Eur. Phys. J. C (2020) 80:783
https://doi.org/10.1140/epjc/s10052-020-8362-3

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Static wormhole solutions and Noether symmetry in modified Gauss–Bonnet gravity

M. Sharif[a], **Iqra Nawazish**[b], **Shahid Hussain**[c]

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore 54590, Pakistan

Received: 29 May 2020 / Accepted: 14 August 2020 / Published online: 27 August 2020

Abstract In this paper, we analyze static traversable wormholes via Noether symmetry technique in modified Gauss–Bonnet $f(\mathcal{G})$ theory of gravity (where $\mathcal{G}$ represents Gauss–Bonnet term). We assume isotropic matter configuration and spherically symmetric metric. We construct three $f(\mathcal{G})$ models, i.e, linear, quadratic and exponential forms and examine the consistency of these models. The traversable nature of wormhole solutions is discussed via null energy bound of the effective stress–energy tensor while physical behavior is studied through standard energy bounds of isotropic fluid. We also discuss the stability of these wormholes inside the wormhole throat and conclude the presence of traversable and physically stable wormholes for quadratic as well as exponential $f(\mathcal{G})$ models.

1 Introduction

The general theory of relativity (GR) not only incorporates information about gravity and matter but also provides foundation for the understanding of black holes and standard big-bang model of cosmology. GR is the simplest relativistic theory of gravity that is consistent with the experimental data but still suffers from some unresolved issues like earlier and current cosmic expansions. The favorable and optimistic approach to unveil the salient features of these dark aspects is to modify the gravity by introducing some extra degrees of freedom in the Einstein–Hilbert action. These modifications are formulated by replacing or adding curvature invariants as well as their corresponding generic functions in Einstein–Hilbert action referred as modified gravitational theories [1].

Recent observational facts of modern cosmology indicate the current accelerated expansion of the universe. This expansion occurs due to the strange force with fascinating anti-gravitational impacts, named as dark energy (DE). One of the approaches to study the nature of DE is the modified theories of gravity. Nojiri and Odintsov [2] proposed $f(\mathcal{G})$ ($\mathcal{G}$ represents Gauss–Bonnet (GB) invariant) gravity by including higher-order correction terms. The inspiration of this theory arises from the string theory at low energy scale which efficiently helps to examine the late-time evolution of the cosmos. The GB invariant is quadratic in nature and is free from spin-2 ghost instabilities acts as a four-dimensional topological term which is the composition of the scalar curvature (R), Ricci ($R_{\alpha\beta}$) and Riemann tensors ($R_{\alpha\beta\mu\nu}$) defined as $\mathcal{G}=R^2-4R_{\alpha\beta}R^{\alpha\beta}+R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$.

This theory has a quite rich cosmological structure which describes fascinating characteristics of early as well as late-time cosmological evolution and is consistent with solar system constraints. The GB invariant gives fascinating results when either comprised of a scalar field or a general function $f(\mathcal{G})$ is included in the Einstein–Hilbert action [3–5]. This theory provides a possibility to study the transformation from non-phantom to phantom phase and from decelerated to an accelerated region. It is observed that $f(\mathcal{G})$ gravity well describes the laws of thermodynamics and many other cosmological issues [6–9]. Sharif and Fatima [10] investigated the spherical interior solutions of this gravity by applying conformal Killing vectors corresponding to isotropic as well as anisotropic fluid configurations and checked the physical consistency via energy conditions.

Noether symmetry is recognized as the most efficient method to investigate the analytic solutions that help to find the conserved parameters of the field equations corresponding to symmetry generators. Capozziello et al. [11] examined the analytic solutions of static spherically symmetric spacetime for the power-law functional form of $f(R)$ theory. The same authors [12,13] extended this work for the non-static case and obtained exact solutions for constant as well as variable curvature scalar. Vakili [14] used this approach for flat FRW model to discuss the current cosmic expansion through

[a] e-mail: msharif.math@pu.edu.pk (corresponding author)
[b] e-mail: iqranawazish07@gmail.com
[c] e-mail: shahidhussaine56@gmail.com

an effective equation of state (EoS) parameter in $f(R)$ gravity. Many researchers [15–20] investigated the current accelerated cosmic expansion through this approach in different modified theories.

A wormhole (WH) is a hypothetical bridge or tunnel that allows a smooth passing through different regions of spacetime. If hypothetical tunnel connects two regions of the same spacetime then intra-universe WH is established whereas inter-universe WH appears for two distinct spacetimes. The existence of exotic matter (matter with negative energy density) encourages observer to move smoothly through tunnel but its sufficient amount leads to controversial existence of a realistic WH. Consequently, the only way to have a physically viable WH model is to minimize the usage of exotic matter in the tunnel. For any static configuration, the most crucial problem is stability analysis which defines their behavior against perturbations as well as enhances physical characterization. A singularity free configuration identifies a stable state which successfully prevents the WH to collapse while a WH can also exist for quite a long time even if it is unstable due to very slow decay.

The study of WH geometries has gained much attention in modified theories of gravity. In $f(R)$ scenario, Lobo and Oliveira [21] assumed distinct fluid distributions with constant shape function to investigate the WH geometry. Jamil et al. [22] examined feasible WH solutions with non-commutative geometry by considering a specific shape function corresponding to power-law $f(R)$ model. Bahamonde et al. [23] used the same gravity for FRW universe model to analyze the cosmological WH solutions with isotropic fluid. Mazharimousavi and Halilsoy [24] discussed the conditions of WH for vacuum/non-vacuum cases and obtained the stable WH geometry for $f(R)$ model along with polynomial evolution. Sharif and Fatima [25,26] explored the non-static solutions of WH as well as static spherically symmetric WH in galactic halo region in $f(\mathcal{G})$ gravity. Bahamonde et al. [27] found definite solutions of shape function and red-shift parameter via Noether symmetry and examined the graphical behavior in the background of non-minimal coupling with torsion scalar in scalar-tensor theory.

Recently, Sharif and Nawazish investigated the static WH solutions using Noether symmetry technique in both $f(R)$ [28] as well as $f(R,T)$ gravity [29] and found stable structure for two different values of red-shift function. In this paper, we study the physical presence of WH via Noether symmetry technique in $f(\mathcal{G})$ theory and explore WH properties associated with perfect fluid. The paper is arranged in the following pattern. Section 2 represents the basic formalism of this gravity. We obtain point-like Lagrangian in Sect. 3 which is used in Sect. 4 to estimate WH solutions for variable red-shift function. Section 5 explores the stable structure of developed WH geometries and summary of our results is given in the last section.

2 Basic formalism of $f(\mathcal{G})$ gravity

The action of $f(\mathcal{G})$ gravity in 4-dimensions with matter Lagrangian is presented by

$$S=\frac{1}{2k^2}\int[R+f(\mathcal{G})]\sqrt{-g}d^4x+\int\sqrt{-g}\mathcal{L}_m d^4x, \quad (1)$$

where k is the coupling constant and $\mathcal{L}_m$ defines matter Lagrangian. Varying this action with respect to metric tensor, the corresponding field equations are

$$\begin{aligned}G_{\alpha\beta}=&\frac{1}{2}g_{\alpha\beta}f(\mathcal{G})-(2RR_{\alpha\beta}-4R^{\mu}_{\alpha}R_{\mu\beta}-4R_{\alpha\mu\beta\nu}R^{\mu\nu}\\&+2R^{\mu\nu\gamma}_{\alpha}R_{\beta\mu\nu\gamma})f_{\mathcal{G}}\\&-(2R\nabla^2g_{\alpha\beta}-2\nabla_\alpha\nabla_\beta R-4R^{\mu\nu}g_{\alpha\beta}\nabla_\mu\nabla_\nu\\&-4\nabla^2R_{\alpha\beta}+4\nabla_\beta\nabla_\mu R^{\mu}_{\alpha}+4\nabla_\alpha\nabla_\mu R^{\mu}_{\beta}\\&+4\nabla^\mu\nabla^\nu R_{\alpha\mu\beta\nu})f_{\mathcal{G}}+k^2T_{\alpha\beta},\end{aligned} \quad (2)$$

where $\nabla^2=\nabla_\alpha\nabla^\alpha$ is d'Alembert operator, ∇_α indicates the covariant derivative and $f_{\mathcal{G}}$ denotes differentiation of generic function with respect to $\mathcal{G}$. The stress–energy tensor is determined by the following form

$$T_{\alpha\beta}=\frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta\left(g^{\alpha\beta}\right)}. \quad (3)$$

Here the metric tensor depends only upon the distribution of matter yielding

$$T_{\alpha\beta}=g_{\alpha\beta}\mathcal{L}_m-2\frac{\delta\mathcal{L}_m}{\delta g^{\alpha\beta}}. \quad (4)$$

The energy–momentum tensor for perfect fluid configuration is

$$T^{(m)}_{\alpha\beta}=(\rho_m+p_m)u_\alpha u_\beta+p_m g_{\alpha\beta}, \quad (5)$$

where p_m and ρ_m characterize pressure and energy density, respectively and u_α represents the four velocity of the fluid.

The static spherically symmetric line element [30] is given by

$$ds^2=-e^{a(r)}dt^2+e^{b(r)}dr^2+M(r)(d\theta^2+\sin^2\theta d\phi^2), \quad (6)$$

where the triplet (M,a,b) indicates generic radial functions. For $M(r)=r^2$ [31], the spherical symmetry (6) characterizes Morris–Thorne WH in which $a(r)$ is identified as the red-shift function as it determines gravitational red-shift of WH whereas $e^{b(r)}=(1-\frac{h(r)}{r})^{-1}$ with $h(r)$ being the shape function as it specifies spacial shape of WH. The radial coordinate r possesses non-monotonic behavior as it decreases from infinity to minimum radius when a traveler moves from one part of WH. The space occupying minimum radius is known as throat of WH. The radial coordinate starts increasing from minimum radius to infinity as traveler comes out of the throat and entered into another region of WH. The basic property of WH is the flaring-out condition for which $\frac{h(r)-h(r)'r}{h(r)^2} > 0$. At the throat or near the throat, the traversable WH demands $0 \leq h'(r) < 1$, where prime represents differentiation with respect to r. The sufficient condition of traversable WH is the finite red-shift function throughout the whole space of WH. This condition ensures the absence of horizons and consequently, allows a traveler to move into a WH as well as appreciates a smooth exit.

For spherically symmetric spacetime (5) and perfect fluid (6), we formulate the field equations corresponding to Eqs. (1) and (2) as follows

$$\frac{e^{a}(-4M''M+2b'M'M+M'^2+4Me^b)}{4e^bM^2} = \rho_m e^a k^2 - \frac{1}{2} e^a f(\mathcal{G}) + e^{a-2b}\Big[a'^4 - \frac{M'a'b'}{2M^2} + \frac{M'a'b'^2}{M} - \frac{3M'^2a'b'}{4M^2} - 2\frac{M'a''b'}{M} + 4\frac{M''a''}{M} - \frac{e^b a'^2}{M} + \frac{e^b a'b'}{M} - 3\frac{e^b a'M'}{M^2} - 2\frac{e^b a''}{M} + 2\frac{M''a'^2}{M} - 2\frac{M''a'b'}{M} + 3\frac{M''a'M'}{M^2} - \frac{3b'a'^3}{4} - \frac{a'M'^3}{2M^3} - \frac{3a''M'^2}{2M^2} + 2a''^2 - \frac{a'^2b'M'}{2M} + 2a''a'^2 - \frac{3a''a'b'}{2} + \frac{a'^2b'^2}{8}\Big] f_{\mathcal{G}} - e^{a-2b}\Big(\frac{4a''M'}{M} - \frac{b'M'^2}{M^2} + 4\frac{M'M''}{M^2} - \frac{M'^3}{M^3} - \frac{a'^3}{2} + \frac{3a'^2b'}{2} - a'a'' + 2a''b' + \frac{a'b'M'}{M} - \frac{a'^2M'}{M} + 2\frac{a'M'^2}{M^2} - a'b'\Big) - e^{a-b}\Big(a'^3 - b'a'^2 + 2\frac{M'a'^2}{M} + 2a''a' - e^{2a-2b}a'b'^2 - 2\frac{M'^2}{M^2}\Big) f'_{\mathcal{G}} + \Big[\frac{4}{M}e^{a-b} + e^{a-2b}\Big(\frac{M'^2}{M^2} + 4\frac{M''}{M} + 3a'^2 - 4a'b' + 5a''\Big)\Big] f''_{\mathcal{G}}, \tag{7}$$

$$-\frac{(M'^2+2a'M'M+M'^2-4Me^b)}{4M^2} = k^2 p_m e^b + 2p_m e^b + \frac{1}{2} e^b f(\mathcal{G}) - e^{-b}\Big[\Big(\frac{a'b'^2M'}{2M} + \frac{a'^2M'^2}{4M^2} - 7\frac{a'b'M'^2}{M^2} + \frac{a'^3M'}{M} + 2\frac{a'M'a''}{M} + 2\frac{a'M'M''}{M} - \frac{3a'M'^3}{2M^3} + \frac{11b'M'^3}{4M^3} + \frac{a''M'^2}{2M^2} - \frac{a'^2e^b}{M} + \frac{a'b'e^b}{M^2} + \frac{4b'M'e^b}{M^2} - \frac{2a''e^b}{M} - \frac{4M''e^b}{M^2} + \frac{4M'^2e^b}{M^3} + \frac{4M''^2}{M^2} - \frac{4M''M'^2}{M^3} - \frac{M'^4}{2M^4} - \frac{a'b'^2M'}{2M} - \frac{2b'M'^3}{M^2} + \frac{4M'^3M''}{M^3} + \frac{a'^4}{4} - \frac{a'^3b'}{2} - a'b'a'' + a''^2 + \frac{a'^2b'^2}{4} + \frac{b'^2M'^2}{M^2} - \frac{4b'M'M''}{M^2} + \frac{a'M'M''}{M^2} - \frac{4M''e^b}{M^2} - 2M''M'^2\Big) f_{\mathcal{G}} + \frac{a'^3}{2} + a'^2M + a''a'M + \Big(\frac{2M'^3}{M^3} - \frac{a'^2M'}{M} - \frac{5a'M'^2}{2M^2} + \frac{3b'M'^2}{2} + \frac{2a'e^b}{M} + \frac{4M'e^b}{M^2} - \frac{2M'M''}{M^2} - \frac{2b'M''}{M} + \frac{4MM''}{M^2} - \frac{a'^3}{2} - a'a''\Big) f'_{\mathcal{G}}\Big], \tag{8}$$

$$\frac{M'M(a'-b') + 2M''M + M^2a'^2 - M^2a'b' - M'^2 + 2M^2a''}{4Me^b} = k^2p_mM + \frac{1}{2} f(\mathcal{G}) - e^{-2b}\Big(\frac{a'^3M'}{2} - \frac{3a'^2b'M'}{4} - a'^2e^b + \frac{a'^2M''}{2} + \frac{a'b'^2M'}{4} + a'b'e^b - \frac{a'b'M''}{2} + \frac{a'^2M'^2}{M} - \frac{3a'b'M'^2}{4M} + \frac{a'M'a''}{M} + \frac{b'^2M'^2}{2M} + \frac{2b'M'e^b}{M} + a''M'' + a'M'a'' - \frac{a''b'M'}{2} - 2a''e^b - \frac{2b'M'M''}{M} - \frac{2a'M'e^b}{M} + \frac{8e^{2b}}{M} - \frac{4M''e^b}{M} + \frac{2M''^2}{M} + \frac{b'M'^3}{2M^2} - \frac{M''M'^2}{M^2} + \frac{M'^4}{2M^3} - \frac{2M'^2e^b}{M^2}\Big) f_{\mathcal{G}} - e^{-2b}\Big[\frac{a'^3M^2}{2} + a'^2M'M - \frac{a'^2b'M^2}{2} + a''a'M^2 + \Big(-\frac{a'^3M}{2} + \frac{3a'b'M'}{2} - \frac{a'M'^2}{2M} + \frac{13b'M'^2}{8M} - a''a'M - a''M' + \frac{11M'e^b}{2M} - a'M'' + \frac{9a'^2M'}{8} - \frac{a'^2b'M}{2} + \frac{5b'^2M'}{8} - \frac{4M'}{M} - \frac{5b'M''}{4} + \frac{M'^3}{8M^2}\Big) f'_{\mathcal{G}} + \Big(a'M' + \frac{5M'^2}{4M} + \frac{5b'M'}{4} - \frac{5M''}{2}\Big) f''_{\mathcal{G}}\Big]. \tag{9}$$

The energy bounds indicate the nature of matter incorporated by astrophysical configurations. If the well-defined bounds are preserved then the configurations are said to be supported by an ordinary matter. In case of WH geometry, a realistic WH configuration may exist if these energy bounds violate. In order to define such energy bounds, Raychaudhari equations are considered to be the most fundamental ingredients given as

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \Theta_{\mu\nu}\Theta^{\mu\nu} - R_{\mu\nu}l^{\mu}l^{\nu}, \tag{10}$$

$$\frac{d\theta}{d\tau}=-\frac{1}{2}\theta^2-\sigma_{\mu\nu}\sigma^{\mu\nu}+\Theta_{\mu\nu}\Theta^{\mu\nu}-R_{\mu\nu}k^{\mu}k^{\nu}, \tag{11}$$

where θ, l^{μ}, k^{μ}, σ and Θ represent expansion scalar, timelike vector, null vector, shear and rotation tensors. These equations are defined for both timelike (first equation) and null (second equation) congruence. In both equations, the positivity of last term demands attractive gravity. For the Einstein–Hilbert action, these energy bounds are split into null (NEC) ($\rho_m+p_m\geq 0$), weak (WEC) ($\rho_m\geq 0,\ \rho_m+p_m\geq 0$), strong (SEC) ($\rho_m+p_m\geq 0,\ \rho_m+3p_m\geq 0$) and dominant (DEC) ($\rho_m\geq 0,\ \rho_m\pm p_m\geq 0$) energy conditions [32]. These conditions originate from the Raychaudhari equations purely on geometric arguments, hence are valid for any modified theory implying that $T^{(m)}_{\mu\nu}k^{\mu}k^{\nu}\geq 0$ can be replaced with $T^{eff}_{\mu\nu}k^{\mu}k^{\nu}\geq 0$. For detailed study of energy conditions in modified gravity, see the literature [33,34]. In modified Gauss–Bonnet gravity, these energy constraints become

- **NEC:** $\rho_{eff}+p_{eff}\geq 0$,
- **SEC:** $\rho_{eff}+p_{eff}\geq 0,\quad \rho+3p_{eff}\geq 0$,
- **DEC:** $\rho_{eff}\geq 0,\quad \rho_{eff}\pm p_{eff}\geq 0$,
- **WEC:** $\rho_{eff}+p_{eff}\geq 0,\quad \rho_{eff}\geq 0$.

where $\rho_{eff}=\rho_m+\rho_c$ and $p_{eff}=p_m+p_c$. With the help of Eqs. (7) and (8), we obtain

$$\begin{aligned}
p_m=&-\frac{\left(M'^2+2a'M'M-4Me^b\right)}{4M^2e^b(2+k^2)}-\frac{f(\mathcal{G})}{2(2+k^2)}\\
&+\frac{e^{-2b}}{2+k^2}\Big[\Big(\frac{a'b'^2M'}{2M}+\frac{a'^2M'^2}{4M^2}-\frac{7a'b'M'^2}{M^2}\\
&+\frac{M'a'^3}{M}+\frac{2a'a''M'}{M}+\frac{2a'M''M'}{M}\\
&-\frac{3a'M'^3}{2M^3}\,\frac{11b'M'^3}{4M^3}+\frac{a''M'^2}{2M^2}-\frac{a'^2e^b}{M}\\
&+\frac{a'b'e^b}{M}+\frac{4b'M'e^b}{M^2}-\frac{2a''e^b}{M^2}\,\frac{4M''e^b}{M^2}\\
&+\frac{4M'^2e^b}{M^3}+\frac{4M''^2}{M^2}-\frac{4M''M'^2}{M^3}\\
&-\frac{M'^4}{2M^4}-\frac{a'b'^2M'}{2M}-\frac{2b'M'^3}{M^2}\\
&+\frac{a'^4}{4}+\frac{4M'^3M''}{M^3}-\frac{a'^3b'}{2}-a'b'a''+a''^2\\
&+\frac{a'^2b'^2}{4}+\frac{b'^2M'^2}{M^2}-\frac{4b'M'M''}{M^2}\\
&+\frac{a'M'M''}{M^2}-\frac{4M''e^b}{M^2}-2M''M'^2\Big)f_{\mathcal{G}}\\
&+\frac{a'^3}{2}+a'^2M+a''a'M\\
&+\Big(\frac{2M'^3}{M^3}-\frac{a'^2M'}{M}-\frac{5a'M'^2}{2M^2}+\frac{3b'M'^2}{2}\\
&+\frac{2a'e^b}{M}+\frac{4M'e^b}{M^2}-\frac{2M'M''}{M^2}\\
&-\frac{2b'M''}{M}+\frac{4MM''}{M^2}-\frac{a'^3}{2}-a'a''\Big)f'_{\mathcal{G}}\Big],
\end{aligned}\tag{12}$$

$$\begin{aligned}
\rho_m=&\frac{\left(-4M''M+2b'M'M+M'^2+4Me^b\right)}{4k^2e^bM^2}+\frac{f_{\mathcal{G}}}{2k^2}\\
&-\frac{e^{-2b}}{k^2}\Big(\frac{a'^4}{2}-\frac{a'b'M'}{2M^2}\\
&+\frac{M'a'b'^2}{M}-\frac{M'^2a'b'}{M^2}-\frac{2M'a''b'}{M}+2a''^2-\frac{e^ba'^2}{M}\\
&+\frac{e^ba'b'}{M}-\frac{3M'a'e^b}{M^2}-\frac{2e^ba''}{M}+\frac{2M''a'^2}{M}\\
&-\frac{2M''a'b'}{M}+\frac{3M''a'M'}{M^2}+\frac{4M''a''}{M}\\
&+\frac{M'^2a'b'}{4M^2}+\frac{a'^4}{2}-\frac{a'M'^3}{2M^3}-\frac{a''M'^2}{2M^2}-\frac{3a'^3b'}{4}\\
&-\frac{a'^2b'M'}{2M}+2a''a'^2-\frac{3a''a'b'}{2}-\frac{a''M'^2}{M^2}\\
&+\frac{b'^2a'^2}{8}\Big)f_{\mathcal{G}}+\frac{e^{-2b}}{k^2}\Big(2b'a''-a'a''+\frac{4M'a''}{M}\\
&-\frac{M'^2b'}{M^2}+\frac{4M''M'}{M^2}-\frac{M'^3}{M^3}+\frac{a'b'M'}{M}\\
&-\frac{a'^3}{2}+\frac{3a'^2b'}{2}-\frac{a'^2M'}{M}+\frac{2a'M'^2}{M^2}-a'b'\Big)\\
&+\frac{e^{-b}}{k^2}\Big(\frac{2a'^2M'}{M}+a'^3-a'^2b'+2a'a''-e^{2a-2b}b'^2a'\\
&-\frac{2M'^2}{M^2}\Big)f'_{\mathcal{G}}+\Big[\frac{4e^{-b}}{k^2M}+\frac{e^{-2b}}{k^2}\Big(\frac{M'^2}{M^2}-4a'b'\\
&+\frac{4M''}{M}+3a'^2+5a''\Big)\Big]f''_{\mathcal{G}}.
\end{aligned}\tag{13}$$

For the traversability of WH, the basic property is the violation of NEC in GR. This violation prevents the WH throat to shrink and leads to the physically unrealistic WH solutions. The modified theories of gravity provide $T^{eff}_{\alpha\beta}$ as an alternative source to meet the violation of NEC. In this regard, these theories may have an opportunity for usual matter configuration to fulfill the energy constraints. Simplifying Eqs. (7) and (8), we obtain NEC with respect to the effective stress–energy tensor as follows

$$\rho_{eff}+p_{eff}=\frac{1}{2e^b}\left(\frac{M'a'}{M}-\frac{2M''}{M}+\frac{M'^2}{M^2}+\frac{M'b'}{M}\right). \tag{14}$$

3 Point-like Lagrangian and Noether symmetry approach

Here we use Lagrange multiplier technique to formulate the Lagrangian for the action (1). We tak

$$A=\int dr\sqrt{-g}[R+f(\mathcal{G})-\mu_1(\mathcal{G}-\bar{\mathcal{G}})+\mathcal{L}_m], \tag{15}$$

where $\sqrt{-g}=Me^{\frac{a}{2}}e^{\frac{b}{2}}$ and $\mathcal{L}_m=p_m$ while curvature scalar and GB invariant are

$$\begin{aligned}R&=\frac{1}{e^b}\left(-\frac{a'^2}{2}+\frac{a'b'}{2}-\frac{a'M'}{M}-\frac{2M''}{M}\right.\\&\left.+\frac{b'M'}{M}+\frac{M'^2}{2M^2}-a''+\frac{2e^b}{M}\right)\\ \bar{\mathcal{G}}&=\frac{2e^{-2b}}{M^2}\left(a'^2M'^2-3b'a'M'^2-e^ba'^2\right.\\&\left.+e^ba'b'+2a''M'^2+4a'M'M''-2e^ba''\right).\end{aligned}$$

Varying the action (15) relative to $\mathcal{G}$, we obtain $\mu_1=f_{\mathcal{G}}(\mathcal{G})$ whereas the conservation of energy–momentum tensor relative to perfect fluid gives $p_m(r)=\rho_0e^{\frac{-a(1+w)}{2w}}$, ($\rho_o$ is integration constant while w denotes EoS parameter). Putting all these values in Eq. (15), it follows that

$$\begin{aligned}A&=\int\left[Me^{\frac{b+a}{2}}\left\{R+f(\mathcal{G})-\mathcal{G}f_{\mathcal{G}}+\rho_0e^{\frac{-a(1+w)}{2w}}\right.\right.\\&+\frac{2e^{-2b}f_{\mathcal{G}}}{M^2}\left(a'^2M'^2-3b'a'M'^2\right.\\&-e^ba'^2+e^ba'b'+2a''M'^2\\&\left.\left.\left.+4a'M'M''-2e^ba''\right)\right\}\right]dr.\end{aligned} \tag{16}$$

In order to eliminate second order derivative, we integrate these terms by parts and neglect boundary terms which leads to

$$\begin{aligned}&\mathcal{L}\left(r,a,b,M,\mathcal{G},a',b',M',\mathcal{G}'\right)\\&=e^{\frac{a+b}{2}}M\left(R+f-\mathcal{G}f_{\mathcal{G}}+\rho_0e^{\frac{-a(1+w)}{2w}}\right)\\&+\frac{2e^{\frac{a-3b}{2}}}{M}\\&\times\left((4a'b'+2a'M')(M'^2-e^b)+2a'b'e^b\right)f_{\mathcal{G}}\\&-\frac{4e^{\frac{a-3b}{2}}}{M}\left(M'^2-e^b\right)a'\mathcal{G}'f_{\mathcal{G}\mathcal{G}}.\end{aligned} \tag{17}$$

For static spherically symmetric metric, Hamiltonian of the dynamical system and the Euler–Lagrange equation corresponding to point-like Lagrangian are characterized as

$$\mathcal{H}=\sum_i q_i'p^i-\mathcal{L},\quad \frac{\partial\mathcal{L}}{\partial q_i}-\frac{d}{dr}\left(\frac{\partial\mathcal{L}}{\partial q_i'}\right)=0, \tag{18}$$

where q^i are generalized coordinates. The differential of Lagrangian with respect to the configuration space $(a, b, M, \mathcal{G})$ gives

$$\begin{aligned}&\frac{e^{\frac{a}{2}}e^{\frac{b}{2}}}{2}(-\mathcal{G}Mf_{\mathcal{G}}+RM+Mf_{\mathcal{G}}-Mp)\\&+\frac{1}{2}\left(4a'b'e^{\frac{a-b}{2}}+\frac{a'M'^3}{M^2}e^{\frac{a-3b}{2}}\right.\\&\left.-\frac{a'^2M'^2}{M}e^{\frac{a-3b}{2}}+\frac{3a'b'M'^2}{M}e^{\frac{a-3b}{2}}\right)f_{\mathcal{G}}\\&+\frac{1}{2}f_{\mathcal{G}\mathcal{G}}\left(\frac{-3\mathcal{G}'a'M'^2}{M}e^{\frac{a-3b}{2}}-4a'\mathcal{G}'\right.\\&\left.\times e^{\frac{a-b}{2}}\right)-\frac{1}{2}a'f_{\mathcal{G}}\left(4b'e^{\frac{a-b}{2}}\right.\\&\left.+\frac{M'^3e^{\frac{a-3b}{2}}}{M^2}-\frac{2a'M'^2e^{\frac{a-3b}{2}}}{M}+\frac{3b'M'^2e^{\frac{a-3b}{2}}}{M}\right)\\&-\frac{1}{2}b'\left(4b'e^{\frac{a-b}{2}}+\frac{M'^3e^{\frac{a-3b}{2}}}{M^2}\right.\\&\left.-\frac{2a'M'^2e^{\frac{a-3b}{2}}}{M}+\frac{3b'M'^2e^{\frac{a-3b}{2}}}{M}\right)f_{\mathcal{G}}-\left(-4b'^2\right.\\&\times e^{\frac{a-b}{2}}-\frac{2M'^4e^{\frac{a-3b}{2}}}{M^3}\\&-\frac{2M'^3b'e^{\frac{a-3b}{2}}}{M^2}+\frac{3M'^2M''e^{\frac{a-3b}{2}}}{M^2}-\frac{2a''M'^2e^{\frac{a-3b}{2}}}{M}+4\\&\times b''e^{\frac{a-b}{2}}+\frac{2a'M'^3e^{\frac{a-3b}{2}}}{M^2}\\&-\frac{4a'M'M''e^{\frac{a-3b}{2}}}{M}+\frac{4a'b'M'^2e^{\frac{a-3b}{2}}}{M}\\&+\frac{3b''M'^2e^{\frac{a-3b}{2}}}{M}+\frac{6b'M''M'e^{\frac{a-3b}{2}}}{M}\\&\left.-\frac{6M'^2b'^2e^{\frac{a-3b}{2}}}{M}-\frac{3M'^3b'e^{\frac{a-3b}{2}}}{M^2}\right)f_{\mathcal{G}}-\frac{1}{2}a'\left(-3\mathcal{G}'e^{\frac{a-b}{2}}\right.\\&\left.-\frac{4\mathcal{G}'M'^2e^{\frac{a-3b}{2}}}{M}\right)f_{\mathcal{G}\mathcal{G}}-\frac{1}{2}b'\left(\frac{-3\mathcal{G}'M'^2e^{\frac{a-3b}{2}}}{M}-4\mathcal{G}'e^{\frac{a-b}{2}}\right)\\&f_{\mathcal{G}\mathcal{G}}-\mathcal{G}'\left(-4\mathcal{G}'e^{\frac{a-b}{2}}-\frac{3\mathcal{G}'M'^2e^{\frac{a-3b}{2}}}{M}\right)f_{\mathcal{G}\mathcal{G}\mathcal{G}}\\&-\left(\frac{-3\mathcal{G}''M'^2e^{\frac{a-3b}{2}}}{M}-\frac{6\mathcal{G}'M'M''e^{\frac{a-3b}{2}}}{M}-4\right.\\&\times\mathcal{G}''e^{\frac{a-b}{2}}+\frac{3\mathcal{G}'M'^3e^{\frac{a-3b}{2}}}{M^2}\\&\left.+\frac{6\mathcal{G}'b'M'^2e^{\frac{a-3b}{2}}}{M}+4\mathcal{G}'b'e^{\frac{a-b}{2}}\right)f_{\mathcal{G}\mathcal{G}}=0,\end{aligned} \tag{19}$$

$$\frac{1}{2M^4}\left[-\left(12a'M'M''M^3e^{\frac{a-3b}{2}}\right.\right.$$

$$+9a'b'M'^2M^3e^{\frac{a-3b}{2}} - 9a'b'M'^2M^3e^{\frac{a-3b}{2}} -M^5e^{\frac{a+b}{2}} + 4a'^2M^4e^{\frac{a-b}{2}} + 8a''M^4e^{\frac{a-b}{2}} +\mathcal{G}M^5e^{\frac{a+b}{2}} + 3a'M'^3M^2e^{\frac{a-3b}{2}} +pM^5e^{\frac{a+b}{2}} - RM^5e^{\frac{a+b}{2}} - 6a'M'^3M^2e^{\frac{a-3b}{2}} +6a''M'^2M^3e^{\frac{a-3b}{2}}\Big)f_{\mathcal{G}} + \Big(+4a'\mathcal{G}'M^4e^{\frac{a-b}{2}} - 9a'M'^2M^3e^{\frac{a-3b}{2}} +6a'\mathcal{G}'M'^2M^3e^{\frac{a-3b}{2}}\Big)f_{\mathcal{G}\mathcal{G}}\Big] = 0, \quad (20)$$

$$e^{\frac{a}{2}}e^{\frac{b}{2}}\left(-\mathcal{G}f_{\mathcal{G}} + R + f_{\mathcal{G}} - p\right) +e^{\frac{a-3b}{2}}\Big[\Big(\frac{4M'^3a'}{M^3} - \frac{5M'^2a'^2}{2M^2} + \frac{15M'^2a'b'}{2M^2} +\frac{a'^3M'}{M} - \frac{3M'a'b'}{M} -\frac{3M'a'^2b'}{M} - \frac{9M'a'b'^2}{M} - \frac{3M'^2a''}{M^2} - \frac{6M''M'a'}{M^2} +\frac{4M'a''a'}{M} + \frac{2M''a'^2}{M} - \frac{6M'b'a''}{M} -\frac{6M'b''a'}{M} - \frac{6M''b'a'}{M}\Big)f_{\mathcal{G}} + \Big(\frac{3\mathcal{G}'a'^2M'}{M} -\frac{9a'\mathcal{G}'b'M'}{M} + \frac{6a'\mathcal{G}''M'}{M} + \frac{6a''\mathcal{G}'M'}{M} + \frac{6a'\mathcal{G}'M''}{M} -\frac{6a'M'^2\mathcal{G}'}{M^2}\Big)f_{\mathcal{G}\mathcal{G}} + \frac{6\mathcal{G}'^2a'M'}{M}f_{\mathcal{G}\mathcal{G}\mathcal{G}}\Big] = 0, \quad (21)$$

$$\frac{1}{2M^2}\Big[-2M^3e^{\frac{a}{2}}e^{\frac{b}{2}}f_{\mathcal{G}} -\Big(-3a'^2M'^2Me^{\frac{a-3b}{2}} - 4a'^2M^2e^{\frac{a-b}{2}} + 6a'M'^3e^{\frac{a-3b}{2}} +4a'b'M^2e^{\frac{a-b}{2}} + 9a'b'M'^3e^{\frac{a-3b}{2}} - 12a'M'M''Me^{\frac{a-3b}{2}} -6a''M'^2Me^{\frac{a-3b}{2}} -8a''M^2e^{\frac{a-b}{2}}\Big)f_{\mathcal{G}\mathcal{G}} +\Big(6a'M'^2\mathcal{G}'Me^{\frac{a-3b}{2}} + 8a'\mathcal{G}'M^2e^{\frac{a-b}{2}}\Big)f_{\mathcal{G}\mathcal{G}\mathcal{G}}\Big] = 0. \quad (22)$$

In order to solve the system of non-linear differential equations, Noether symmetry is recognized as a significant tool. The physical properties of any dynamical structure can be illustrated by the respective Lagrangian which narrates the energy density as well as the presence of symmetries of the system. Noether theorem can be stated as a group generator that provides conserved quantity only if point-like Lagrangian shows constant behavior under a continuous group. To analyze the associated conserved quantity as well as the existence of Noether symmetry for the spherical system, we take a vector field K [35,36]

$$K = \tau\left(r, q^i\right)\frac{\partial}{\partial r} + \zeta^i\left(r, q^i\right)\frac{\partial}{\partial q^i}, \quad (23)$$

where τ and ζ^i are unknown coefficients while r acts as an affine parameter of K. This leads to uniqueness of the vector field in the tangent space.

The corresponding invariance condition is characterized by

$$K^{[1]}\mathcal{L} + (D\tau)\mathcal{L} = DB(r, q^i). \quad (24)$$

Here B signifies the boundary term, $K^{[1]}$ and D represent the first order expansion and total derivative, respectively given by

$$K^{[1]} = K + \left(D\zeta^i - q'^iD\tau\right)\frac{\partial}{\partial q'^i}, \quad D = q'^i\frac{\partial}{\partial q^i} + \frac{\partial}{\partial r}. \quad (25)$$

Invariance condition (24) leads to the Noether symmetries which represent the related conserved parameters in terms of first integral. Under translation with respect to time as well as position, if the Lagrangian shows constant behavior, then the first integral describes conservation of energy as well as the linear momentum whereas rotationally symmetric Lagrangian provides angular momentum conservation [37]. The first integral for invariance condition (24) is expressed in the form

$$\Sigma = B - \tau\mathcal{L} - \left(\zeta^i - q'^i\tau\right)\frac{\partial\mathcal{L}}{\partial q'^i}. \quad (26)$$

The vector field and first order expansion for the configuration space become

$$K = \tau\frac{\partial}{\partial r} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial M} + \delta\frac{\partial}{\partial\mathcal{G}},$$
$$K^{[1]} = \tau\frac{\partial}{\partial r} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial M} + \delta\frac{\partial}{\partial\mathcal{G}} +\alpha'\frac{\partial}{\partial a'} + \beta'\frac{\partial}{\partial b'} + \gamma'\frac{\partial}{\partial M'} + \delta'\frac{\partial}{\partial\mathcal{G}'}, \quad (27)$$

where the unknown parameters of vector field having the radial derivative are given by

$$\sigma'_j = D\sigma_j - q'^iD\tau, \quad j = 1, \ldots, 4, \quad (28)$$

where $\sigma_j(j = 1, 2, 3, 4)$ denote α, β, γ and δ, respectively. Comparing the coefficients of a'^2b', $a'b'^2M'^2$, $a'b'M'^3$ and $M'^2\mathcal{G}'^2a'$, we obtain

$$\tau_{,a}f_{\mathcal{G}} = 0, \quad \tau_{,b}f_{\mathcal{G}} = 0, \quad \tau_{,M}f_{\mathcal{G}} = 0, \quad \tau_{,\mathcal{G}}f_{\mathcal{G}\mathcal{G}} = 0, \quad (29)$$

This leads to a trivial solution for $f_{\mathcal{G}} = 0$. For non-trivial solution, we assume $f_{\mathcal{G}} \neq 0$ and compare the coefficients of $a', b', M', \mathcal{G}', b'^2, M'^2, \mathcal{G}'^2, \mathcal{G}'M'^2, a'b'M', a'^2M'\mathcal{G}', a'M'b'^2$ and $a'M'\mathcal{G}'^2$ leading to following equations

$$4e^{a-b/2}[f_{\mathcal{G}}(-\beta_{,r}-\gamma_{,r})+\delta_{,r}f_{\mathcal{G}\mathcal{G}}]=MB_{,a},$$
$$-4e^{a-b/2}\alpha_{,r}f_{\mathcal{G}}=MB_{,b}, \tag{30}$$
$$-4e^{a-b/2}\alpha_{,r}f_{\mathcal{G}}=MB_{,M},$$
$$4e^{a-b/2}\alpha_{,r}f_{\mathcal{G}\mathcal{G}}=MB_{,\mathcal{G}}, \tag{31}$$
$$\alpha_{,b}f_{\mathcal{G}}=0,\quad \alpha_{,M}f_{\mathcal{G}}=0,$$
$$\alpha_{,\mathcal{G}}f_{\mathcal{G}\mathcal{G}}=0,\quad \alpha_{,r}f_{\mathcal{G}\mathcal{G}}=0, \tag{32}$$
$$4\gamma_{,r}f_{\mathcal{G}}=0,\quad 4\gamma_{,a}f_{\mathcal{G}\mathcal{G}}=0,$$
$$8\gamma_{,b}f_{\mathcal{G}}=0,\quad 4\gamma_{,\mathcal{G}}f_{\mathcal{G}\mathcal{G}}=0. \tag{33}$$

For $f_{\mathcal{G}}\neq 0$, we compare remaining coefficients and obtain over determined system of equations as follows

$$\tau_{,a}=0,\quad \tau_{,b}=0,\quad \tau_{,M}=0,\quad \tau_{,\mathcal{G}}=0,$$
$$\gamma_{,a}=0,\quad \gamma_{,\mathcal{G}}=0 \tag{34}$$
$$B_{,b}=0,\quad B_{,M}=0,\quad B_{,\mathcal{G}}=0,$$
$$\gamma_{,r}=0,\quad \gamma_{,b}=0 \tag{35}$$
$$\alpha_{,r}=0,\quad \alpha_{,b}=0,\quad \alpha_{,M}=0,$$
$$\alpha_{,\mathcal{G}}=0, \tag{36}$$
$$4e^{a-b/2}[-f_{\mathcal{G}}\beta_{,r}+\delta_{,r}f_{\mathcal{G}\mathcal{G}}]=MB_{,a}, \tag{37}$$
$$\beta_{,r}f_{\mathcal{G}}-\delta_{,r}f_{\mathcal{G}\mathcal{G}}=0,$$
$$\beta_{,a}f_{\mathcal{G}}-\delta_{,a}f_{\mathcal{G}\mathcal{G}}=0, \tag{38}$$
$$(-\alpha+\beta+M^{-1}\gamma-\delta-\alpha_{,a}-\beta_{,b}+\tau_{,r})f_{\mathcal{G}}$$
$$+\delta_{,b}f_{\mathcal{G}\mathcal{G}}=0, \tag{39}$$
$$(-\alpha+\beta+M^{-1}\gamma-\alpha_{,a}-\beta_{,M}-\gamma_{,M}+\tau_{,r})f_{\mathcal{G}}$$
$$-(\delta-\delta_{,M})f_{\mathcal{G}\mathcal{G}}=0, \tag{40}$$
$$2(\alpha-3\beta-M^{-1}\gamma+\alpha_{,a}+\beta_{,b}+2\gamma_{,M}-2\tau_{,r})f_{\mathcal{G}}$$
$$+(2\delta-\delta_{,b})f_{\mathcal{G}\mathcal{G}}=0, \tag{41}$$
$$(\alpha-3\beta-M^{-1}\gamma+\alpha_{,a}+2\beta_{,M}+3\gamma_{,M}-3\tau_{,r})f_{\mathcal{G}}$$
$$+(\delta-\delta_{,M})f_{\mathcal{G}\mathcal{G}}=0, \tag{42}$$
$$-\beta_{,\mathcal{G}}f_{\mathcal{G}}+f_{\mathcal{G}\mathcal{G}}(\alpha-\beta-M^{-1}\gamma+\alpha_{,a}+\delta_{,\mathcal{G}}-\tau_{,r})$$
$$+\delta f_{\mathcal{G}\mathcal{G}\mathcal{G}}=0, \tag{43}$$
$$2\beta_{,\mathcal{G}}f_{\mathcal{G}}$$
$$+f_{\mathcal{G}\mathcal{G}}(\alpha+3\beta+M^{-1}\gamma-\alpha_{,a}-2\gamma_{,M}-\delta_{,\mathcal{G}}+2\tau_{,r})$$
$$+2\delta f_{\mathcal{G}\mathcal{G}\mathcal{G}}=0, \tag{44}$$
$$e^{a+b/2}M\left[(R+f-\mathcal{G}f_{\mathcal{G}})\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{M}+\tau_{,r}\right)\right.$$
$$\left.+\rho_0 e^{\frac{-a(1+w)}{2w}}\left(\frac{\alpha}{2w}+\frac{\beta}{2}\frac{\gamma}{M}+\tau_{,r}\right)-\delta\mathcal{G}f_{\mathcal{G}\mathcal{G}}\right]=B_{,r}\,. \tag{45}$$

Here we solve Eqs. (34)–(45) for three different choices of parameters given by

- $\beta(r,a,b,M,\mathcal{G})=0,\quad \delta(r,a,b,M,\mathcal{G})=0,$
- $\beta(r,a,b,M,\mathcal{G})=0,\quad \delta(r,a,b,M,\mathcal{G})\neq 0$ or vice versa.
- $\beta(r,a,b,M,\mathcal{G})\neq 0,\quad \delta(r,a,b,M,\mathcal{G})\neq 0.$

4 $f(\mathcal{G})$ models and wormhole solutions

In order to evaluate unknown parameters of symmetry generators and explicit solution of f, we consider above mentioned possibilities of β and δ.

Case I: $\beta=\delta=0$

In this case, we obtain

$$\alpha=\xi_1+\xi_2e^{-a},\quad \gamma=M(\xi_3-\xi_4),\quad \tau=\xi_5r+\xi_6,$$
$$f(\mathcal{G})=\xi_1\mathcal{G}+\xi_2,\quad B(r,a)_{,a}=0, \tag{46}$$

where $\xi_{i's}$ denotes integration constants and explicit form of f corresponds to linear model which is compatible with Gauss–Bonnet gravity. The coefficient of boundary term, symmetry generator and $f(\mathcal{G})$ solution satisfy Eqs. (34)-(44). Consequently, the symmetry generator and the first integral yield

$$K=\xi_1\frac{\partial}{\partial r}+\xi_2\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right),$$
$$\Sigma=\xi_3+\xi_4r^3-\xi_0$$
$$\left[e^{\frac{a}{2}}e^{\frac{b}{2}}\left(r^2R+r^2f_{\mathcal{G}}-r^2\mathcal{G}f_{\mathcal{G}}-r^2p\right)\right.$$
$$+e^{\frac{a}{2}}e^{\frac{b}{2}}\left(\frac{4a'b'}{e^b}+\frac{8a'}{re^{2b}}-\frac{4a'^2}{e^{2b}}+\frac{12a'b'}{e^{2b}}\right)f_{\mathcal{G}}$$
$$\left.+\left(\frac{-12a'\mathcal{G}'}{e^{2b}}-\frac{4a'\mathcal{G}'}{e^b}\right)f_{\mathcal{G}\mathcal{G}}\right]-\xi_2e^{\frac{a}{2}}e^{\frac{b}{2}}$$
$$\times\left[\left(\frac{4b'}{e^b}+\frac{8}{re^{2b}}-\frac{8a'}{e^{2b}}+\frac{12b'}{e^{2b}}\right)f_{\mathcal{G}}\right.$$
$$\left.+\left(\frac{-12\mathcal{G}'}{e^{2b}}-\frac{4\mathcal{G}'}{e^b}\right)f_{\mathcal{G}\mathcal{G}}\right].$$

We insert symmetry generators, $f(\mathcal{G})$ model in Eq. (45) with $B_{,r}=\xi_7$, $M=r^2$, $e^b=(1-h(r)/r)^{-1}$ and $a(r)=-k/r$ (where k is positive constant) leading to

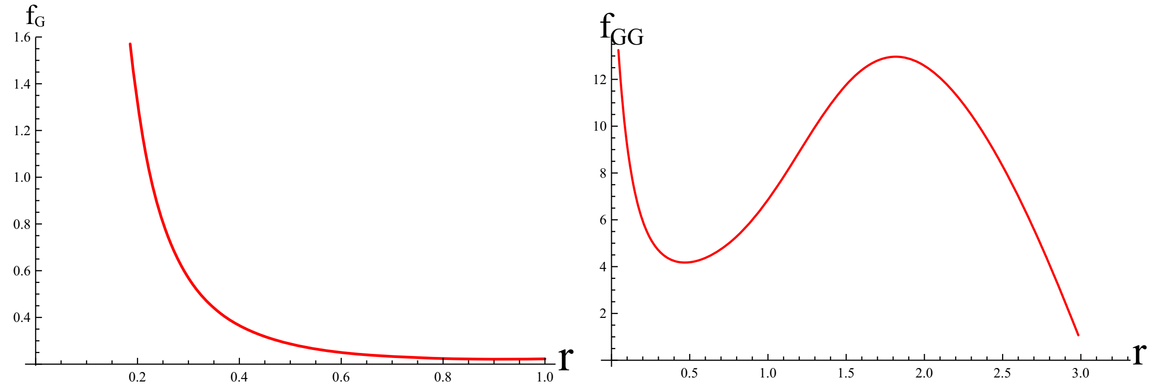

Fig. 1 Evolution of $f_{\mathcal{G}}$ and $f_{\mathcal{G}\mathcal{G}}$ versus r for $k = 0.005$, $\xi_1 = 0.01$, $\xi_2 = 0.85$, $\xi_3 = 0.1$, $\xi_7 = 0.5$, $w = -1$ and $\rho_0 = 0.5$

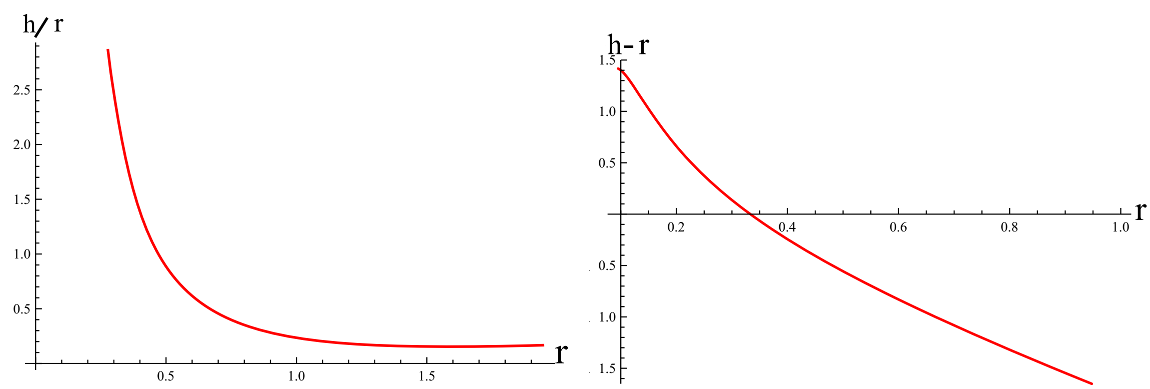

Fig. 2 Variation of the shape function versus r

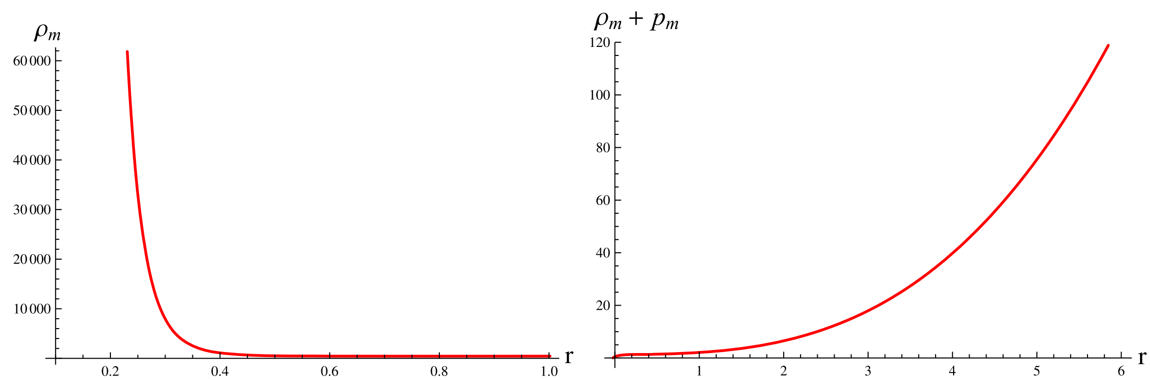

Fig. 3 Evolution of energy bounds versus r for $w = -1$

$$\frac{1}{2w}\left(-\sqrt{\frac{r}{r-h(r)}}e^{\frac{k(2w+1)}{2rw}}\xi_2\rho_0 r^2+2\rho_0\sqrt{\frac{r}{r-h(r)}}r^2\left(w-\frac{1}{2}\right)\xi_3 e^{\frac{k}{2rw}}+3\right.$$
$$\times\, w\left(\left(\frac{r^2(k+8r)h'(r)+\left(k^2+3kr+24r^2\right)h(r)-k^2r-4kr^2-32r^3+4r}{2r^3}\right.\right.$$
$$\left.\left.\left.+\;r^2\xi_1\right)\left(e^{-\frac{k}{2r}}\xi_3+\frac{e^{\frac{k}{2r}}\xi_2}{3}\right)\sqrt{\frac{r}{r-h(r)}}-\frac{2\xi_7}{3}\right)\right)=0. \tag{47}$$

In order to study the geometry, traversability and physical viability of WH in the presence of phantom energy, we consider $w=-1$ and solve this non-linear equation numerically to construct graphical analysis of the shape function. This analysis leads to measure compatibility of linear $f(\mathcal{G})$ model with viable models under the condition of regular and positive derivatives of $f(\mathcal{G})$ function [38]. Furthermore, we explore the possibility of traversable WHs through graphical interpretation of effective NEC. The graphical analysis of energy bounds, i.e., NEC, WEC, SEC and DEC help to explore the presence/absence of ordinary matter.

In both plots of Fig. 1, the positively evolving curves represent that $f(\mathcal{G})$ satisfies viability constraints as $f_{\mathcal{G}}>0$ and $f_{\mathcal{G}\mathcal{G}}>0$. For linear modified GB function, the WH geometry is analyzed in the context of accelerated expansions ($w=-1$) in Fig. 2. The left plot indicates WH geometry to be asymptotically flat in a very short interval of r as $h/r\rightarrow 0$ as $r\rightarrow 1$. In the right plot, the trajectory identifies throat of WH at $r_0=0.34$ and the derivative of the shape function at this point remains positive, i.e., $\frac{dh(r_0)}{dr}<1$. In the presence of accelerated expansion of cosmos, the graphical analysis of WH geometry shows that the configuration is compatible with Morris–Thorne WH proposal.

The WH configuration is more significant if it is supported by ordinary matter, i.e., the normal matter that satisfies energy bounds. The criteria of energy bounds indicates that NEC is the weakest condition as the violation of NEC leads to inconsistent behavior of WEC, SEC and DEC. If a matter distribution follows DEC then WEC and NEC holds trivially while SEC needs to be checked separately. In order to examine realistic nature of WH, we discuss the evolution of energy density and pressure of normal matter in Fig. 3. The graphical interpretation indicates that $\rho_m\geq 0$ and $\rho_m+p_m\geq 0$. This behavior of matter variables indicates that the WH is physically viable inside the throat ($r_0=0.34$). To study traversable behavior of WH, we substitute $a(r)=-k/r$ and Eq. (47) in (14) leading to

$$p_{eff}+\rho_{eff}=-\frac{h(r)}{r^3}+\frac{h'(r)}{r^2}+\frac{k(r-h(r))}{r^4}.$$

For traversable WH, the violation of effective NEC ($\rho_{eff}+p_{eff}<0$) is required which also fulfills the flaring-out condition. Figure 4 shows negatively increasing curve which indicates that $p_{eff}+\rho_{eff}<0$ implying existence of traversable WH solution. For linear $f(\mathcal{G})$ model, the WH is found to be traversable as well as physically viable in the presence of accelerating phases of cosmos.

Case II: $\beta=0,\quad \delta\neq 0$

For $\beta=0$, we solve the Eqs. (38)–(44) and obtain

$$\alpha=\chi_1+\chi_2e^{-a},\quad \gamma=M(\chi_3Y(M)-\chi_4),$$

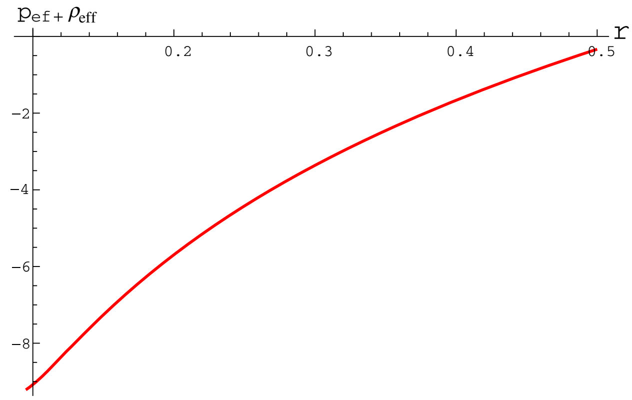

Fig. 4 Evolution of effective NEC versus r for $w=-1$

$$\tau=\chi_5r+\chi_6,$$
$$f(\mathcal{G})=\chi_7\mathcal{G}^2+\chi_8\mathcal{G}+\xi_1,\quad \delta=\chi_3Y(M),\tag{48}$$

where $\chi_{i's}$ are constants of integration while the explicit form of f corresponds to quadratic model. Solving Eq. (37) for above solutions, we get

$$B_{,a}=0,\quad Y(M)=\frac{\chi_4}{\chi_3}.$$

Now, we insert symmetry generators, quadratic form of $f(\mathcal{G})$ model in Eq. (45) with $B=\frac{\chi_1 r}{\chi_3}+\chi_8$, $M=r^2$, $e^b=(1-h(r)/r)^{-1}$, $a(r)=-k/r$ and obtain a non-linear equation given by

$$\begin{aligned}
&2h(r)^4\chi_7w(\{k(r(4r+k(-1+4r^2))\\
&\quad+h(r)(k-5r-8kr^2+12r^3+4rh(r)\\
&\quad\times(k-3r))+r^2(1-12r^2+12rh(r))h'(r))\}\\
&\quad\{r^3(r-h(r))^2\}^{-1})^2-8rh(r)^3\\
&\quad\times\chi_7w(\{k(r(4r+k(-1+4r^2))\\
&\quad+h(r)(k-5r-8kr^2+12r^3+4(k-3r)\\
&\quad\times rh(r))+r^2(1-12r^2+12rh(r))h'(r))\}\\
&\quad\{r^3(r-h(r))^2\}^{-1})^2+12r^2h(r)^2\\
&\quad\times\chi_7w(\{k(r(4r+k(-1+4r^2))\\
&\quad+h(r)(k-5r-8kr^2+12r^3+4(k-3r)\\
&\quad\times rh(r))+r^2(1-12r^2+12rh(r))h'(r))\}\\
&\quad\{r^3(r-h(r))^2\}^{-1})^2-8r^3h(r)\\
&\quad\times\chi_7w(\{k(r(4r+k(-1+4r^2))\\
&\quad+h(r)(k-5r-8kr^2+12r^3+4(k-3r)\\
&\quad\times rh(r))+r^2(1-12r^2+12rh(r))h'(r))\}\\
&\quad\{r^3(r-h(r))^2\}^{-1})^2+2r^4\chi_7w\\
&\quad(\{k(r(4r+k(-1+4r^2))\\
&\quad+h(r)(k-5r-8kr^2+12r^3+4(k-3r)rh(r))\\
&\quad+r^2(1-12r^2+12rh(r))h'(r))\}\{r^3(r-h(r))^2\}^{-1})^2
\end{aligned}$$

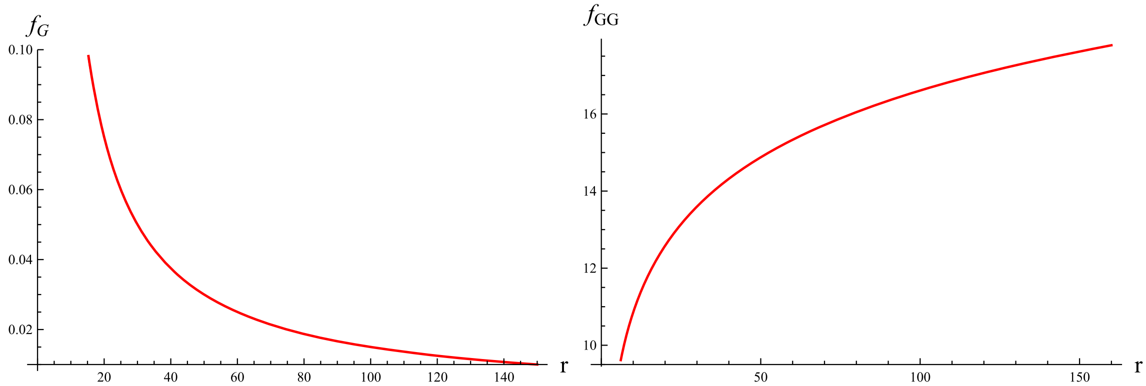

Fig. 5 Evolution of quadratic $f(\mathcal{G})$ model versus r for $\chi_7 = -0.1$, $\rho_0 = -1.5$, $k = 0.05$ and $w = -1$

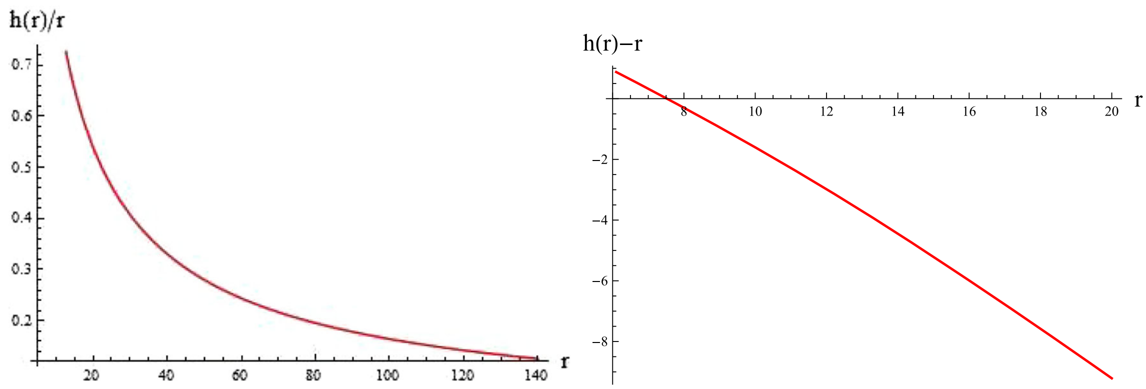

Fig. 6 Variation of the shape function versus r

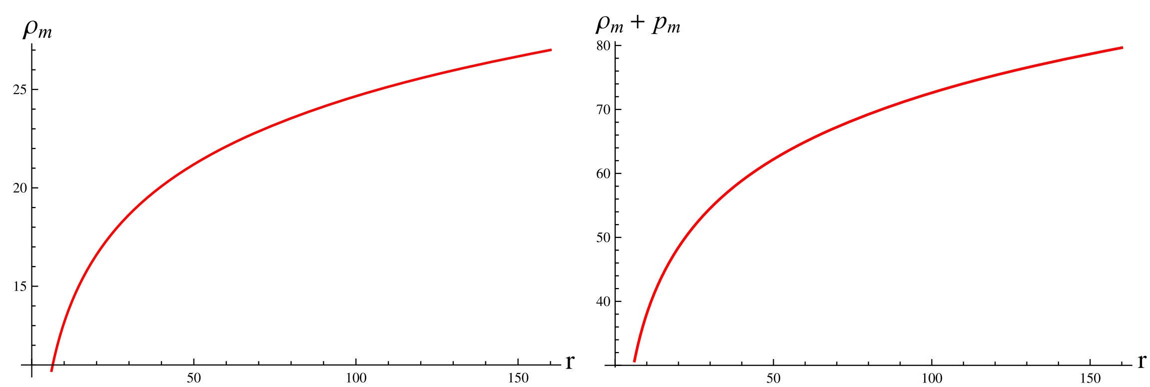

Fig. 7 Evolution of energy bounds versus r for $w = -1$

$$
\begin{aligned}
&+\{-(k^2+3kr\\
&+24r^2)h(r)+r(-4+k^2+4kr\\
&+32r^2-r(k+8r)h'(r))\}\{2r^5\}^{-1}\\
&+\rho_0 e^{k(1+w)/2rw}r^{12}=0.
\end{aligned}
\tag{49}
$$

The numerical solution of this equation leads to analyze the behavior of viability of quadratic $f(\mathcal{G})$ model, geometrical properties of shape function, presence/absence of ordinary and exotic matter graphically. In Fig. 5, we explore the consistency of quadratic model with standard models of modified GB gravity. In both plots, the positively decreasing (left) and increasing (right) curves preserve the viability constraints as $f_{\mathcal{G}} > 0$ and $f_{\mathcal{G}\mathcal{G}} > 0$. In Fig. 6, we study the geometry of WH constructed by quadratic $f(\mathcal{G})$ model and corresponding shape function. The left plot demonstrates the asymptotically flat shape of WH as $h/r \to 0$ when $r \to \infty$. In the right plot, the trajectory of $h(r) - r$ locates WH throat at $r_0 = 7$ and at this point, the derivative of the shape function is found to be positive but greater than 1. This analysis defines a horizon-free asymptotically flat WH whose throat

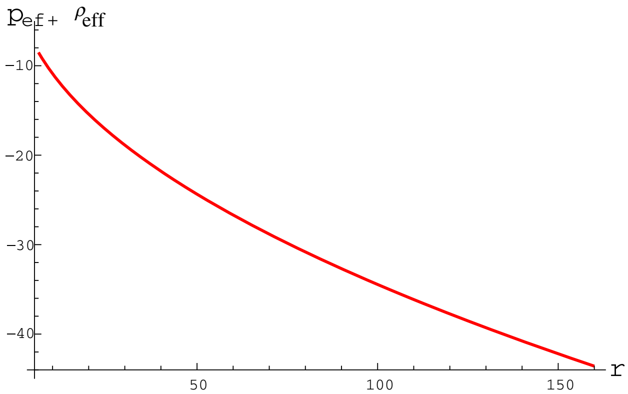

Fig. 8 Evolution of effective NEC versus r for $w = -1$

is located at $r_0 = 7$ and $h(r_0) = r_0$ in the background of phantom energy ($w = -1$).

Now, we examine physical and traversable behavior of WH via energy conditions for ordinary matter and effective NEC, respectively. Figure 7 explores the nature of matter variables. In both plots, the matter variables are found to be increasing positively ensuring that the presence of ordinary matter is confirmed as $\rho_m \geq 0$ and $\rho_m + p_m \geq 0$. The behavior of effective NEC versus r is shown in Fig. 8. The trajectory of effective matter variables is found to be negative increasing as r increases. This behavior indicates that at the throat, the effective NEC is violated ensuring the presence of exotic matter leading to traversable WH solution. In this regard, the realistic horizon-free asymptotically flat WH solution admits traversable behavior for quadratic $f(\mathcal{G})$ model.

Case III: $\beta = \delta \neq 0$

In this case, we solve the system of over determined equations (38)–(44) and get

$$\alpha = \phi_1 + \phi_2 e^{-a}, \quad \gamma = M(\phi_3 - \phi_4), \quad \tau = \phi_5 r + \phi_6,$$
$$f(\mathcal{G}) = \phi_7 e^{\phi_7 \mathcal{G} + \phi_8} + \phi_9,$$
$$\delta = \phi_3 e^{b} T_1(M) T_2(\mathcal{G}), \quad \beta = \phi_3 e^{b/\phi_7} T_1(M) T_2(\mathcal{G}),$$

where $\phi_{i's}$ are arbitrary constants and the explicit form of f defines exponential model of modifies GB gravity. Using above solutions in Eq. (37), we have

$$B_{,a} = 0, \quad T_1(M) = \phi_7 e^{\phi_9 M}, \quad T_2(\mathcal{G}) = \phi_7 e^{-\phi_{10}\mathcal{G}}.$$

Now, we insert symmetry generators, exponential $f(\mathcal{G})$ model, $B = \frac{\phi_1 r}{+}\frac{\phi_8}{\phi_3}$, $M = r^2$, $e^b = (1 - h(r)/r)^{-1}$ and $a(r) = -k/r$ in Eq. (45) which leads to the following nonlinear equation

$$e^{-\frac{k}{2r}} + \sqrt{r/(r-h(r))}\Big(e^{\frac{k(1+w)}{2rw}} \rho_0 \Big(- \frac{\phi_3 e^{k/r}}{2w}$$
$$+ \frac{1}{2}\phi_7\phi_9\phi_{10} \exp\Big\{ r^2 - \Big((r-h(r))^2$$
$$\times 2\Big[\Big(\Big(k\Big(r(4r + k(-1+4r^2))$$
$$+ h(r)(k - 5r - 8kr^2 + 12r^3 + 4(k-3r)rh(r))$$
$$+ r^2(1 - 12r^2 + 12rh(r))h'(r)\Big)\Big)$$
$$\{r^3(r-h(r))^2\}^{-1}\Big]\Big) r^{-6}\Big\}(r - h(r)/r)\Big)$$
$$+ \Big\{ 2r^{-5}\phi_7\phi_9\phi_{10} e^{\phi_1 + r^2}(-r + h(r))$$
$$\Big[\Big\{ k\Big(r(4r + k(-1+4r^2)) + h(r)\Big(k - 5r$$
$$-8kr^2 + 12r^3 + 4(k-3r)rh(r)\Big)$$
$$+ r^2(1 - 12r^2 + 12rh(r))h'(r)\Big)\Big\}\{r^3$$
$$\times (r-h(r))^2\}^{-1}\Big]\Big\} + \Big(\phi_3 e^{k/r}/2 - \Big\{\phi_7\phi_9\phi_{10}$$
$$\exp\Big\{ r^2 - \Big(2(r-h(r))^2\Big[\Big\{k\Big(r\Big(4r$$
$$+ k(-1+4r^2)\Big)$$
$$+ h(r)(k - 5r - 8kr^2 + 12r^3 + 4(k-3r)rh(r))$$
$$+ r^2\Big(12rh$$
$$-12r^2 - 1\Big)h'(r)\Big)\Big\}\{r^3(r-h(r))^2\}^{-1}\Big]\Big)$$
$$r^{-6}\Big\} r\Big\}\{2(-r + h(r))\}^{-1}\Big)\Big(\exp\Big\{\phi_1$$
$$+ \Big(2(r-h(r))^2\Big[\Big\{k(r(4r + k(-1+4r^2)) + h(r)$$
$$\Big(k - 5r - 8kr^2 + 12r^3 + 4r$$
$$\times (k-3r)h(r)\Big) + r^2(1 - 12r^2 + 12rh(r))h'(r))\Big\}$$
$$\{r^3(r-h(r))^2\}^{-1}\Big]\Big) r^{-6}\Big\}$$
$$-\Big\{2\exp\Big\{\phi_1$$
$$+ \Big(2(r-h(r))^2\Big[\Big\{k\Big(r(4r + k(-1+4r^2))$$
$$+ h(r)\Big(k - 5r - 8kr^2$$
$$+ 12r^3 + 4(k-3r)rh(r)\Big)$$

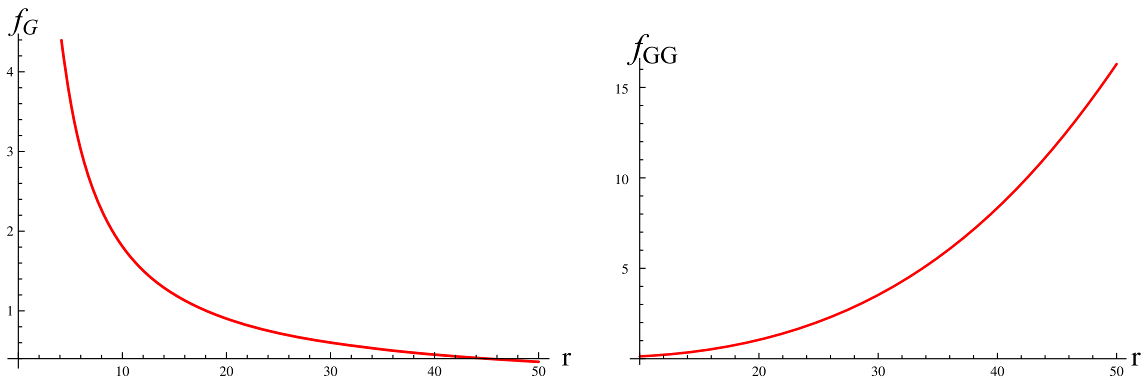

Fig. 9 Evolution of $f(\mathcal{G})$ model versus r for $\phi_1 = 0.01$, $\phi_7 = -1.25$, $\phi_9 = 1$, $\phi_{10} = 1.5$, $k = 0.5$, $w = -1$, and $\rho_0 = -1.5$

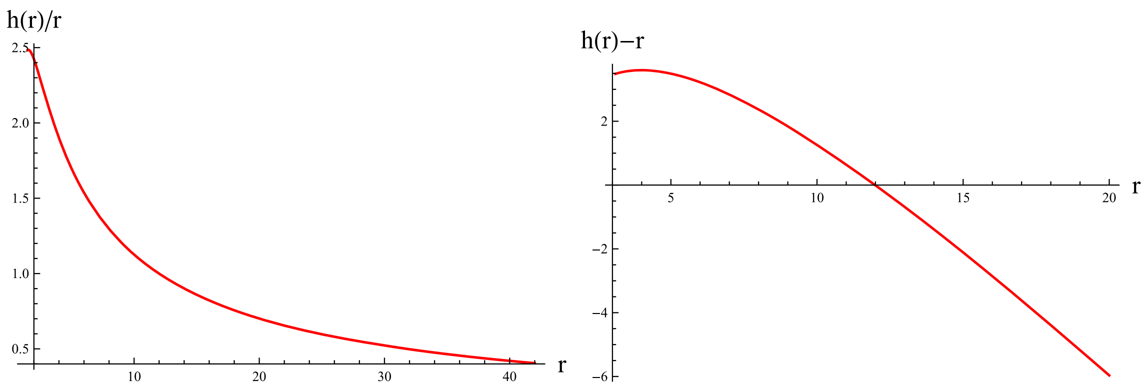

Fig. 10 Variation of the shape function versus r

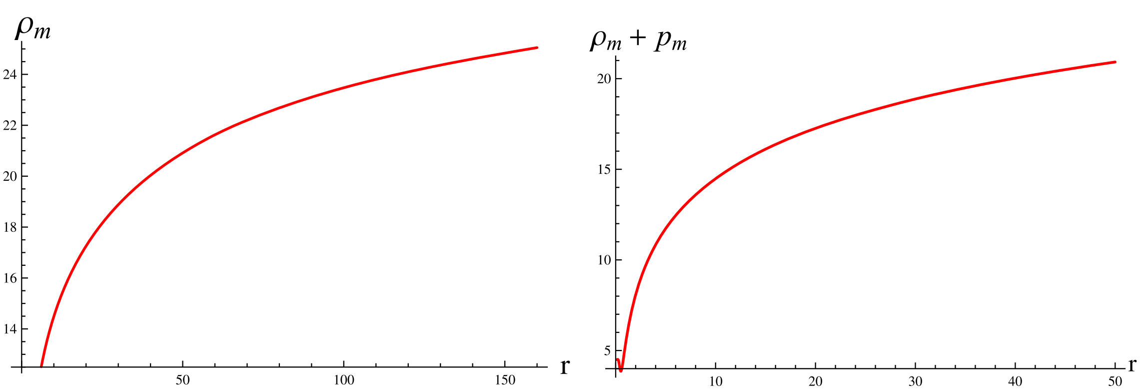

Fig. 11 Evolution of energy bounds versus r for $w = -1$

$$+r^2(1-12r^2+12rh(r))h'(r)\Big)\Big\}(r^3(r-h(r))^2)^{-1}\Big]\Big)$$
$$\times r^{-6}\Big\}(-r+h(r))^2\Big[\Big\{k\Big(r(4r+k(-1+4r^2))$$
$$+h(r)\Big(k-5r-8kr^2+12r^3+4(k-3r)rh(r)\Big)$$
$$+r^2(1-12r^2+12rh(r))h'(r)\Big)\Big\}$$
$$(r^3(r-h(r))^2)^{-1}\Big]\Big\}r^{-6}+\Big((k^2+3kr+24r^2)h(r)$$
$$+r(4-k^2-4kr-32r^2+r(k+8r)h'(r))\Big)$$
$$\times\{2r^5\}^{-1}\Big)\Big)r^2=0. \qquad (50)$$

The numerical solution of this equation leads to study viability of exponential $f(\mathcal{G})$ model and geometry of WH configuration. We also establish graphical analysis to explore the exotic/ordinary nature of matter that defines physically acceptable and traversable WH configuration. In Fig. 9, we discuss the viable behavior of exponential model of modi-

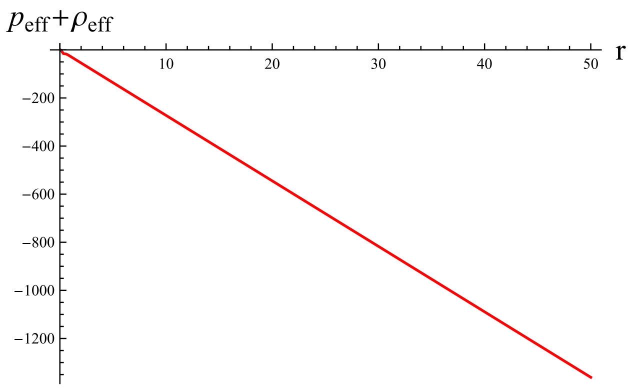

Fig. 12 Evolution of effective NEC versus r for $w=-1$

fied GB gravity. In both plots, the positively decreasing (left) and increasing (right) curves show that $f_{\mathcal{G}}>0$ and $f_{\mathcal{G}\mathcal{G}}>0$ implying consistency with viable GB models. Figure 10 elaborates the geometry of numerically constructed WH. In the left plot, positively decreasing curve follows asymptotically flat shape as $r\to\infty$. The right plot locates WH throat at $r_0=12$ and at this point, the derivative of the shape function remains greater than 1. This analysis defines horizon-free asymptotically flat WH that possesses a throat at $r_0=12$ such that $h(r_0)=r_0$.

To explore the existence of physically viable and traversable WH, we establish graphical analysis of ordinary as well as exotic matter variables in Figs. 11 and 12. In plots of Fig. 11, the trajectories are found to be positively increasing justifying the existence of realistic WH supported by ordinary matter inside the throat. Figure 12 shows the traversable behavior of WH due to violation of effective NEC that introduces repulsive effects into the WH throat. In case of exponential $f(\mathcal{G})$ model, the realistic as well as traversable WH exists in the background of accelerating cosmos.

5 Stability analysis

Here, we examine the stability of WH solutions through Tolman–Oppenheimer–Volkoff (TOV) equation for linear, quadratic and exponential $f(\mathcal{G})$ models in the context of accelerated ($w=-1$) as well as decelerated ($w=0.3$) expanding cosmos. For perfect fluid configuration, the radial function of Bianchi identity ($\nabla_\alpha T^{\alpha\beta}=0$) characterizes TOV equation as

$$\frac{a'}{2}(p_m+\rho_m)+\frac{dp_m}{dr}=0. \quad (51)$$

The divergence of stress–energy tensor with respect to modified terms and Eq. (51) leads to define modified TOV equation given by

$$p'_{eff}+\mathcal{M}_{eff}(p_{eff}+\rho_{eff})+\frac{M'}{M}\left(T^c_{11}-\frac{T^c_{22}e^b}{M}\right)=0, \quad (52)$$

where $p_{eff}=T^{(c)}_{11}+p_m$, $\rho_{eff}=T^{(c)}_{00}+\rho_m$ and $\mathcal{M}_{eff}=\frac{a'e^{b-a}}{2}$ defines effective gravitational mass. The expressions for gravitational $\mathcal{F}_g$ and hydrostatic $\mathcal{F}_h$ forces can be written as

$$\mathcal{F}_h=\frac{d}{dr}(T^{(c)}_{11}+p_m),$$

$$\mathcal{F}_g=\mathcal{M}_{eff}\left(p_{eff}+\rho_{eff}\right)+\frac{M'}{M}\left(T^c_{11}-\frac{T^c_{22}e^b}{M}\right).$$

These dynamical forces significantly explore the stable/unstable state of static configuration. Here, we discuss the stability/instability of static traversable and physically viable WH solutions corresponding to linear, quadratic and exponential $f(\mathcal{G})$ models. The stable WH may exists if these dynamical forces counterbalance each other effect, i.e., $\mathcal{F}_h+\mathcal{F}_g=0$ or $\mathcal{F}_g$=-$\mathcal{F}_h$.

Figure 13 shows the behavior of gravitational and hydrostatic forces for both linear (left plot) as well as exponential (right plot) models in the context of accelerated cosmos ($w=-1$). In the left plot, the trajectories corresponding to hydrostatic and gravitational forces are found to be positively decreasing leading to stable state of WH solution due to null effect of these forces. The analysis of right plot indicates that the stable state of WH solution can be achieved as gravitational and hydrostatic forces are evolving positively but in opposite direction and consequently, canceling the effects of each other. In Fig. 14, we study the stable state of WHs for quadratic GB model when universe experiences accelerated phase of expansion ($w=-1$). This analysis indicates that the horizon-free asymptotically flat traversable and physically viable WHs are stable against accelerated expanding cosmos for both quadratic as well as exponential models of $f(\mathcal{G})$ gravity.

6 Final remarks

In Einstein's gravity, the violation of NEC is the basic requirement for the existence of traversable WH. The violation of NEC defines exotic nature of matter that should be minimized for a physically viable WH. For modified theories, the stress–energy tensor relative to ordinary matter fulfills energy bounds ensuring the presence of a viable WH while the existence of exotic matter is confirmed by the effective matter variables which do not obey energy bounds like effective NEC. In this paper, we have used Noether symmetry technique to evaluate some exact solutions that helps to construct

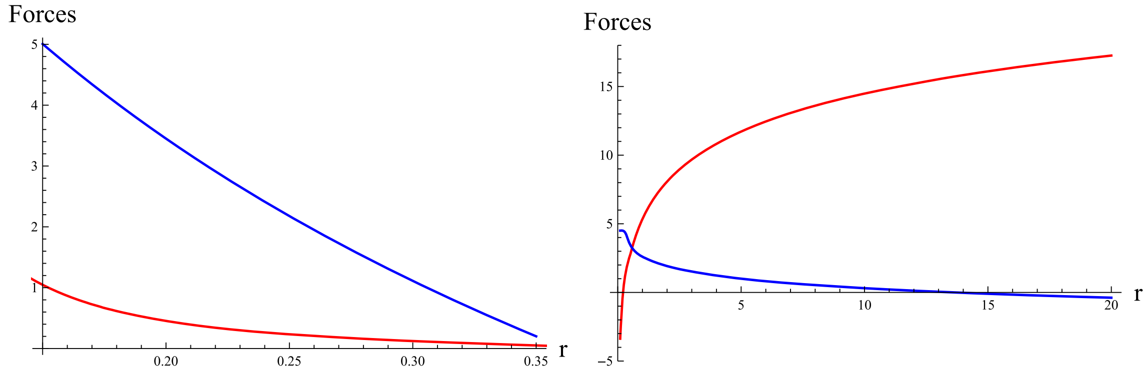

Fig. 13 Plots of $\mathcal{F}_h$ (blue) and $\mathcal{F}_g$ (red) versus r for $f(\mathcal{G}) = \xi_1\mathcal{G} + \xi_2$ (left) and $f(\mathcal{G}) = \phi_7 e^{\phi_7\mathcal{G}+\phi_8} + \phi_9$ (right), $w = -1$

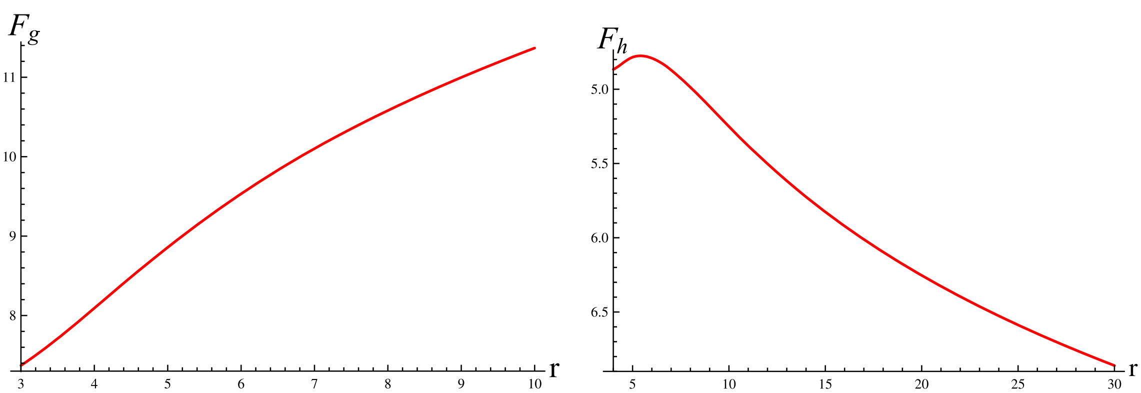

Fig. 14 Plots of $\mathcal{F}_g$ and $\mathcal{F}_h$ versus r for $f(\mathcal{G}) = \chi_7\mathcal{G}^2 + \chi_8\mathcal{G} + \chi_1$ with $w = -1$

static WHs in $f(\mathcal{G})$ theory. We have discussed the presence of exotic and normal matter in WHs through effective and ordinary energy bounds. We have also examined stable/unstable state of constructed WHs via modified TOV equation.

We have used the invariance condition to solve over determined system of equations and evaluated symmetry generator, related conserved quantities and three different $f(\mathcal{G})$ models such as linear, quadratic and exponential models. In the context of these models, we have formulated WH solutions in the background of accelerated expanding cosmos ($w = -1$) and analyzed the WH geometry for variable red-shift function $a(r) = -k/r$. For linear $f(\mathcal{G})$ model, we have found horizon-free WH which is found to be asymptotically flat in a very short interval of r. The throat of this WH is located at $r_0 = 0.34$ with $h'(r_0) < 1$ implying flaring-out condition is preserved. For both quadratic and exponential models, the WH geometry is compatible with Morris–Thorne's suggested geometry, i.e., the finite red-shift function introduces horizon-free ($h(r) < r$), asymptotically flat WH as $r \to \infty$ while flaring-out condition violates ($h(r_0) = r_0$ but $h'(r_0) > 1$) for both models. Using numerical solution of shape function, the viability of new $f(\mathcal{G})$ models is examined graphically. The graphical interpretation indicates that the derivative of $f(\mathcal{G})$ models are positive ensuring viable state of these models.

A WH is traversable if there exists strong repulsive effects or exotic matter near WH throat while physically viable WH is defined by ordinary matter. The violation of effective NEC ($p_{eff} + \rho_{eff} \leq 0$) confirms the presence of repulsive force inside the throat. The positivity of matter variables like $\rho_m \geq 0$, $\rho_m + p_m \geq 0$, $\rho_m - p_m \geq 0$ and $\rho_m + 3p_m \geq 0$ preserve consistency with energy conditions, i.e., NEC, WEC, DEC and SEC relative to ordinary matter and consequently, supports physically viable WH. For all formulated $f(\mathcal{G})$ models, the violation of effective NEC inside WH throat confirms the presence of traversable WH while fulfillment of ordinary bounds leads to physically viable WHs in the background of accelerated expansion.

The stability/instability of these traversable and physically viable WHs is examined via modified TOV equation. For linear $f(\mathcal{G})$ model, the WH configuration surrounded by accelerated expanding cosmos is found to be unstable due to unbalanced state of hydrostatic and gravitational forces. In case of quadratic and exponential models with $w = -1$, the WH solutions preserves equilibrium state as the dynamical forces counterbalance each other effect. Sharif and Nawazish [28] have constructed traversable and realistic WH solution in $f(R)$ gravity for constant as well as variable forms of red-shift function. They have formulated exponential form of $f(R)$ and also considered a standard power-law $f(R)$ models. The stability analysis of both models indicates that WH solutions are stable when universe experiences decelerated rate of expansion while in the presence of accelerated expansion, these configurations become unstable. In the present work, we have evaluated three viable $f(\mathcal{G})$ models, i.e., linear, quadratic and exponential models that yield traversable and physically viable WHs. For quadratic and exponential models, these configurations are stable whereas in case of linear model, this stability is disturbed in the presence of phantom energy.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The datasets generated during analysis are not publicly available but will be provided from the corresponding author on reasonable request.]

Funded by SCOAP[3].

References

1. S. Capozziello, M. de Laurentis, Phys. Rep. **509**, 167 (2011)
2. S. Nojiri, S.D. Odintsov, Phys. Lett. B **631**, 1 (2005)
3. R.R. Metsaev, A.A. Tseytlin, Nucl. Phys. B **293**, 385 (1987)
4. S. Nojiri, S.D. Odintsov, M. Sami, Phys. Rev. D **74**, 046004 (2006)
5. L. Amendola, C. Charmousis, S.C. Davis, J. Cosmol. Astropart. Phys. **10**, 004 (2007)
6. H.M. Sadjadi, Phys. Scr. **83**, 055006 (2011)
7. S. Chatterjee, M. Parikh, Class. Quantum Gravity **31**, 155007 (2014)
8. M. Sharif, H.I. Fatima, Astrophys. Space Sci. **354**, 2124 (2014)
9. M. Sharif, A. Ikram, Int. J. Mod. Phys. D **24**, 1550003 (2015)
10. M. Sharif, H.I. Fatima, Int. J. Mod. Phys. D **25**, 1650083 (2016)
11. S. Capozziello, A. Stabile, A. Troisi, Class. Quantum Gravity **24**, 2153 (2007)
12. S. Capozziello, A. Stabile, A. Troisi, Class. Quantum Gravity **25**, 085004 (2008)
13. S. Capozziello, A. Stabile, A. Troisi, Class. Quantum Gravity **27**, 165008 (2010)
14. B. Vakili, Phys. Lett. B **16**, 664 (2008)
15. M. Sharif, I. Nawazish, J. Exp. Theor. Phys. **120**, 49 (2014)
16. M. Sharif, I. Shafique, Phys. Rev. D **90**, 084033 (2014)
17. M. Sharif, H.I. Fatima, J. Exp. Theor. Phys. **122**, 104 (2016)
18. M. Sharif, I. Nawazish, Gen. Relativ. Gravit. **49**, 76 (2017)
19. M. Sharif, I. Nawazish, Eur. Phys. J. C **77**, 198 (2017)
20. M. Sharif, I. Nawazish, Mod. Phys. Lett. A **32**, 1750136 (2017)
21. F.S.N. Lobo, M.A. Oliveira, Phys. Rev. D **80**, 104012 (2009)
22. M. Jamil et al., J. Korean Phys. Soc. **65**, 917 (2014)
23. S. Bahamonde et al., Phys. Rev. D **94**, 044041 (2016)
24. S.H. Mazharimousavi, M. Halilsoy, Mod. Phys. Lett. A **31**, 1650203 (2016)
25. M. Sharif, H.I. Fatima, Gen. Relativ. Gravit. **48**, 148 (2016)
26. M. Sharif, H.I. Fatima, Astrophys. Space Sci. **361**, 127 (2016)
27. S. Bahamonde et al., Phys. Rev. D **94**, 084042 (2016)
28. M. Sharif, I. Nawazish, Ann. Phys. **389**, 283 (2018)
29. M. Sharif, I. Nawazish, Ann. Phys. **400**, 37 (2019)
30. M.S. Morris, K.S. Thorne, Am. J. Phys. **56**, 395 (1988)
31. G.F.R. Ellis, R. Maartens, M.A.H. MacCallum, *Relativistic Cosmology* (Cambridge University Press, Cambridge, 2012)
32. S.M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison Wesley, Boston, 2004)
33. S. Capozziello, F.S.N. Lobo, J.P. Mimoso, Phys. Lett. B **730**, 280 (2014)
34. S. Capozziello, F.S.N. Lobo, J.P. Mimoso, Phys. Rev. D **91**, 124019 (2015)
35. S. Capozziello, M. De Laurentis, S.D. Odintsov, Eur. Phys. J. C **72**, 2068 (2012)
36. S. Bahamonde, K. Dialektopoulos, U. Camci, Symmetry **12**, 68 (2020)
37. J. Hanc, S. Tuleja, M. Hancova, Am. J. Phys. **72**, 428 (2004)
38. A. De Felice, S. Tsujikawa, Phys. Lett. B **675**, (2009)