Brane gravity in 4D from Chern–Simons gravity theory

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Abstract We evaluate a 5-dimensional Randall Sundrum type metric in the Lagrangian of the Einstein–Chern–Simons gravity, and then we derive an action and its corresponding field equations, for a 4-dimensional brane embedded in the 5-dimensional space-time of the theory, which in the limit $l \rightarrow 0$ leads to the 4-dimensional general relativity with cosmological constant. An interpretation of the h^a matter field present in the Einstein–Chern–Simons gravity action is given. As an application, we find some Friedmann–Lemaître–Robertson–Walker cosmological solutions that exhibit accelerated behavior.

1 Introduction

The study of the so called brane world models has introduced completely new ways of looking upon standard problems in many areas of theoretical physics. The existence of those new dimensions may have non trivial effects in our understanding of the cosmology of the early Universe, among many other issues.

The idea dates back to the 1920s, to the works of Kaluza and Klein [1,2] who tried to unify electromagnetism with Einstein gravity by assuming that the photon originates from the fifth component of the metric. By convention, it has always been assumed that such extra dimensions should be compactified to manifolds of small radii with sizes of the order of the Planck length.

It was only in the last years of the twentieth century when people started to ask the question of how large could these extra dimensions be without getting into conflict with observations.

Of particular interest in this context are the Randall and Sundrum models [3,4] for warped backgrounds, with compact or even infinite extradimensions. Randall and Sundrum proposed that the metric of the space-time is given by

$$ds^2 = e^{-2kr_c} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2, \quad (1)$$

i.e. a 4-dimensional metric multiplied by a “warp factor” which is a rapidly changing function of an additional dimension, where k is a scale of order the Planck scale, x^μ are coordinates for the familiar four dimensions, while $0 \leq \phi \leq \pi$ is the coordinate for an extra dimension, which is a finite interval whose size is set by r_c , known as “compactification radius”. Randall and Sundrum showed that this metric is a solution to Einstein’s equations.

The Einstein–Chern–Simons gravity [5] is a gauge theory whose Lagrangian density is given by a 5-dimensional Chern–Simons form for the so called $\mathfrak{B}$ algebra. This algebra can be obtained from the Anti-de Sitter algebra and a particular semigroup S by means of the S -expansion procedure introduced in Refs. [6,7]. The field content induced by the $\mathfrak{B}$ algebra includes the vielbein e^a , the spin connection ω^{ab} , and two extra bosonic fields h^a and k^{ab} . The Einstein–Chern–Simons gravity has the interesting property that the 5-dimensional Chern–Simons Lagrangian for the $\mathcal{B}$ algebra, given by [5]

$$L_{ChS}^{(5)}[e, \omega, h, k] = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right), \quad (2)$$

where $R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}$ and $T^a = de^a + \omega_c^a e^c$, leads to the standard general relativity without cosmological constant in the limit where the coupling constant l tends to zero while keeping the Newton’s constant fixed. It should be noted that there is an absence of kinetic terms for the fields h^a and k^{ab} in the Lagrangian $L_{ChS}^{(5)}$ [9] (for detail see appendix).

The main purpose of this letter is to make the 5-dimensional Einstein–Chern–Simons theory consistent with the idea of a 4-dimensional space-time, through the replace-

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ment of a Randall Sundrum type metric in the Lagrangian (2). It is also a purpose of this paper to get an interpretation of the h^a matter field present in the Einstein–Chern–Simons gravity action.

The article is organized as follows: in Sect. 5, we evaluate a 5-dimensional Randall Sundrum type metric in the Lagrangian (2), and then we derive an action for a 4-dimensional brane embedded in the 5-dimensional space-time of the Einstein–Chern–Simons gravity, which in the limit $l \rightarrow 0$ leads to the 4-dimensional general relativity with cosmological constant. In Sect. 3, we write the action for the brane in tensorial language and then we find the corresponding Friedmann equations for homogeneous and isotropic cosmology, and we obtain some De Sitter accelerated solutions. Finally, concluding remarks are presented in Sect. 4.

2 Action for a 4-dimensional brane embedded in the 5-dimensional space-time

Chern–Simons theories of gravity are valid only in odd-dimensions and, in order to have a well defined even-dimensional theory, it would be necessary to carry out some kind of dimensional reduction or compactification.

In Refs. [3,4] a mechanism was introduced that allows to connect the Einstein–Chern–Simons action with an action for a 4-dimensional brane embedded in the 5-dimensional space-time.

In Ref. [5], the 5-dimensional Chern–Simons Lagrangian for gravity (2) was developed. The corresponding action is invariant under the so-called $\mathfrak{B}$ algebra, which induces two geometrical fields, the vielbein e^a and the spin connection ω^{ab} , and two bosonic matter fields, h^a and k^{ab} . In order to find an action and its corresponding field equations for a 4-dimensional brane embedded in the 5-dimensional space-time Σ_5 of the Einstein–Chern–Simons gravity, we will consider the following 5-dimensional Randall Sundrum type metric

$$\begin{aligned} ds^2 &= e^{2f(\phi)} \tilde{g}_{\mu\nu}(\tilde{x}) d\tilde{x}^\mu d\tilde{x}^\nu + r_c^2 d\phi^2 \\ &= e^{2f(\phi)} \tilde{\eta}_{mn} \tilde{e}^m \tilde{e}^n + r_c^2 d\phi^2, \end{aligned} \quad (3)$$

where $e^{2f(\phi)}$ is the so-called “warp factor”, and r_c is the so-called “compactification radius” of the extra dimension, which is associated with the coordinate $0 \leq \phi < 2\pi$. The symbol $\sim$ denotes 4-dimensional quantities related to the space-time Σ_4 . We will use the usual notation,

$$\begin{aligned} x^\alpha &= (\tilde{x}^\mu, \phi); \quad \alpha, \beta = 0, \dots, 4; \quad a, b = 0, \dots, 4; \\ \mu, \nu &= 0, \dots, 3; \quad m, n = 0, \dots, 3; \\ \eta_{ab} &= \text{diag}(-1, 1, 1, 1, 1); \\ \tilde{\eta}_{mn} &= \text{diag}(-1, 1, 1, 1). \end{aligned} \quad (4)$$

This allows us, for example, to write

$$\begin{aligned} e^m(\phi, \tilde{x}) &= e^{f(\phi)} \tilde{e}^m(\tilde{x}) = e^{f(\phi)} \tilde{e}_\mu^m(\tilde{x}) d\tilde{x}^\mu; \\ e^4(\phi) &= r_c d\phi. \end{aligned} \quad (5)$$

From the vanishing torsion condition

$$T^a = de^a + \omega_b^a e^b = 0, \quad (6)$$

we obtain

$$\omega_{b\alpha}^a = -e_b^\beta \left(\partial_\alpha e_\beta^a - \Gamma_{\alpha\beta}^\gamma e_\gamma^a \right), \quad (7)$$

where $\Gamma_{\alpha\beta}^\gamma$ is the Christoffel symbol.

From equations (5) and (6), we find

$$\omega_4^m = \frac{e^f f'}{r_c} \tilde{e}^m, \quad \text{with } f' = \frac{\partial f}{\partial \phi}, \quad (8)$$

and the 4-dimensional vanishing torsion condition

$$\begin{aligned} \tilde{T}^m &= \tilde{d}\tilde{e}^m + \tilde{\omega}_n^m \tilde{e}^n = 0, \quad \text{with } \tilde{\omega}_n^m = \omega_n^m \text{ and} \\ \tilde{d} &= d\tilde{x}^\mu \frac{\partial}{\partial \tilde{x}^\mu}. \end{aligned} \quad (9)$$

From (8), (9) and the Cartan’s second structural equation, $R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}$, we obtain the components of the 2-form curvature

$$\begin{aligned} R^{m4} &= \frac{e^f}{r_c} \left(f'^2 + f'' \right) d\phi \tilde{e}^m, \\ R^{mn} &= \tilde{R}^{mn} - \left(\frac{e^f f'}{r_c} \right)^2 \tilde{e}^m \tilde{e}^n, \end{aligned} \quad (10)$$

where the 4-dimensional 2-form curvature is given by

$$\tilde{R}^{mn} = \tilde{d}\tilde{\omega}^{mn} + \tilde{\omega}_p^m \tilde{\omega}^{pn}. \quad (11)$$

The torsion-free condition implies that the third term in the Eq. (2) vanishes. This means that the Lagrangian (2) is no longer dependent on the field k^{ab} . So that the Lagrangian (2) has two independent fields, e^a and h^a , and it is given by

$$\begin{aligned} L_{ChS}^{(5)}[e, h] &= \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e \\ &\quad + \alpha_3 \varepsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + l^2 R^{ab} R^{cd} h^e \right). \end{aligned} \quad (12)$$

From Eq. (12) we can see that the Lagrangian contains the Gauss–Bonnet term L_{GB} , the Einstein–Hilbert term L_{EH} and a term L_H which couples geometry and matter. In fact, replacing (5) and (10) in (12), and using $\tilde{\varepsilon}_{mnpq} = \varepsilon_{mnpq4}$,

we obtain

$$\begin{aligned} L_{GB} &= \varepsilon_{abcde} R^{ab} R^{cd} e^e \\ &= r_c d\phi \left\{ \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} - \left(\frac{2e^{2f}}{r_c^2} \right) \right. \\ &\quad \left(3f'^2 + 2f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q \\ &\quad + \left(\frac{e^{4f}}{r_c^4} f'^2 \right) \left(5f'^2 + 4f'' \right) \\ &\quad \left. \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right\}, \end{aligned} \quad (13)$$

$$\begin{aligned} L_{EH} &= \varepsilon_{abcde} R^{ab} e^c e^d e^e \\ &= r_c d\phi \left\{ \left(3e^{2f} \right) \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q \right. \\ &\quad \left. - \left(\frac{e^{4f}}{r_c^2} \right) \left(5f'^2 + 2f'' \right) \right. \\ &\quad \left. \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right\}, \end{aligned} \quad (14)$$

$$\begin{aligned} L_H &= \varepsilon_{abcde} R^{ab} R^{cd} h^e \\ &= \tilde{\varepsilon}_{mnpq} \left\{ \tilde{R}^{mn} - \left(\frac{e^f f'}{r_c} \right)^2 \tilde{e}^m \tilde{e}^n \right\} \\ &\quad \times \left\{ \left[\tilde{R}^{pq} - \left(\frac{e^f f'}{r_c} \right)^2 \tilde{e}^p \tilde{e}^q \right] h^4 - \right. \\ &\quad \left. \left(\frac{4e^f}{r_c} \right) \left(f'^2 + f'' \right) d\phi \tilde{e}^p h^q \right\}. \end{aligned} \quad (15)$$

Note that in Eq. (13) there is a quadratic term in the curvature given by $r_c d\phi \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq}$. The action associated with this term can be directly integrated in ϕ . In fact,

$$\begin{aligned} r_c \int_{\Sigma_5} d\phi \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} &= r_c \int_0^{2\pi} d\phi \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} \\ &= 2\pi r_c \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq}, \end{aligned} \quad (16)$$

which represents the 4-dimensional Gauss–Bonnet term. This term is a topological term, so that it does not contribute to the dynamics, and it can be eliminated.

There is also a quadratic term in the curvature in Eq. (15): $\tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} h^4$. Following Ref. [3], we assume that matter has only nonzero components on the 4-dimensional manifold Σ_4 , so that we postulate that the components of the h^a -field are given by

$$h^m(\phi, \tilde{x}) = e^{g(\phi)} \tilde{h}^m(\tilde{x}), \quad h^4 = 0, \quad (17)$$

where $g(\phi)$ is a well behaved function in $0 \leq \phi < 2\pi$. This means that the quadratic term $\tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} h^4$ vanishes.

By replacing (13), (14) and (15) in (12) we find that the action takes the form

$$\begin{aligned} \tilde{S}[\tilde{e}, \tilde{h}] &= \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \left(A \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + B \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q + \right. \\ &\quad \left. C \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q + E \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q \right), \end{aligned} \quad (18)$$

where

$$A = 2r_c \int_0^{2\pi} d\phi e^{2f} \left[\alpha_3 - \frac{\alpha_1 l^2}{r_c^2} (3f'^2 + 2f'') \right], \quad (19)$$

$$\begin{aligned} B &= -\frac{1}{r_c} \int_0^{2\pi} d\phi e^{4f} \\ &\quad \left[\frac{2\alpha_3}{3} (5f'^2 + 2f'') - \frac{\alpha_1 l^2}{r_c^2} f'^2 (5f'^2 + 4f'') \right], \end{aligned} \quad (20)$$

$$C = -\frac{4\alpha_3 l^2}{r_c} \int_0^{2\pi} d\phi e^f e^g (f'^2 + f''), \quad (21)$$

$$E = \frac{4\alpha_3 l^2}{r_c^3} \int_0^{2\pi} d\phi e^{3f} e^g f'^2 (f'^2 + f''). \quad (22)$$

Since $f(\phi)$ and $g(\phi)$ are arbitrary and continuously differentiable functions, and since we are working with a cylindrical variety, if we choose (non-unique choice) $f(\phi) = g(\phi) = \ln(K + \sin \phi)$ with $K = \text{constant} > 1$, we find that (19), (20), (21) and (22) lead to

$$A = \frac{2\pi}{r_c} \left[\alpha_3 r_c^2 (2K^2 + 1) + \alpha_1 l^2 \right], \quad (23)$$

$$B = \frac{\pi}{2r_c} \left[\alpha_3 (4K^2 + 1) - \frac{\alpha_1 l^2}{2r_c^2} \right], \quad (24)$$

$$C = -4r_c^2 E = \frac{4\pi \alpha_3 l^2}{r_c}. \quad (25)$$

Taking into account that $L_{ChS}^{(5)}[e, h]$ flows into $L_{EH}^{(5)}$ when $l \rightarrow 0$ [5], we have that action (18) should lead to the action of Einstein–Hilbert when $l \rightarrow 0$. From (18) it is direct to see that this occurs when $A = -1/2$ and $B = \Lambda/12$, where Λ is the cosmological constant. In this case, from Eqs. (23), (24) and (25), it is direct to see that

$$\alpha_1 = -\frac{r_c}{4\pi l^2} \left[1 + \frac{(4r_c^2 \Lambda - 3)(2K^2 + 1)}{3(10K^2 + 3)} \right], \quad (26)$$

$$\alpha_3 = \frac{1}{12\pi r_c} \frac{(4r_c^2 \Lambda - 3)}{(10K^2 + 3)}, \quad (27)$$

$$C = -4r_c^2 E = \frac{l^2}{3r_c^2} \frac{(4r_c^2 \Lambda - 3)}{(10K^2 + 3)}, \quad (28)$$

and, therefore the action (18) takes the form

$$\begin{aligned} \tilde{S}[\tilde{e}, \tilde{h}] &= \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \left(-\frac{1}{2} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + \frac{\Lambda}{12} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right. \\ &\quad \left. + C \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q - \frac{C}{4r_c^2} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q \right), \end{aligned} \quad (29)$$

corresponding to a 4-dimensional brane embedded in the 5-dimensional space-time of the Einstein–Chern–Simons gravity. We can see that, when $l \rightarrow 0$ then $C \rightarrow 0$, and hence (29) becomes the 4-dimensional Einstein–Hilbert action with cosmological constant.

3 An application to cosmology

In this section, we will find De Sitter cosmological solutions associated with the action (29). We start our analysis by writing the action (29) in tensorial language. The first two terms of the Lagrangian (29) can be written as

$$\begin{aligned}\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{e}^p\tilde{e}^q &= -2\sqrt{-\tilde{g}}\tilde{R}d^4\tilde{x}, \\ \tilde{\varepsilon}_{mnpq}\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q &= -24\sqrt{-\tilde{g}}d^4\tilde{x},\end{aligned}\quad (30)$$

where $\tilde{g}$ is the determinant of the 4-dimensional metric tensor $\tilde{g}_{\mu\nu}$, and $\tilde{R}$ is the 4-dimensional Ricci scalar. The two remaining terms are obtained after some calculations. The results are given by

$$\begin{aligned}\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{e}^p\tilde{h}^q &= 2\sqrt{-\tilde{g}}\left(\tilde{R}\tilde{h} - 2\tilde{R}_\nu^\mu\tilde{h}_\mu^\nu\right)d^4\tilde{x}, \\ \tilde{\varepsilon}_{mnpq}\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{h}^q &= 6\sqrt{-\tilde{g}}\tilde{h}d^4\tilde{x},\end{aligned}\quad (31)$$

where $\tilde{h}^m = \tilde{h}_\mu^m d\tilde{x}^\mu$ and $\tilde{h} = \tilde{h}_\mu^\mu$. Introducing (30, 31) in the Lagrangian (29) we obtain

$$\begin{aligned}\tilde{S}[\tilde{g}, \tilde{h}] &= \int d^4\tilde{x}\sqrt{-\tilde{g}} \\ &\quad \left[\left(\tilde{R} - 2\Lambda\right) + 2C\left(\tilde{R}\tilde{h} - 2\tilde{R}_\nu^\mu\tilde{h}_\mu^\nu\right) + 6E\tilde{h}\right].\end{aligned}\quad (32)$$

If we consider a maximally symmetric space-time (for instance, the de Sitter's space), the equation 13.4.6 of Ref. [8] allows us to write

$$\tilde{h}_{\mu\nu} = \frac{\tilde{F}(\tilde{\varphi})}{4}\tilde{g}_{\mu\nu},\quad (33)$$

where $\tilde{F}$ is an arbitrary function of the 4-scalar field $\tilde{\varphi} = \tilde{\varphi}(\tilde{x})$. This means

$$\tilde{R}_\nu^\mu\tilde{h}^\nu_\mu = \frac{\tilde{F}(\tilde{\varphi})}{4}\tilde{R}, \quad \tilde{h} = \tilde{h}_{\mu\nu}\tilde{g}^{\mu\nu} = \tilde{F}(\tilde{\varphi}),\quad (34)$$

so that the action (32) takes the form

$$\begin{aligned}\tilde{S}[\tilde{g}, \tilde{\varphi}] &= \int d^4\tilde{x}\sqrt{-\tilde{g}} \\ &\quad \times \left[\left(\tilde{R} - 2\Lambda\right) + C\tilde{R}\tilde{F}(\tilde{\varphi}) + 6E\tilde{F}(\tilde{\varphi})\right],\end{aligned}\quad (35)$$

which corresponds to an action for the 4-dimensional gravity coupled non-minimally to a scalar field. Note that this action has the form

$$\tilde{S}_B = \tilde{S}_g + \tilde{S}_{g\varphi} + \tilde{S}_\varphi,\quad (36)$$

where, $\tilde{S}_g$ is a pure gravitational action term, $\tilde{S}_{g\varphi}$ is a non-minimal interaction term between gravity and a scalar field, and $\tilde{S}_\varphi$ represents a kind of scalar field potential. In order to write down the action in the usual way, we define the constant ε and the potential $V(\varphi)$ as (removing the symbols $\sim$ in (35))

$$\varepsilon = -\frac{\kappa C}{3E}, \quad V(\varphi) = -\frac{3E}{\kappa}F(\varphi),\quad (37)$$

where κ is the gravitational constant. This permits to rewrite the action for a 4-dimensional brane non-minimally coupled to a scalar field, immersed in a cylindrical 5-dimensional space-time as

$$S[g, \varphi] = \int d^4x\sqrt{-g} \{(R - 2\Lambda) + \varepsilon R V(\varphi) - 2\kappa V(\varphi)\}.\quad (38)$$

The corresponding field equations are obtained by varying the action (38) and equalizing it to zero. In fact

$$\begin{aligned}\delta\mathcal{L} &= \sqrt{-g}\left[G_{\mu\nu}(1 + \varepsilon V) + \Lambda g_{\mu\nu} + \kappa g_{\mu\nu}V\right]\delta g^{\mu\nu} \\ &\quad - 2\kappa\sqrt{-g}\frac{\partial V}{\partial\varphi}\left(1 - \frac{\varepsilon R}{2\kappa}\right)\delta\varphi = 0,\end{aligned}\quad (39)$$

where no surface terms have been included. This means that equations describing the behavior of the 4-dimensional brane in the presence of the scalar field φ are given by

$$G_{\mu\nu}(1 + \varepsilon V) + \Lambda g_{\mu\nu} = -\kappa g_{\mu\nu}V,\quad (40)$$

$$\frac{\partial V}{\partial\varphi}\left(1 - \frac{\varepsilon R}{2\kappa}\right) = 0.\quad (41)$$

Note that if $\varepsilon \rightarrow 0$, then the scalar field potential behaves as a constant V_0 , and the Einstein's vacuum field equations with (a modified) cosmological constant are recovered:

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} = 0,\quad (42)$$

where $\Lambda_0 = (\Lambda + \kappa V_0) / (1 + \varepsilon V_0) \approx (\Lambda + \kappa V_0)$.

In order to construct a model of universe based on Eqs. (40)–(41), we consider the Friedmann–Lemaître–Robertson–Walker metric

$$\begin{aligned}ds^2 &= -dt^2 + a(t)^2 \\ &\quad \times \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\psi^2)\right],\end{aligned}\quad (43)$$

where $a(t)$ is the so called “scale factor” and $k = 0, +1, -1$ describes flat, spherical and hyperbolic spatial geometries, respectively. Following the usual procedure, we find that the Friedmann–Lemaître–Robertson–Walker type equations, are given by

$$3\frac{\dot{a}^2 + k}{a^2}(1 + \varepsilon V) - \Lambda + 3\varepsilon\frac{\dot{a}}{a}\dot{\varphi}\frac{\partial V}{\partial\varphi} = \kappa V,\quad (44)$$

$$\begin{aligned}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)(1 + \varepsilon V) - \Lambda \\ + \varepsilon\left[\dot{\varphi}^2\frac{\partial^2 V}{\partial\varphi^2} + \left(\ddot{\varphi} + 2\frac{\dot{a}}{a}\dot{\varphi}\right)\frac{\partial V}{\partial\varphi}\right] = \kappa V,\end{aligned}\quad (45)$$

$$\frac{\partial V}{\partial\varphi}\left[1 - \frac{3\varepsilon}{\kappa}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)\right] = 0.\quad (46)$$

Assuming that $\partial V/\partial\varphi \neq 0$, Eq. (46) has the solutions

$$a(t) = \begin{cases} \sqrt{\frac{6\varepsilon}{\kappa}} \sinh\left(\sqrt{\frac{\kappa}{6\varepsilon}}t\right), & k = -1; \\ C_1 \exp\left(\sqrt{\frac{\kappa}{6\varepsilon}}t\right), & k = 0; \\ \sqrt{\frac{6\varepsilon}{\kappa}} \cosh\left(\sqrt{\frac{\kappa}{6\varepsilon}}t\right), & k = +1; \end{cases} \quad (47)$$

representing the maximally symmetric De Sitter space-time. Here, $\varepsilon > 0$ and C_1 is an arbitrary constant, which can be chosen equal to 1.

In the flat case $k = 0$, Eqs. (44) and (45) can be solved exactly for φ . Replacing $a = e^{\sqrt{\kappa/6\varepsilon}t}$ in (44), and assuming that V has the form

$$V(\varphi) = \lambda\varphi^n, \quad (48)$$

where λ and n are constants, we obtain the following equation

$$\left(\frac{\kappa}{2\varepsilon} - \Lambda\right) + \sqrt{\frac{\kappa}{2}}V \left[\sqrt{3\varepsilon n} \frac{\dot{\varphi}}{\varphi} - \sqrt{\frac{\kappa}{2}} \right] = 0, \quad (49)$$

which has as solution

$$\Lambda = \frac{\kappa}{2\varepsilon}, \quad \frac{\dot{\varphi}}{\varphi} = \frac{1}{n} \sqrt{\frac{\kappa}{6\varepsilon}} = \frac{1}{n} \sqrt{\frac{\Lambda}{3}}, \quad (50)$$

and therefore

$$\varphi(t) = C_2 e^{(1/n)\sqrt{\Lambda/3}t}, \quad (51)$$

where C_2 is an arbitrary constant which can be chosen equal to 1.

On the other hand, if we replace $a = e^{\sqrt{\kappa/6\varepsilon}t}$, (48), (50) and (51) in Eq. (45), we see that the equality $0 = 0$ is satisfied, which means that no constraints are imposed on the constants n and λ in the scalar field potential V .

Consequently, the solution of the system of equations (44)–(46) is given by

$$a(t) = e^{\sqrt{\Lambda/3}t}, \quad \varphi(t) = e^{(1/n)\sqrt{\Lambda/3}t}. \quad (52)$$

This solution describes an accelerated flat expanding universe in the presence of a scalar field that exhibits an exponential behavior.

Observe that solution (51) is also valid for late time cosmology in the cases with spherical and hyperbolic spatial geometries, namely $k = \pm 1$, where $\frac{\dot{a}}{a} \approx \sqrt{\frac{\kappa}{6\varepsilon}} = \sqrt{\frac{\Lambda}{3}}$.

4 Concluding remarks

In this work, we have obtained an action and its corresponding field equations, for a 4-dimensional brane embedded in the 5-dimensional space-time of the Einstein–Chern–Simons theory of gravity. This framework leads to the 4-dimensional general relativity with cosmological constant in the limit $l \rightarrow 0$. We have also obtained an interpretation of the h^a

matter field present in the Einstein–Chern–Simons gravity action, which was related to a scalar field. As an application, the Friedmann–Lemaître–Robertson–Walker equations for cosmology were found and some De Sitter accelerated solutions were obtained, considering a scalar field potential with the form $V = \lambda\varphi^n$, where λ and n represent arbitrary constants.

In Ref. [9] it was found that the Lagrangian (2) can be written in the form (90) (see appendix). From (90) we can see that kinetic terms corresponding to the fields h^a and k^{ab} , absent in the Lagrangian (2), are present in the surface term of the Lagrangian (90) through equation (91). This situation is common to all Chern–Simons theories. This has the consequence that the action (38) does not have the kinetic term for the scalar field φ .

It could be interesting to add a kinetic term to the 4-dimensional brane action. In this case, the action (38) takes the form

$$S[g, \varphi] = \int d^4x \sqrt{-g} \left\{ (R - 2\Lambda) + \varepsilon R V(\varphi) - 2\kappa \left[\frac{1}{2} (\nabla_\mu \varphi) (\nabla^\mu \varphi) + V(\varphi) \right] \right\}. \quad (53)$$

The corresponding field equations are given by

$$G_{\mu\nu} (1 + \varepsilon V) + \Lambda g_{\mu\nu} + \varepsilon H_{\mu\nu} = \kappa T_{\mu\nu}^\varphi, \quad (54)$$

$$\nabla_\mu \nabla^\mu \varphi - \frac{\partial V}{\partial \varphi} \left(1 - \frac{\varepsilon R}{2\kappa} \right) = 0, \quad (55)$$

where $T_{\mu\nu}^\varphi$ is the energy-momentum tensor of the scalar field

$$T_{\mu\nu}^\varphi = \nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} \nabla^\lambda \varphi \nabla_\lambda \varphi + V \right), \quad (56)$$

and the rank-2 tensor $H_{\mu\nu}$ is defined as

$$H_{\mu\nu} = g_{\mu\nu} \nabla^\lambda \nabla_\lambda V - \nabla_\mu \nabla_\nu V. \quad (57)$$

Note that if $\varepsilon \rightarrow 0$, then the Einstein field equations with cosmological constant, in the presence of a scalar field, are recovered,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^\varphi, \quad \nabla_\mu \nabla^\mu \varphi - \frac{\partial V}{\partial \varphi} = 0. \quad (58)$$

Following the usual procedure, we find that the Friedmann–Lemaître–Robertson–Walker type equations are

$$3 \left(\frac{\dot{a}^2 + k}{a^2} \right) (1 + \varepsilon V) - \Lambda + 3\varepsilon \frac{\dot{a}}{a} \dot{\varphi} \frac{\partial V}{\partial \varphi} = \kappa \left(\frac{1}{2} \dot{\varphi}^2 + V \right), \quad (59)$$

$$\left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right) (1 + \varepsilon V) - \Lambda + \varepsilon \left[\dot{\varphi}^2 \frac{\partial^2 V}{\partial \varphi^2} + \left(\ddot{\varphi} + 2 \frac{\dot{a}}{a} \dot{\varphi} \right) \frac{\partial V}{\partial \varphi} \right] = -\kappa \left(\frac{1}{2} \dot{\varphi}^2 - V \right), \quad (60)$$

$$\ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} + \frac{\partial V}{\partial \varphi} \left[1 - \frac{3\varepsilon}{\kappa} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right) \right] = 0. \quad (61)$$

From here, we can see that in the limit $\varepsilon \rightarrow 0$, the Friedmann equations with a scalar field are recovered (see e.g. Ref. [10]):

$$3 \left(\frac{\dot{a}^2 + k}{a^2} \right) - \Lambda = \kappa \left(\frac{1}{2} \dot{\varphi}^2 + V \right), \quad (62)$$

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} - \Lambda = -\kappa \left(\frac{1}{2} \dot{\varphi}^2 - V \right), \quad (63)$$

$$\ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0. \quad (64)$$

In the case of a constant scalar field $\varphi = \varphi_0$, we have that $\ddot{\varphi} = \dot{\varphi} = 0$ and $V(\varphi_0) = V_0$. Then, we find the following solution for the system (59)–(61):

$$a(t) = \begin{cases} \sqrt{\frac{6\varepsilon}{\kappa}} \sinh \left(\sqrt{\frac{\kappa}{6\varepsilon}} t \right), & k = -1, \\ C \exp \left(\sqrt{\frac{\kappa}{6\varepsilon}} t \right), & k = 0, \\ \sqrt{\frac{6\varepsilon}{\kappa}} \cosh \left(\sqrt{\frac{\kappa}{6\varepsilon}} t \right), & k = +1, \end{cases} \quad (65)$$

where C is an arbitrary constant, $\varepsilon > 0$ and the constant scalar field potential takes the form

$$V_0 = \frac{\kappa - 2\varepsilon\Lambda}{\kappa\varepsilon}. \quad (66)$$

If we wish to interpret the scalar field φ as a Klein–Gordon particle, with potential $V(\varphi) = (m^2/2)\varphi^2$, then the mass of this field has to be given by

$$m = \frac{1}{|\varphi_0|} \sqrt{\frac{2(\kappa - 2\varepsilon\Lambda)}{\kappa\varepsilon}}, \quad (67)$$

where $\kappa - 2\varepsilon\Lambda > 0$.

On the other hand, in the case that $k = 0$ and the scalar field is non-constant, equations (59)–(61) take the form

$$3 \frac{\dot{a}^2}{a^2} (1 + \varepsilon V) - \Lambda + 3\varepsilon \frac{\dot{a}}{a} \dot{\varphi} \frac{\partial V}{\partial \varphi} = \kappa \left(\frac{1}{2} \dot{\varphi}^2 + V \right), \quad (68)$$

$$\left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) (1 + \varepsilon V) - \Lambda + \varepsilon \left[\dot{\varphi}^2 \frac{\partial^2 V}{\partial \varphi^2} + \left(\ddot{\varphi} + 2 \frac{\dot{a}}{a} \dot{\varphi} \right) \frac{\partial V}{\partial \varphi} \right] = -\kappa \left(\frac{1}{2} \dot{\varphi}^2 - V \right), \quad (69)$$

$$\ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} + \frac{\partial V}{\partial \varphi} \left[1 - \frac{3\varepsilon}{\kappa} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] = 0. \quad (70)$$

A solution for this system can be found by considering first

$$1 - \frac{3\varepsilon}{\kappa} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = 0 \quad (71)$$

in equation (70). The solution of (71) for the scale factor is given by

$$a(t) = C_1 e^{\sqrt{\kappa/6\varepsilon}t}, \quad (72)$$

where C_1 is an arbitrary constant and $\varepsilon > 0$.

Conversely, the replacement of (71) and (72) in (70), leads to the equation for the scalar field

$$\ddot{\varphi} + \frac{\kappa}{2\varepsilon} \dot{\varphi} = 0, \quad (73)$$

whose solution is

$$\varphi(t) = C_2 e^{-\sqrt{3\kappa/2\varepsilon}t} + C_3, \quad (74)$$

where C_2 and C_3 are arbitrary constants.

Choosing $C_1 = C_2 = 1$ and $C_3 = 0$, we arrive at the following solution for equation (70),

$$a(t) = e^{\sqrt{\kappa/6\varepsilon}t}, \quad \varphi(t) = e^{-\sqrt{3\kappa/2\varepsilon}t}, \quad (75)$$

which describes a De Sitter universe in exponential accelerated expansion, in presence of a scalar field that exhibits an exponential decay.

For consistency, the solution (75) must also be a solution of the equations (68) and (69). By replacing (75) in (68), and assuming that the potential of the scalar field is given by

$$V(\varphi) = \lambda \varphi^n, \quad (76)$$

with λ and n constants, we obtain the following equation

$$\left(\frac{\kappa}{2\varepsilon} - \Lambda \right) - \frac{\kappa}{2} \left[\frac{3\kappa}{2\varepsilon} \varphi^2 + (1 + 3n) V \right] = 0, \quad (77)$$

which is satisfied choosing the cosmological constant and the potential as follows

$$\Lambda = \frac{\kappa}{2\varepsilon},$$

$$V(\varphi) = -\frac{3\kappa}{2\varepsilon(1+3n)}\varphi^2 = -\frac{3\Lambda}{(1+3n)}\varphi^2. \quad (78)$$

According to equations (76) and (78), we see that

$$n = 2, \quad \lambda = -\frac{3\Lambda}{7}. \quad (79)$$

Observe that $\Lambda = \kappa/2\varepsilon > 0$, then $\lambda < 0$. Therefore, the scalar field φ , with potential

$$V(\varphi) = \lambda\varphi^n = -\frac{3\Lambda}{7}\varphi^2 = \frac{m^2}{2}\varphi^2, \quad (80)$$

might be identified with a tachyon, a hypothetical particle with imaginary mass, whose speed is higher than the speed of light. In this case, the mass of the particle is

$$m = i\sqrt{\frac{6\Lambda}{7}}. \quad (81)$$

Note that equation (69) is consistent with the results found. The solution (75) can be rewritten in terms of the cosmological constant as

$$a(t) = e^{\sqrt{\Lambda/3}t}, \quad \varphi(t) = e^{-\sqrt{3\Lambda}t}. \quad (82)$$

Finally, it is worth noting that in Ref. [11] a 5-dimensional Chern–Simons action for gravity, invariant under the Poincaré group enlarged by an Abelian ideal, was found. Equations and solutions found in the present article, obtained from Einstein–Chern–Simons gravity [5], can be easily replicated for the theory in Ref. [11], through the replacement $\alpha_1 = 0$ and $\alpha_3 = l = 1$.

It might be of interest to note some similarities and differences between the four-dimensional Lagrangian of the action (29) and the Lagrangian of the gravity theory based on the Maxwell algebra in four dimensions proposed by Azcarraga, Kamimura and Lukierski in Ref. [12].

The Lagrangian of the action (29) is obtained from the so called Einstein–Chern–Simons Lagrangian (90) for the $\mathfrak{B}$ Lie algebra (86), whose generators are given by $\{J_{ab}, P_a, Z_{ab}, Z_a\}$ [5].

The gravity theory of Ref. [12] is based on the Maxwell algebra whose generators are given by $\{J_{ab}, P_a, Z_{ab}\}$. This algebra can also be obtained from the anti de Sitter (AdS) algebra via the mentioned S-expansion procedure.

The corresponding one-form gauge connection and the two-form curvature are given by

$$A = e^a P_a + \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{2}k^{ab} Z_{ab},$$

$$F = T^a P_a + \frac{1}{2}R^{ab} J_{ab} + \frac{1}{2}\mathcal{F}^{ab} Z_{ab}, \quad (83)$$

where $T^a = de^a + \omega^a_c e^c = D_\omega e^a$, $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$, $\mathcal{F}^{ab} = D_\omega k^{ab} + \Lambda e^a e^b$, which allows us to construct the Lagrangians (29) of the Ref. [12]:

$$\mathcal{L} = -\frac{1}{2\kappa}\varepsilon_{abcd}R^{ab}e^ce^d + \frac{\lambda}{4\kappa}\varepsilon_{abcd}e^ae^be^ce^d$$

$$+ \frac{\mu}{2\kappa}\varepsilon_{abcd}(DA)^{ab}e^ce^d$$

$$+ \frac{\mu^2}{4\kappa\lambda}\varepsilon_{abcd}(DA)^{ab}(DA)^{cd}, \quad (84)$$

where $\mathcal{F}^{ab} \equiv F^{ab}$ and $k^{ab} \equiv A^{ab}$.

Since Maxwell's algebra is contained in algebra $\mathfrak{B}$, it would be reasonable to expect that the action (84) should be a particular case of the action (29). However, this fact is not observed when the corresponding Lagrangians are compared. The reason for this discrepancy is due to the fact that we have imposed the freedom of torsion condition on Lagrangian (2). The imposition of this condition eliminates from the Lagrangian (2) the field $k^{ab} \equiv A^{ab}$. Comparison of the Lagrangians (29) and (84) shows that they have in common only the Einstein–Hilbert and the cosmological terms. An interesting problem could be consider equation (2) without imposing the torsion-free condition, carry out the same procedure that leads to a Lagrangian analogous to Lagrangian (29) and then comparing it with the Lagrangian (84).

It may be also of interest to note that although the two Lagrangians contain a cosmological term, they have different origins. The cosmological term of the Lagrangian (84) has its origin in the commutation relation $[P_a, P_b] = Z_{ab}$, this means in the field k^{ab} , while the cosmological term of the Lagrangian (29) appears when the 5-dimensional Randall–Sundrum metric is introduced.

It might be of interest to note that the fields k^{ab} , associated with the generators Z_{ab} , plays a role similar to that played by the field A^{ab} in Ref. [12], that is, to introduce a cosmological term in an alternative way that generalizes standard gravity. However, as we have already mentioned, this term is not present in our approach. On the other hand, the h^a field, associated with the generators Z_a , plays the role of a scalar field in the case of maximally symmetric space-times.

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5 Appendix

A ChS Lagrangian in $d = 2n + 1$ dimensions is defined to be the following local function of a one-form gauge connection A :

$$\begin{aligned} L_{ChS}^{(2n+1)}(A) &= (n+1)k \int_0^1 dt \left\langle A \left(t dA + t^2 A^2 \right)^n \right\rangle, \\ &= (n+1)k \int_0^1 dt \left\langle A \left(t F + (t^2 - t) A^2 \right)^n \right\rangle \end{aligned} \quad (85)$$

where $\langle \dots \rangle$ denotes an invariant tensor for the corresponding Lie algebra, $F = dA + AA$ is the corresponding two-form curvature and k is a constant [13].

Some time ago was shown that the standard, five-dimensional General Relativity can be obtained from Chern–Simons gravity theory for a certain Lie algebra $\mathfrak{B}$ [5], whose generators $\{J_{ab}, P_a, Z_{ab}, Z_a\}$ satisfy the commutation relationships

$$\begin{aligned} [J_{ab}, J_{cd}] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \\ [J_{ab}, P_c] &= \eta_{cb} P_a - \eta_{ca} P_b \\ [P_a, P_b] &= Z_{ab} \\ [J_{ab}, Z_{cd}] &= \eta_{cb} Z_{ad} - \eta_{ca} Z_{bd} + \eta_{db} Z_{ca} - \eta_{da} Z_{cb} \\ [J_{ab}, Z_c] &= \eta_{cb} Z_a - \eta_{ca} Z_b \\ [Z_{ab}, P_c] &= \eta_{cb} Z_a - \eta_{ca} Z_b \end{aligned} \quad (86)$$

This algebra was obtained from the anti de Sitter (AdS) algebra and a particular semigroup S by means of the S-expansion procedure introduced in Refs. [6], [7].

In order to write down a Chern–Simons lagrangian for the $\mathfrak{B}$ algebra, we start from the one-form gauge connection

$$A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a, \quad (87)$$

and the two-form curvature

$$\begin{aligned} F &= \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2} \left(D_\omega k^{ab} + \frac{1}{l^2} e^a e^b \right) Z_{ab} \\ &\quad + \frac{1}{l} \left(D_\omega h^a + k^a_b e^b \right) Z_a. \end{aligned} \quad (88)$$

A Chern–Simons lagrangian in $d = 5$ dimensions is defined to be the following local function of a one-form gauge connection A :

$$L_{ChS}^{(5)}(A) = k \left\langle A F^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right\rangle, \quad (89)$$

where $\langle \dots \rangle$ denotes an invariant tensor for the corresponding Lie algebra, $F = dA + AA$ is the corresponding two-form curvature and k is a constant [13].

Using theorem VII.2 of Ref. [6], and the extended Cartan's homotopy formula as in Ref. [14], and integrating by parts, it is possible to write down the Chern–Simons Lagrangian in five dimensions for the $\mathfrak{B}$ algebra as [5], [9]

$$\begin{aligned} L_{EChS}^{(5)} &= \alpha_1 l^2 \epsilon_{abcde} R^{ab} R^{cd} e^e \\ &\quad + \alpha_3 \epsilon_{abcde} \\ &\quad \left(\frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right) \\ &\quad + dB_{EChS}^{(4)} \end{aligned} \quad (90)$$

where the surface term $B_{EChS}^{(4)}$ is given by

$$\begin{aligned} B_{EChS}^{(4)} &= \alpha_1 l^2 \epsilon_{abcde} e^a \omega^{bc} \left(\frac{2}{3} d\omega^{de} + \frac{1}{2} \omega_f^d \omega^{fe} \right) \\ &\quad + \alpha_3 \epsilon_{abcde} \\ &\quad \left[l^2 \left(h^a \omega^{bc} + k^{ab} e^c \right) \left(\frac{2}{3} d\omega^{de} + \frac{1}{2} \omega_f^d \omega^{fe} \right) \right. \\ &\quad \left. + l^2 k^{ab} \omega^{cd} \left(\frac{2}{3} de^e + \frac{1}{2} \omega_f^d e^e \right) + \frac{1}{6} e^a e^b e^c \omega^{de} \right] \end{aligned} \quad (91)$$

and where α_1, α_3 are parameters of the theory, l is a coupling constant, $R^{ab} = d\omega^{ab} + \omega_b^a$ corresponds to the curvature 2-form in the first-order formalism related to the 1-form spin connection, and e^a, h^a and k^{ab} are other gauge fields presents in the theory.

Finally it might be necessary to notice that:

(a) The lagrangian is split into two independent pieces, one proportional to α_1 and the other to α_3 . The piece proportional to α_1 corresponds to the Inönü–Wigner contraction of the Chern–Simons Lagrangian corresponding to AdS -algebra, and therefore it is the Chern–Simons Lagrangian for the Poincaré–Lie algebra $ISO(4, 1)$. The piece proportional to α_3 contains the Einstein–Hilbert term $\epsilon_{abcde} R^{ab} e^c e^d e^e$ plus non-linear couplings between the curvature and the bosonic “matter” fields k^{ab} and h^a , where the parameter l^2 can be interpreted as a kind of coupling constant.

(b) When the constant α_1 vanishes, the lagrangian (90) almost exactly matches the one given in Ref. [11], the only difference being that in our case the coupling constant l^2 appears explicitly in the last two terms.

The presence or absence of the coupling constant l in the lagrangian could seem like a minor or trivial matter, but it is not. As the authors of Ref. [11] clearly state, the presence of the Einstein–Hilbert term in this kind of action does not guarantee that the dynamics will be that of general relativity. In general, extra constraints on the geometry do appear, even around a “vacuum” solution with $k^{ab} = h^a = 0$. In fact, the variation of the lagrangian, modulo boundary terms, can be written as

$$\begin{aligned} \delta L_{CS}^{(5)} = & \epsilon_{abcde} \\ & \left(2\alpha_3 R^{ab} e^c e^d + \alpha_1 l^2 R^{ab} R^{cd} + 2\alpha_3 l^2 D_\omega k^{ab} R^{cd} \right) \delta e^e \\ & + \alpha_3 l^2 \epsilon_{abcde} R^{ab} R^{cd} \delta h^e + 2\epsilon_{abcde} \delta \omega^{ab} \\ & \left(\alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D k^{cd} T^e + \alpha_3 e^c e^d T^e \right. \\ & \left. + \alpha_3 l^2 R^{cd} D h^e + \alpha_3 l^2 R^{cd} k_f^e e^f \right. \\ & \left. + 2\alpha_3 l^2 \epsilon_{abcde} \delta k^{ab} R^{cd} T^e \right). \end{aligned} \quad (92)$$

Therefore, when the condition $\alpha_1 = 0$ is chosen, the torsionless condition imposed, and a solution without matter ($k^{ab} = h^a = 0$) is picked out, we are left with

$$\delta L_{CS}^{(5)} = 2\alpha_3 \epsilon_{abcde} R^{ab} e^c e^d \delta e^e + \alpha_3 l^2 \epsilon_{abcde} R^{ab} R^{cd} \delta h^e \quad (93)$$

In this way, besides general relativity equations of motions $\epsilon_{abcde} R^{ab} e^c e^d = 0$, the equations of motion of pure Gauss–Bonnet theory $\epsilon_{abcde} R^{ab} R^{cd} = 0$ do also appear as an anomalous constraint on the geometry.

It is at this point where the presence of the coupling constant l makes the difference. In the present approach, it does play the role of a coupling constant between geometry and “matter”. For this reason, in this case the limit $l \rightarrow 0$ leads to the Einstein–Hilbert term in the Lagrangian,

$$L_{CS}^{(5)} = \frac{2}{3} \alpha_3 \epsilon_{abcde} R^{ab} e^c e^d e^e. \quad (94)$$

In the same way, when we impose the weak limit of coupling constant, $l \rightarrow 0$, the extra constraints just vanish, and $\delta L_{CS}^{(5)} = 0$ lead us to just the Einstein–Hilbert dynamics in the vacuum,

$$\delta L_{CS}^{(5)} = 2\alpha_3 \epsilon_{abcde} R^{ab} e^c e^d \delta e^e + 2\alpha_3 \epsilon_{abcde} \delta \omega^{ab} e^c e^d T^e. \quad (95)$$

It is interesting to observe that the argument given here is not just a five-dimensional accident; at every odd dimension, it is possible to realize the S -expansion in the way sketched here, to take the weak limit of coupling constant, $l \rightarrow 0$, and recover Einstein–Hilbert gravity.

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