\beta)_{V-A} + V_{us}(\bar{s}_\alpha u_\beta)_{V-A}](\bar{c}_\beta b_\alpha)_{V-A}, \qquad (8)
\end{aligned}
$$

where $(\bar{q}_1 q_2)_{V-A} = \bar{q}_1\gamma^\mu(1-\gamma_5)q_2$.

Here we apply the so-called naive factorization to H_{eff} [47], the nonleptonic two-body decay amplitude T can be reduced to a product of a transition matrix element of a weak current $\langle X|J^\mu|B_c\rangle$ and an annihilation matrix element of another weak current $\langle M_2|J_\mu|0\rangle$:

$$
\begin{aligned}
T &= \langle XM_2|H_{eff}|B_c\rangle \\
&\approx \frac{G_F}{\sqrt{2}}V_{bc}V_{ij}a_1\langle X|J^\mu|B_c\rangle\langle M_2|J_\mu|0\rangle, \qquad (9)
\end{aligned}
$$

$a_1 = c_1 + \frac{1}{N_c}c_2$ and $N_c = 3$ is the number of colors. The annihilation matrix element $\langle M_2|J_\mu|0\rangle$ is related to the decay constant of M_2. When M_2 is a pseudoscalar meson [48],

$$\langle M_2|J_\mu|0\rangle = if_{M_2}P_{M_2\mu},$$

where f_{M_2} is the decay constant of meson M_2, and P_{M_2} is the momentum of M_2. When M_2 is a vector meson [49],

$$\langle M_2|J_\mu|0\rangle = \epsilon_\mu f_{M_2}M_{M_2},$$

where M_{M_2}, f_{M_2} and ϵ are the mass, decay constant and polarization vector of the vector meson M_2, respectively. The decay constant of the meson can be obtained either by theoretical model or by indirect experiment measurement.

In Eqs. (6) and (9), we find that the most important things to get the decay width of the corresponding decay are to calculate hadronic weak-current matrix elements $\langle X(P_f)|J^\mu|B_c(P)\rangle$. We will give the detailed calculation of the hadronic weak-current matrix elements in the Sect. 3.

3 The hadronic weak-current matrix elements

The calculation of the hadronic weak-current matrix element are different for different models. In this paper, we combine the B–S method which is based on relativistic B–S equation with Mandelstam formalism [50] and relativistic wave functions to calculate the hadronic matrix element. The numerical values of wave functions have been obtained by solving the full Salpeter equation which we will introduce in Appendix. As an example, we consider the semileptonic decay $B_c \to X\ell^+\nu_\ell$ in Fig. 1. In this way, at the leading order the hadronic matrix element can be written as an overlap integral over the wave functions of initial and final mesons [51],

$$\langle X(P_f,\varepsilon)|J^\mu|B_c(P)\rangle = \int \frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr} \times\left[\bar{\varphi}^{++}_{P_f}(\vec{q}_f)\frac{\not{P}}{M}\varphi^{++}_{P}(\vec{q})\gamma^\mu(1-\gamma_5)\right], \tag{10}$$

where $\vec{q}$ ($\vec{q}_f$) is the relative three-momentum between the quark and anti-quark in the initial (final) meson and $\vec{q}_f = \vec{q} - \frac{m_1'}{m_1'+m_2'}\vec{P}_f$. $\vec{P}_f$ is the three dimensional momentum of X, $\varphi^{++}_P(\vec{q})$ is the positive Salpeter wave function of B_c meson and $\varphi^{++}_{P_f}(\vec{q}_f)$ is the positive Salpeter wave function of X meson, $\bar{\varphi}^{++}_{P_f} = \gamma_0(\varphi^{++}_{P_f})^\dagger\gamma_0$. The detailed calculation of the hadronic matrix element Eq. (10) which is a function of final meson momentum P_f were discussed by Ref. [51], so the Eq. (10) is suitable for the whole kinetic region. We have calculated B_c weak decays to S-wave and P-wave mesons [36,37,42] with this hadronic matrix element in previous work, and the results were consistent with the results of some other different models. So the B_c weak decays to $Z(3930)$ and $X(4160)$ are calculated by the same metnod in this work. The corresponding Salpeter wave functions for the different mesons are shown in the next section.

4 The relativistic wave functions of meson

4.1 For B_c meson with quantum number $J^P = 0^-$

The general form for the relativistic wave function of pseudoscalar meson B_c can be written as [52]:

$$\varphi_{0^-}(\vec{q}) = \left[f_1(\vec{q})\not{P} + f_2(\vec{q})M + f_3(\vec{q})\not{q}_\perp + f_4(\vec{q})\frac{\not{P}\not{q}_\perp}{M}\right]\gamma_5, \tag{11}$$

where M is the mass of the pseudoscalar meson, and $f_i(\vec{q})$ are functions of $|\vec{q}|^2$. Due to the last two equations of Eq. (A7): $\varphi^{+-}_{0^-} = \varphi^{-+}_{0^-} = 0$, we have:

$$f_3(\vec{q}) = \frac{f_2(\vec{q})M(-\omega_1+\omega_2)}{m_2\omega_1+m_1\omega_2},$$
$$f_4(\vec{q}) = -\frac{f_1(\vec{q})M(\omega_1+\omega_2)}{m_2\omega_1+m_1\omega_2}, \tag{12}$$

where m_1, m_2 and $\omega_1 = \sqrt{m_1^2+\vec{q}^2}, \omega_2 = \sqrt{m_2^2+\vec{q}^2}$ are the masses and the energies of quark and anti-quark in B_c mesons, $q_\perp = q - (q\cdot P/M^2)P$, and $q_\perp^2 = -|\vec{q}|^2$.

The numerical values of radial wave functions f_1, f_2 and eigenvalue M can be obtained by solving the first two Salpeter equations in Eq. (A7). According to the Eq. (A6) the relativistic positive wave function of pseudoscalar meson B_c in C.M.S can be written as [52]:

$$\varphi^{++}_{0^-}(\vec{q}) = b_1\left[b_2 + \frac{\not{P}}{M} + b_3\not{q}_\perp + b_4\frac{\not{q}_\perp\not{P}}{M}\right]\gamma_5, \tag{13}$$

where the b_is ($i = 1, 2, 3, 4$) are related to the original radial wave functions f_1, f_2, quark masses m_1, m_2, quark energy w_1, w_2, and meson mass M:

$$b_1 = \frac{M}{2}\left(f_1(\vec{q}) + f_2(\vec{q})\frac{m_1+m_2}{\omega_1+\omega_2}\right), \quad b_2 = \frac{\omega_1+\omega_2}{m_1+m_2},$$
$$b_3 = -\frac{(m_1-m_2)}{m_1\omega_2+m_2\omega_1}, \quad b_4 = \frac{(\omega_1+\omega_2)}{(m_1\omega_2+m_2\omega_1)}.$$

4.2 For $Z(3930)$ and $X(4160)$ mesons with quantum number $J^P = 2^{++}$

Considering $Z(3930)$ and $X(4160)$ as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$, the general expression of the relativistic wave function can be written as [53]

$$\begin{aligned}\varphi_{2^{++}}(\vec{q}_f) = \varepsilon_{\mu\nu}q^\nu_{f\perp}\Bigg\{&q^\mu_{f\perp}\Bigg[f_1'(\vec{q}_f) + \frac{\not{P}_f}{M_f}f_2'(\vec{q}_f) + \frac{\not{q}_{f\perp}}{M_f}f_3'(\vec{q}_{f\perp}) + \frac{\not{P}_f\not{q}_{f\perp}}{M_f^2}f_4'(\vec{q}_f)\Bigg]\\ &+\gamma^\mu[M_f f_5'(\vec{q}_f) + \not{P}_f f_6'(\vec{q}_f) + \not{q}_{f\perp}f_7'(\vec{q}_f)]\\ &+\frac{i}{M_f}f_8'(\vec{q}_f)\epsilon^{\mu\alpha\beta\delta}P_{f\alpha}q_{f\perp\beta}\gamma_\delta\gamma_5\Bigg\},\end{aligned} \tag{14}$$

with the constraint on the components of the wave function:

$$f_1'(\vec{q}_f) = \frac{[q^2_{f\perp}f_3'(\vec{q}_f) + M_f^2 f_5'(\vec{q}_f)]}{M_f m_1'},$$
$$f_2'(\vec{q}_f) = 0, \quad f_7'(\vec{q}_f) = 0, \quad f_8' = \frac{f_6'(\vec{q}_f)M_f}{m_1'}.$$

Then we have the reduced wave function $\varphi_{2^{++}}(\vec{q}_f)$ as:

$$\begin{aligned}\varphi^{++}_{\chi_{c2}}(\vec{q}_f) = \varepsilon_{\mu\nu}q^\nu_{f\perp}\Bigg\{&q^\mu_{f\perp}\Bigg[a_1 + a_2\frac{\not{P}_f}{M_f} + a_3\frac{\not{q}_{f\perp}}{M_f} + a_4\frac{\not{q}_{f\perp}\not{P}_f}{M_f^2}\Bigg]\\ &+\gamma^\mu\Bigg[a_5 + a_6\frac{\not{P}_f}{M_f} + a_7\frac{\not{q}_{f\perp}}{M_f} + a_8\frac{\not{P}_f\not{q}_{f\perp}}{M_f^2}\Bigg]\Bigg\},\end{aligned} \tag{15}$$

with

$$a_1 = \frac{q^2_{f\perp}}{2M_f m_1'}n_1 + \frac{(f_5'(\vec{q}_f)w_2' - f_6'(\vec{q}_f)m_2')M_f}{2m_1'w_2'},$$

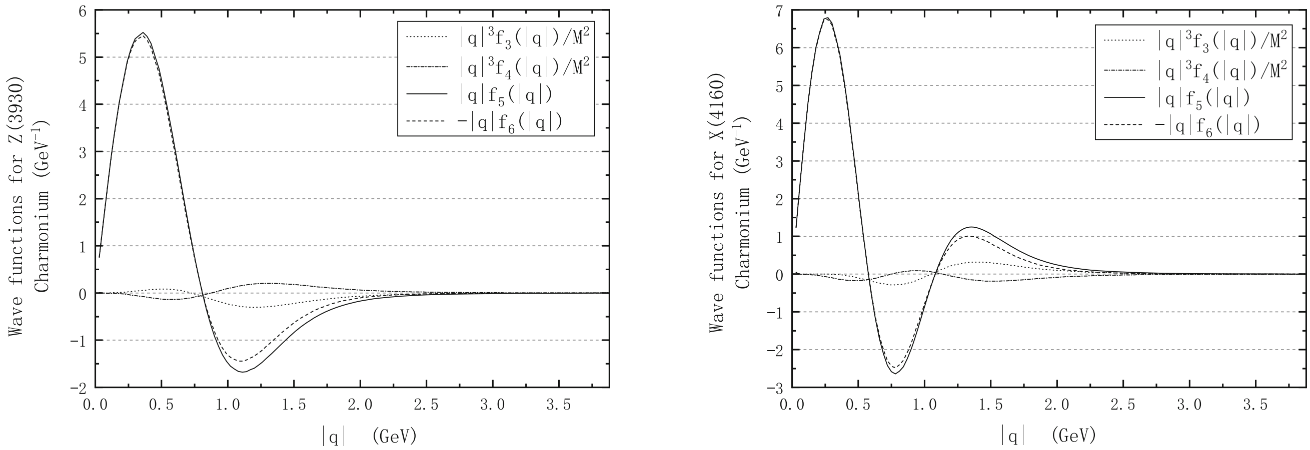

Fig. 3 The wave functions of $Z(3930)$ and $X(4160)$

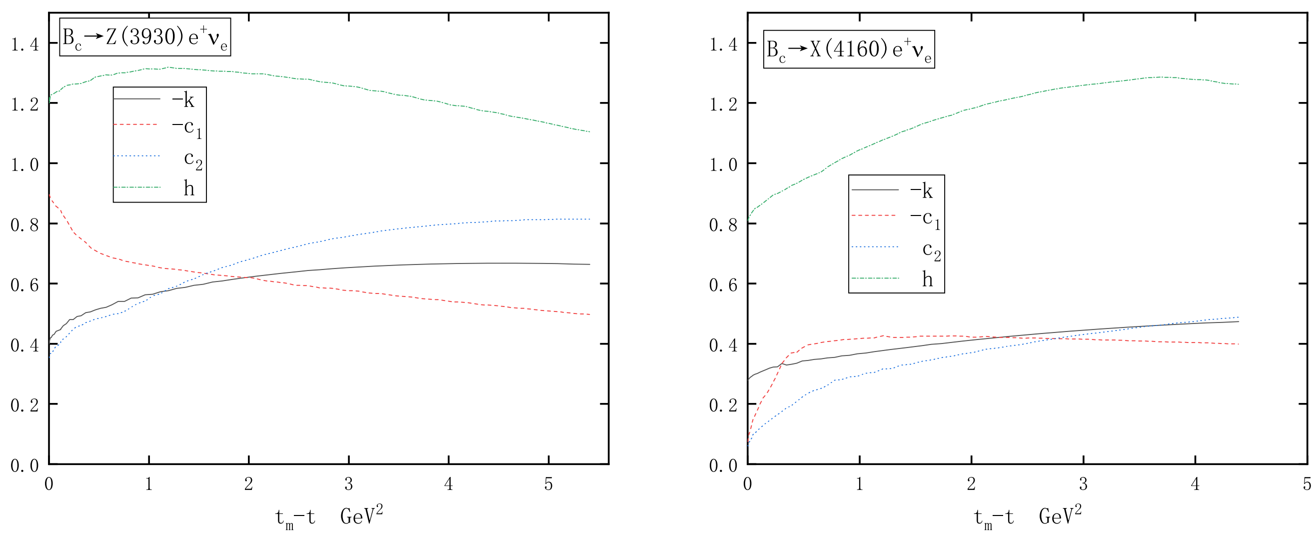

Fig. 4 The form factor of semileptonic decay B_c to $Z(3930)$ and $X(4160)$

$$a_2 = \frac{(f_6'(\vec{q}_f)w_2' - f_5'(\vec{q}_f)m_2')M_f}{2m_1'w_2'},$$

$$a_3 = \frac{1}{2}n_1 + \frac{f_6'(\vec{q}_f)M_f^2}{2m_1'w_2'}, \quad a_4 = \frac{1}{2}\left(-\frac{w_1'}{m_1'}\right)n_1 + \frac{f_5'(\vec{q}_f)M_f^2}{2m_1'w_2'},$$

$$a_5 = \frac{M_f}{2}n_2, \quad a_6 = \frac{M_f m_1'}{2w_1'}n_2, \quad a_7 = 0, \quad a_8 = \frac{M_f^2}{2w_1'}n_2,$$

$$n_1 = \frac{1}{2}\left(f_3'(\vec{q}_f) + f_4'(\vec{q}_f)\frac{m_1'}{w_1'}\right),$$

$$n_2 = \frac{1}{2}\left(f_5'(\vec{q}_f) - f_6'(\vec{q}_f)\frac{w_1'}{m_1'}\right),$$

where M_f, P_f, $f_i'(\vec{q}_f)$ are the mass, momentum and the radial wave functions of $Z(3930)$ and $X(4160)$, respectively. m_1', m_2' and $\omega_1' = \sqrt{m_1'^2 + \vec{q}_f^2}, \omega_2' = \sqrt{m_2'^2 + \vec{q}_f^2}$ are the masses and the energies of quark and anti-quark in $Z(3930)$ and $X(4160)$. To show the numerical results of wave functions explicitly, we plot the wave functions of $Z(3930)$ and $X(4160)$ states in Fig. 3.

5 Number results and discussions

In order to fix Cornell potential in Eq. (A11) and masses of quarks, we take these parameters: $a = e = 2.7183$, $\lambda = 0.21$ GeV^2, $\Lambda_{QCD} = 0.27$ GeV, $\alpha = 0.06$ GeV, $m_b = 4.96$ GeV, $m_c = 1.62$ GeV, etc. [53], which are best to fit the mass spectra of B_c and other heavy meson states. Taking these parameters to B–S equation, and solving the B–S equation numerically, we get the masses of $Z(3930)$, $X(4160)$ and B_c as: $M_{Z(3930)} = (3.926 \pm 0.167)$ GeV, $M_{X(4160)} = (4.156 \pm 0.170)$ GeV, $M_{B_c} = (6.276 \pm 0.303)$ GeV, varying all the input parameters (λ, Λ_{QCD}, α, etc) simultaneously within $\pm 5\%$ of the central values, we also obtain the uncertainties of masses, and the corresponding wave functions were obtained

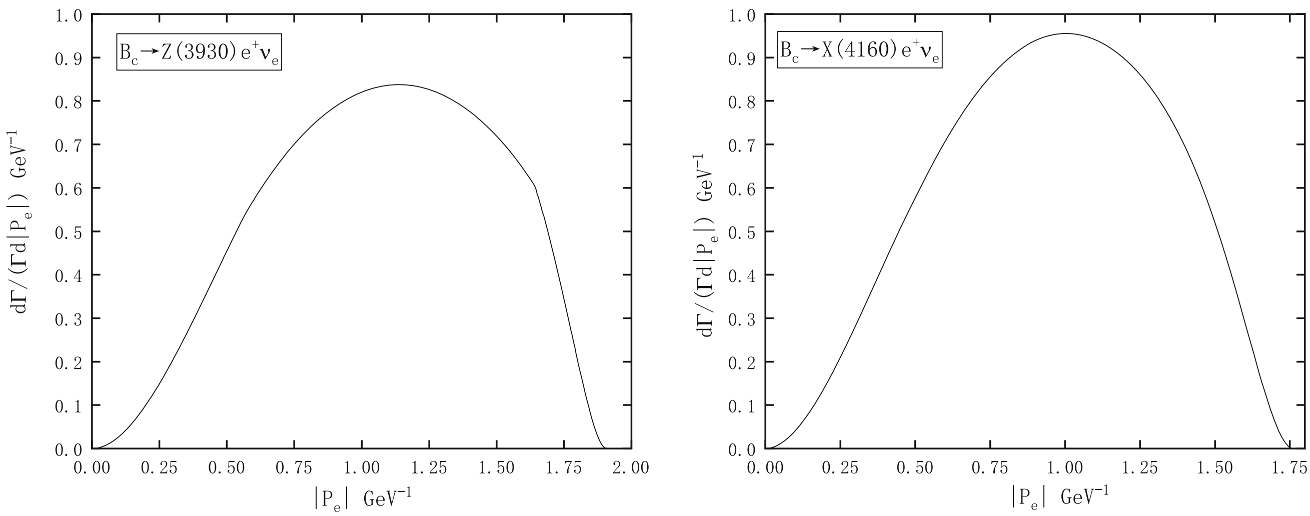

Fig. 5 The leptonic energy spectra of semileptonic decay B_c to $Z(3930)$ and $X(4160)$

in Sect. 4. Then we can calculate the semileptonic decays and nonleptonic decays of B_c to $Z(3930)$ and $X(4160)$.

5.1 The semileptonic decays

In order to calculate the semileptonic decays of B_c to $Z(3930)$ and $X(4160)$, we use the central values of the CKM matrix elements: $V_{cb} = 0.0406$, and other constants: $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, which are taken from PDG [4]. Taking the masses and the corresponding wave functions to Eq. (10), we represent the hadronic transition weak-current matrix elements as proper integrations of the components of the B–S wave functions. And the hadronic weak-current matrix element can be written as the form factors k, c_1, c_2, h. The form factors are related to four-momentum transfer squared $t = (P - P_f)^2 = M^2 + M_f^2 - 2ME_f$ which provides the kinematic range for the semileptonic decay of B_c. It varies from $t = 0$ to $t = 5.52$ GeV^2 for the decays to $Z(3930)$ and from $t = 0$ to $t = 4.48$ GeV^2 for the decays to $X(4160)$. In Fig. 4 we give the relations of $(t_m - t)$($t_m = (M - M_f)^2$ is the maximum of t) and the form factors. Taking the form factor to the Eq. (6), then we will get the leptonic energy spectra $\frac{d\Gamma}{\Gamma dP_e}$ for semileptonic B_c decay to $Z(3930)$ and $X(4160)$, the leptonic energy spectra are plotted in Fig. 5 which are related to the momentum of the final mesons.

Using the leptonic energy spectra, we calculate the decay widths of the semileptonic $B_c \to X\ell^+\nu_\ell$ ($X = Z(3930)$ or $X(4160)$, $\ell = e, \mu, \tau$) and give the results in Table 1. Since m_τ is very large and $m_e \simeq m_\mu$ is quite a good approximation for the B_c meson decays, thus only the cases where the lepton is an electron or τ are given in Table 1. Because of the larger kinematic ranges and the different wave functions in Fig. 3, the corresponding decay widths of $B_c^+ \to Z(3930)$ are larger than these of $B_c^+ \to X(4160)$.

Table 1 The decay widths of exclusive semileptonic decays of B_c to $Z(3930)$, $X(4160)$ (in 10^{-15}GeV)

Mode	Ours
$B_c^+ \to Z(3930)\, e^+\bar{\nu}_e$	$(4.39^{+0.13}_{-0.24}) \times 10^{-1}$
$B_c^+ \to Z(3930)\, \tau^+\bar{\nu}_\tau$	$(0.78^{+0.31}_{-0.42}) \times 10^{-3}$
$B_c^+ \to X(4160)\, e^+\bar{\nu}_e$	$(5.14^{+0.83}_{-0.49}) \times 10^{-3}$
$B_c^+ \to X(4160)\, \tau^+\bar{\nu}_\tau$	$(3.80^{+0.45}_{-0.38}) \times 10^{-6}$

Table 2 The decay widths of exclusive nonleptonic decays of B_c to $Z(3930)$, $X(4160)$ (in 10^{-15} GeV)

Mode	Ours
$B_c^+ \to Z(3930) + \pi$	$(1.88^{+0.49}_{-0.66}) \times 10^{-3}a_1^2$
$B_c^+ \to Z(3930) + K$	$(1.38^{+0.37}_{-0.51}) \times 10^{-4}a_1^2$
$B_c^+ \to Z(3930) + \rho$	$(6.26^{+1.42}_{-1.48}) \times 10^{-3}a_1^2$
$B_c^+ \to Z(3930) + K^*$	$(3.82^{+0.61}_{-0.86}) \times 10^{-4}a_1^2$
$B_c^+ \to X(4160) + \pi$	$(6.89^{+1.50}_{-1.15}) \times 10^{-5}a_1^2$
$B_c^+ \to X(4160) + K$	$(4.71^{+0.82}_{-0.79}) \times 10^{-6}a_1^2$
$B_c^+ \to X(4160) + \rho$	$(2.37^{+0.56}_{-0.42}) \times 10^{-3}a_1^2$
$B_c^+ \to X(4160) + K^*$	$(1.64^{+0.45}_{-0.33}) \times 10^{-4}a_1^2$

5.2 The nonleptonic decays

We only consider two-body nonleptonic decays of B_c^+ to $Z(3930)$ and $X(4160)$, and another meson is light meson. Thus, the hadronic transition matrix elements of weak currents have a fixed momentum transfer. To calculate the decay widths basis on Eq. 9, we only need to calculate the annihilation matrix element $\langle M_2|J_\mu|0\rangle$ which is related to the decay constant of M_2. The masses and decay constants are: $M_\pi = 0.140$ GeV, $f_\pi = 0.130$ GeV, $M_\rho = 0.775$ GeV,

Table 3 The branching ratio(in %) of exclusive semileptonic decay B_c to $Z(3930)$, $X(4160)$ with the lifetime of B_c:$\tau_{B_c} = 0.453$ ps

Mode	Results	Mode	Results
$B_c^+ \to Z(3930)e^+\bar{\nu}_e$	$(3.03^{+0.09}_{-0.16}) \times 10^{-2}$	$B_c^+ \to X(4160)\ e^+\bar{\nu}_e$	$(3.55^{+0.83}_{-0.35}) \times 10^{-4}$
$B_c^+ \to Z(3930)\tau^+\bar{\nu}_\tau$	$(0.55^{+0.22}_{-0.30}) \times 10^{-4}$	$B_c^+ \to X(4160)\ \tau^+\bar{\nu}_\tau$	$(2.62^{+0.31}_{-0.26}) \times 10^{-7}$
$B_c^+ \to Z(3930) + \pi$	$(1.68^{+0.44}_{-0.58}) \times 10^{-4}$	$B_c^+ \to X(4160) + \pi$	$(6.17^{+1.35}_{-1.02}) \times 10^{-6}$
$B_c^+ \to Z(3930) + K$	$(1.24^{+0.33}_{-0.46}) \times 10^{-5}$	$B_c^+ \to X(4160) + K$	$(4.21^{+0.75}_{-0.70}) \times 10^{-7}$
$B_c^+ \to Z(3930) + \rho$	$(5.61^{+1.28}_{-1.33}) \times 10^{-4}$	$B_c^+ \to X(4160) + \rho$	$(2.12^{+0.42}_{-0.37}) \times 10^{-4}$
$B_c^+ \to Z(3930) + K^*$	$(3.43^{+0.54}_{-0.78}) \times 10^{-5}$	$B_c^+ \to X(4160) + K^*$	$(1.47^{+0.41}_{-0.29}) \times 10^{-5}$

$f_\rho = 0.205$ GeV, $M_K = 0.494$ GeV, $f_K = 0.156$ GeV, $M_{K^*} = 0.892$ GeV, $f_{K^*} = 0.217$ GeV [4,54], respectively. And the corresponding CKM matrix elements are: $V_{ud} = 0.974$ and $V_{us} = 0.2252$. Using the form factors of B_c nonleptonic decays and the decay corresponding constants, we show the nonleptonic decay widths which are related to the parameter a_1 in Table 2. The results of B_c nonleptonic decay are affected by the CKM matrix elements, so the results of light mesons π, ρ are larger than the ones of light mesons K, K^* in Table 2, respectively.

In order to compare the numerical values with experimental measurements in the future, Taking the values $a_1 = 1.14$ for nonleptonic decays [45,46], combining the life time of B_c meson, we calculate the branching ratios of the decays and list them in Table 3. Because of $B_c \to Z(3930)$, $X(4160)$ have small kinematic ranges and the wave functions have some minus parts in $Z(3930)$, and $X(4160)$, comparing our results with B_c decays to $\chi_{c2}(1P)$ in Ref. [36], the results are smaller than the results of B_c decay to $\chi_{c2}(1P)$. The uncertainties of decay widths and branching ratios shown in Tables 1, 2 and 3, which are very large. The large uncertainties not only come from the phase spaces, but also from the variation of the node of the $2P$ and $3P$ wave functions, which means that a small change of node location will result in large uncertainties.

In summary, considering $Z(3930)$ and $X(4160)$ as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, respectively, we study the semileptonic and nonleptonic B_c decays to $Z(3930)$ and $X(4160)$ by the improved B–S method which consider the relativistic correction. According to the Mandelstam formalism and the relativistic wave functions of heavy mesons, we get the corresponding decay form factors, and obtain the corresponding decay widths and branching ratios. Because of the minus value in the wave functions of $Z(3930)$ and $X(4160)$ and the small CKM V_{bc}, the decay widths and branching ratios are very small. But now the large hadron collider (LHC) will produce as many as 5×10^{10} B_c events per year [43,44]. If sufficient events can be observed, some channels will provide us a sizable ratios, such as the branching ratios of the order of (10^{-6}) could be measured precisely at the LHC, and maybe they will detect the productions of $Z(3930)$ and $X(4160)$ in B_c exclusive weak semileptonic and nonleptonic decay. Then our results will provide a new way to observe the $Z(3930)$ and $X(4160)$ and the necessary information for the study of B_c meson.

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Appendix A: Instantaneous Bethe–Salpeter equation

In this section, we briefly review the Bethe-Salpeter (B–S) equation and its instantaneous one, the Salpeter equation.

The B–S equation is read as [55]:

$$(\not{p}_1 - m_1)\chi(q)(\not{p}_2 + m_2) = i \int \frac{d^4k}{(2\pi)^4} V(P, k, q)\chi(k), \tag{A1}$$

where $\chi(q)$ is the B–S wave function, $V(P, k, q)$ is the interaction kernel between the quark and antiquark, and p_1, p_2 are the momentum of the quark 1 and anti-quark 2.

We divide the relative momentum q into two parts, $q_{\parallel}$ and $q_{\perp}$,

$$q^{\mu} = q^{\mu}_{\parallel} + q^{\mu}_{\perp},$$
$$q^{\mu}_{\parallel} \equiv (P \cdot q/M^2)P^{\mu}, \quad q^{\mu}_{\perp} \equiv q^{\mu} - q^{\mu}_{\parallel}.$$

B–S equation Eq. (A1) is a four dimension covariant equation, in order to solve the Eq. (A1), we will take the instantaneous approximation in the interaction kernel $V(P, k, q)$, then the B–S equation will lose the covariance. The effect of instantaneous approximation in $V(P, k, q)$ could be corrected by the retardation effects in $V(P, k, q)$. But the retardation effects in $V(P, k, q)$ are very small for the heavy mesons [56–58], this means that the influence of the instantaneous approximation on the covariance of B–S equation are very small for the heavy mesons. The instantaneous approximation in $V(P, k, q)$ almost don't influence the wave functions, and the decay matrix elements which involve the heavy mesons mostly unchanged [56]. Our model mostly keeps the covariance in the calculation, and the weak decay results also satisfy the Lorentz-covariance.

In instantaneous approach, the kernel $V(P, k, q)$ takes the simple form [59]:

$$V(P, k, q) \Rightarrow V(|\vec{k} - \vec{q}|).$$

Let us introduce the notations $\varphi_p(q^{\mu}_{\perp})$ and $\eta(q^{\mu}_{\perp})$ for three dimensional wave function as follows:

$$\varphi_p(q^{\mu}_{\perp}) \equiv i \int \frac{dq_p}{2\pi} \chi(q^{\mu}_{\parallel}, q^{\mu}_{\perp}),$$
$$\eta(q^{\mu}_{\perp}) \equiv \int \frac{dk_{\perp}}{(2\pi)^3} V(k_{\perp}, q_{\perp})\varphi_p(k^{\mu}_{\perp}). \tag{A2}$$

Then the BS equation can be rewritten as:

$$\chi(q_{\parallel}, q_{\perp}) = S_1(p_1)\eta(q_{\perp})S_2(p_2). \tag{A3}$$

The propagators of the two constituents can be decomposed as:

$$S_i(p_i) = \frac{\Lambda^{+}_{ip}(q_{\perp})}{J(i)q_p + \alpha_i M - \omega_i + i\epsilon} + \frac{\Lambda^{-}_{ip}(q_{\perp})}{J(i)q_p + \alpha_i M + \omega_i - i\epsilon}, \tag{A4}$$

with

$$\omega_i = \sqrt{m_i^2 + q_T^2}, \quad \Lambda^{\pm}_{ip}(q_{\perp}) = \frac{1}{2\omega_{ip}}\left[\frac{\not{P}}{M}\omega_i \pm J(i)(m_i + \not{q}_{\perp})\right], \tag{A5}$$

where $i = 1, 2$ for quark and anti-quark, respectively, and $J(i) = (-1)^{i+1}$.

Introducing the notations $\varphi^{\pm\pm}_p(q_{\perp})$ as:

$$\varphi^{\pm\pm}_p(q_{\perp}) \equiv \Lambda^{\pm}_{1p}(q_{\perp})\frac{\not{P}}{M}\varphi_p(q_{\perp})\frac{\not{P}}{M}\Lambda^{\pm}_{2p}(q_{\perp}). \tag{A6}$$

With contour integration over q_p on both sides of Eq. (A3), we obtain:

$$\varphi_p(q_{\perp}) = \frac{\Lambda^{+}_{1p}(q_{\perp})\eta_p(q_{\perp})\Lambda^{+}_{2p}(q_{\perp})}{(M - \omega_1 - \omega_2)} - \frac{\Lambda^{-}_{1p}(q_{\perp})\eta_p(q_{\perp})\Lambda^{-}_{2p}(q_{\perp})}{(M + \omega_1 + \omega_2)},$$

and the full Salpeter equation:

$$(M - \omega_1 - \omega_2)\varphi^{++}_p(q_{\perp}) = \Lambda^{+}_{1p}(q_{\perp})\eta_p(q_{\perp})\Lambda^{+}_{2p}(q_{\perp}),$$
$$(M + \omega_1 + \omega_2)\varphi^{--}_p(q_{\perp}) = -\Lambda^{-}_{1p}(q_{\perp})\eta_p(q_{\perp})\Lambda^{-}_{2p}(q_{\perp}),$$
$$\varphi^{+-}_p(q_{\perp}) = \varphi^{-+}_p(q_{\perp}) = 0. \tag{A7}$$

For the different J^{PC} (or J^P) states, we give the general form of wave functions. Reducing the wave functions by the last equation of Eq. (A7), then solving the first and second equations in Eq. (A7) to get the wave functions and mass spectrum. We have discussed the solution of the Salpeter equation in detail in Refs. [52,53].

The normalization condition for BS wave function is:

$$\int \frac{q_T^2 dq_T}{2\pi^2} Tr\left[\overline{\varphi}^{++}\frac{\not{P}}{M}\varphi^{++}\frac{\not{P}}{M} - \overline{\varphi}^{--}\frac{\not{P}}{M}\varphi^{--}\frac{\not{P}}{M}\right] = 2P_0. \tag{A8}$$

In our model, the instantaneous interaction kernel V is Cornell potential, which is the sum of a linear scalar interaction and a vector interaction:

$$V(r) = V_s(r) + V_0 + \gamma_0 \otimes \gamma^0 V_v(r) = \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3}\frac{\alpha_s}{r}, \tag{A9}$$

where λ is the string constant and $\alpha_s(\vec{q})$ is the running coupling constant. In order to fit the data of heavy quarkonia, a constant V_0 is often added to confine potential. We introduce a factor $e^{-\alpha r}$ to avoid the infrared divergence in the momentum space:

$$V_s(r) = \frac{\lambda}{\alpha}(1 - e^{-\alpha r}), \quad V_v(r) = -\frac{4}{3}\frac{\alpha_s}{r}e^{-\alpha r}. \tag{A10}$$

It is easy to know that when $\alpha r \ll 1$, the potential becomes to Eq. (A9). In the momentum space and the C.M.S of the bound state, the potential reads :

$$V(\vec{q}) = V_s(\vec{q}) + \gamma_0 \otimes \gamma^0 V_v(\vec{q}),$$
$$V_s(\vec{q}) = -\left(\frac{\lambda}{\alpha} + V_0\right)\delta^3(\vec{q}) + \frac{\lambda}{\pi^2}\frac{1}{(\vec{q}^2 + \alpha^2)^2},$$
$$V_v(\vec{q}) = -\frac{2}{3\pi^2}\frac{\alpha_s(\vec{q})}{(\vec{q}^2 + \alpha^2)}, \tag{A11}$$

where the running coupling constant $\alpha_s(\vec{q})$ is:

$$\alpha_s(\vec{q}) = \frac{12\pi}{33 - 2N_f} \frac{1}{\log\left(a + \frac{\vec{q}^2}{\Lambda_{QCD}^2}\right)}.$$

We introduce a small parameter a to avoid the divergence in the denominator. The constants λ, α, V_0 and Λ_{QCD} are the parameters that characterize the potential. $N_f = 3$ for $\bar{b}q$ (and $\bar{c}q$) system.

References

1. S. Uehara et al., Belle Collaboration, Phys. Rev. Lett. **104**, 092001 (2010)
2. S. Uehara et al., Belle Collaboration, Phys. Rev. Lett. **96**, 082003 (2006)
3. B. Aubert et al., BABAR Collaboration, Phys. Rev. D **81**, 092003 (2010)
4. K. A. Olive et al., (Partile Data Group), Chin. Phys. C. **38**, 090001 (2015)
5. B.Q. Li, K.T. Chao, Phys. Rev. D **79**, 094004 (2009)
6. X. Liu, Chin. Sci. Bull. **59**, 3815 (2014)
7. H.X. Chen, W. Chen, X. Liu, S.L. Zhu, Phys. Rep. **639**, 1 (2016)
8. P. Pakhlov et al., Belle Collaboration, Phys. Rev. Lett. **100**, 202001 (2008)
9. N. Ikeno, M. Bayar, E. Oset, Eur. Phys. J. C **78**, 429 (2018)
10. R. Molina, E. Oset, Phys. Rev. D **80**, 114013 (2009)
11. D. Y. Chen, X. Liu, T. Matsuki, Prog. Theor. Exp. Phys. 043B05 (2015)
12. T.H. Wang, G.L. Wang, H.F. Fu, W.L. Ju, JHEP **07**, 120 (2013)
13. W. Chen, H.X. Chen, X. Liu, T.G. Steele, S.L. Zhu, Phys. Rev. D **96**, 114017 (2017)
14. S.L. Olsen, Front. Phys. **10**, 101401 (2015)
15. H.C. Kim, K.S. Kim, M.K. Cheoun, D. Jido, M. Oka, Eur. Phys. J. A **52**(7), 184 (2016)
16. Y.M. Wang, C.D. Lu, Phys. Rev. D **77**, 054003 (2008)
17. Z.G. He, B.Q. Li, Phys. Lett. B **693**, 36–43 (2010)
18. R.L. Zhu, Phys. Rev. D **92**, 074017 (2015)
19. L.P. He, D.Y. Chen, X. Liu, T. Matsuki, Eur. Phys. J. C **74**, 3028 (2014)
20. Y.C. Yang, Z.R. Xia, J.L. Ping, Phys. Rev. D **81**, 094003 (2010)
21. H. Wang, Z.Z. Yan, J.L. Ping, Eur. Phys. J. C **75**, 196 (2015)
22. L.C. Gui, L.S. Lu, Q.F. Lu, X.H. Zhong, Q. Zhao, Phys. Rev. D **98**, 016010 (2018)
23. Z.H. Wang, Y. Zhang, L.B. Jiang, T.H. Wang, Y. Jiang, G.L. Wang, Eur. Phys. J. C **77**(1), 43 (2017)
24. C.H. Chang, Y.Q. Chen, Phys. Rev. D **49**, 3399 (1994)
25. C.H. Chang, Y.Q. Chen, Phys. Rev. D **46**, 3845 (1992)
26. M. Beneke, G. Buchalla, Phys. Rev. D **53**, 4991 (1996)
27. J.F. Liu, K.T. Chao, Phys. Rev. D **56**, 4133 (1997)
28. Y.S. Dai, D.S. Du, Eur. Phys. J. C **9**, 557 (1999)
29. D. Ebert, R.N. Faustov, V.O. Galkin, Phys. Rev. D **82**, 034019 (2010)
30. D. Ebert, R.N. Faustov, V.O. Galkin, Phys. Rev. D **82**, 034032 (2010)
31. M.A. Ivanov, J.G. Körner, P. Santorelli, Phys. Rev. D **63**, 074010 (2001)
32. M.A. Ivanov, J.G. Körner, P. Santorelli, Phys. Rev. D **73**, 054024 (2006)
33. X.X. Wang, W. Wang, C.D. Lu, Phys. Rev. D **79**, 114018 (2009)
34. W. Wang, Y.L. Shen, C.D. Lu, Phys. Rev. D **79**, 054012 (2009)
35. X. Liu, Z.J. Xiao, Phys. Rev. D **81**, 074017 (2010)
36. Z.H. Wang, G.L. Wang, C.H. Chang, J. Phys. G **39**, 015009 (2012)
37. Z.H. Wang, G.L. Wang, H.F. Fu, Y. Jiang, Int. J. Mod. Phys. A **27**, 1250049 (2012)
38. H.F. Fu, Y. Jiang, C.S. Kim, G.L. Wang, JHEP **1106**, 015 (2011)
39. W.L. Ju, G.L. Wang, H.F. Fu, T.H. Wang, Y. Jiang, JHEP **1404**, 065 (2014)
40. W.L. Ju, G.L. Wang, H.F. Fu, Z.H. Wang, Y. Li, JHEP **1509**, 171 (2015)
41. W.L. Ju, T.H. Wang, Y. Jiang, H. Yuan, G.L. Wang, J. Phys. G **43**, 045004 (2016)
42. C.H. Chang, H.F. Fu, G.L. Wang, J.M. Zhang, Sci. Chin. Phys. Mech. Astron. **58**, 071001 (2015)
43. N. Brambilla et al., CERN Yellow Report, CERN-2005-005, and references therein
44. N. Brambilla et al., Eur. Phys. J. C **71**, 1534 (2011)
45. G. Buchalla, A.J. Buras, M.E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996)
46. A. Ali, J. Chay, C. Greub, P. Ko, Phys. Lett. B **424**, 161 (1998)
47. M. Bauer, B. Stech, M. Wirbel, Z. Phys. C **34**, 103 (1987)
48. G. Cvetic, C.S. Kim, G.L. Wang, W. Namgung, Phys. Lett. B **596**, 84 (2004)
49. G.L. Wang, Phys. Lett. B **633**, 492 (2006)
50. S. Mandelstam, Proc. R. Soc. Lond. **233**, 248 (1955)
51. C.H. Chang, J.K. Chen, G.L. Wang, Commun. Theor. Phys. **46**, 467 (2006)
52. C.S. Kim, G.L. Wang, Phys. Lett. B **584**, 285 (2004)
53. C.H. Chang, G.L. Wang, Sci. China Ser. G **53**, 2005 (2010)
54. P. Ball, R. Zwicky, Phys. Rev. D **71**, 014029 (2005)
55. E.E. Salpeter, H.A. Bethe, Phys. Rev. **84**, 1232 (1951)
56. C.F. Qiao, H.W. Huang, K.T. Chao, Phys. Rev. D **54**, 2273 (1996)
57. C.F. Qiao, H.W. Huang, K.T. Chao, Phys. Rev. D **60**, 094004 (1999)
58. D. Ebert, R.N. Faustov, V.O. Galkin, Phys. Rev. D **62**, 034014 (2000)
59. E.E. Salpeter, Phys. Rev. **87**, 328 (1952)