



Upper bounds on the compactness at the innermost light ring of anisotropic horizonless spheres

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Received: 14 July 2020 / Accepted: 14 August 2020 / Published online: 19 August 2020
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Abstract In the background of isotropic horizonless spheres, Hod recently provided an analytical proof of a bound on the compactness at the innermost light ring with the dominant energy condition. In this work, we extend the discussion of isotropic spheres to anisotropic spheres. With the dominant energy and non-negative trace conditions, we prove that Hod's bound also holds in the case of anisotropic horizonless spheres.

1 Introduction

According to general relativity, highly curved spacetimes may possess closed light rings (null circular geodesics), on which massless particles can orbit in a circle [1, 2]. It is well known that closed light rings are usually related to black hole spacetimes. In fact, closed light rings may also exist in the horizonless ultra-compact spacetime. From theoretical and astrophysical aspects, the light rings have been extensively studied in various gravity backgrounds [3–13].

The closed light ring plays an important role in understanding properties of curved spacetimes. For example, the interesting phenomenon of strong gravitational lensing in highly curved spacetimes is closely related to the existence of light rings [14]. In addition, the light ring can be used to describe the distribution of exterior matter fields outside black holes [15–20]. And it was also proved that the innermost light ring provides the fastest way to circle a central black hole as measured by observers at the infinity [21–23]. Moreover, the existence of stable light rings suggests that the central compact stars may suffer from nonlinear instabilities [24–30]. And unstable light rings can be used to determine the characteristic resonances of black holes [31–38].

Recently, the compactness at the innermost closed light ring was investigated. The compactness can be described by the parameter $\frac{m(r)}{r}$, where $m(r)$ is the gravitational mass

within the radius r . In the case of black holes, the compactness parameter at the innermost light ring is characterized by the lower bound $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \geq \frac{1}{3}$ with r_γ^{in} as the innermost light ring radius. However, very differently in the horizonless case, numerical data in [39] suggests that the compactness parameter may satisfy an upper bound $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{1}{3}$ for spherically symmetric ultra-compact isotropic spheres. Hod has provided compact analytical proofs of the characteristic intriguing bound $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{1}{3}$ for the spherically symmetric spatially regular spheres with isotropic tensor ($p = p_\tau$), where p and p_τ are interpreted as the radial pressure and the tangential pressure respectively [40].

In the present paper, we study the compactness at the innermost light ring of horizonless spheres. We shall prove the bound $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{1}{3}$ in the case of anisotropic sphere with $p \neq p_\tau$. We point out that this bound in the isotropic case of $p = p_\tau$ has been proved in [40]. Our main results are included in the last section.

2 Investigations on the compactness at the innermost light ring

We study the closed light ring of spherically symmetric configurations. In the standard Schwarzschild coordinate, these spacetimes are expressed by line element [40]

$$ds^2 = -e^{-2\delta} \mu dt^2 + \mu^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where the metric has two functions $\delta(r)$ and $\mu(r) = 1 - \frac{2m(r)}{r}$. For horizonless asymptotically flat spacetimes, the metric functions are characterized by the near origin behavior [40–42]

$$\mu(r \rightarrow 0) = 1 + O(r^2) \quad \text{and} \quad \delta(0) < \infty \quad (2)$$

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and the far region behavior [40–42]

$$\mu(r \rightarrow \infty) = 1 \quad \text{and} \quad \delta(r \rightarrow \infty) = 0. \tag{3}$$

We denote the components of the energy–momentum tensor as

$$\rho = -T_t^t, \quad p = T_r^r \quad \text{and} \quad p_\tau = T_\theta^\theta = T_\phi^\phi, \tag{4}$$

where ρ , p and p_τ are respectively the energy density, the radial pressure and the tangential pressure of the horizonless configurations [17,43]. For the case of isotropic energy-momentum tensor, there is the relation $p = p_\tau$ [40]. In this work, our discussion also covers the case of $p \neq p_\tau$. The unknown metric functions are determined by the Einstein equations $G_\nu^\mu = 8\pi T_\nu^\mu$. With the energy density and pressures (4), one can express the Einstein field equations in the form

$$\mu' = -8\pi r \rho + \frac{1 - \mu}{r}, \tag{5}$$

$$\delta' = \frac{-4\pi r(\rho + p)}{\mu}. \tag{6}$$

Using the Einstein field Eqs. (5) and (6), it has been explicitly proved that the closed light rings are characterized by the relation [41]

$$\mathcal{R}(r) = 3\mu(r) - 1 - 8\pi r^2 p(r) = 0 \quad \text{for} \quad r = r_\gamma, \tag{7}$$

where r_γ is the radius of the closed light ring.

From Eq. (2) and the regular condition $p(0) < \infty$, the function $\mathcal{R}(r)$ is characterized by asymptotical behaviors

$$\mathcal{R}(r) = 3\mu(r) - 1 - 8\pi r^2 p(r) \rightarrow 2 \quad \text{for} \quad r \rightarrow 0. \tag{8}$$

We label r_γ^{in} as the radius of the innermost closed light ring, which corresponds to the smallest positive root of $\mathcal{R}(r) = 0$. In the range $[0, r_\gamma^{in}]$, the function $\mathcal{R}(r)$ satisfies the relation

$$\mathcal{R}(r) \geq 0 \quad \text{for} \quad r \in [0, r_\gamma^{in}]. \tag{9}$$

In particular, at the innermost closed light ring, there is the relation

$$\mathcal{R}(r) = 3\mu(r) - 1 - 8\pi r^2 p(r) = 0 \quad \text{for} \quad r = r_\gamma^{in}. \tag{10}$$

Substituting Eqs. (5) and (6) into the conservation equation $T_{r;\mu}^\mu = 0$, one obtains a relation

$$P'(r) = \frac{r}{2g} [\mathcal{R}(\rho + p) + 2\mu T], \tag{11}$$

where $P(r) = r^4 p(r)$, $\mathcal{R} = 3\mu - 1 - 8\pi r^2 p$ and $T = -\rho + p + 2p_\tau$.

In proving the following bound (22), Hod has imposed the dominant energy condition [40]. In this work, we impose the same dominant energy condition

$$\rho \geq |p|, \quad |p_\tau| \geq 0. \tag{12}$$

And we also take the non-negative trace condition, which is [39–42]

$$T = -\rho + p + 2p_\tau \geq 0. \tag{13}$$

In fact, for a polytropic pressure density equation of the form $p = p_\tau = k_p \rho$, Hod has obtained a bound $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{k_p + 2}{6(k_p + 1)}$, where ρ , p and p_τ are interpreted as the energy density, the radial pressure and the tangential pressure respectively [40]. In the case of $k_p \geq \frac{1}{3}$, Hod’s bound is $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{1}{3}$, which is the same as (22). So Hod proved the bound (22) in the case of $p = p_\tau$. In the present work, we generalize the discussion to cover the case of $p \neq p_\tau$.

Relations (9–13) yield that the function $P(r)$ satisfies the inequality

$$P'(r) \geq 0 \quad \text{for} \quad r \in [0, r_\gamma^{in}]. \tag{14}$$

Near the origin, the pressure function $P(r)$ has the asymptotical behavior

$$P(r \rightarrow 0) = 0. \tag{15}$$

With relations (14) and (15), one obtains

$$P(r) \geq 0 \quad \text{for} \quad r \in [0, r_\gamma^{in}]. \tag{16}$$

The relation (16) and $P(r) = r^4 p(r)$ yield that

$$p(r) \geq 0 \quad \text{for} \quad r \in [0, r_\gamma^{in}]. \tag{17}$$

In particular, at the innermost light ring, there is the relation

$$p(r_\gamma^{in}) \geq 0. \tag{18}$$

According to (10) and (18), one finds that

$$3\mu(r) - 1 = 8\pi r^2 p(r) \geq 0 \quad \text{for} \quad r = r_\gamma^{in}. \tag{19}$$

The relation (19) yields that

$$\mu(r_\gamma^{in}) \geq \frac{1}{3}. \tag{20}$$

The inequality (20) can be expressed as

$$1 - \frac{2m(r_\gamma^{in})}{r_\gamma^{in}} \geq \frac{1}{3}. \tag{21}$$

Then we obtain an upper bound on the compactness at the innermost light ring

$$\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{1}{3}. \tag{22}$$

3 Conclusions

We studied the compactness at the innermost light ring of anisotropic horizonless spheres. We assumed the dominant energy and non-negative trace conditions. At the innermost light ring, we obtained an upper bound on the compactness expressed as $\frac{m(r_\gamma^{in})}{r_\gamma^{in}} \leq \frac{1}{3}$, where r_γ^{in} is the innermost light ring and $m(r_\gamma^{in})$ is the gravitational mass within the sphere of radius r_γ^{in} . In fact, Hod firstly proved this bound in the spacetime of horizonless spheres with isotropic tensor $p = p_\tau$ [40]. In the present work, we proved the same bound in the background of horizonless spheres with generalized anisotropic tensor covering the case of $p \neq p_\tau$.

Acknowledgements We would like to thank the anonymous referee for the constructive suggestions to improve the manuscript. This work was supported by the Shandong Provincial Natural Science Foundation of China under Grant No. ZR2018QA008. This work was also supported by a grant from Qufu Normal University of China under Grant No. xkjjc201906.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: I would like to emphasize that all relevant physical and mathematical calculations are explicitly presented in this paper.]

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Funded by SCOAP³.

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