



Dragging of inertial frames in the composed black-hole-particle system and the weak cosmic censorship conjecture

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Abstract We analyze a gedanken experiment in which a spinning particle that also possesses an extrinsic orbital angular momentum is captured by a spinning Kerr black hole. The gravitational spin-orbit interaction *decreases* the energy of the particle, thus allowing one to test the validity of the Penrose weak cosmic censorship conjecture in extreme situations that have not been analyzed thus far. It is explicitly shown that, to leading order in the black-hole-particle interactions, the *linearized* test particle can over-spin the black hole, thus exposing its inner spacetime singularity to external observers. However, we prove that the general relativistic effect of dragging of inertial frames by the orbiting particle contributes to the energy budget of the system a *non-linear* black-hole-particle interaction term that ultimately ensures the validity of the Penrose cosmic censorship conjecture in this type of gedanken experiments.

1 Introduction

Singularities in curved spacetimes represent extreme physical situations in which general relativity, Einstein's theory of gravity, loses its predictive power. In order to preserve the deterministic nature of classical general relativity in the presence of spacetime singularities, Penrose [1] has suggested that spacetime singularities that arise in gravitational collapse are always hidden inside of black holes. This intriguing idea, known as the weak cosmic censorship *conjecture*, has attracted the attention of physicists and mathematicians over the last five decades (see e.g., [2–14] and references therein).

The elegant singularity theorems of Hawking and Penrose [15, 16] have revealed the physically interesting fact that generic solutions of the Einstein field equations may contain curvature singularities which, according to the cosmic censorship conjecture [1], should be hidden inside of black

holes, invisible to distant (external) observers. Physical processes that threaten to remove the shielding horizon of a black hole and to expose its inner spacetime singularity to external observers are therefore forbidden by the Penrose weak cosmic censorship conjecture [1]. For the advocates of the cosmic censorship conjecture, the mathematically challenging (and physically interesting) task is to find out how such 'dangerous' physical processes, which threaten to violate the weak cosmic censorship conjecture, eventually fail to remove the black-hole horizon [2–14].

One may try to transform a near-extremal spinning Kerr black hole of mass M and angular momentum per unit mass a , which is characterized by the relation [17]¹

$$M^2 - a^2 \geq 0, \quad (1)$$

into a naked (*horizonless*) singularity by sending into the black hole particles that carry large amounts of angular momenta. In this context, it should be remembered that a Kerr spacetime with $M^2 - a^2 < 0$ does not contain an event horizon and it therefore represents a naked singularity. Thus, the composed black-hole-particle gedanken experiment challenges the integrity of the black-hole shielding horizon. This type of gedanken experiments therefore allow one to test the validity of the celebrated Penrose cosmic censorship conjecture [1].

In the present paper we shall inquire into the physical mechanism that protects the horizon of a spinning Kerr black hole from being eliminated by the absorption of spinning particles that also possess extrinsic orbital angular momenta. Intriguingly, the gravitational spin-orbit interaction between the intrinsic and extrinsic angular momenta of the absorbed particle² can *decrease* the energy which is delivered to the

¹ We shall use natural units in which $G = c = \hbar = 1$.

² The gravitational spin-orbit interaction is analogous to the more familiar electromagnetic spin-orbit interaction [see Eq. (3) below].

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black hole for a given amount of the particle's total angular momentum [see Eq. (3) below]. Thus, as will become evident below, the gedanken experiment that we shall analyze in the present paper poses a physical challenge to the weak cosmic censorship conjecture which is *greater* than the ones studied in former gedanken experiments of this type [2–14]. In particular, former studies of the composed black-hole-particle system in the context of the Penrose weak cosmic censorship conjecture have not analyzed the influence of the gravitational spin-orbit interaction³ on the final outcome of the physical absorption process.

Below we shall explicitly show that *linearized* test particles that carry orbital angular momenta and intrinsic (spin) angular momenta can over-spin⁴ an absorbing Kerr black hole, thus exposing its inner spacetime singularity to external observers. However, we shall then prove that the general relativistic effect of dragging of inertial frames by the orbiting particle (the *non*-linear interaction between the particle's total angular momentum and the black-hole horizon generators) produces a non-linear black-hole-particle interaction term which, for a given value of the particle's total angular momentum, *increases* the energy of the absorbed particle. In particular, below we shall explicitly show that this intriguing general relativistic effect ultimately ensures the validity of the Penrose weak cosmic censorship conjecture [1] in the composed black-hole-particle gedanken experiment.

2 Description of the composed black-hole-particle system

We shall analyze a gedanken experiment in which a spinning particle, which also possesses an extrinsic orbital angular momentum, is captured by a near-extremal rotating Kerr black hole. As mentioned above, and will be evident below, in this composed black-hole-particle system it is essential to take into account the *non*-linear interaction term between the absorbing black hole and the captured particle. In particular, the intriguing non-linear effect of dragging of inertial frames, which is caused by the orbital motion of the particle in the black-hole spacetime, makes the horizon generators rotate faster as the orbiting particle approaches the black-hole horizon [18]. Below we shall take into account the small non-linear correction to the angular velocity Ω of the black hole

³ It should be emphasized that, contrary to the gedanken experiment that we shall analyze in the present paper, former studies of the weak cosmic censorship conjecture [2–14] in the composed black-hole-particle system have not considered captured particles that possess *both* intrinsic and extrinsic angular momenta. Thus, the spin-orbit interaction, that will play a key role in our analysis [see Eq. (3) below], was irrelevant in these former studies of the weak cosmic censorship conjecture.

⁴ That is, the final configuration, after the absorption of the test particle by the black hole, may violate the black-hole condition (1).

[see Eq. (11) below], which reflects the dragging of inertial frames by the orbiting particle.

We consider a spinning particle of rest mass μ , total angular momentum J , intrinsic angular momentum (spin) s , and proper cylindrical radius R , which moves in the equatorial plane of a spinning Kerr black-hole spacetime. We shall assume that the intrinsic spin of the particle is orthogonal to the plane of motion. In addition, we shall assume that the physical parameters of the composed black-hole-particle system are characterized by the strong inequalities

$$\mu \ll R \ll M, \quad (2)$$

which imply that the particle has negligible self-gravity and that it is much smaller than the geometric size of the central absorbing black hole.

Our main goal is to challenge the validity of the Penrose weak cosmic censorship conjecture [1] in the most extreme physical situation. The greatest challenge to the conjecture is achieved when, for a given value J of the particle's total angular momentum, the energy $\mathcal{E}(J)$ delivered to the black hole by the captured particle is as small as possible. We shall therefore consider a particle which is released to fall freely into the central black hole from a radial turning point of its motion,⁵ a proper distance R outside the horizon [19–21]⁶

The conserved energy \mathcal{E} of the spinning particle in the rotating Kerr black-hole spacetime is given by the characteristic quadratic equation $\tilde{\alpha}\mathcal{E}^2 - 2\tilde{\beta}\mathcal{E} + \tilde{\gamma} = 0$, where the mathematically cumbersome coefficients $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are explicitly given in [22]. In particular, for a given set $\{\mu, R, J, s\}$ of the particle's physical parameters, the minimum energy (as measured by asymptotic observers) delivered to the rotating black hole by the captured particle is given by the expression [22, 23]:

$$\mathcal{E}_c^{\min} = \Omega_c J + \frac{R(r_+ - r_-)}{2\alpha} \sqrt{\mu^2 + J^2 \frac{r_+^2}{\alpha^2} - \frac{J s (r_+ - r_-) r_+}{2\mu\alpha^2}}, \quad (3)$$

where

$$r_{\pm} = M \pm (M^2 - a^2)^{1/2} \quad \text{and} \quad \alpha \equiv r_+^2 + a^2 \quad (4)$$

are respectively the radii of the Kerr black-hole horizons and the rationalized surface area of the black hole. The physical parameter Ω_c is the characteristic *J*-dependent angular

⁵ Note that such particles are characterized by zero radial momenta at the point of capture.

⁶ As originally discussed by Bekenstein [20], the expression (3) for the total conserved energy of the particle (as measured by asymptotic observers) in the black-hole spacetime is physically valid when every part of the particle (which is characterized by a finite proper cylindrical radius R) is still outside the black-hole horizon.

velocity [18] of the black-hole horizon at the point of capture [see Eq. (11) below].

The last term on the right-hand-side of the energy expression (3) represents the above mentioned gravitational spin-orbit interaction between the extrinsic (orbital) angular momentum of the particle and its intrinsic (spin) angular momentum. It is important to stress the fact that, for $J \cdot s > 0$, this spin-orbit interaction term *decreases* the total energy which is delivered to the black hole by the (spinning and orbiting) captured particle. In particular, in the $J \cdot s > 0$ case, the energy (3) of the spinning particle in the black-hole spacetime is a decreasing function of the particle’s spin s . Thus, the energy delivered to the black hole can be minimized by substituting into the energy expression (3) the maximally allowed value [24]

$$s \leq s^{\max} = \mu R \tag{5}$$

of the particle’s intrinsic spin. Interestingly, it has been explicitly proved by Moller [24] that a finite-size spinning particle which respects the weak (positive) energy condition must conform to the upper bound (5).

Taking cognizance Eqs. of (3) and (5), one obtains the expression

$$\mathcal{E}_c^{\min} = \Omega_c J + \frac{R(r_+ - r_-)}{2\alpha} \left(\sqrt{\mu^2 + J^2 \frac{r_+^2}{\alpha^2}} - \frac{Jr_+}{\alpha} \right) \tag{6}$$

for the minimum energy of the spinning particle at the point of capture. Interestingly, in the regime

$$\frac{J}{M\mu} \gg 1 \tag{7}$$

of large angular momenta, one finds from (6) that, for a given value J of the particle’s total angular momentum, the simple expression⁷

$$\mathcal{E}_c^{\min} = \Omega_c J \cdot \{1 + O[\mu^2 R(r_+ - r_-)/J^2]\} \tag{8}$$

provides a lower bound on the energy which is delivered to the spinning Kerr black hole by the captured particle.

3 Dragging of inertial frames and the Penrose weak cosmic censorship conjecture in the composed Kerr-black-hole-orbiting-particle system

In the present section we shall analyze the validity of the Penrose weak cosmic censorship conjecture [1] in the con-

⁷ Note that, in the $J/M\mu \gg 1$ regime, one finds the series of strong inequalities $\mu^2 R(r_+ - r_-)/J^2 \ll R(r_+ - r_-)/r_+^2 \ll 1$. Here we have used the inequalities $(r_+ - r_-)/r_+ \leq 1$ and $R \ll r_+$.

text of our gedanken experiment, in which a spinning black hole absorbs a spinning and orbiting particle which is characterized by the minimized energy expression (8).

It is important to stress the fact that, to leading order in the black-hole-particle interaction, the linearized particle moves on a fixed (unperturbed) background described by the Kerr spacetime. In particular, the zeroth order angular velocity of the spinning Kerr black hole is given by [17]

$$\Omega^{(0)} = \frac{a}{\alpha} . \tag{9}$$

As we shall explicitly show below, in the context of our gedanken experiment, it is also important to take into account *higher-order* (that is, *non-linear*) interactions between the central black hole and the orbiting particle. In particular, as discussed above, the intriguing physical mechanism of dragging of inertial frames, which is caused by the angular momentum of the orbiting particle, makes the horizon generators rotate faster as the orbiting particle approaches the black-hole horizon. At the assimilation point of the particle, the black-hole angular velocity Ω has changed from its linearized (unperturbed) value $\Omega^{(0)}$ to $\Omega^{(0)} + \Omega_c$.

In particular, in the regime

$$\frac{J}{M^2} \ll 1 \tag{10}$$

of small angular momenta,⁸ one can use the perturbative expansion [18]

$$\Omega_c = \Omega^{(0)} + \varpi_1 \frac{J}{M^3} + \varpi_2 \frac{J^2}{M^5} + \dots \tag{11}$$

for the angular velocity of the black-hole horizon at the capture point of the particle, where $\{\varpi_i = \varpi_i(M, a)\}$ are *J-independent* dimensionless coefficients. It is interesting to mention that Will [18] has proved that, in a physical system composed of a slowly rotating central black hole coupled to an axisymmetric ring of orbiting particles, the leading-order expansion coefficient in (11) is given by the simple value $\varpi_1(M, a = 0) = \frac{1}{4}$.

Intriguingly, as explicitly shown in [25], one may use a continuity argument, which is based on the characteristic smooth evolution of the black-hole angular-velocity during an adiabatic assimilation process of an orbiting particle, in order to obtain the universal (spin-independent) value

$$\varpi_1(M, a) = \frac{1}{4} \tag{12}$$

for the non-linear expansion coefficient ϖ_1 .

⁸ Note that the two inequalities (7) and (10) can be satisfied simultaneously in the $\mu/M \ll 1$ regime [see Eq. (2)].

In the present paper we shall go beyond the linearized test particle approximation by taking into account the leading-order non-linear contribution to the energy of the captured particle in the composed black-hole-particle system. In particular, taking cognizance of Eqs. (8), (9), and (11), one finds the non-linear expression⁹

$$\mathcal{E}_c^{\min} = \frac{aJ}{\alpha} + \varpi_1 \frac{J^2}{M^3} + O\left(\frac{aJ^3}{M^6}\right) \tag{13}$$

for the minimum energy which is delivered to the spinning Kerr black hole by the captured particle.

The assimilation of a particle with energy \mathcal{E}_c^{\min} [see Eq. (13)] and total angular momentum J by the black hole produces the following changes in the black-hole parameters¹⁰:

$$M \rightarrow M + \mathcal{E}_c^{\min} \quad ; \quad Ma \rightarrow Ma + J . \tag{14}$$

Hence, the condition $M_{\text{new}}^2 - a_{\text{new}}^2 \geq 0$ [see (1)] for the black hole to preserve its integrity after the absorption of the particle now reads

$$(M + \mathcal{E}_c^{\min})^2 - \left(\frac{Ma + J}{M + \mathcal{E}_c^{\min}}\right)^2 \geq 0 . \tag{15}$$

Taking cognizance of Eqs. (13), (14), and (15), one finds that the black-hole condition can be expressed in the form

$$\left(M + \frac{aJ}{\alpha} + \varpi_1 \frac{J^2}{M^3}\right)^4 - (Ma + J)^2 \geq 0 . \tag{16}$$

It is convenient to define

$$r_{\pm} \equiv M \pm \epsilon , \tag{17}$$

which yields the relations

$$r_+ r_- = M^2 - \epsilon^2 = a^2 . \tag{18}$$

⁹ It is worth mentioning that there are also non-linear self-interaction contributions of order $O(\mu^2/M)$ to the energy of the particle in the black-hole spacetime. Note, however, that in the regime $J/M\mu \gg 1$ of large angular momenta [see (7)], this higher-order self-interaction contribution to the energy of the particle is negligible as compared to the non-linear interaction term $O(J^2/M^3)$ which stems from the general relativistic effect of dragging of inertial frames by the angular momentum of the orbiting particle.

¹⁰ Note that these characteristic changes in the black-hole physical parameters also imply $a \rightarrow (Ma + J)/(M + \mathcal{E}_c^{\min})$.

Substituting (18) into (16), one can express the black-hole condition in the form¹¹

$$\left[M + \frac{aJ}{(M + \epsilon)^2 + a^2} + \varpi_1 \frac{J^2}{M^3}\right]^4 - (Ma + J)^2 \geq 0 . \tag{19}$$

We shall henceforth assume that the absorbing Kerr black hole is a near-extremal one with $\epsilon/M \ll 1$. Keeping terms up to order $O(\epsilon^2/M^2, J\epsilon/M^3, J^2/M^4)$, one obtains from (19) the black-hole condition

$$\left(\frac{J}{M} - \epsilon\right)^2 + \left(\frac{4\varpi_1}{M^2} - \frac{1}{2M^2}\right)J^2 \geq 0 . \tag{20}$$

Inspecting the inequality (20), one immediately realizes that the black-hole condition can be violated by a *linearized* test particle (for which $\varpi_1 = 0$) [10, 11].

We therefore learn from the black-hole condition (20) that one must take into account the non-linear interaction between the angular momentum of the captured particle and the black-hole generators¹² in order to insure the validity of the Penrose weak cosmic censorship conjecture [1] in the present gedanken experiment. In particular, taking cognizance of the dimensionless non-linear expansion coefficient (12), one can express the black-hole condition (20) in the form

$$\left(\frac{J}{M} - \epsilon\right)^2 + \frac{J^2}{2M^2} \geq 0 . \tag{21}$$

It is clear that the black-hole condition (21) is respected. We therefore conclude that the non-linear interaction between the angular momentum of the particle and the black-hole generators (namely, the general relativistic effect of dragging of inertial frames by the orbiting particle) is essential for ensuring the validity of the cosmic censorship conjecture in this type of gedanken experiments.

4 Summary

In the present paper we have inquired into the physical mechanism that protects the horizon of a spinning Kerr black hole from being removed due to the absorption of finite-size massive particles that possess *both* intrinsic (*spin*) and extrinsic (*orbital*) angular momenta.

¹¹ Here we have used the relation $\alpha = r_{\pm}^2 + a^2 = (M + \epsilon)^2 + a^2$ [see Eqs. (4) and (17)].

¹² It is worth emphasizing again that this non-linear interaction term [see Eq. (13)], which characterizes the composed black-hole-particle system, stems from the general relativistic effect of dragging of inertial frames by the orbital motion of the particle in the curved black-hole spacetime.

It has been shown that the gravitational spin-orbit interaction experienced by the spinning and orbiting particle in the curved black-hole spacetime [an energy term proportional to $\frac{J \cdot s}{M^3}$, see Eq. (3)] can *decrease* the energy delivered to the black hole by the captured particle. This important physical fact implies that the composed black-hole-particle system studied in the present paper poses a challenge to the weak cosmic censorship conjecture which is greater¹³ than the ones considered in former gedanken experiments of this type [2–14].

Interestingly, it has been demonstrated that, to leading order in the black-hole-particle physical interactions, the *linearized* test particle can over-spin the spinning Kerr black hole [see Eq. (20)], thus exposing its inner spacetime singularity to external observers.

However, we have pointed out that the intriguing physical mechanism of dragging of inertial frames by the orbiting particle contributes an additional (*non-linear*) positive term of order $O(J^2/M^3)$ to the energy budget of the particle in the curved black-hole spacetime [see Eq. (13)]. In particular, we have explicitly shown that, by *increasing* the energy of the captured particle, this general relativistic effect (namely, the non-linear interaction between the angular momentum of the orbiting particle and the black-hole generators) ultimately ensures the validity of the Penrose weak cosmic censorship conjecture [1] in this type of gedanken experiments.

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¹³ It is worth emphasizing again that, to the best of our knowledge, former black-hole-particle gedanken experiments that have been designed to challenge the validity of the Penrose weak cosmic censorship conjecture [2–14] have not considered the case of captured particles that carry *both* intrinsic (spin) and extrinsic (orbital) angular momenta. Thus, the gravitational spin-orbit interaction was physically irrelevant in former studies of the weak cosmic censorship conjecture. As explicitly demonstrated in the present paper, this intriguing physical mechanism is important in the context of the cosmic censorship conjecture since, for a given value J of the total angular momentum of the particle, it decreases the energy $\mathcal{E}_c^{\min}(J)$ which is delivered to the black by the captured particle.