



Dynamics and exact Bianchi I spacetimes in Einstein–æther scalar field theory

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Abstract We determine exact and analytic solutions of the gravitational field equations in Einstein–æther scalar model field with a Bianchi I background space. In particular, we consider nonlinear interactions of the scalar field with the æther field. For the model under consideration we can write the field equations by using the minisuperspace description. The point-like Lagrangian of the field equations depends on three unknown functions. We derive conservation laws for the field equations for specific forms of the unknown functions such that the field equations are Liouville integrable. Furthermore, we study the evolution of the field equations and the evolution of the anisotropies by determining the equilibrium points and analyzing their stability.

1 Introduction

According to the cosmological principle, the universe is homogeneous and isotropic in large scales. Indeed, the evolution of the universe from the radiation dominant epoch till the present cosmic acceleration can be well-explained by the homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) model [1]. However, FLRW fails to explain the early history of the universe starting from the origin and pre-inflation epoch where quantum effects should be taken into account.

Inflation is the main mechanism to explain today isotropization of the observable universe. The mechanism of inflation is often based on the existence of a scalar field known as inflaton [2]. The scalar field energy density temporarily dominates the dynamics and drives the universe towards a locally isotropic and homogeneous form that leaves only very small residual anisotropies at the end of a brief inflaton-

dominated period. These anisotropies are observed in the cosmic microwave background, which support the idea that the spacetimes become isotropic ones by evolving in time [3–5]. In addition a recent detailed study by using the X-ray clusters challenged the isotropic scenario [6] and supported the anisotropic cosmological scenario.

The spatial homogeneous but anisotropic spacetimes are known as either Kantowski-Sachs or Bianchi cosmologies. The isometry group of Kantowski-Sachs spacetime is $\mathbb{R} \times SO(3)$, and does not act simply transitively on spacetime, nor does it possess a subgroup with simple transitive action. This model isotropizes to close FLRW models [7–10]. On the other hand, Bianchi spacetimes contain many important cosmological models including the standard FLRW model in the limit of the isotropization, e.g., Bianchi III isotropizes to open FLRW models, and Bianchi I isotropizes to flat FLRW models. In Bianchi models, the spacetime manifold is foliated along the time axis with three dimensional homogeneous hypersurfaces. The Bianchi classification provides a list of all real 3-dimensional Lie algebras up to isomorphism. The classification contains eleven classes, nine of which contain a single Lie algebra and two of which contain a continuum-sized family of Lie algebras, but two of the groups are often included in the infinite families, giving nine types of Bianchi spatially homogeneous spacetimes instead of eleven classes. Bianchi spacetimes contain several important cosmological models that have been used for the discussion of anisotropies of primordial universe and for its evolution towards the observed isotropy of the present epoch [11–14]. There is an interesting hierarchy of Bianchi models. In particular, the LRS Bianchi I model naturally appears as a boundary subset of the LRS Bianchi III model. The last one is an invariant boundary of the LRS Bianchi type VIII model as well. Additionally, LRS Bianchi type VIII can be viewed as an invariant boundary of the LRS Bianchi type IX models [15–20].

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Bianchi spacetimes in the presence of a scalar field were studied in [21]; where it has been found that an initial anisotropic universe can end into a FLRW universe (i.e., it isotropizes) for specific initial conditions whenever the scalar field potential has a large positive value. For exponential scalar field the exact solution of the field equations have been found for some particular Bianchi spacetimes [22–24]. These exact solutions lead to isotropic homogeneous spacetimes as it was found in [25, 26].

An exact anisotropic solution of special interest is the Kasner universe. The Kasner spacetime is the exact solution of the field equations in general relativity in the vacuum for the Bianchi I spacetime, where the space directions are isometries, that is, the three-dimensional space admits three translation symmetries. Kasner universe has various applications in gravitation. One of the most important application is that Kasner solution can describe the evolution of the Mixmaster universe when the contribution of the Ricci scalar of the three-dimensional spatial hypersurface in the field equations is negligible [27]. Hence, Kasner solution is essential for the description of the BKL singularity. For other applications of the Kasner universe and in general of the Bianchi I spacetimes in gravitational physics we refer the reader to [28–35] and references therein.

In this work, we are interested on the study of the gravitational field equations in a Lorentz-violating theory known as Einstein–aether theory [36–40]. Specifically, it is introduced a unit vector, the aether, in the gravitational action. The existence of the aether spontaneously breaks the boost sector of the Lorentz symmetry by picking out a preferred frame at each point in spacetime. The action for Einstein–aether theory is the most general covariant functional of the spacetime metric g_{ab} and aether field u^a involving no more than two derivatives, excluding total derivatives [41, 42].

There are few known exact solutions of the field equations in Einstein field equations. Exact solutions in the Vacuum for the Bianchi I, the Bianchi III, the Bianchi V and the isotropic FLRW spacetime were derived recently in [43, 44]. In [45] the authors presented a generic static spherical symmetric solution in Einstein–aether theory, where it has been shown that the Schwarzschild spacetime is recovered. Other inhomogeneous exact solutions have been studied previously in [46–48]. The spherical collapse in Einstein–aether theory is studied in [49] where a comparison with the Hořava gravity is presented. We remark that Einstein–aether theory can be seen as the classical limit of Hořava gravity. Moreover, Gödel-type spacetimes are investigated in [50, 51].

Furthermore, there are various studies of Einstein–aether models with a matter source. The general evolution in the presence of modified Chaplygin gas was studied in [53], while an analysis with the presence of a Maxwell field was performed in [54]. Exact inhomogeneous spacetimes without

any isometry in Einstein–aether theory with a matter source were derived recently in [52].

It has been proposed that a scalar field contributes to the field equations of Einstein–aether theory where the scalar field can interact with the aether [55]. Such model can describe the so-called Lorentz-violated inflation [56]. The dynamics of spatially homogeneous Einstein–aether cosmological models with scalar field with generalized harmonic potential in which the scalar field is coupled to the aether field expansion and shear scalars were studied in [57, 58], with emphasis on homogeneous Kantowski–Sachs models in [59–61]. A similar analysis on the equilibrium points of the field equations was performed for isotropic FLRW spacetimes in [62–64]. Exact and analytic solutions of isotropic and homogeneous spacetimes in Einstein–aether scalar field cosmology are presented in [56, 65–67].

In the following we are interested on the exact solutions of Bianchi I spacetimes in Einstein–aether theory with a scalar field interacting with the aether field. We consider a nonlinear interaction, and we are able to write the field equations by using the minisuperspace approach. The existence of a point-like Lagrangian which can describe the field equations is essential for our analysis because we can apply techniques of Analytic Mechanics to study the dynamics and determine exact solutions for field equations. We are interested on the dynamical systems analysis of the equilibrium points for the gravitational field equations. From such analysis we can extract information for the evolution of the field equations and for the main phases of the cosmological history. For this analysis one can apply linearization around equilibrium points, monotonic principle [68], the invariant manifold theorem [69–73], the center manifold theorem [69–71, 74], and normal forms theory [69–71].

The plan of the paper is as follows. In Sect. 2, we present the model of our consideration which is the Einstein–aether scalar field theory in Bianchi I spacetime. We write the field equations and the specific form of the interaction term between scalar field and aether field. We write the point-like Lagrangian of the field equations. In Sect. 3, we present analytic solutions of the field equations, the method that we use to constraint the unknown functions of the model and determine that analytic solutions is based on the existence of conservation laws. In particular, we investigate the Liouville integrability of the field equations. In Sect. 4, we perform a detailed analysis of the equilibrium points for the gravitational field equations by using Hubble-normalized variables. Additionally, we use the center manifold theorem and the normal forms calculations to analyze the stability of sets of nonhyperbolic equilibrium points. It is well-known that the procedure based on the formal series of polynomial changes of coordinates devised by Poincaré [75–79] to integrate linearizable dynamical systems in the neighbourhood of a equilibrium point. It can also be used to normalize the system

in the neighborhood of a equilibrium point for nonlinearizable dynamical systems, which are systems whose linearization at the equilibrium point present resonances. This procedure is the basis of the normal forms calculations to be implemented in Sect. 4.1.2. In Sect. 5, we use an alternative dynamical system’s formulation which leads to the evolution of anisotropies decouples; and we study a reduced two-dimensional dynamical system with local and with Poincaré variables. Section 6 is devoted to conclusions.

2 Einstein–aether scalar field model

In this work, we consider the Einstein–aether theory with a scalar field interacting with aether field, with action integral [55]:

$$S = \int dx^4 \left(\sqrt{-g} \frac{R}{2} \right) - S_\phi - S_{Aether}, \tag{1}$$

S_{Aether} corresponding to the aether field u^μ as follows:

$$S_{Aether} = \int d^4x \sqrt{-g} \left(K^{\alpha\beta\mu\nu} u_{\mu;\alpha} u_{\nu;\beta} - \lambda (u^c u_c + 1) \right), \tag{2}$$

and S_ϕ to the action integral of the scalar field

$$S_\phi = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + V(\phi, g^{\alpha\beta}, u_{a;\beta}, u^\alpha) \right). \tag{3}$$

The interaction of the scalar field $\phi(x^\mu)$ with the aether field u^μ , is introduced in the potential function of the scalar field $V = V(\phi, g^{\alpha\beta}, u_{a;\beta}, u^\alpha)$, function λ is a Lagrange multiplier which has been introduced to ensure the unitarity of the aether field u^μ , i.e. $u^\mu u_\mu + 1 = 0$. Moreover, tensor $K^{\alpha\beta\mu\nu}$ is defined by the metric tensor $g^{\mu\nu}$ as follows

$$K^{\alpha\beta\mu\nu} \equiv c_1 g^{\alpha\beta} g^{\mu\nu} + c_2 g^{\alpha\mu} g^{\beta\nu} + c_3 g^{\alpha\nu} g^{\beta\mu} + c_4 g^{\mu\nu} u^\alpha u^\beta,$$

in which c_1, c_2, c_3 and c_4 are the coupling constants of the aether field with the gravitational field. Consequently, since the scalar field $\phi(x^\mu)$ is interacting with the aether field, and the latter is interacting with the gravitational fields, we can say that the scalar field $\phi(x^\mu)$ is not minimally coupled to gravity. However, our proposal is rather different from the so-called scalar tensor theory.

2.1 Bianchi I spacetime

For the underlying space in our consideration, we assume the locally rotational symmetric Bianchi I spacetime with line element

$$ds^2 = -N(t) dt^2 + e^{2\lambda(t)} \left(e^{\sqrt{2}\beta_+(t)} dx^2 + e^{-\frac{1}{\sqrt{2}}\beta_+(t)} + \sqrt{\frac{3}{2}}\beta_-(t) dy^2 + e^{-\frac{1}{\sqrt{2}}\beta_+(t) + \sqrt{\frac{3}{2}}\beta_-(t)} dy^2 \right), \tag{4}$$

where $e^{\lambda(t)}$ is the radius of the three dimensional space, and $\beta_+(t), \beta_-(t)$ are the anisotropy parameters. In the limit $\beta_+(t) \rightarrow 0$ and $\beta_-(t) \rightarrow 0$, the line element (4) reduces to that of the spatially flat FLRW spacetime.

The Kasner spacetime as we discussed in the introduction is an exact solution of the field equations of general relativity for the line element (4) where the anisotropic parameters $\beta_+(t), \beta_-(t)$ are always different from zero. While Kasner spacetime can describe the BKL singularity, in the presence of additional matter source the behaviour of the spacetime is different.

Specifically, in the presence of a cosmological, the evolution of the spacetime can describe an early anisotropic space where the isotropic de Sitter universe is a future attractor. Bianchi I spacetime is the simplest anisotropic model and it is one the basic models to study small anisotropies in the universe. It has been proposed that the small anisotropies of the CMB can be related with the early anisotropies of the universe [80], while Bianchi models can reproduce anisotropies of that kind [81]. In [82] it has been show that Bianchi I models with anisotropic fluid sources can reproduce measurable anisotropy in the CMB without effects upon the primordial nucleosynthesis of helium-4.

In addition a detailed study of dynamics of Bianchi I universe with anisotropic source and an isotropic dark energy component performed in [83], where it was found that the asymptotic behaviour of the universe mimics the de Sitter universe, while anisotropic fluid source contributes in the CMB quadrupole. In the presence of a homogeneous scalar field with an exponential potential the late time attractor is also the isotropic FLRW spacetime for a specific values of the exponent for the potential where the Kitada and Maeda no-hair theorem is applied [26,84].

Bianchi I spacetime admits three isometries which are the three translations of the Euclidean space, that is, the vector fields $\{\partial_x, \partial_y, \partial_z\}$. Furthermore, we assume that the scalar field $\phi(x^\mu) = \phi(t, x, y, z)$, inherits the symmetries of the spacetime which means that the scalar field is homogeneous and depends only on the variable t , that is, $\phi(x^\mu) = \phi(t)$.

However, the field equations of the Bianchi I spacetime can describe the evolution of the anisotropic parameters in inhomogeneous spacetimes, such are the Szekeres spacetimes [85]. Specifically, in the case of silent universes, spacetimes with zero magnetic part of the Weyl tensor, and with an inhomogeneous pressureless fluid source, the dynamics of the anisotropic parameters of the inhomogeneous spacetime admit as past attractors two Kasner universes. However, in

the same model in the presence of additional isotropic fluid source, the anisotropic inhomogeneous Kasner-like attractors reduce to anisotropic and homogeneous spacetimes [85–87].

For the aether field we choose the comoving observer: $u^\mu = \frac{1}{N(t)}\delta_t^\mu$. For this selection, the aether field inherits the symmetries of the spacetime, while the limit of the FLRW spacetime can be recovered [43]. Moreover, as we shall see in the following, with this specific selection for aether field u^μ the field equations can be derived by minisuper-space approach for a specific form of the potential function $V(\phi, g^{\alpha\beta}, u_{\alpha;\beta}, u^\alpha)$.

For the Bianchi I spacetime the kinematic quantities $\{\theta, \sigma^2, \omega_{\mu\nu}, \alpha^\mu\}$ for aether field of our consideration, i.e. $u^\mu = \frac{1}{N(t)}\delta_t^\mu$, are derived

$$\theta = \frac{3}{N}\dot{\lambda}, \quad \sigma^2 = \frac{3}{8N^2} \left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2 \right), \tag{5}$$

and

$$\omega_{\mu\nu} = 0, \quad \alpha^\mu = 0. \tag{6}$$

The kinematic quantities $\{\theta, \sigma^2, \omega_{\mu\nu}, \alpha^\mu\}$ are the expansion rate, the shear, the vorticity and the acceleration for the aether field u^μ , as they are defined in the 1+3 decomposition; that is,

$$\theta = u_{\mu;\nu}h^{\mu;\nu}, \quad a_\mu = u_{\mu;\nu}u^\nu, \quad \sigma^2 = \sigma^{\mu\nu}\sigma_{\mu\nu}, \tag{7}$$

in which

$$\begin{aligned} \sigma_{\mu\nu} &= u_{(\kappa;\lambda)} \left(h_\mu^\kappa h_\nu^\lambda - \frac{1}{3} h^{\kappa\lambda} h_{\mu\nu} \right) \quad \text{and} \\ \omega_{\mu\nu} &= u_{[\kappa;\lambda]} h_\mu^\kappa h_\nu^\lambda. \end{aligned} \tag{8}$$

while $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$.

For a potential function of the form $V = V(\phi, \theta, \sigma^2)$, variation with respect to the metric tensor of (1) produce gravitational field equations, which are

$$G_{\mu\nu} = T_{\mu\nu}^{Aether} + T_{\mu\nu}^\phi, \tag{9}$$

where $G^{\mu\nu}$ is the Einstein tensor, $T_{Aether}^{\mu\nu}$ is the energy–momentum tensor of the aether field defined as [55]:

$$\begin{aligned} T_{\mu\nu}^{Aether} &= 2c_1 \left(u_{\mu;\alpha} u_{\nu;\beta} g^{\alpha\beta} \right) \\ &\quad + 2\lambda u_\mu u_\nu + g_{\mu\nu} \Phi_u \\ &\quad - 2 \left[(u_{(\mu} J_{\nu)}^\alpha)_{;\alpha} + (u^\alpha J_{(\mu\nu)})_{;\alpha} - (u_{(\mu} J_{\nu)}^\alpha)_{;\alpha} \right] \\ &\quad - 2c_4 (u_{\mu;\alpha} u^\alpha) (u_{\nu;\beta} u^\beta), \end{aligned} \tag{10}$$

with $J^\mu{}_\nu = -K^{\mu\beta}{}_{\alpha\nu} u_{;\beta}^\alpha$, $\Phi_u = -K^{\alpha\beta}{}_{\mu\nu} u_{;\alpha}^\mu u_{;\beta}^\nu$, while $T_{\mu\nu}^\phi$ is the energy momentum tensor of the scalar field [57,58]:

$$\begin{aligned} T_{\mu\nu}^\phi &= \phi_{;\mu}\phi_{;\nu} - \left(\frac{1}{2} g^{\alpha\beta} \phi_{;\alpha}\phi_{;\beta} + V(\phi, \theta, \sigma^2) \right) g_{\mu\nu} \\ &\quad + \theta V_{,\theta} g_{\mu\nu} + (V_{,\theta})_{;\alpha} u^\alpha h_{\mu\nu} \\ &\quad + \left(\theta V_{,\sigma^2} + (V_{,\sigma^2})_{;\alpha} u^\alpha \right) \sigma_{\mu\nu} \\ &\quad + V_{,\sigma^2} \left(\sigma_{\mu\nu;\alpha} u^\alpha - 2\sigma^2 u_\mu u_\nu \right). \end{aligned} \tag{11}$$

2.2 Energy–momentum tensors

In [57,58] the dynamical analysis of the field equations for the locally rotational Bianchi I spacetime studied for the potential of the form

$$V(\phi, \theta, \sigma^2) = V_0(\phi) + V_1(\phi)\theta + V_2(\phi)\sigma, \tag{12}$$

for specific functions of $V_I(\phi)$. In particular, for exponential functions $V_I(\phi)$ in [58] or for power-law functions $V_I(\phi)$ in [57]. In the case of FLRW spacetime, where $\sigma = 0$, scalar field potentials with more general nonlinear dependence on parameter θ , have been proposed and studied in the literature [62–66].

In the case of FLRW spacetime, in [56] the authors proposed an Einstein–aether scalar field where the interaction between the aether and the scalar fields is introduced in the coupling coefficients of the aether field with the gravitational field. That leads to an equivalent theory with that [55], where the scalar field potential is quadratic in the expansion rate θ . The theory has been proposed as an alternative Lorentz violating inflationary model. In this theoretical framework the field equations can be described by a canonical point-like Lagrangian. Because of that property, various techniques from analytic mechanics applied in [67] can be used to determine new exact solutions.

Hence, in this work we consider the scalar field potential to be quadratic on θ and σ , that is,

$$V(\phi, \theta, \sigma^2) = V_0(\phi) + V_1(\phi)\theta^2 + V_2(\phi)\sigma^2, \tag{13}$$

in order to be in agreement with the Einstein–aether scalar field model proposed in [56].

For the line element (4) with $N(t) = 1$ and for the aether field $u^\mu = \frac{1}{N(t)}\delta_t^\mu$ the energy–momentum tensor $T_{\mu\nu}^{Aether}$ is diagonal with the following nonzero components:

$$\begin{aligned} T_t^t \text{ }^{Aether} &= 3(c_1 + 3c_2 + c_3)\dot{\lambda}^2 \\ &\quad + \frac{3}{4}(c_1 + c_3) \left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2 \right), \end{aligned} \tag{14a}$$

$$\begin{aligned} T_x^x \text{ }^{Aether} &= (c_1 + 3c_2 + c_3) \left(2\ddot{\lambda} + 3\dot{\lambda}^2 \right) \\ &\quad + \frac{c_1 + c_3}{4} \left(4\sqrt{2}\dot{\beta}_+ - 3(\dot{\beta}_+)^2 + 12\sqrt{2}\dot{\lambda}\dot{\beta}_+ \right) \\ &\quad - \frac{3(c_1 + c_2)}{4} (\dot{\beta}_-)^2, \end{aligned} \tag{14b}$$

$$T_y^y{}^{Aether} = (c_1 + 3c_2 + c_3) (2\ddot{\lambda} + 3\dot{\lambda}^2) - \frac{c_1 + c_3}{4} (2\sqrt{2}\dot{\beta}_+ + 3(\dot{\beta}_+)^2 + 6\sqrt{2}\dot{\lambda}\dot{\beta}_+) + \frac{c_1 + c_3}{4} (2\sqrt{6}\dot{\beta}_- - 3(\dot{\beta}_-)^2 + 6\sqrt{6}\dot{\lambda}\dot{\beta}_+), \tag{14c}$$

$$- \frac{\sqrt{6}}{4} V_2(\phi)_{,\phi} \dot{\beta}_- \dot{\phi}. \tag{15d}$$

2.3 Minisuperspace description

Similarly with the case of FLRW in [56], the field equations of the gravitation Action Integral (1) can be derived from the point-like Lagrangian of the form

$$T_z^z{}^{Aether} = (c_1 + 3c_2 + c_3) (2\ddot{\lambda} + 3\dot{\lambda}^2) - \frac{c_1 + c_3}{4} (2\sqrt{2}\dot{\beta}_+ + 3(\dot{\beta}_+)^2 + 6\sqrt{2}\dot{\lambda}\dot{\beta}_+) - \frac{c_1 + c_3}{4} (2\sqrt{6}\dot{\beta}_- + 3(\dot{\beta}_-)^2 + 6\sqrt{6}\dot{\lambda}\dot{\beta}_+). \tag{14d}$$

$$L(y^A, \dot{y}^A) = L_{GR}(y^A, \dot{y}^A) + L_\phi(y^A, \dot{y}^A) + L_{Aether}(y^A, \dot{y}^A), \tag{16}$$

Similarly, for the potential (13) the energy-momentum tensor $T_{\mu\nu}^\phi$ have the following nonzero components

where the vector fields y^A is $y^A = (N, \lambda, \dot{\lambda}, \beta_+, \dot{\beta}_+, \beta_-, \dot{\beta}_-, \phi, \dot{\phi})$, while a dot demotes total derivative with respect to the variable t , that is $\dot{y}^A = \frac{dy^A}{dt}$.

$$T_t^t{}^\phi = 9V_1(\phi)\dot{\lambda}^2 + \frac{3}{8}V_2(\phi)((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) - \frac{1}{2}\dot{\phi}^2 - V_0(\phi), \tag{15a}$$

Function $L_{GR}(y^A, \dot{y}^A)$ describes the point-like Lagrangian of General Relativity,

$$T_x^x{}^\phi = 3V_1(\phi)(2\ddot{\lambda} + 3\dot{\lambda}^2) + 6V_1(\phi)_{,\phi}\dot{\lambda}\dot{\phi} + \frac{1}{2}\dot{\phi}^2 - V_0(\phi) + V_2(\phi) \times \left(\frac{\sqrt{2}}{2}\dot{\beta}_+ - \frac{3}{8}((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) + \frac{3\sqrt{2}}{2}\dot{\lambda}\dot{\beta}_+ \right) + \frac{\sqrt{2}}{2}V_2(\phi)_{,\phi}\dot{\beta}_+\dot{\phi}, \tag{15b}$$

$$L_{GR}(y^A, \dot{y}^A) = \frac{e^{3\lambda}}{N} \left(-3\dot{\lambda}^2 + \frac{3}{8}((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) \right), \tag{17}$$

that term $L_\phi(y^A, \dot{y}^A)$ describes the point-like Lagrangian of the scalar field, that is,

$$T_y^y{}^\phi = 3V_1(\phi)(2\ddot{\lambda} + 3\dot{\lambda}^2) + 6V_1(\phi)_{,\phi}\dot{\lambda}\dot{\phi} + \frac{1}{2}\dot{\phi}^2 - V_0(\phi) - \frac{1}{8}V_2(\phi) \times (2\sqrt{2}\dot{\beta}_+ + 3(\dot{\beta}_+)^2 + 6\sqrt{2}\dot{\lambda}\dot{\beta}_+) - \frac{\sqrt{2}}{4}V_2(\phi)_{,\phi}\dot{\beta}_+\dot{\phi} + \frac{1}{8}V_2(\phi) \times (2\sqrt{6}\dot{\beta}_- - 3(\dot{\beta}_-)^2 + 6\sqrt{6}\dot{\lambda}\dot{\beta}_-) + \frac{\sqrt{6}}{4}V_2(\phi)_{,\phi}\dot{\beta}_-\dot{\phi}, \tag{15c}$$

$$L_\phi(y^A, \dot{y}^A) = -\frac{e^{3\lambda}}{N} \left(9V_1(\phi)\dot{\lambda}^2 + \frac{3}{8}V_2(\phi)((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) + \frac{1}{2N}\dot{\phi}^2 - NV_0(\phi) \right), \tag{18}$$

while $L_{Aether}(y^A, \dot{y}^A)$ includes the terms which correspond to the aether field u^μ , which is given by the following expression

and

$$T_z^z{}^\phi = 3V_1(\phi)(2\ddot{\lambda} + 3\dot{\lambda}^2) + 6V_1(\phi)_{,\phi}\dot{\lambda}\dot{\phi} + \frac{1}{2}\dot{\phi}^2 - V_0(\phi) - \frac{1}{8}V_2(\phi) \times (2\sqrt{2}\dot{\beta}_+ + 3(\dot{\beta}_+)^2 + 6\sqrt{2}\dot{\lambda}\dot{\beta}_+) - \frac{\sqrt{2}}{4}V_2(\phi)_{,\phi}\dot{\beta}_+\dot{\phi} - \frac{1}{8}V_2(\phi) \times (2\sqrt{6}\dot{\beta}_- + 3(\dot{\beta}_-)^2 + 6\sqrt{6}\dot{\lambda}\dot{\beta}_-)$$

$$L_{Aether}(y^A, \dot{y}^A) = -\frac{3}{N}e^{3\lambda} \left((c_1 + 3c_2 + c_3)\dot{\lambda}^2 + \left(\frac{c_1 + c_3}{4} \right) ((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) \right). \tag{19}$$

Therefore, the point-like Lagrangian (16) is written as follows

$$L(y^A, \dot{y}^A) = \frac{e^{3\lambda}}{N} \left(-3F(\phi)\dot{\lambda}^2 + \frac{3}{8}M(\phi)((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) + \frac{1}{2}\dot{\phi}^2 \right) - Ne^{3\lambda}U(\phi), \tag{20}$$

in which $U(\phi) = V_0(\phi)$, $F(\phi) = (1 + c_1 + 3c_2 + c_3 + 3V_1(\phi))$ and $M(\phi) = (1 - 2(c_1 + c_3) - V_2(\phi))$.

Variation with respect to the lapse function N gives the constraint equation

$$\left(-3F(\phi)\dot{\lambda}^2 + \frac{3}{8}M(\phi)((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2) + \frac{1}{2}\dot{\phi}^2 \right) + U(\phi) = 0, \tag{21}$$

where we have set $N(t) = 1$. Moreover, from the variation with respect to the variables $\{\lambda, \beta_+, \beta_-, \phi\}$, we find the second-order field equations:

$$F(\phi) \left(2\ddot{\lambda} + 3\dot{\lambda}^2 \right) + \frac{3}{8} M(\phi) \left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2 \right) + \left(\frac{1}{2} \dot{\phi}^2 - U(\phi) - 6F(\phi)_{,\phi} \dot{\lambda} \dot{\phi} \right) = 0, \tag{22a}$$

$$\ddot{\phi} + 3\dot{\lambda} \dot{\phi} + U(\phi)_{,\phi} + 3F(\phi)_{,\phi} \dot{\lambda}^2 - \frac{3}{8} M(\phi)_{,\phi} \left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2 \right) = 0, \tag{22b}$$

$$\ddot{\beta}_+ + 3\dot{\lambda} \dot{\beta}_+ + \ln(M(\phi))_{,\phi} \dot{\beta}_+ \dot{\phi} = 0, \tag{22c}$$

$$\ddot{\beta}_- + 3\dot{\lambda} \dot{\beta}_- + \ln(M(\phi))_{,\phi} \dot{\beta}_- \dot{\phi} = 0. \tag{22d}$$

The latter two equations can be integrated as follows

$$M(\phi) e^{3\lambda} \dot{\beta}_+ - I_1 = 0, \quad M(\phi) e^{3\lambda} \dot{\beta}_- - I_2 = 0, \tag{23}$$

where I_1, I_2 are integration constants. The first-order differential equations (23) are two conservation laws for the field equations.

In addition we can construct the third conservation law

$$M(\phi) e^{3\lambda} (\beta_- \dot{\beta}_+ - \beta_+ \dot{\beta}_-) - I_3 = 0, \tag{24}$$

which is the angular momentum in the plane $\{\beta_+, \beta_-\}$.

Lagrangian function (20) describes a singular dynamical system, because $\frac{\partial L}{\partial N} = 0$. However, without loss of generality we can select $N(t) = N(\lambda(t), \beta_+(t), \beta_-(t), \phi(t))$, such that Lagrangian (20) describes the equation of motion of a point particle which motion takes place into the four-dimensional manifold with line element

$$ds_{(4)}^2 = \frac{e^{3\lambda}}{N} \left(-3F(\phi) d\lambda^2 + \frac{3}{8} M(\phi) \left((d\beta_+)^2 + (d\beta_-)^2 \right) + \frac{1}{2} d\phi^2 \right), \tag{25}$$

under the action of the potential function $V_{eff} = Ne^{3\lambda}U(\phi)$. The line element (25) is called the minisuperspace of the gravitational system.

The minisuperspace description is very helpful because techniques and results from Analytic Mechanics can be applied to study the dynamics and the general evolution of the field equations; and also determine exact and analytic solutions of the field equations.

We proceed our analysis by constructing analytic solutions of the gravitational field equations. We assume $N(t) = N(\lambda(t), \beta_+(t), \beta_-(t), \phi)$.

3 Analytic solutions

In this Section, we present some analytic solutions of the field equations for specific forms of the unknown functions

$U(\phi), F(\phi)$ and $M(\phi)$. As we mentioned before, the point-like Lagrangian (20) describes the motion of a point in a four-dimensional space with conservation laws: the quantities I_1, I_2, I_3 and the constraint equation (21), which can be seen as the Hamiltonian function $h(\lambda, \dot{\lambda}, \beta_+, \dot{\beta}_+, \beta_-, \dot{\beta}_-, \phi, \dot{\phi})$, with Hamiltonian constraint $h = 0$. The four conservation laws are independent and not all, but only three of them, are in involution. They are $\{I_1, I_2, h\}$. Therefore, in order to infer about the integrability of the field equations and to be able to write an analytic solution we need to determine at least an additional conservation law.

In order to specify the unknown functions $U(\phi), F(\phi)$ and $M(\phi)$ such that the field equations admit additional conservation laws, we apply the analysis presented before in [88–90]. We use the theory of point transformations to provide a geometric criteria to constrain the unknown functions of the gravitational theory and construct conservation laws.

We focus on the construction of conservation laws linear in the momentum. In order to have the latter true, two main requirements should be satisfied: the minisuperspace (25) to admit isometries and the effective potential $V_{eff} = Ne^{3\lambda}U(\phi)$ to be invariant under the action of a point transformation with generator and isometry of (25).

We define the new scalar field $d\phi = \sqrt{K(\psi)}d\psi$, such that the minisuperspace (25) takes the form

$$ds_{(4)}^2 = \frac{e^{3\lambda}}{N} \left(-3F(\psi) d\lambda^2 + \frac{3}{8} M(\psi) \left((d\beta_+)^2 + (d\beta_-)^2 \right) + \frac{1}{2} K(\psi) d\psi^2 \right). \tag{26}$$

For arbitrary functions $F(\psi), M(\psi)$ the latter line element admits only three isometries, which form the E^2 group in the plane $\{\beta_+, \beta_-\}$. The corresponding conservation laws are the I_1, I_2 and I_3 . There are two cases in which we classify the existence of solutions. These are Case A: $F(\psi)$ arbitrary and Case B: $M(\psi)$ arbitrary.

3.1 Case A: $F(\psi)$ arbitrary

Without loss of generality we assume $K(\psi) = F(\psi)$ and $N = e^{3\lambda}F(\psi), M(\psi) = F(\psi)\bar{M}(\psi), U(\psi) = \frac{\bar{U}(\psi)}{F(\psi)}$, hence the point-like Lagrangian (20) is written

$$L(y^A, \dot{y}^A) = \left(-3\dot{\lambda}^2 + \frac{3}{8} \bar{M}(\psi) \left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2 \right) + \frac{1}{2} \dot{\psi}^2 \right) - e^{6\lambda} \bar{U}(\psi). \tag{27}$$

The field equations which are derived from the point-like Lagrangian (27) admit additional conservation laws linear in the momentum when $\{\bar{M}(\psi) = M_1 e^{M_0 \psi}, \bar{U}(\psi) = U_0\}, \{\bar{M}(\psi) = M_1 e^{M_0 \psi}, \bar{U}(\psi) = U_0 e^{-6\kappa \phi}\}$.

3.1.1 $\bar{M}(\psi) = M_1 e^{M_0 \psi}$, $\bar{U}(\psi) = U_0$

For $\bar{M}(\psi) = M_1 e^{M_0 \psi}$ and $\bar{U}(\psi) = U_0$, the gravitational field equations admit the additional conservation laws

$$\Phi_1 = e^{M_0 \psi} (\beta_+ \dot{\beta}_+ + \beta_- \dot{\beta}_-), \tag{28a}$$

$$\Phi_2 = \frac{3}{16} M_0^2 e^{M_0 \psi} \left(((\beta_+)^2 - (\beta_-)^2) \dot{\beta}_- - \beta_+ \beta_- \dot{\beta}_+ \right) + (\dot{\beta}_- + M_0 \beta_- \dot{\phi}), \tag{28b}$$

$$\Phi_2 = \frac{3}{16} M_0^2 e^{M_0 \psi} \left(((\beta_+)^2 - (\beta_-)^2) \dot{\beta}_+ - \beta_+ \beta_- \dot{\beta}_- \right) - (\dot{\beta}_+ + M_0 \beta_+ \dot{\phi}), \tag{28c}$$

By using the conservation laws I_1, I_2 the field equations are described by the point-like Lagrangian

$$L(y^A, \dot{y}^A) = \left(-3\dot{\lambda}^2 + \frac{1}{2}\dot{\psi}^2 \right) - U_0 e^{6\lambda} - \frac{3((I_1)^2 + (I_2)^2)}{8} e^{-M_0 \psi}, \tag{29}$$

where the reduced field equations are

$$\ddot{\lambda} - U_0 e^{6\lambda} = 0, \tag{30a}$$

$$\ddot{\psi} - \frac{3((I_1)^2 + (I_2)^2)}{8} \frac{M_0}{M_1} e^{-M_0 \psi} = 0, \tag{30b}$$

with constraint equation

$$\left(-3\dot{\lambda}^2 + \frac{1}{2}\dot{\psi}^2 \right) + U_0 e^{6\lambda} + \frac{3((I_1)^2 + (I_2)^2)}{8M_1} e^{-M_0 \psi} = 0. \tag{30c}$$

Hence, the field equations are reduced to the following system

$$3\dot{\lambda}^2 - U_0 e^{6\lambda} = \lambda_0, \tag{31a}$$

$$\frac{1}{2}\dot{\psi}^2 + \frac{3((I_1)^2 + (I_2)^2)}{8M_1} e^{-M_0 \psi} = \lambda_0. \tag{31b}$$

We find that the latter two equations are conservation laws for the field equations, but they are nonlinear in the momentum and are hidden symmetries [91–93].

For $\lambda_0 = 0$ the analytic solution is

$$\lambda(t) = -\frac{1}{6} \ln \left(3U_0 (t - t_0)^2 \right),$$

$$\psi(t) = \frac{1}{M_0} \ln \left(-\left(\frac{3((I_1)^2 + (I_2)^2)}{16M_1} \right) (t - t_1)^2 \right). \tag{32}$$

On the other hand for $\lambda_0 \neq 0$ the analytic solution is

$$\lambda(t) = \frac{1}{6} \ln \left(\frac{\lambda_0}{U_0} \left(\tanh \left(\sqrt{3\lambda_0} (t - t_0) \right)^2 - 1 \right) \right), \tag{33a}$$

$$\psi(t) = -\frac{1}{M_0} \ln \left(M_1 \frac{8\lambda_0 \left(1 - \tanh \left(\frac{\sqrt{2\lambda_0}}{2M_0} (t - t_1) \right)^2 \right)}{3((I_1)^2 + (I_2)^2)} \right), \tag{33b}$$

with

$$\beta_+(t) = \frac{8I_1 M_1 \sqrt{2\lambda_0}}{3M_0((I_1)^2 + (I_2)^2)} \times \tanh \left(\left(M_0 \frac{\sqrt{2\lambda_0}}{2} (t - t_1) \right)^2 \right) + \beta_{+0}, \tag{33c}$$

$$\beta_+(t) = \frac{8I_2 M_1 \sqrt{2\lambda_0}}{3M_0((I_1)^2 + (I_2)^2)} \times \tanh \left(\left(M_0 \frac{\sqrt{2\lambda_0}}{2} (t - t_1) \right)^2 \right) + \beta_{-0}. \tag{33d}$$

In the latter solution if $U_0 = 0$, it follows $\lambda(t) = \frac{\sqrt{3\lambda_0}}{3} (t - t_0)$.

We remark that the line element of the underlying space has the following form

$$ds^2 = -\left(e^{3\lambda(t)} F(\psi(t)) \right)^2 dt^2 + e^{2\lambda(t)} \times \left(e^{\sqrt{2}\beta_+(t)} dx^2 + e^{-\frac{1}{\sqrt{2}}\beta_+(t) + \sqrt{\frac{3}{2}}\beta_-(t)} dy^2 + e^{-\frac{1}{\sqrt{2}}\beta_+(t) - \sqrt{\frac{3}{2}}\beta_-(t)} dz^2 \right), \tag{34}$$

where $F(\psi(t))$ is an arbitrary function.

3.1.2 Analytic solution for arbitrary $\bar{M}(\psi)$

We observe that using the conservation law I_1, I_2 in (27) and for $\bar{U}(\psi) = U_0$, the point-like Lagrangian of the reduced field equations is written

$$L(y^A, \dot{y}^A) = \left(-3\dot{\lambda}^2 + \frac{1}{2}\dot{\psi}^2 \right) - U_0 e^{6\lambda} - M'(\psi), \tag{35}$$

where $M'(\psi) = \frac{3((I_1)^2 + (I_2)^2)}{8} (M(\psi))^{-1}$.

The reduced gravitational field equations are

$$\ddot{\lambda} - U_0 e^{6\lambda} = 0, \quad \ddot{\psi} + (M'(\psi))_{,\psi} = 0, \tag{36}$$

with constraint $(-3\dot{\lambda}^2 + \frac{1}{2}\dot{\psi}^2) + U_0 e^{6\lambda} + M'(\psi) = 0$, and hidden conservation laws

$$3\dot{\lambda}^2 - U_0 e^{6\lambda} = \lambda_0, \quad \frac{1}{2}\dot{\psi}^2 + M'(\psi) = \lambda_0. \tag{37}$$

from which it follows

$$\lambda(t) = \frac{1}{6} \ln \left(\frac{\lambda_0}{U_0} \left(\tanh \left(\sqrt{3\lambda_0} (t - t_0) \right)^2 - 1 \right) \right) \quad \text{or}$$

$$\lambda(t) = \frac{\sqrt{3\lambda_0}}{3}(t - t_0) \text{ for } U_0 = 0, \tag{38}$$

while $\psi(t)$ is given in terms of quadratures.

Some functions of $M'(\psi)$ where ψ is expressed in closed form are presented in [94]. Recall that the conservation laws I_1, I_2 are $I_1 = \bar{M}(\psi)\dot{\beta}_+, I_2 = \bar{M}(\psi)\dot{\beta}_-$.

3.1.3 $\bar{M}(\psi) = M_1 e^{M_0\psi}, \bar{U}(\psi) = U_0 e^{-6\kappa\phi}$

When $\bar{M}(\psi) = M_1 e^{M_0\psi}, \bar{U}(\psi) = U_0 e^{-6\kappa\phi}$, then the gravitational field equations admit the additional conservation law

$$\Phi_4 = -6\left(\dot{\lambda} + \frac{\dot{\phi}}{6\kappa}\right) + \frac{3}{8}\frac{M_1 M_0}{\kappa} e^{M_0\phi} (\beta_+ \dot{\beta}_+ + \beta_- \dot{\beta}_-), \tag{39}$$

The five conservation laws $\{h, I_1, I_2, I_3, \Phi_4\}$ do not provide any set of four-conservation laws which are in involution except from the case where $M_0 = 0$, that is $\bar{M}(\psi) = M_1$ i.e. $F(\psi) \simeq M(\psi)$. Thus, the anisotropic parameters $\dot{\beta}_+$ and $\dot{\beta}_-$ are linear functions of t , that is

$$\beta_+(t) = I_1 t + I_+, \quad \beta_-(t) = I_2 t + I_-, \tag{40}$$

while the other field equations are generated by the point-like Lagrangian

$$L(y^A, \dot{y}^A) = \left(-3\dot{\lambda}^2 + \frac{1}{2}\dot{\psi}^2\right) - U_0 e^{6(\lambda - \kappa\phi)} - \frac{3}{8} \left((I_1)^2 + (I_2)^2\right). \tag{41}$$

We define the new scalars $\lambda = u + v$ and $\psi = \frac{1}{\kappa}u + \sqrt{6}v$, where the gravitational field equations are simplified to

$$\left(3 - \frac{1}{2\kappa^2}\right)\dot{u}^2 + 6\left(1 - \frac{1}{\sqrt{6}\kappa}\right)\dot{u}\dot{v} - U_0 e^{6(1-\sqrt{6}\kappa)v} - \frac{3}{8} \left((I_1)^2 + (I_2)^2\right) = 0, \tag{42}$$

$$\ddot{u} + \frac{U_0 \kappa (6\kappa - \sqrt{6})}{\sqrt{6}\kappa - 1} e^{6(1-\sqrt{6}\kappa)v} = 0, \tag{43}$$

$$\ddot{v} - \frac{U_0 (6\kappa^2 - 1)}{\sqrt{6}\kappa - 1} e^{6(1-\sqrt{6}\kappa)v} = 0.$$

The latter system can be easily integrated and written the analytic solution by using closed-form functions.

3.2 Case B: $M(\psi)$ arbitrary

We define a new field $d\zeta = \sqrt{M}d\phi$, such that the point-like Lagrangian (20) to be written as

$$L(y^A, \dot{y}^A) = \left(-3\bar{F}(\zeta)\dot{\lambda}^2 + \frac{3}{8}\left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2\right) + \frac{1}{2}\zeta^2\right) - e^{6\lambda}\bar{U}(\zeta), \tag{44}$$

where we have set $N = e^{3\lambda}M(\zeta)$ and the new functions are defined as $F(\zeta) = \bar{F}(\zeta)M(\zeta), U(\zeta) = \frac{\bar{U}(\zeta)}{M(\zeta)}$. In addition, we apply the conservation-laws I_1, I_2 such that the remaining field equations are simplified to

$$F\ddot{\lambda} + F_{,\zeta}\dot{\lambda}\dot{\phi} - e^{6\lambda}U(\zeta) = 0, \tag{45a}$$

$$\ddot{\phi} + 3F_{,\zeta}\dot{\lambda}^2 + e^{6\lambda}U_{,\zeta} = 0, \tag{45b}$$

$$\left(-3\bar{F}(\zeta)\dot{\lambda}^2 + \frac{3}{8}\left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2\right) + \frac{1}{2}\zeta^2\right) + e^{6\lambda}\bar{U}(\zeta) + \frac{3}{8}\left((I_1)^2 + (I_2)^2\right) = 0. \tag{45c}$$

We apply the same procedure as before, where we find that the reduced dynamical system admits linear conservation laws for the following sets of the unknown functions $\{F(\zeta) = F_0, U(\zeta) = U_0\}, \{F(\zeta) = F_0 e^{-F_1\zeta}, U(\zeta) = 0\}$ and $\left\{F(\zeta) = F_0\zeta^2, U(\zeta) = U_0\zeta^{-\sqrt{\frac{6}{F_0}}}\right\}$. The two first sets are covered in case A; therefore, we continue with the presentation of the new analytic solution for the power-law functions.

3.2.1 $F(\zeta) = F_0\zeta^2, U(\zeta) = U_0\zeta^{-\sqrt{\frac{6}{F_0}}}$

For $F(\zeta) = F_0\zeta^2, U(\zeta) = U_0\zeta^{-\sqrt{\frac{6}{F_0}}}$ the reduced field equations (45a), (45b) and (45c) admit the extra conservation law $\Phi_5 = \frac{d}{dt}\left(e^{-\sqrt{6F_0}\lambda}\zeta\right)$.

Using the new canonical variables $x = e^{\sqrt{6F_0}\lambda}\zeta$ and $y = e^{-\sqrt{6F_0}\lambda}\zeta$ or $\zeta^2 = xy, \lambda = \frac{1}{2\sqrt{6F_0}}\ln\left(\frac{x}{y}\right)$, the field equations are written as

$$\ddot{x} - 2\sqrt{\frac{6}{F_0}}y^{-1-\sqrt{\frac{6}{F_0}}}, \quad \ddot{y} = 0, \tag{46}$$

$$\frac{1}{2}\dot{x}\dot{y} + U_0 y^{-\sqrt{\frac{6}{F_0}}} + \frac{3}{8}\left((I_1)^2 + (I_2)^2\right) = 0,$$

where the conservation law Φ_5 becomes $\Phi_5 = \dot{y}$.

Consequently, we find the analytic solution

$$x = -\frac{2U_0}{\Phi_5}\left(1 - \sqrt{\frac{6}{F_0}}\right)^{-1} (\Phi_5(t - t_0))^{-\sqrt{\frac{6}{F_0}}} + x_1(t - t_0) + x_0, \quad y = \Phi_5(t - t_0), \tag{47}$$

with constraint equation $x_1\Phi_5 + \frac{3}{8}\left((I_1)^2 + (I_2)^2\right) = 0$.

Recalling that at this case, the line element is of the form

$$ds^2 = - \left(e^{3\lambda(t)} M(\zeta(t)) \right)^2 dt^2 + e^{2\lambda(t)} \left(e^{\sqrt{2}\beta_+(t)} dx^2 + e^{-\frac{1}{\sqrt{2}}\beta_+(t) + \sqrt{\frac{3}{2}}\beta_-(t)} dy^2 + e^{-\frac{1}{\sqrt{2}}\beta_+(t) - \sqrt{\frac{3}{2}}\beta_-(t)} dz^2 \right),$$

where $M(\zeta(t))$ is an arbitrary function and for the latter solution the anisotropic functions β_+, β_- are linear functions on t and the scale factor $\lambda(t)$ is expressed as

$$\exp(\lambda) = \left(\frac{x}{y} \right)^{2\sqrt{6F_0}} = \left(\bar{U}_1(t-t_0)^{-1-\frac{1}{\sqrt{6F_0}}} + x_1 \right)^{2\sqrt{6F_0}} \tag{48}$$

where $\bar{U}_1 = \bar{U}_1(U_0, \Phi_5, F_0)$.

Considering now the case where $M(\zeta(t))$ is a constant function, then for large values of $(t-t_0)$ we have $e^\lambda \simeq x_1$, from where we find the exact solution

$$ds^2 = - (x_1)^6 dt^2 + x_1^2 \left(e^{\sqrt{2}I_1 t} dx^2 + e^{-\frac{1}{\sqrt{2}}I_1 t + \sqrt{\frac{3}{2}}I_2 t} dy^2 + e^{-\frac{1}{\sqrt{2}}I_1 t - \sqrt{\frac{3}{2}}I_2 t} dz^2 \right), \tag{49}$$

the latter is an anisotropic solution with constant volume. On the other hand, for small values of $t-t_0$ it follows that the dominant term is $e^\lambda \simeq \bar{U}_1(t-t_0)^{-2\sqrt{6F_0}-2}$ from which we write

$$ds^2 = - \left(\bar{U}_1(t-t_0)^{-2\sqrt{6F_0}-2} \right)^6 dt^2 + (\bar{U}_1)^2 (t-t_0)^{-4\sqrt{6F_0}-4} \left(e^{\sqrt{2}I_1 t} dx^2 + e^{-\frac{1}{\sqrt{2}}I_1 t + \sqrt{\frac{3}{2}}I_2 t} dy^2 + e^{-\frac{1}{\sqrt{2}}I_1 t - \sqrt{\frac{3}{2}}I_2 t} dz^2 \right), \tag{50}$$

or under the change of coordinates $(t-t_0) \simeq \tau^{\frac{1}{k}}$, where $C-1 = -6\sqrt{6F_0}-6$, the spacetime metric is written as

$$ds^2 = -d\tau^2 + \tau^{\frac{2(K-1)}{3k}} \left(e^{\sqrt{2}I_1 \tau^{\frac{1}{k}}} dx^2 + e^{\left(-\frac{1}{\sqrt{2}}I_1 + \sqrt{\frac{3}{2}}I_2\right)\tau^{\frac{1}{k}}} dy^2 + e^{\left(-\frac{1}{\sqrt{2}}I_1 - \sqrt{\frac{3}{2}}I_2\right)\tau^{\frac{1}{k}}} dz^2 \right), \tag{51}$$

where we have removed the non-essential constants.

4 Dynamical systems analysis

We continue our study by performing a detailed analysis of the equilibrium points for the gravitational field equations. From such analysis we can extract information of the evolution of solutions of field equations and for the description of the main phases of the cosmological history. This approach

has been widely applied before in various cosmological models with many interesting results, for example, we refer the reader to [95–104] and references therein.

The equilibrium points of a spherically symmetric cosmology in Einstein–aether theory were studied before in [48]; specifically, non-comoving perfect fluid has been considered. Static gravitational models in Einstein–aether theory with a perfect fluid with a barotropic equations of state and a scalar field were studied in [60,61]. In addition in [52] it was performed a detailed study of the stability for inhomogeneous and anisotropic models of generalized Szekeres spacetimes. Moreover, isotropic and homogeneous models in Einstein–aether theory with scalar field were considered before in [65–67]. The equilibrium points of Einstein–aether scalar field theory in Bianchi I spacetimes were studied in [57,58].

We continue by defining new variables in the so-called H -normalization (recall $H = \frac{\dot{\theta}}{3} = \frac{\dot{\lambda}}{N}$) [105]:

$$x = \frac{\dot{\phi}}{\sqrt{6F\dot{\lambda}}}, \quad y^2 = \frac{U}{3F\dot{\lambda}^2}, \quad \Sigma_+ = \frac{1}{2}\sqrt{\frac{M}{2F}} \frac{\beta_+}{\dot{\lambda}}, \tag{52}$$

$$\Sigma_- = \frac{1}{2}\sqrt{\frac{M}{2F}} \frac{\beta_-}{\dot{\lambda}},$$

With the use of the new variables the gravitational field equations (21)–(22d) are written as follows

$$\frac{dx}{d\lambda} = -\frac{3}{2}(1-x^2)x - \frac{1}{2}(\sqrt{6}\mu + 3x)y^2 - \sqrt{\frac{3}{2}}\frac{F,\phi}{F}(1-x^2) + \frac{M(3Mx + \sqrt{6FM},\phi)}{2F^2} \left((\Sigma_+)^2 + (\Sigma_-)^2 \right), \tag{53a}$$

$$\frac{dy}{d\lambda} = \frac{1}{2}y \left((3 + \sqrt{6}\mu x + 3(x^2 - y^2)) + \sqrt{\frac{3}{2}}\frac{F,\phi}{F}xy + 3\left(\frac{M}{F}\right)^2 \left((\Sigma_+)^2 + (\Sigma_-)^2 \right) \right), \tag{53b}$$

$$\frac{d\Sigma_+}{d\lambda} = -\frac{3}{2}\sqrt{\frac{M}{F}}(1-x^2+y^2)\Sigma_+ + \frac{3}{2}\left(\frac{M}{F}\right)^{\frac{5}{2}}\Sigma_+ \left((\Sigma_+)^2 + (\Sigma_-)^2 \right) + \sqrt{6}\sqrt{M}\frac{F,\phi}{F}x\Sigma_+ - \sqrt{6}\frac{M,\phi}{\sqrt{M}}x\Sigma_+, \tag{53c}$$

$$\frac{d\Sigma_-}{d\lambda} = -\frac{3}{2}\sqrt{\frac{M}{F}}(1-x^2+y^2)\Sigma_- + \frac{3}{2}\left(\frac{M}{F}\right)^{\frac{5}{2}}\Sigma_- \left((\Sigma_+)^2 + (\Sigma_-)^2 \right)$$

$$+ \sqrt{6}\sqrt{M}\frac{F,\phi}{F}x\Sigma_- - \sqrt{6}\frac{M,\phi}{\sqrt{M}}x\Sigma_-, \tag{53d}$$

along with the algebraic equation

$$1 - (x^2 + y^2) - ((\Sigma_+)^2 + (\Sigma_-)^2) = 0. \tag{54}$$

Given $\mu = \sqrt{F}\frac{U,\phi}{U}$ we can express ϕ as a function of μ through $\phi = \phi(\mu)$. The evolution equation for μ is given by the first-order ordinary differential equation

$$\frac{d\mu}{d\lambda} = \sqrt{\frac{3}{2}}x \left(2F\frac{U,\phi\phi}{U} + \mu\frac{F,\phi}{\sqrt{F}} - 2\mu^2 \right). \tag{55}$$

For any equilibrium solution of the field equations, $P = (x_P, y_P, \Sigma_{+P}, \Sigma_{-P})$, Eq. (22a) becomes

$$\frac{\ddot{\lambda}}{\lambda^2} = -\frac{1}{\lambda_0}, \quad (\lambda_0)^{-1} = \frac{1}{2} \left(4\sqrt{6}x_P + 3x_P^2 + 3 \left(1 - y_P^2 + (\Sigma_{+P})^2 + (\Sigma_{-P})^2 \right) \right) \tag{56}$$

which means that $\lambda(t) = \lambda_0 \ln(t - t_0)$ for $(\lambda_0)^{-1} \neq 0$. Similarly, for the anisotropic parameters β_+, β_- we find

$$\ddot{\beta}_{\pm} = -\sigma_{\pm}\dot{\lambda}^2, \quad \sigma_{\pm} = 2 \left(3\sqrt{2} + 4\sqrt{3}x_P \right) \Sigma_{\pm P} \tag{57}$$

from which it follows $\beta_{\pm}(t) = (\lambda_0)^2 \sigma_{\pm} \ln(t - t_0) + \beta_{\pm 1}(t - t_{0\pm})$.

Finally, the exact solution for the scalar field at the critical point P is

$$\phi(t) = \phi_1 t^{\frac{1}{2}(1+\sqrt{1-4\phi_0})} + \phi_2 t^{\frac{1}{2}(1-\sqrt{1-4\phi_0})}, \tag{58}$$

where $\phi_0 = 3(\lambda_0)^2 \left(\sqrt{6}x + \mu y^2 - 2 \left((\Sigma_{+P})^2 + (\Sigma_{-P})^2 - 1 \right) \right)$.

4.1 $F(\phi) = \phi^2, M(\phi) = \phi^2$ and $U(\phi) = U_0\phi^\mu$

We proceed our analysis by considering $F(\phi) = \phi^2, M(\phi) = \phi^2$ and $U(\phi) = U_0\phi^\mu$, where the system (53) is simplified as

$$\frac{dx}{d\lambda} = \frac{1}{2} \left(2\sqrt{6} + 3x \right) \left(x^2 + (\Sigma_+)^2 + (\Sigma_-)^2 - 1 \right), \tag{59a}$$

$$\frac{dy}{d\lambda} = \frac{1}{2}y \left(3x^2 + \sqrt{6}(\mu + 2)x + 3x^2 + 3 \left(1 - y^2 + (\Sigma_+)^2 + (\Sigma_-)^2 \right) \right), \tag{59b}$$

$$\frac{d\Sigma_+}{d\lambda} = \frac{3}{2}\Sigma_+ \left(x^2 + (\Sigma_+)^2 + (\Sigma_-)^2 - y^2 - 1 \right), \tag{59c}$$

$$\frac{d\Sigma_-}{d\lambda} = \frac{3}{2}\Sigma_- \left(x^2 + (\Sigma_+)^2 + (\Sigma_-)^2 - y^2 - 1 \right), \tag{59d}$$

where now $\mu = const.$ Using constraint (54) the system (59) becomes

$$\begin{aligned} \frac{dx}{d\lambda} &= -\frac{1}{2} \left(3x + 2\sqrt{6} \right) y^2, \\ \frac{dy}{d\lambda} &= \frac{1}{2}y \left(3x^2 + \sqrt{6}(\mu + 2)x - 6y^2 + 6 \right), \\ \frac{d\Sigma_+}{d\lambda} &= -3\Sigma_+y^2, \quad \frac{d\Sigma_-}{d\lambda} = -3\Sigma_-y^2, \end{aligned} \tag{60}$$

where the evolution equation for Σ_- is decoupled, therefore the system's dimensionality can be reduced in one-dimension. We restrict the analysis to the reduced system in the three dimensional manifold $\{x, y, \Sigma_+\}$, where the equilibrium points of (60), have the following coordinates

$$P_A = (x_A, 0, \Sigma_{+A}), \quad P_B = \left(-\frac{\mu + 2}{\sqrt{6}}, \sqrt{\frac{2 - \mu(4 + \mu)}{6}}, 0 \right). \tag{61}$$

Point P_B describes an isotropic FLRW universe, the exact solution at the equilibrium point. It is a scaling solution with an equation of state parameter $w_{\phi B} = -1 + \frac{(\mu+2)^2}{3}$. Point P_B exists when $|\mu + 2| < \sqrt{6}$. On the other hand, P_A describes a two-dimensional surface, that is, a family of nonhyperbolic equilibrium points, which in general describe an anisotropic universe when $\Sigma_{+A} \neq 0$. At the family of points P_A only the kinetic part of the scalar field contributes in the cosmological solution.

We determine the eigenvalues of the linearized system around the critical points. For the family of points P_A the eigenvalues are

$$e_1(P_A) = 3 + \sqrt{\frac{3}{2}}(\mu + 2)x_A, \quad e_2(P_A) = e_3(P_A) = 0. \tag{62}$$

Eigenvalues $e_1(P_A)$ can be negative when $\left\{ \mu < -2 - \sqrt{6}, -\frac{\sqrt{6}}{\mu+2} < x_A \leq 1 \right\}$ or $\left\{ \mu > -2 + \sqrt{6}, -1 \leq x_A - \frac{\sqrt{6}}{\mu+2} \right\}$. However, because two of the eigenvalues are zero the center manifold theorem should be applied (see Sect. 4.1.1).

For point P_B the three eigenvalues are

$$e_1(P_B) = \mu(4 + \mu) - 2, \quad e_2(P_B) = e_3(P_B) = \frac{1}{2}e_1(P_B). \tag{63}$$

Consequently, point P_B , whenever it exists, is always an attractor.

4.1.1 Center manifold theorem for P_A

Introducing the new variables

$$x_1 = x - x_A, \quad x_2 = \Sigma_+ - \Sigma_{+A}, \quad x_3 = y, \tag{64}$$

we obtain the evolution equations

$$\frac{dx_1}{d\lambda} = -\frac{1}{2} \left(\sqrt{6}(\mu + 2) + 6x_1 + 6x_A \right) x_3^2, \tag{65a}$$

$$\frac{dx_2}{d\lambda} = -3(x_2 + \Sigma_{+A}) x_3^2, \tag{65b}$$

$$\frac{dx_3}{d\lambda} = \frac{1}{2} \left(\sqrt{6}(\mu + 2)x_1 - 6x_3^2 + \sqrt{6}(\mu + 2)x_A + 6 \right) x_3. \tag{65c}$$

The center manifold is therefore given by the graph

$$\left\{ (x_2, x_1, x_3) \in \mathbb{R}^3 : x_3 = h(x_1, x_2), Dh = 0, h(0, 0) = 0, x_1^2 + x_2^2 \leq \delta \right\}, \tag{66}$$

where h satisfies the partial differential equation

$$\begin{aligned} h \left(h \left(\left(\sqrt{6}(\mu + 2) + 6x_1 + 6x_A \right) \frac{\partial h}{\partial x_1} \right. \right. \\ \left. \left. + 6(\Sigma_{+A} + x_2) \frac{\partial h}{\partial x_2} - 6h \right) \right. \\ \left. + \sqrt{6}(\mu + 2)x_1 + \sqrt{6}(\mu + 2)x_A + 6 \right) = 0. \end{aligned} \tag{67}$$

The above equation admits the three solutions:

$$h(x_1, x_2) = 0,$$

$$h(x_1, x_2) = \pm$$

$$\frac{\sqrt{6(x_2 + \Sigma_{+A})^2 c_1(\xi) - \mu^2 - 4\mu + 2\sqrt{6}(\mu + 2)\xi(x_2 + \Sigma_{+A}) + 2}}{\sqrt{6}}, \tag{68}$$

where c_1 is an arbitrary function of $\xi = \frac{\sqrt{6}(\mu+2)+6x_1+6x_A}{6(x_2+\Sigma_{+A})}$. Only the first solution satisfies $Dh = 0, h(0, 0) = 0$. Therefore, the center manifold is given locally by

$$\left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : w = 0, x_1^2 + x_2^2 \leq \delta \right\}. \tag{69}$$

The evolution on the center manifold is given by

$$x'_1 = 0, \quad x'_2 = 0. \tag{70}$$

That is x_1 and x_2 are constants at the center manifold.

Introducing the time rescaling $\frac{df}{d\tau} = \frac{1}{x_3^2} \frac{df}{d\lambda}$, the equations become

$$x'_1(\tau) = \frac{1}{2} \left(-\sqrt{6}(\mu + 2) - 6x_1(\tau) - 6x_A \right), \tag{71a}$$

$$x'_2(\tau) = -3x_2(\tau) - 3\Sigma_{+A}, \tag{71b}$$

$$x'_3(\tau) = \frac{\sqrt{6}(\mu + 2)x_1(\tau) - 6x_3(\tau)^2 + \sqrt{6}(\mu + 2)x_A + 6}{2x_3(\tau)}, \tag{71c}$$

whose general solutions are

$$\begin{aligned} x_1(\tau) &= c_1 e^{-3\tau} - \frac{\mu + 2}{\sqrt{6}} - x_A, \quad x_2(\tau) = c_2 e^{-3\tau} - \Sigma_{+A}, \\ x_3(\tau) &= \frac{\sqrt{2\sqrt{6}c_1(\mu + 2)e^{-3\tau} + 6c_3 e^{-6\tau} - \mu^2 - 4\mu + 2}}{\sqrt{6}}. \end{aligned} \tag{72}$$

Hence, $x_1(\tau) \rightarrow -\frac{\mu+2}{\sqrt{6}} - x_A, \quad x_2(\tau) \rightarrow -\Sigma_{+A}, \quad x_3(\tau) \rightarrow \frac{\sqrt{-\mu^2-4\mu+2}}{\sqrt{6}}$, as $\tau \rightarrow \infty$.

4.1.2 Normal forms

In this section we show normal form of expansions for the vector field (65) defined in a vicinity of P_A , expressed in the form of Proposition 1. In general, let $\mathbf{X} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth vector field satisfying $\mathbf{X}(\mathbf{0}) = \mathbf{0}$. We can formally construct the Taylor expansion of \mathbf{x} about $\mathbf{0}$, namely, $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_k + O(|\mathbf{x}|^{k+1})$, where $\mathbf{X}_r \in H^r$, the real vector space of vector fields whose components are homogeneous polynomials of degree r , $\mathbf{X}_1 = \mathbf{D}\mathbf{X}(\mathbf{0})\mathbf{x} \equiv \mathbf{A}\mathbf{x}$, i.e., the matrix of derivatives. For $r = 1$ to k we write

$$\mathbf{X}_r(\mathbf{x}) = \sum_{m_1=1}^r \dots \sum_{m_n=1}^r \sum_{j=1}^n \mathbf{X}_{m,j} \mathbf{x}^{m_j} \mathbf{e}_j, \quad \sum_i m_i = r, \tag{73}$$

Let denote the vector space $B_r = \{ \mathbf{x}^m \mathbf{e}_i := x_1^{m_1} x_2^{m_2} x_3^{m_3} \mathbf{e}_i | m_j \in \mathbb{N}, \sum m_j = r, i, j = 1, 2, 3 \} \subset H^r$.

Let $\mathbf{L}_A^{(r)}$ be the linear operator that assigns to $\mathbf{h}(\mathbf{y}) \in H^r$ the Lie bracket of the vector fields $\mathbf{A}\mathbf{y}$ and $\mathbf{h}(\mathbf{y})$:

$$\mathbf{L}_A^{(r)} : H^r \rightarrow H^r, \quad \mathbf{h} \rightarrow \mathbf{L}_A^{(r)} \mathbf{h}(\mathbf{y}) = \mathbf{D}\mathbf{h}(\mathbf{y})\mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{h}(\mathbf{y}). \tag{74}$$

Applying this operator to monomials $\mathbf{x}^m \mathbf{e}_i$, where m is a multiindex of order r and \mathbf{e}_i basis vector of \mathbb{R}^3 , we find

$$\mathbf{L}_J^{(r)} \mathbf{x}^m \mathbf{e}_i = \{ (\mathbf{m} \cdot \lambda) - \lambda_i \} \mathbf{x}^m \mathbf{e}_i. \tag{75}$$

The eigenvectors in B_r for which $\Lambda_{\mathbf{m},i} \equiv (\mathbf{m} \cdot \lambda) - \lambda_i \neq 0$ form a basis of $B^r = \mathbf{L}_J(H^r)$ and those such that $\Lambda_{\mathbf{m},i} = 0$, associated to the resonant eigenvalues, form a basis for the complementary subspace, G^r , of B^r in H^r . Obtaining the normal form we must look for resonant terms, i.e., those terms of the form $\mathbf{x}^m \mathbf{e}_i$ with \mathbf{m} and i such that $\Lambda_{\mathbf{m},i} = 0$ for the available \mathbf{m}, i .

Theorem 1 (Theorem 2.3.1 in [69]) *Giving a smooth vector field $\mathbf{X}(\mathbf{x})$ on \mathbb{R}^n with $\mathbf{X}(\mathbf{0}) = \mathbf{0}$, there is a polynomial transformation to new coordinates, \mathbf{y} , such that the differential equation $\mathbf{x}' = \mathbf{X}(\mathbf{x})$ takes the form $\mathbf{y}' = \mathbf{J}\mathbf{y} + \sum_{r=1}^N \mathbf{w}_r(\mathbf{y}) + O(|\mathbf{y}|^{N+1})$, where \mathbf{J} is the real Jordan form of $\mathbf{A} = \mathbf{D}\mathbf{X}(\mathbf{0})$ and $\mathbf{w}_r \in G^r$, a complementary subspace of H^r on $B^r = \mathbf{L}_A(H^r)$, where $\mathbf{L}_A^{(r)}$ is the linear operator that assigns to*

$\mathbf{h}(\mathbf{y}) \in H^r$ the Lie bracket of the vector fields $\mathbf{A}\mathbf{y}$ and $\mathbf{h}(\mathbf{y})$, $\mathbf{L}_A^{(r)} : H^r \rightarrow H^r$, $\mathbf{h} \rightarrow \mathbf{L}_A^{(r)} \mathbf{h}(\mathbf{y}) = \mathbf{Dh}(\mathbf{y})\mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{h}(\mathbf{y})$.

Let $\mathbf{x}^* = (x_A, \Sigma_{+A}, 0)^T \in P_A$. By taking the linear transformation $x_1 = x - x_A$, $x_2 = \Sigma_+ - \Sigma_{+A}$, $x_3 = y$, as in Sect. 4.1.1, we obtain the vector field \mathbf{X} given by (65) which is C^∞ in a neighborhood of the origin. Let $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \frac{1}{2}(\sqrt{6}(\mu + 2)x_A + 6) \notin \mathbb{Z}$.

Proposition 1 (Leon and Paliathanasis 2020) *Let be the vector field \mathbf{X} given by (65). Then, there exist a transformation to new coordinates $x \rightarrow z$, such that (65), defined in a vicinity of $\mathbf{0}$, has normal form*

$$z'_1 = -\frac{3\sqrt{6}(\mu + 2)(\mu(\mu + 4) - 2)z_1^2 z_3^2}{(\sqrt{6}(\mu + 2)x_A + 6)^2} - \frac{3z_3^4(\sqrt{6}(\mu + 2)(\mu(\mu + 4) + 10) + 12\sqrt{6}(\mu + 2)x_A^2 + 18(\mu(\mu + 4) + 6)x_A)}{2(\sqrt{6}(\mu + 2)x_A + 6)^2} + \mathcal{O}(|z|^5), \tag{76a}$$

$$z'_2 = -\frac{18(\mu + 2)^2 \Sigma_{+A} z_1^2 z_3^2}{(\sqrt{6}(\mu + 2)x_A + 6)^2} + \frac{3(\mu + 2)z_1 z_2 z_3^2}{(\mu + 2)x_A + \sqrt{6}} - \frac{9\Sigma_{+A} z_3^4(\mu(\mu + 4) + 2\sqrt{6}(\mu + 2)x_A + 10)}{(\sqrt{6}(\mu + 2)x_A + 6)^2} + \mathcal{O}(|z|^5), \tag{76b}$$

$$z'_3 = \frac{1}{2}z_3(\sqrt{6}(\mu + 2)(x_A + z_1) + 6) + \frac{3(\mu + 2)z_1 z_3^3(\sqrt{6}(\mu(\mu + 4) + 7) + 9(\mu + 2)x_A)}{(\sqrt{6}(\mu + 2)x_A + 6)^2} + \mathcal{O}(|z|^5). \tag{76c}$$

Proof The system (65) can be written as

$$\mathbf{x}' = \mathbf{J}\mathbf{x} + \mathbf{X}_2(\mathbf{x}) + \mathbf{X}_3(\mathbf{x}) \tag{77}$$

where \mathbf{x} stands for the phase vector $\mathbf{x} = (x_1, x_2, x_3)^T$, and

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}x_A(\mu + 2) + 3 \end{pmatrix},$$

$$\mathbf{X}_2(\mathbf{x}) = \begin{pmatrix} x_3^2 \left(-\sqrt{\frac{3}{2}}(\mu + 2) - 3x_A \right) \\ -3\Sigma_{+A}x_3^2 \\ \sqrt{\frac{3}{2}}(\mu + 2)x_1x_3 \end{pmatrix},$$

$$\mathbf{X}_3(\mathbf{x}) = \begin{pmatrix} -3x_1x_3^2 \\ -3x_2x_3^2 \\ -3x_3^3 \end{pmatrix}. \tag{78}$$

Simplifying the quadratic part The linear operator $\mathbf{L}_J^{(2)} : H^2 \rightarrow H^2$ has eigenvectors $\mathbf{x}^m \mathbf{e}_i$ with eigenvalues $\Lambda_{\mathbf{m},i} = \lambda_3 m_3 - \lambda_i, i = 1, 2, 3, m_1, m_2, m_3 \geq 0, m_1 + m_2 + m_3 = 2$. The eigenvalues $\Lambda_{\mathbf{m},i}$ for the allowed \mathbf{m}, i are $\Lambda_{(0,0,2),1} =$

$$\sqrt{6}(\mu + 2)x_A + 6, \Lambda_{(0,0,2),2} = \sqrt{6}(\mu + 2)x_A + 6, \Lambda_{(1,0,1),3} = 0.$$

Eliminating the non-resonant quadratic terms, we implement the quadratic transformation

$$\mathbf{x} \rightarrow \mathbf{y} + \mathbf{h}_2(\mathbf{y}), \quad \mathbf{h}_2 : H^2 \rightarrow H^2,$$

$$\mathbf{h}_2(\mathbf{y}) = \begin{pmatrix} -\frac{y_3^2(\sqrt{6}(\mu + 2) + 6x_A)}{2(\sqrt{6}(\mu + 2)x_A + 6)} \\ \frac{3\Sigma_{+A}y_3^2}{\sqrt{6}(\mu + 2)x_A + 6} \\ 0 \end{pmatrix}, \tag{79}$$

such that the vector field (77) transforms into

$$\mathbf{y}' = \mathbf{J}\mathbf{y} - \mathbf{L}_J^{(2)} \mathbf{h}_2(\mathbf{y}) + \mathbf{X}_2(\mathbf{y}) + \tilde{\mathbf{X}}_3(\mathbf{y}) + \tilde{\mathbf{X}}_4(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^5), \tag{80}$$

where

$$\tilde{\mathbf{X}}_3(\mathbf{y}) = \begin{pmatrix} \frac{3(\mu(\mu + 4) - 2)y_1 y_3^2}{\sqrt{6}(\mu + 2)x_A + 6} \\ \frac{3\sqrt{6}(\mu + 2)\Sigma_{+A}y_1 y_3^2 - 3y_2 y_3^2}{\sqrt{6}(\mu + 2)x_A + 6} \\ \frac{3y_3^3(\mu(\mu + 4) + 3\sqrt{6}(\mu + 2)x_A + 16)}{2\sqrt{6}(\mu + 2)x_A + 12} \end{pmatrix},$$

$$\tilde{\mathbf{X}}_4(\mathbf{y}) = \begin{pmatrix} -\frac{3y_3^4(\sqrt{6}(\mu + 2)(\mu(\mu + 4) + 10) + 12\sqrt{6}(\mu + 2)x_A^2 + 18(\mu(\mu + 4) + 6)x_A)}{2(\sqrt{6}(\mu + 2)x_A + 6)^2} \\ \frac{9\Sigma_{+A}y_3^4(\mu(\mu + 4) + 2\sqrt{6}(\mu + 2)x_A + 10)}{(\sqrt{6}(\mu + 2)x_A + 6)^2} \\ 0 \end{pmatrix}, \tag{81}$$

such as

$$-\mathbf{L}_J^{(2)} \mathbf{h}_2(\mathbf{y}) + \mathbf{X}_2(\mathbf{y}) = \frac{\sqrt{6}}{2}(\mu + 2)y_1 y_3 \mathbf{e}_3 \implies \mathbf{y}' = \mathbf{J}\mathbf{y} + \frac{\sqrt{6}}{2}(\mu + 2)y_1 y_3 \mathbf{e}_3 + \tilde{\mathbf{X}}_3(\mathbf{y}) + \tilde{\mathbf{X}}_4(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^5). \tag{82}$$

Simplifying the cubic part The linear operator $\mathbf{L}_J^{(3)} : H^3 \rightarrow H^3$ has eigenvectors $\mathbf{x}^m \mathbf{e}_i$ with eigenvalues $\Lambda_{\mathbf{m},i} = \lambda_3 m_3 - \lambda_i, i = 1, 2, 3, m_1, m_2, m_3 \geq 0, m_1 + m_2 + m_3 = 3$. The eigenvalues $\Lambda_{\mathbf{m},i}$ for the allowed \mathbf{m}, i are $\Lambda_{(1,0,2),1} = \Lambda_{(1,0,2),2} = \Lambda_{(0,1,2),2} = \Lambda_{(0,0,3),3} = \sqrt{6}(\mu + 2)x_A + 6$.

Eliminating the non-resonant terms of third order, we implement the coordinate transformation

$$\mathbf{y} = \mathbf{z} + \mathbf{h}_3(\mathbf{z}), \quad \mathbf{h}_3 : H^3 \rightarrow H^3,$$

$$\mathbf{h}_3(\mathbf{z}) = \begin{pmatrix} \frac{3(\mu(\mu + 4) - 2)z_1 z_3^2}{(\sqrt{6}(\mu + 2)x_A + 6)^2} \\ \frac{3z_3^2(z_2(\sqrt{6}(\mu + 2)x_A + 6) - \sqrt{6}(\mu + 2)\Sigma_{+A}z_1)}{(\sqrt{6}(\mu + 2)x_A + 6)^2} \\ -\frac{3z_3^3(\mu(\mu + 4) + 3\sqrt{6}(\mu + 2)x_A + 16)}{2(\sqrt{6}(\mu + 2)x_A + 6)^2} \end{pmatrix}, \tag{83}$$

such as

$$\begin{aligned}
 -\mathbf{L}_J^{(3)} \mathbf{h}_3(\mathbf{z}) + \tilde{\mathbf{X}}_3(\mathbf{z}) = 0 &\implies \mathbf{z}' = \mathbf{Jz} \\
 + \frac{\sqrt{6}}{2}(\mu + 2)z_1z_3\mathbf{e}_3 + \tilde{\mathbf{X}}_4(\mathbf{z}) + \mathbf{O}(|\mathbf{z}|^5), & \quad (84)
 \end{aligned}$$

where

$$\tilde{\mathbf{X}}_4(\mathbf{z}) = \begin{pmatrix} -\frac{3z_3^4(\sqrt{6}(\mu+2)(\mu(\mu+4)+10)+12\sqrt{6}(\mu+2)x_A^2+18(\mu(\mu+4)+6)x_A)}{2(\sqrt{6}(\mu+2)x_A+6)^2} - \frac{3\sqrt{6}(\mu+2)(\mu(\mu+4)-2)z_1^2z_3^2}{(\sqrt{6}(\mu+2)x_A+6)^2} \\ -\frac{18(\mu+2)^2\Sigma_{+A}z_1^2z_3^2}{(\sqrt{6}(\mu+2)x_A+6)^2} + \frac{3(\mu+2)z_1z_2z_3^2}{(\mu+2)x_A+\sqrt{6}} - \frac{9\Sigma_{+A}z_3^4(\mu(\mu+4)+2\sqrt{6}(\mu+2)x_A+10)}{(\sqrt{6}(\mu+2)x_A+6)^2} \\ \frac{3(\mu+2)z_1z_3^3(\sqrt{6}(\mu(\mu+4)+7)+9(\mu+2)x_A)}{(\sqrt{6}(\mu+2)x_A+6)^2} \end{pmatrix}. \quad (85)$$

Then, the result of the proposition follows. \square

Finally, the fourth order terms, which all are non-resonant, can be removed under the quartic transformation

$$\begin{aligned}
 z_1 \rightarrow w_1 - \frac{3\sqrt{6}(\mu + 2)(\mu(\mu + 4) - 2)w_1^2w_3^2}{(\sqrt{6}(\mu + 2)x_A + 6)^3} \\
 - \frac{3w_3^4(\sqrt{6}(\mu+2)(\mu(\mu+4)+10)+12\sqrt{6}(\mu+2)x_A^2+18(\mu(\mu+4)+6)x_A)}{2(\sqrt{6}(\mu+2)x_A+6)^2(2\sqrt{6}(\mu+2)x_A+12)}, & \quad (86)
 \end{aligned}$$

$$\begin{aligned}
 z_2 \rightarrow w_2 - \frac{18(\mu + 2)^2\Sigma_{+A}w_1^2w_3^2}{(\sqrt{6}(\mu + 2)x_A + 6)^3} \\
 + \frac{3(\mu + 2)w_1w_2w_3^2}{((\mu + 2)x_A + \sqrt{6})(\sqrt{6}(\mu + 2)x_A + 6)} \\
 - \frac{9\Sigma_{+A}w_3^4(\mu(\mu + 4) + 2\sqrt{6}(\mu + 2)x_A + 10)}{(\sqrt{6}(\mu + 2)x_A + 6)^2(2\sqrt{6}(\mu + 2)x_A + 12)}, & \quad (87)
 \end{aligned}$$

$$z_3 \rightarrow w_3 + \frac{3(\mu + 2)w_1w_3^3(\sqrt{6}(\mu(\mu + 4) + 7) + 9(\mu + 2)x_A)}{(\sqrt{6}(\mu + 2)x_A + 6)^3}. \quad (88)$$

Neglecting the higher order terms we obtain the integrable system

$$z'_1 = 0, \quad z'_2 = 0, \quad z'_3 = \frac{1}{2}z_3(\sqrt{6}(\mu + 2)(x_A + z_1) + 6), \quad (89)$$

with general solution

$$\begin{aligned}
 z_1(\lambda) &= z_{10}, \quad z_2(\lambda) = z_{20}, \\
 z_3(\lambda) &= z_{30}e^{\frac{1}{2}\lambda(\sqrt{6}(\mu+2)(x_A+z_{10})+6)}. & \quad (90)
 \end{aligned}$$

5 Alternative dynamical system’s formulation

Using the alternative variables and time derivative

$$\begin{aligned}
 x &= \frac{\dot{\phi}}{\sqrt{6F\dot{\lambda}}}, \quad z = \frac{3F\dot{\lambda}^2}{U}, \quad \Sigma_+ = \frac{1}{2}\sqrt{\frac{M}{2F}}\frac{\beta_+}{\dot{\lambda}}, \\
 \Sigma_- &= \frac{1}{2}\sqrt{\frac{M}{2F}}\frac{\beta_-}{\dot{\lambda}}, \quad \frac{df}{d\tau} = z\frac{df}{d\lambda}, & \quad (91)
 \end{aligned}$$

we obtain the dynamical system

$$\begin{aligned}
 x' &= -\sqrt{\frac{3}{2}}(\mu + 2) - 3x, \quad \Sigma'_+ = -3\Sigma_+, \\
 z' &= -z[z(\sqrt{6}(\mu + 2)x + 6) - 6]. & \quad (92)
 \end{aligned}$$

It is worth noticing that the system (92) is integrable with general solution

$$\begin{aligned}
 x(\tau) &= c_1e^{-3\tau} - \frac{\mu + 2}{\sqrt{6}}, \quad \Sigma_+(\tau) = c_2e^{-3\tau}, \\
 z(\tau) &= \frac{6}{2\sqrt{6}c_1(\mu + 2)e^{-3\tau} + 6c_3e^{-6\tau} - \mu^2 - 4\mu + 2}. & \quad (93)
 \end{aligned}$$

The equilibrium points are

$$\begin{aligned}
 P_B : (x, \Sigma_+, z) &= \left(-\frac{\mu + 2}{\sqrt{6}}, 0, -\frac{6}{\mu(\mu + 4) - 2}\right), \\
 P_C : (x, \Sigma_+, z) &= \left(-\frac{\mu + 2}{\sqrt{6}}, 0, 0\right). & \quad (94)
 \end{aligned}$$

The eigenvalues of P_B are $\{-6, -3, -3\}$, therefore, it is a sink. On the other hand, P_C has eigenvalues $\{6, -3, -3\}$, and it is a saddle. Interestingly, the equation for Σ_+ decouples, and we can study a reduced dynamical system for the variables (x, z) .

In Fig. 1 are presented some orbits of system (92) for some values of parameter μ in the Poincarè variables (U, V) . The point P_B , whenever it exists, it is a sink. Point P_C is always a saddle. The system admits three configurations at infinity $Q_1 : (U, V) = (1, 0)$, $Q_2 : (U, V) = (0, 1)$, and $Q_3 : (U, V) = (-1, 0)$, whose dynamics is shown in the plots. In this coordinates the set $P_A : (x, y, \Sigma_+) = (x_A, 0, \Sigma_{+A})$ is translated to Q_2 due to $\lim_{y \rightarrow 0} z = \infty$, $\lim_{z \rightarrow \infty} (U, V) = (0, 1)$.

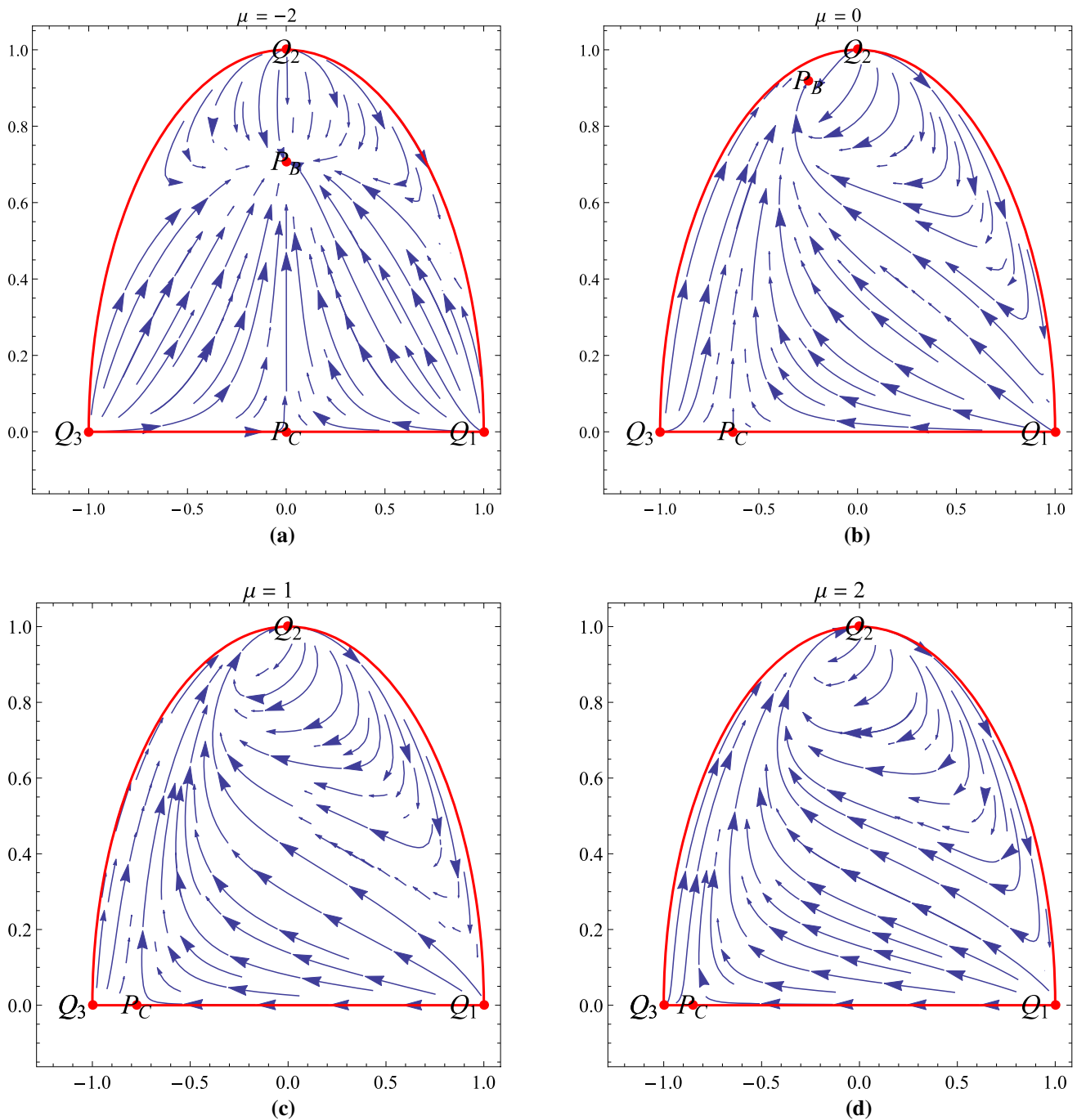


Fig. 1 Orbits of the system (92) for some values of parameter μ in the Poincarè variables $(U, V) = \left(\frac{x}{\sqrt{1+x^2+z^2}}, \frac{z}{\sqrt{1+x^2+z^2}}\right)$. The point P_B , whenever it exists, it is a sink. Point P_C is always a saddle. The

system admits three configurations at infinity $Q_1 : (U, V) = (1, 0)$, $Q_2 : (U, V) = (0, 1)$, and $Q_3 : (U, V) = (-1, 0)$, whose dynamics is shown in the plots

6 Conclusions

In this paper we have investigated a Lorentz violating Einstein-aether theory which contains a scalar field nonminimally coupled with the aether field. For the physical space we

consider the homogeneous but anisotropic Bianchi I space-time.

We have extended previous analyses on the subject by considering an interacting function between the scalar and the aether fields, which is nonlinear on the kinematic quantities of the time-like aether field. In particular we assume that

the interacting function is quadratic on the expansion rate θ and on the shear σ , while in the generic scenario has three unknown functions of the scalar field, as expressed by Eq. (13).

The novelty of the interacting function under consideration is that we can determine a point-like Lagrangian and write the field equations by using the minisuperspace description. Indeed, the field equations can be seen as the motion of a point-like particle in a four-dimensional Riemannian space with coordinates the three scalars of the Bianchi I spacetime and the field ϕ , under the action of a potential function. By using this property, we are able to apply methods from analytic mechanics and study the integrability properties of the field equations. We use Ansätze for conservation laws which are linear in the momentum, such that it is possible to specify the unknown functions of the field equations, which allows for exact or analytic solutions of the field equations by using closed-form functions. Hence, the field equations are Liouville integrable.

In order to study the dynamics and the evolution of the anisotropies, we determine the equilibrium points for the field equations. These points describe some specific physical solutions for the model of our consideration. We perform our analysis by using the Hubble-normalized variables, also by using an alternative dimensionless variables which lead to the evolution of anisotropies with local and with Poincaré variables. From the two sets of variables we conclude that the isotropic spatially flat FLRW spacetime is a future attractor for the physical space. However, anisotropic solutions of Kasner-like are allowed by the theory of our consideration. Additionally, we have used the center manifold theorem and the normal forms calculations to analyze the stability of sets of nonhyperbolic equilibrium points. All these tools lead to system's reductions: center manifold and alternative formulations reduce the system dimensionality; whereas normal forms allow to eliminate non-resonant terms by using a sequence of nearly identity nonlinear transformations, keeping at each step only the terms at perturbation level which are relevant in the dynamics.

Finally, it is worth mentioning that our work contributes to the subject of Lorentz violating theories with a matter source. From the results of our analysis, it follows that real anisotropic physical solutions exist in Einstein-aether scalar field theory, while the generic evolution of the dynamics to an isotropic state in large scales, it is supported by the theory.

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