



# Spontaneously broken 3d Hietarinta/Maxwell Chern–Simons theory and minimal massive gravity

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**Abstract** We show that minimal massive 3d gravity (MMG) of (Bergshoeff et al. in *Class Quantum Grav* 31:145008, 2014), as well as the topological massive gravity, are particular cases of a more general ‘minimal massive gravity’ theory (with a single massive propagating mode) arising upon spontaneous breaking of a local symmetry in a Chern–Simons gravity based on a Hietarinta or Maxwell algebra. Similar to the MMG case, the requirements that the propagating massive mode is neither tachyon nor ghost and that the central charges of an asymptotic algebra associated with a boundary CFT are positive, impose restrictions on the range of the parameters of the theory.

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## 1 Introduction

Three-dimensional gravity theories have attracted great deal of attention since the early 80s as simpler tools for studying features of General Relativity in higher dimensions, its possible consistent modifications and extensions to quantum gravity. Since then a variety of different 3d gravity models with interesting geometric and physical properties have been constructed and analyzed. Among these is the minimal massive 3d gravity (MMG) [1] which will be the focus of our attention in this paper. This gravity model is a particular case of a class of Chern–Simons-like theories [2–4]. In contrast to the genuine 3d Chern–Simons theories which do not have local degrees of freedom in the bulk, the Chern–Simons-like gravities have propagating massive spin-2 modes coupled to a number of other spin-2 fields.<sup>1</sup>

One of the main motivations for constructing modifications of 4d General Relativity which include massive gravitons is to try to explain in this way the nature of dark matter and dark energy. Three-dimensional massive gravities serve as useful toy models for studying peculiar features and issues of these theories regarding e.g. the absence of Ostrogradski ghosts etc. An open fundamental question regarding gravity theories with massive gravitons is whether a spin-2 field mass can be attributed to spontaneous breaking of a space-time symmetry which in general can be an extension

<sup>1</sup> By “spin-2 fields” we somewhat loosely mean 3d Lorentz-vector-valued one-form fields  $a^{ra} = dx^\mu a_\mu^{ra}(x)$  ( $r = 1, 2, \dots, N$ ) which include a dreibein  $e^a(x)$  and a dualized spin connection  $\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc}$ .

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of the Poincaré group. To answer this question one should first individualize such a symmetry and then, ideally, find a mechanism generating mass of a corresponding spin-2 field similar to that of Englert–Brout–Higgs–Guralnik–Hagen–Kibble. By now, such a mechanism is not known for gauge spin-2 fields. In this situation one can resort to old constructions called Phenomenological Lagrangians (see e.g. [5–7]) which have proved useful for understanding the most general structure of symmetry breaking terms with the use of Goldstone fields on which these symmetries are realized non-linearly. A notable example is the first construction of the supergravity action with non-linearly realized local supersymmetry [8] (see [9] for a review and further developments).

In this paper we would like to address the above question for 3d Chern–Simons-like MMG of [1] and, in particular, to understand whether the presence of a massive spin-2 mode therein can be seen as an effect of (partial) spontaneous breaking of a local symmetry containing the 3d Poincaré group as a subgroup. We will show that this is indeed the case.<sup>2</sup>

The MMG contains three ‘spin-2’ fields, the dreibein  $e^a$ , the spin connection  $\omega^a$  and an additional one-form field  $h^a$ . The first two are associated with gauge fields of the local 3d Poincaré group generated by the translations  $P_a$  and Lorentz rotations  $J_a$ . We would also like to treat  $h^a$  as a gauge field associated with an additional vector generator  $Z_a$  that extends the Poincaré group to a larger symmetry which is however broken in the MMG action. We will restore this larger symmetry by coupling the gauge fields  $e^a$  and  $h^a$  to a Stückelberg-like spin-1 Goldstone field associated with spontaneous breaking of  $Z_a$ -symmetry. The symmetry algebra in question is the simplest among algebras constructed by Hietarinta [11], a class of finite-dimensional supersymmetry-like algebras containing higher-spin generators.<sup>3</sup> The commutators of the generators of this algebra are

$$\begin{aligned} [J^a, J^b] &= \epsilon^{abc} J_c, & [J^a, P^b] &= \epsilon^{abc} P_c, \\ [J^a, Z^b] &= \epsilon^{abc} Z_c, \\ [P_a, P_b] &= 0, & [Z^a, Z^b] &= \epsilon^{abc} P_c. \end{aligned} \quad (1.1)$$

Note that the commutator of  $Z_a$  closes on translations, somewhat similar to supersymmetry. Notice also that this algebra is isomorphic (dual) to the three-dimensional Maxwell algebra [21, 22] in which the role of the generators  $P_a$  and  $Z_a$  gets interchanged ( $P_a \leftrightarrow Z_a$ ), namely

$$[Z_a, Z_b] = 0, \quad [P^a, P^b] = \epsilon^{abc} Z_c. \quad (1.2)$$

<sup>2</sup> The broken symmetry under consideration is not a 3d Weyl symmetry which was assumed to be a source of the graviton mass in [10].

<sup>3</sup> The most studied example of the Hietarinta algebras is the one in which the spin-1/2 generators of a supersymmetry algebra are replaced by their spin-3/2 counterparts. This algebra underlies the so-called Hypergravity put forward in  $D = 2 + 1$  by Aragone and Deser [12] (see e.g. [13–20] for further studies of this theory).

The Chern–Simons action for gravity with the local symmetry generated by the 3d Maxwell algebra was constructed and studied in [23–26]<sup>4</sup> while its Hietarinta counterpart was considered in [18, 30]. Since from the algebraic point of view the construction of the action is the same for (1.1) and (1.2) and the only difference between the two is the choice of the physical interpretation of the generators and corresponding gauge fields, in what follows we will call the general model under consideration the Hietarinta/Maxwell Chern–Simons Gravity (HMCSG).

In Sects. 2 and 3 we will show that augmenting the HMCSG action with terms that break linearly realized symmetry (1.1) along  $Z_a$  one gets an extension of the Minimal Massive Gravity. It has, in general, two more coupling terms in comparison with the MMG, but still has a single massive propagating degree of freedom, as we show by performing the Hamiltonian analysis in Sect. 5 and studying linear perturbations of the fields around an  $AdS_3$  background in Sect. 7. In Sect. 4, as a side remark, we demonstrate that when the parameters of the HMCSG are restricted by a certain condition which makes its equations of motion integrable, the model reduces to a pure Chern–Simons theory with the gauge group  $SL(2, R) \times SL(2, R) \times SL(2, R)$ . In Section 6 we compute the central charges of an asymptotic symmetry algebra of the HMCSG with  $AdS_3$  boundary conditions. As in the MMG case, the requirements that the propagating massive mode is neither tachyon nor ghost and that the boundary CFT central charges are positive impose restrictions on the range of the parameters of the HMCSG theory. We analyze these restrictions for some particular cases for which the parameters of the HMCSG differ from those of the original MMG in Sect. 7, and conclude with comments and an outlook in Sect. 8.

## 2 Hietarinta/Maxwell Chern–Simons gravity and its minimal massive extension

Let us start by reviewing the construction of a gravity action which enjoys local symmetry transformations associated to the algebra (1.1). The algebra (1.1) has the following invariant bilinear form

$$\begin{aligned} \langle J_a, Z_b \rangle &= a\eta_{ab}, & \langle J_a, P_b \rangle &= \langle Z_a, Z_b \rangle = -\sigma m\eta_{ab}, \\ \langle J_a, J_b \rangle &= \eta_{ab}, \end{aligned} \quad (2.1)$$

where  $m$  is a parameter of the dimension of mass,  $a$  has the dimension of  $m^{\frac{1}{2}}$ , while  $(-\sigma)$  is an arbitrary dimensionless parameter.

<sup>4</sup> Higher-spin extensions of the Maxwell algebra and corresponding gravity models were considered in [27]. See also [28, 29] for a detailed study of the 3d Maxwell group, its infinite-dimensional extensions, applications and additional references.

sionless constant.<sup>5</sup> The dimensions of the coefficients reflect the canonical dimensions of  $[J_a] = m^0$ ,  $[P_a] = m$  and  $[Z_a] = m^{\frac{1}{2}}$ .

In the case of the Maxwell algebra (1.2) the dimension of  $Z_a$  changes to  $m^2$  and the corresponding bilinear form is

$$\begin{aligned} \langle J_a, P_b \rangle &= a m \eta_{ab}, & \langle J_a, Z_b \rangle &= \langle P_a, P_b \rangle = -\sigma m^2 \eta_{ab}, \\ \langle J_a, J_b \rangle &= \eta_{ab}, \end{aligned} \tag{2.2}$$

where now the parameter  $a$  is dimensionless.

The bilinear form (2.1) is used to construct the Chern–Simons action (in which the wedge product of the differential forms is implicit)

$$S = \frac{1}{2m} \int_{\mathcal{M}_3} \left\langle \mathbf{A} d\mathbf{A} + \frac{2}{3} \mathbf{A}^3 \right\rangle, \tag{2.3}$$

for the gauge field one-form  $\mathbf{A}$  taking values in the algebra (1.1)

$$\mathbf{A} = e^a P_a + \omega^a J_a + h^a Z_a. \tag{2.4}$$

Explicitly, for the components of (2.4) the action (2.3) takes the following form

$$\begin{aligned} S_{HCS} &= \frac{1}{2} \int_{\mathcal{M}_3} \left[ \frac{2a}{m} h^a R_a - \sigma (2e^a R_a + h^a \nabla h_a) \right. \\ &\quad \left. + \frac{1}{m} \left( \omega^a d\omega_a + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c \right) \right], \end{aligned} \tag{2.5}$$

where

$$\nabla h^a = dh^a + \varepsilon^{abc} \omega_b h_c, \quad R^a = d\omega^a + \frac{1}{2} \varepsilon^{abc} \omega_b \omega_c. \tag{2.6}$$

The Hietarinta Chern–Simons (HCS) action (2.5) is invariant (up to a boundary term) under the infinitesimal gauge transformations

$$\begin{aligned} \delta e^a &= \nabla \varepsilon^a_p + \varepsilon^{abc} (h^b \varepsilon^c_z + e^b \varepsilon^c_j), \\ \delta h^a &= \nabla \varepsilon^a_z + \varepsilon^{abc} h^b \varepsilon^c_j, \\ \delta \omega^a &= \nabla \varepsilon^a_j. \end{aligned} \tag{2.7}$$

Note that the term  $\frac{2a}{m} h^a R_a$  in (2.5) can be absorbed by the term  $2e^a R_a$  upon the field redefinition  $e^a \rightarrow e^a - \frac{a}{\sigma m} h^a$ . So, without loss of generality, instead of (2.5) we will deal with the action

$$\begin{aligned} S_{HCS} &= \frac{1}{2} \int_{\mathcal{M}_3} \left[ -\sigma (2e^a R_a + h^a \nabla h_a) \right. \\ &\quad \left. + \frac{1}{m} \left( \omega^a d\omega_a + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c \right) \right]. \end{aligned} \tag{2.8}$$

Also note that if instead of the Hietarinta algebra (1.1), we had used the Maxwell algebra (1.2) and the corresponding

bilinear form (2.2) as the basis for constructing the action (2.3), instead of (2.8) we would get

$$\begin{aligned} S_{MCS} &= \frac{1}{2} \int_{\mathcal{M}_3} \left[ 2a e^a R_a - m \sigma (2h^a R_a + e^a \nabla e_a) \right. \\ &\quad \left. + \frac{1}{m} \left( \omega^a d\omega_a + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c \right) \right]. \end{aligned} \tag{2.9}$$

In this action the role of the dreibein  $e^a$  (associated with the Poincaré translations) and of the additional spin-2 field  $h^a$  get interchanged in comparison to (2.8).<sup>6</sup> Now we can absorb the first term of (2.9) into its second term by redefining  $h^a \rightarrow h^a - \frac{a}{\sigma m} e^a$  and get

$$\begin{aligned} S_{MCS} &= \frac{1}{2} \int_{\mathcal{M}_3} \left[ -m \sigma (2h^a R_a + e^a \nabla e_a) \right. \\ &\quad \left. + \frac{1}{m} \left( \omega^a d\omega_a + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c \right) \right]. \end{aligned} \tag{2.10}$$

So, if one insists on associating the genuine graviton field with the Poincaré generator  $P_a$ , one concludes that the Maxwell Chern–Simons (MCS) gravity based on (2.10) actually does not have the standard Einstein term  $e^a R_a$ . In this respect the Maxwell Chern–Simons gravity (2.10) can be regarded as a deformation of the “exotic” Einstein gravity considered e.g. in [31]. The parity-odd first order action of the latter is obtained from (2.10) by removing its first term.

From the Chern–Simons structure of the action (2.3) it follows that the models under consideration do not have propagating degrees of freedom in the 3d bulk.

### 2.1 Breaking the Hietarinta symmetry

We would now like to generate non-trivial bulk dynamics (and mass) of fields in the above Hietarinta/Maxwell Chern–Simons model by adding to the action (2.8) terms which can be associated with a spontaneous breaking of the Hietarinta symmetry (1.1) down to its Poincaré subalgebra.

#### 2.1.1 Spontaneous breaking of the rigid symmetry

By the Goldstone’s theorem, the spontaneous breaking of a rigid (global) continuous symmetry is characterized by the appearance of massless Nambu–Goldstone fields associated with broken symmetry generators. In the case under consideration these are the vector generators  $Z_a$  and the corresponding Goldstone field is a vector field  $A_a(x)$  of mass dimension  $m$  [18] which should not be confused with the Chern–Simons one-form (2.4). The Goldstone vector field

<sup>5</sup> The minus sign in front of  $\sigma$  was chosen to make our convention closer to that of [1]. We will also set the value of the gravitational constant as  $16\pi G = 1$ .

<sup>6</sup> Notice that in the Maxwell case the dimension of  $h_a$  gets changed in comparison with the Hietarinta case in accordance with the change of the dimension of  $Z_a$  in (2.2).

appears in the Cartan one-form<sup>7</sup>

$$\begin{aligned} \Omega_0 &= g_0^{-1} dg_0 = E_0^a P_a + H_0^a Z_a, \\ E_0^a &= dx^a - \frac{f^{-2}}{2} \varepsilon^{abc} A_b dA_c, \\ H_0^a &= f^{-1} dA^a(x), \end{aligned} \tag{2.11}$$

where

$$g_0 = e^{x^a P_a} e^{f^{-1} A^a(x) Z_a}, \tag{2.12}$$

is a Hietarinta group element with  $x^a$  being a flat 3d space-time coordinate and  $f$  being a symmetry breaking parameter of mass-dimension  $m^{\frac{3}{2}}$ . The subscript 0 indicates that, at this moment, we are dealing with a rigid symmetry with respect to which the one-form (2.11) is invariant under the transformation

$$g_0 \rightarrow e^{\varepsilon^a J_a} e^{\varepsilon^p P_a} e^{\varepsilon^a Z_a} g_0, \tag{2.13}$$

where the parameters are  $x$ -independent. The spontaneously broken symmetry associated with  $\varepsilon_Z^a Z_a$  is realized on  $x^a$  and the Goldstone field  $A_a(x)$  infinitesimally as a non-linear transformation [18]

$$\begin{aligned} \delta x^a &= \frac{f^{-1}}{2} \varepsilon^{abc} \varepsilon_b Z_a(x), \\ \delta A^a &= f \varepsilon_Z^a - \frac{f^{-1}}{2} \varepsilon_{dbc} \varepsilon_Z^b A^c \partial^d A^a. \end{aligned} \tag{2.14}$$

The unique Lagrangian for  $A_a(x)$  with the minimal number of derivatives (up to two) which is invariant under (2.14) is of the Volkov–Akulov type and has the following form

$$\begin{aligned} S_1 &= \frac{\mu_1 f^2}{3!} \int \varepsilon_{abc} E_0^a E_0^b E_0^c \\ &= \mu_1 \int d^3x \left( f^2 + \frac{1}{2} \varepsilon^{abc} A_a \partial_b A_c \right. \\ &\quad \left. - \frac{f^{-2}}{8} \varepsilon^{abc} \varepsilon^{def} A_a A_d \partial_e A_b \partial_f A_c \right), \end{aligned} \tag{2.15}$$

where  $\mu_1$  is a dimensionless constant parameter.

Note that a would be third-order derivative term in (2.15) vanishes. Interestingly, the action (2.15) contains the Abelian Chern–Simons term for  $A_a$ , while the presence of the quartic term breaks  $U(1)$  gauge invariance of the CS action and makes propagating a scalar mode of  $A_a$  which happens to be of a Galileon type (see [18] for details). Therefore, the spontaneous breaking of the Hietarinta symmetry produces the vector Goldstone field which has only one dynamical degree of freedom.

Using the components of the Cartan form (2.11) one can also construct a Hietarinta-invariant term which is of the third

order in derivatives of  $A_a(x)$

$$S_2 = \mu_2 f^{\frac{5}{3}} \int \varepsilon_{abc} H_0^a E_0^b E_0^c, \tag{2.16}$$

where  $\mu_2$  is a dimensionless parameter. Modulo total derivatives, it has the following explicit form

$$S_2 = -\frac{\mu_2 f^{-\frac{10}{3}}}{8} \int \varepsilon_{abc} dA^a dA^b dA^c A^2.$$

Also note that two more possible contributions to the Goldstone field action are actually total derivatives

$$\begin{aligned} S_{3,4} &= \int \varepsilon_{abc} \left( \mu_3 f^{\frac{4}{3}} H_0^a H_0^b E_0^c + \mu_4 f H_0^a H_0^b H_0^c \right) \\ &= \int \varepsilon_{abc} \left( \mu_3 f^{-\frac{2}{3}} d(A^a dA^b E_0^c) + \mu_4 f^{-2} dA^a dA^b dA^c \right). \end{aligned} \tag{2.17}$$

To recapitulate, the actions (2.15)–(2.17) are manifestly invariant under Lorentz rotations, Poincaré translations and rigid Hietarinta symmetry (2.14). The last one acts as a (non-linear) shift on the Goldstone field  $A_a$  and thus is spontaneously broken by the vacuum solution  $A_a = 0$ .

### 2.1.2 Gauging the non-linearly realized symmetry

To couple the Goldstone field  $A_a(x)$  to the gauge fields (2.4), we should covariantize the Cartan form (2.11) which makes it invariant under the transformation (2.13) whose parameters are promoted to functions of the space-time coordinates  $x^\mu$ . The result is

$$\Omega = g^{-1} (d + \mathbf{A})g = E^a P_a + \omega^a J_a + H^a Z_a, \tag{2.18}$$

where now

$$g = e^{\phi^a(x) P_a} e^{f^{-1} A^a(x) Z_a}, \tag{2.19}$$

with  $\phi^a(x)$  being an arbitrary 3d vector function and

$$\begin{aligned} E^a &= e^a + \nabla \phi^a + f^{-1} \varepsilon^{abc} h_b A_c - \frac{f^{-2}}{2} \varepsilon^{abc} A_b \nabla A_c, \\ H^a &= h^a + f^{-1} \nabla A^a. \end{aligned} \tag{2.20}$$

The gauge group acts on  $\phi^a$  and  $A^a$  as follows

$$\begin{aligned} \delta \phi^a &= -\varepsilon_P^a - \varepsilon^{abc} (\varepsilon_{Zb} h_c + \varepsilon_{Jb} \phi_c), \\ \delta A^a &= -f \varepsilon_Z^a - \varepsilon^{abc} \varepsilon_{Jb} A_c. \end{aligned} \tag{2.21}$$

Combined with the variations of the gauge fields (2.7), the action of the gauge transformations on (2.20) reduces to their Lorentz rotations

$$\delta_J E^a = -\varepsilon^{abc} \varepsilon_{Jb} E_c, \quad \delta_J H^a = -\varepsilon^{abc} \varepsilon_{Jb} H_c, \tag{2.22}$$

leaving the one-forms (2.20) invariant under the action of the transformations along  $P_a$  and  $Z_a$ .

<sup>7</sup> For the details of the model see [18] which in turn is based on the Volkov–Akulov construction [32,33] of Lagrangians with spontaneously broken and non-linearly realized supersymmetry.

The following comment is now in order. The vector  $\phi^a(x)$  might be thought of as a Goldstone (Stückelberg) field associated with breaking of the local Poincaré translations. However, this “breaking” does not result in changing the number of the physical (on-shell) degrees of freedom of the dreibein  $e^a$ . The reason is that, in addition to the invariance under local Hietarinta symmetry, the Chern–Simons gravity action (2.5) is invariant under the 3d diffeomorphisms

$$x^\mu \rightarrow x^\mu + \zeta^\mu(x). \tag{2.23}$$

Under the diffeomorphisms the dreibein transforms as follows

$$\delta e^a = \nabla(\xi^\mu e^a_\mu) - \varepsilon^{abc}(\xi^\mu \omega_{\mu b})e_c + i_\xi \nabla e^a. \tag{2.24}$$

Comparing (2.24) with (2.7) we see that the first and the second term in (2.24) can be associated, respectively, with local Poincaré translations and Lorentz rotations. Regarding the third term, since on the mass shell  $\nabla e^a = -\varepsilon^{abc}h_b h_c$  this term can be associated with an  $\varepsilon^a_Z$  variation of  $e^a$ . Therefore, on the mass shell, the local Poincaré translations are a redundant symmetry and can be completely substituted with the 3d diffeomorphisms, while off the mass shell the local Poincaré translations can be used to set  $\phi^a = 0$ . Note that once this is done the flat space one-forms (2.11) are obtained from (2.20) by simply setting  $e^a = dx^a$  and  $h^a = 0$ .

We are now ready to generalize the actions (2.15)–(2.17) to describe gauge-invariant couplings of the Goldstone field  $A_a(x)$  to the spin-2 fields  $e^a$ ,  $h^a$  and  $\omega^a$  by replacing  $E^a_0$  and  $H^a_0$  with  $E^a$  and  $H^a$  defined in (2.20). We thus get the following symmetry breaking action

$$S_{sym.br.} = \frac{1}{2} \int_{\mathcal{M}_3} \varepsilon_{abc} \left( \frac{\Lambda_0}{3} E^a E^b E^c + \tilde{\beta} E^a E^b H^c + \tilde{\alpha} E^a H^b H^c + \frac{\tilde{\rho}}{3} H^a H^b H^c \right), \tag{2.25}$$

where  $\Lambda_0$ ,  $\tilde{\beta}$ ,  $\tilde{\alpha}$  and  $\tilde{\rho}$  are arbitrary coupling constants whose dimensions are determined by appropriate powers of the symmetry breaking parameter  $f$ . Note that the first Volkov–Akulov-like term in (2.25) generates a cosmological constant. Note also that, in contrast to (2.17), the last two terms in (2.25) are not total derivatives.

We will now show that the theory described by the sum of the actions (2.8) and (2.25) contains the Minimal Massive Gravity of [1]. The MMG and HMCSG actions are related to each other by a linear transformation of the three spin-2 fields when certain parameters in the latter are set to zero.

### 3 From spontaneously broken HMCSG to MMG

The action (2.25) contains the Goldstone fields  $\phi^a$  and  $A^a$  which, as usual, can be gauge fixed to zero by the correspond-

ing local symmetry transformations (2.21) with the parameters  $\varepsilon^a_P = -\phi^a$  and  $\varepsilon^a_Z = -A^a$ . In this (unitary) gauge the one-forms  $E^A$  and  $H^a$  reduce, respectively, to  $e^a$  and  $h^a$ , and we get the gauge-fixed action

$$S_{HMCSG} = \frac{1}{2} \int_{\mathcal{M}_3} \left( -\sigma (2e^a R_a + h^a \nabla h_a) + \frac{1}{m} (\omega^a d\omega_a + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c) + \frac{1}{2} \int_{\mathcal{M}_3} \varepsilon_{abc} \left( \frac{\Lambda_0}{3} e^a e^b e^c + \tilde{\beta} e^a e^b h^c + \tilde{\alpha} e^a h^b h^c + \frac{\tilde{\rho}}{3} h^a h^b h^c \right) \right), \tag{3.1}$$

whose residual symmetries are the 3d local Lorentz transformations and the diffeomorphisms.

On the other hand, in our conventions and notation the MMG action [1] has the following form

$$S_{MMG} = \frac{1}{2} \int_{\mathcal{M}_3} \left( -2\sigma e^a R_a + 2h^a \nabla e_a + \frac{1}{m} \left( \omega^a d\omega_a + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c \right) + \frac{1}{2} \int_{\mathcal{M}_3} \varepsilon_{abc} \left( \frac{\Lambda_0}{3} e^a e^b e^c + \alpha e^a h^b h^c \right) \right), \tag{3.2}$$

where again  $\sigma = \pm 1$ , and  $m$ ,  $\Lambda_0$  and  $\alpha$  are arbitrary (dimensionful) parameters, and the spin-2 fields are formally denoted in the same way as in (3.1) to simplify notation, though now  $h^a$  is dimensionless. Note that when  $\alpha = 0$  in (3.2), the action reduces to the first-order action for the Topologically Massive Gravity (TMG) [34,35] for which the requirement of positive energy of the massive spin-2 mode singles out the sign  $\sigma = -1$ , while for General Relativity  $\sigma = 1$  (see the discussion in [1]).

The difference between the actions (3.1) and (3.2) is obvious. However, we will now show that the MMG action is a particular case of (3.1) with three independent parameters. To this end, it is useful to notice that the actions are of a Chern–Simons-like type [1,2,4], i.e. they can be written in the following form

$$S = \int_{\mathcal{M}_3} \left( \frac{1}{2} g_{rs} a^r \cdot da^s + \frac{1}{6} f_{rst} a^r \cdot a^s \times a^t \right), \tag{3.3}$$

where  $a^r = (e^a, h^a, \omega^a)$  (i.e.  $r = 1, 2, 3$  stand, respectively, for  $r = e, h, \omega$ ), and  $g_{rs}$  and  $f_{rst}$  are symmetric tensors with constant components. In (3.3) we used the convenient 3d Lorentz-vector algebra notation [2]

$$(a^r \times a^s)^a = \varepsilon^{abc} a^r_b a^s_c, \quad a^r \cdot a^s = \eta^{ab} a^r_a a^s_b. \tag{3.4}$$

In the case of (3.1)  $g_{rs}$  and  $f_{rst}$  have the following non-zero components

$$g_{e\omega} = -\sigma, \quad g_{\omega\omega} = \frac{1}{m}, \quad g_{hh} = -\sigma,$$



$$\begin{aligned}
 f_{eee} &= \Lambda_0, & f_{\omega\omega\omega} &= \frac{1}{m}, & f_{hhh} &= \tilde{\rho}, \\
 f_{eeh} &= \tilde{\beta}, & f_{e\omega\omega} &= -\sigma, & f_{ehh} &= \tilde{\alpha}, & f_{\omega hh} &= -\sigma,
 \end{aligned}
 \tag{3.5}$$

while for MMG (3.2)

$$\begin{aligned}
 g_{e\omega} &= -\sigma, & g_{\omega\omega} &= \frac{1}{m}, & g_{eh} &= 2, \\
 f_{e\omega\omega} &= -\sigma, & f_{\omega\omega\omega} &= \frac{1}{m}, & f_{eh\omega} &= 1, \\
 f_{eee} &= \Lambda_0, & f_{ehh} &= \alpha.
 \end{aligned}
 \tag{3.6}$$

The matrix of the linear transformation of the fields

$$\tilde{a}^p = T^p_q a^q,
 \tag{3.7}$$

which relates (modulo a total derivative) the HMCSG tensor  $g_{pr}$  in (3.5) to the MMG one in (3.6)

$$g_{MMG} = T^T g_{HMCSG} T,
 \tag{3.8}$$

has the following form

$$T^p_q = \begin{pmatrix} 1 & -\frac{1}{m} & 0 \\ 0 & \frac{1}{\sqrt{-m\sigma}} & 0 \\ 0 & -\sigma & 1 \end{pmatrix}.
 \tag{3.9}$$

Note that the form of the matrix  $T$  requires  $m\sigma$  to be negative. This is related to the sign of  $g_{hh} = -2\sigma$  in the HMCSG case. This sign can be flipped by performing the parity transformation  $e^a \rightarrow -e^a$  and  $\sigma \rightarrow -\sigma$  in the HMCSG action (3.1).

Thus, upon performing the transformation (3.7) one brings the action (3.1) to the following form (in which, for simplicity, we remove ‘tilde’ over the redefined fields)

$$\begin{aligned}
 S_{HMCSG} &= \frac{1}{2} \int_{\mathcal{M}_3} \left( -2\sigma e^a R_a + 2h^a \nabla e_a + \frac{1}{m} (\omega^a d\omega_a \right. \\
 &\quad \left. + \frac{1}{3} \varepsilon_{abc} \omega^a \omega^b \omega^c) \right) \\
 &\quad + \frac{1}{2} \int_{\mathcal{M}_3} \varepsilon_{abc} \left( \frac{\Lambda_0}{3} e^a e^b e^c + \alpha e^a h^b h^c + \beta e^a e^b h^c \right. \\
 &\quad \left. + \frac{\rho}{3} h^a h^b h^c \right),
 \end{aligned}
 \tag{3.10}$$

where

$$\begin{aligned}
 \beta &= \frac{\tilde{\beta}}{\sqrt{-m\sigma}} - \frac{\Lambda_0}{m}, & \alpha &= -\frac{2\tilde{\beta}}{m\sqrt{-m\sigma}} - \frac{\tilde{\alpha}\sigma}{m} + \frac{\Lambda_0}{m^2} \sigma, \\
 \rho &= -\frac{\tilde{\rho}\sigma}{m\sqrt{-m\sigma}} + \frac{3\tilde{\beta}}{m^2\sqrt{-m\sigma}} + \frac{3\tilde{\alpha}\sigma}{m^2} - \frac{\Lambda_0 + m^2\sigma}{m^3}
 \end{aligned}
 \tag{3.11}$$

and the values of  $g_{rs}$  and  $f_{rqs}$  are

$$g_{e\omega} = -\sigma, \quad g_{\omega\omega} = \frac{1}{m}, \quad g_{eh} = 1,$$

$$\begin{aligned}
 f_{e\omega\omega} &= -\sigma, & f_{\omega\omega\omega} &= \frac{1}{m}, & f_{eh\omega} &= 1, \\
 f_{eee} &= \Lambda_0, & f_{ehh} &= \alpha, & f_{eeh} &= \beta, & f_{hhh} &= \rho.
 \end{aligned}
 \tag{3.12}$$

The action (3.10) reduces to the MMG action (3.2) when  $\beta = \rho = 0$ .

The equations of motion which follow from (3.10) are

$$\begin{aligned}
 -2\sigma R + 2\nabla h + \Lambda_0 e \times e + \alpha h \times h + 2\beta e \times h &= 0, \\
 2\nabla e + 2\alpha e \times h + \beta e \times e + \rho h \times h &= 0, \\
 -2\sigma \nabla e + \frac{2}{m} R + 2e \times h &= 0.
 \end{aligned}
 \tag{3.13}$$

Note that in order to have three independent dynamical equations, the coefficient of the gravitational Chern–Simons term, i.e.  $1/m$ , should be non-zero. A linear combination thereof brings the above equations to the form

$$\begin{aligned}
 2R + 2m(1 + \sigma\alpha)e \times h + \sigma m\beta e \times e + \sigma m\rho h \times h &= 0, \\
 2\nabla h + 2(m\sigma(1 + \sigma\alpha) + \beta)e \times h \\
 + (m\beta + \Lambda_0)e \times e + (m\rho + \alpha)h \times h &= 0, \\
 2\nabla e + 2\alpha e \times h + \beta e \times e + \rho h \times h &= 0.
 \end{aligned}
 \tag{3.14}$$

Upon the redefinition of the connection

$$\Omega = \omega + \alpha h + \frac{\beta}{2} e,
 \tag{3.15}$$

we have

$$\begin{aligned}
 2R(\Omega) + C_1 e \times e + C_2 e \times h + \frac{\rho}{2} C_3 h \times h &= 0, \\
 2\nabla(\Omega)h + C_3 e \times h + (\Lambda_0 + m\beta)e \times e + (m\rho - \alpha)h \times h &= 0, \\
 2\nabla(\Omega)e + \rho h \times h &= 0,
 \end{aligned}
 \tag{3.16}$$

where

$$\begin{aligned}
 C_1 &= \frac{1}{4} (\beta + 2m\sigma)^2 + \alpha(\Lambda_0 + m\beta) - m^2, \\
 C_2 &= 2 (\alpha (\beta + 2m\sigma) + m(1 + \alpha^2)) \\
 C_3 &= (\beta + 2m\sigma) + 2m\alpha.
 \end{aligned}
 \tag{3.17}$$

Note that in (3.16) and (3.17)  $\beta$  always appears in the combinations  $\Lambda_0 + m\beta$  and  $\beta + 2m\sigma$ . So effectively  $\beta$  shifts  $\Lambda_0$  and promotes  $\sigma = \pm 1$  to a fully-fledged continuous parameter that cannot be scaled away and may take zero value.

Taking the covariant derivative of these equations and comparing the results one finds that for consistency either

$$\alpha (\beta + 2m\sigma) + m(1 + \alpha^2) - \rho(\Lambda_0 + m\beta) = 0,
 \tag{3.18}$$

or

$$h \cdot e = 0.
 \tag{3.19}$$

The latter implies that  $h^a_\mu e^a_\nu$  is a symmetric tensor as in the MMG theory [1], for which the first option (3.18) reduces to

$$1 + \alpha\sigma = 0.
 \tag{3.20}$$

For  $\rho = 0$  the equations (3.16) take the form

$$\begin{aligned} 2R(\Omega) + C_1 e \times e + C_2 e \times h &= 0, \\ 2\nabla(\Omega)h + C_3 e \times h + (\Lambda_0 + m\beta)e \times e - \alpha h \times h &= 0, \\ 2\nabla(\Omega)e &= 0, \end{aligned} \tag{3.21}$$

Note that now among the five coefficients  $C_i$  ( $i = 1, 2, 3$ ),  $\Lambda + m\beta$  and  $\alpha$  only four are functionally independent and expressed in terms of four independent parameters  $\Lambda_0 + m\beta$ ,  $\alpha$ ,  $\beta + 2m\sigma$  and  $m$ .

We see that when  $\rho = 0$  the geometry is torsionless and, in addition, the first equation can be solved for  $h$  as in MMG (provided that  $C_2 = \alpha(\beta + 2m\sigma) + m(1 + \alpha^2) \neq 0$  and hence (3.19) is satisfied), but in our case there is still one more independent coupling constant  $\beta$  like in [36] [equations (A5)–(A8) therein].<sup>8</sup> Alternatively, if we would like to treat  $h$  as the dreibein, we can arrive at the torsionless condition by modifying the connection starting from the second equation in (3.14) and setting  $\Lambda_0 = 0$ .

As in the MMG case [1], solving the first equation in (3.21) for  $h$  we get

$$\begin{aligned} h_{\mu\nu} &= h_{\mu}^a e_{\nu}^b \eta_{ab} = -\frac{2}{C_2} \left( S_{\mu\nu} + \frac{C_1}{2} g_{\mu\nu} \right), \\ g_{\mu\nu} &= e_{\mu}^a e_{\nu}^b \eta_{ab}, \end{aligned} \tag{3.22}$$

where  $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$  is the 3d Schouten tensor. Substituting this solution into the second equation of (3.21) and expressing  $\Omega^a$  through  $e^a$  by solving the torsionless condition in (3.21) we get

$$\begin{aligned} C_{\mu\nu} + \left( \frac{C_3}{2} + \frac{\alpha C_1}{C_2} \right) G_{\mu\nu} \\ - \frac{1}{2} \left( C_3 C_1 - (\Lambda_0 + m\beta)C_2 + \frac{\alpha C_1^2}{C_2} \right) g_{\mu\nu} &= \frac{2\alpha}{C_2} J_{\mu\nu}, \end{aligned} \tag{3.23}$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $C_{\mu\nu} = \frac{1}{\sqrt{-\det g}} \varepsilon_{\mu}^{\tau\rho} \nabla_{\tau} S_{\rho\nu}$  is the Cotton tensor and  $J_{\mu\nu} = \frac{1}{2 \det g} \varepsilon_{\mu}^{\rho\sigma} \varepsilon_{\nu}^{\tau\eta} S_{\rho\tau} S_{\sigma\eta}$ . The above equation has the same form as the MMG metric field Eq. [1] containing three coefficients, which are now composed of four continuous parameters  $\Lambda_0 + m\beta$ ,  $\alpha$ ,  $m$  and  $\beta + 2m\sigma$ .

<sup>8</sup> Also, when  $\rho$  is non-zero, one can make a shift  $e \rightarrow e + ch$  (with an appropriate constant  $c$ ) such that for a certain range of the parameters the term  $h \times h$  disappears from the first equation of (3.16). Thus, one can solve it for  $h$ , but the geometry, in general, remains torsionful, due to the structure of the last two equations in (3.16). So it is not possible, in general, to solve these equations for  $\Omega$  in terms of the dreibein  $e$ . Still, as we will see below, also in the case with  $\rho \neq 0$  the theory has a single propagating bulk degree of freedom and can be studied perturbatively around an  $AdS_3$  vacuum, like the MMG.

#### 4 $SL(2, R) \times SL(2, R) \times SL(2, R)$ CS theory as a degenerate case of MMG and HMCSG

Though the main subject of this paper is the massive gravity theory whose fields satisfy the consistency condition (3.19), in this section we would like to elucidate the structure of the model for which the Eq. (3.18) holds, so the model has only five independent parameters. Then the Eqs. (3.14) or (3.16) are integrable in the sense that their covariant derivatives are identically zero without imposing the additional constraint (3.19) on the fields. This means that (3.14) become the Maurer–Cartan equations for the one-forms  $e^a$ ,  $h^a$  and  $\omega^a$  which should thus be the components of a Cartan form associated with a gauge group of rank 9. This group is semi-simple and should contain the 3d Lorentz group  $SL(2, R)$  as a subgroup. As such, the most reasonable candidate is  $SL(2, R) \times SL(2, R) \times SL(2, R)$ . A Chern–Simons gravity based on this group was considered in [37–39].

To show that this is indeed so, let us consider a simpler case in which  $\rho = 0$ . Then in Eq. (3.14), in which the remaining parameters satisfy the condition (3.18), we redefine the fields  $e^a$  and  $\omega^a$  as follows

$$\begin{aligned} h &\rightarrow \frac{1}{\alpha} h + \frac{m}{\alpha} (\alpha^2 - 1) (4\Lambda_0 \alpha^3 - m^2(1 + \alpha\sigma)^3(3\alpha\sigma - 1))^{-1/2} e, \\ \omega &\rightarrow \omega + 2m(1 + \alpha\sigma) (4\Lambda_0 \alpha^3 - m^2(1 + \alpha\sigma)^3(3\alpha\sigma - 1))^{-1/2} e - h, \\ e &\rightarrow 2\alpha(4\Lambda_0 \alpha^3 - m^2(1 + \alpha\sigma)^3(3\alpha\sigma - 1))^{-1/2} e, \end{aligned} \tag{4.1}$$

where we assume, without loss of generality, that the expression under the square root is positive. Then the Eq. (3.14) [with  $\rho = 0$  and  $\beta = -\frac{m}{\alpha}(1 + \sigma\alpha)^2$ ] take the following form

$$\begin{aligned} R + \frac{1}{2} e \times e &= 0, \\ \nabla e &= 0, \\ \nabla h - \frac{1}{2} h \times h + \frac{1}{2} e \times e &= 0. \end{aligned} \tag{4.2}$$

As one can easily check, these are the Maurer–Cartan equations for the one-form  $\mathbf{A} = \omega^a J_a + e^a P_a + h^a Z_a$  associated with the following linear combinations of the three sets  $T_1$ ,  $T_2$  and  $T_3$  of the generators of  $SL_1(2, R) \times SL_2(2, R) \times SL_3(2, R)$ , respectively:

$$J = T_1 + T_2 + T_3, \quad P = T_1 - T_2, \quad Z = -T_3. \tag{4.3}$$

In the general case (i.e. when  $\rho \neq 0$ ) the transformation of the fields to the form which results in Eq. (4.2) is much more cumbersome and we will not give it here.

We have thus found that the action (3.10) which produce the equations of motion (3.14) with the parameters satisfying the condition (3.18) is similar to that of [39]. Therefore in this case all the bulk degrees of freedom are pure gauge, as e.g. in the case of Gravity based on  $SL(2, R) \times SL(2, R)$ . Here we just have an additional  $SL(2, R)$  field. Of course,

the physical content of the theory depends on the boundary conditions which can be imposed on the components of  $\mathbf{A}$ . These boundary conditions determine for us which is the true dreibein and connection and asymptotic symmetries. For instance, we can associate them with those belonging to  $SL_1(2, R) \times SL_2(2, R)$  and then the third  $SL_3(2, R)$  gauge field completely decouples (see [37–39] for more details).

In summary, the particular choice of the parameters (3.18) in the HMCSG action does not break the Hietarinta/Maxwell symmetry but deforms it to  $SL(2, R) \times SL(2, R) \times SL(2, R)$ . This is similar to how the Poincaré symmetry gets deformed to the (A)dS symmetry by adding the cosmological term to the Einstein gravity action. On the other hand, since the Hietarinta/Maxwell algebra is a contraction of the  $sl(2, R) \times sl(2, R) \times sl(2, R)$  algebra, the HMCS action (2.5) can be obtained as the contraction limit of the  $SL(2, R) \times SL(2, R) \times SL(2, R)$  Chern–Simons action.

### 5 Hamiltonian analysis

We shall now sketch, following [1–3], the Hamiltonian analysis of the system described by the action (3.1) and show that it has one propagating degree of freedom as in the particular case of the MMG model.

Let us assume that the manifold  $\mathcal{M}_3$  on which the theory is defined can be presented as the product  $\mathbb{R} \times \Sigma$ , where  $\Sigma$  is a two-dimensional manifold with boundary parametrized by the coordinates  $x^i, i = 1, 2$ , while  $\mathbb{R}$  defines the temporal direction parametrized by  $x^0$ . Upon this splitting the general Chern–Simons-like action (3.3) takes the following form

$$S = \int_{\mathbb{R}} dx_0 \int_{\Sigma} d^2x \varepsilon^{ij} \left[ g_{rs} \dot{a}_i^r \cdot a_j^s + a_0^r \cdot (g_{rs} \partial_i a_j^s + \frac{1}{2} f_{rst} a_i^s \times a_j^t) \right], \tag{5.1}$$

where dot denotes the derivative with respect to  $x^0$  and  $\varepsilon^{0ij} \equiv \varepsilon^{ij}$ .

From the form of this action we see that the canonical momenta  $p_{ar}^i$  associated to  $a_i^{ar}$  are constrained to be linear combinations of the fields themselves

$$p_{ar}^i = \varepsilon^{ij} g_{rs} a_j^{as}.$$

Upon solving these constraints, one gets the equal-time Poisson (actually Dirac) brackets for the fields  $a_i^{ar}$

$$\{a_i^{ar}(x), a_j^{bq}(y)\} = \varepsilon_{ij} \eta^{ab} g^{rq} \delta^2(x - y),$$

where  $g^{rq}$  is the inverse of  $g_{rq}$ .

From the structure of (5.1) we also see that  $a_0^s$  plays the role of a Lagrange multiplier giving rise to 9 constraints

$$\varphi_r^a = \varepsilon^{ij} \left( g_{rs} \partial_i a_j^s + \frac{1}{2} f_{rst} a_i^s \times a_j^t \right)^a.$$

The corresponding constraint functional for arbitrary fields  $\chi_a^r$  with well defined variation has the following form

$$\varphi[\chi] = \int_{\Sigma} d^2x \chi^r \cdot \varepsilon^{ij} \left( g_{rs} \partial_i a_j^s + \frac{1}{2} f_{rst} a_i^s \times a_j^t \right) + \int_{\partial\Sigma} dx^i \chi^r \cdot a_r. \tag{5.2}$$

The Poisson brackets of these constraints have the following structure

$$\begin{aligned} \{\varphi[\chi], \varphi[\xi]\} &= \varphi[[\chi, \xi]] + \int_{\Sigma} d^2x \chi_a^r \xi_b^s \mathcal{P}_{rs}^{ab} \\ &\quad - \int_{\partial\Sigma} d\phi \chi^r \cdot (g_{rs} \partial_\phi \xi^s + f_{rst} a_\phi^s \times \xi^t), \end{aligned} \tag{5.3}$$

with  $[\chi, \xi]_t = f_{rst} \chi^r \times \xi^s$  and

$$\begin{aligned} \mathcal{P}_{rs}^{ab} &= f_{q[r}^t f_{s]p]t} \eta^{ab} \Delta^{pq} + 2 f_{r[s}^t f_{q]p]t} V^{ab,pq}, \\ V_{ab}^{pq} &= \varepsilon^{ij} a_{ia}^p a_{jb}^q, \quad \Delta^{pq} = V_{ab}^{pq} \eta^{ab}. \end{aligned} \tag{5.4}$$

The integration variable  $\phi$  parametrises a (compact) boundary  $\partial\Sigma$ .

The number of first- and second-class constraints for the model under consideration can be read off from the rank of the matrix  $\mathcal{P}$ , Eq. (5.4), in which one should insert the explicit expressions (3.12) for the tensors  $g_{rs}$  and  $f_{rqs}$ . Note that in (5.4) the indices are raised with the matrix  $g^{rs}$ . If we assume that (3.19) holds, we have an additional (secondary) constraint

$$\Delta^{eh} = 0. \tag{5.5}$$

Taking this into account, a straightforward computation shows that the first term in (5.4) vanishes and  $\mathcal{P}$  becomes degenerate

$$\mathcal{P} = (\alpha\beta - \rho(\Lambda_0 + m\beta) + m(1 + \alpha\sigma)^2) \begin{pmatrix} -V_{ab}^{hh} & V_{ab}^{he} & 0 \\ V_{ab}^{eh} & -V_{ab}^{ee} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.6}$$

Now one should also compute the Poisson brackets of the constraint (5.5) with  $\varphi(\chi)$ . Using a general formula of [2] one gets

$$\begin{aligned} \{\Delta^{eh}, \varphi[\chi]\} &= \varepsilon^{ij} \left( \nabla_i \chi^e \cdot h_j - \nabla_i \chi^h \cdot e_j \right) + \varepsilon^{ij} e_i \times h_j \\ &\quad \cdot \left( m\rho \chi^h + m\sigma(1 + \alpha\sigma) \chi^e \right) \\ &\quad - \varepsilon^{ij} h_i \times h_j \cdot \left( \alpha \chi^e + \rho \chi^h \right) + \varepsilon^{ij} e_i \times e_j \\ &\quad \cdot \left( (\Lambda_0 + m\beta + m\sigma(1 + \alpha\sigma)) \chi^e + \beta \chi^h \right) \end{aligned} \tag{5.7}$$

with

$$\nabla_i \chi = \partial_i \chi + \omega_i \times \chi. \tag{5.8}$$

As in the MMG [1] we thus have the  $(10 \times 10)$  matrix of the Poisson brackets of 10 constraints, i.e.  $\varphi_r^a$  and  $\Delta^{eh}$ , which has rank four. This implies that, if the coefficient in



front of the matrix (5.6) is non-zero, system has 6 first-class and 4 second class constraints which reduce the number of the phase-space physical degrees of freedom in  $a_i^{ar}$  to 2, i.e the system has a single bulk degree of freedom in the Lagrangian formulation.

When the coefficient in (5.6) is zero, which is equivalent to the choice (3.18), the constraint (5.5) is absent and one has 9 first-class constraints  $\varphi_r^a$  which reduce the number of bulk physical degrees of freedom to zero. In this case, as we discussed in Sect. 4, the considered system reduces to the Chern–Simons theory with the gauge group  $SL(2, R) \times SL(2, R) \times SL(2, R)$ .

### 6 $AdS_3$ background and the central charges of the asymptotic symmetry algebra

We shall now study properties of the HMCSG theory for field configurations whose geometry is asymptotically  $AdS_3$  and compute the corresponding centrally extended asymptotic symmetry algebra which underlies a dual  $CFT_2$ .

#### 6.1 $AdS_3$ solution of the HMCSG field equations

For the  $AdS_3$  background to satisfy the field Eq. (3.16) we take the following ansatz for the vevs of  $e$ ,  $h$  and  $\Omega$

$$\langle e \rangle := \bar{e}, \quad \langle h \rangle := mC\bar{e}, \quad \langle \Omega \rangle := \bar{\Omega} - \frac{\rho m^2 C^2}{2} \bar{e}, \tag{6.1}$$

where  $\bar{e}$  and  $\bar{\Omega}$  are  $AdS_3$  dreibein and connection, and  $C$  is a real dimensionless parameter.

Substituting this ansatz into Eq. (3.16) we find that, provided that  $C$  satisfies the cubic equation

$$\rho m^3 C^3 - (m\rho - \alpha)m^2 C^2 - (\beta + 2m\sigma(1 + \alpha\sigma))mC - (\Lambda_0 + m\beta) = 0, \tag{6.2}$$

which always has at least one real root, Eq. (3.16) reduce to those describing the  $AdS_3$  space

$$\bar{R}(\bar{\Omega}) + \frac{l^{-2}}{2} \bar{e} \times \bar{e} = 0, \quad \bar{\nabla} \bar{e} = 0, \tag{6.3}$$

where

$$\begin{aligned} l^{-2} \equiv -\Lambda &= \frac{\rho^2 m^4 C^4}{4} + \frac{\rho m^2 C^2}{2} (\beta + 2m\sigma(1 + \alpha\sigma)) \\ &\quad + 2mC(\beta\alpha + m(1 + \alpha\sigma)^2) \\ &\quad + \left( \frac{\beta^2}{4} + \Lambda_0\alpha + m\beta\sigma(1 + \alpha\sigma) \right) \\ &= \frac{1}{4} \left( \rho m^2 C^2 + \beta + 2m\sigma(1 + \alpha\sigma) \right)^2 \end{aligned}$$

$$\begin{aligned} &+ 2mC(\beta\alpha + m(1 + \alpha\sigma)^2) \\ &+ \Lambda_0\alpha - m^2(1 + \alpha\sigma)^2. \end{aligned} \tag{6.4}$$

$l^{-2}$  is assumed to be positive so that the cosmological constant  $\Lambda$  is negative.<sup>9</sup>

#### 6.2 Asymptotic symmetries and central charges

In [48] Brown and Henneaux studied asymptotic symmetry properties of the pure 3d GR with  $AdS_3$  boundary conditions. The local 3d Lorentz symmetry and 3d diffeomorphisms of GR give rise to six first-class constraints generating these symmetries. These can be split into two mutually commuting sets of generators corresponding to the  $SL(2, R) \times SL(2, R)$  group of the Chern–Simons formulation of the theory. When evaluated on an asymptotically  $AdS_3$  space, each set was shown to generate the Virasoro algebra with a nontrivial central extension. This analysis was generalized to 3D massive theories of gravity in [1, 4, 49–51] and to the Maxwell–Chern–Simons gravity in [52].

We will now carry out the computation of the centrally extended asymptotic symmetry algebra for the HMCSG theory, following closely the steps explained in detail in [49] and [1]. Consider the following combination of the constraints (5.2)

$$L_{\pm}[\chi] = \varphi_e[\chi^\mu e_\mu] + \varphi_h[\chi^\mu h_\mu] + a_{\pm} \varphi_\omega[\chi^\mu e_\mu], \tag{6.5}$$

in which the parameters in the brackets are field-dependent and  $\chi^\mu(x)$  are associated with the parameter of 3d diffeomorphisms.

For convenience we have defined  $\varphi_\omega$  for the spin connection  $\omega$  in (3.10). The constant parameters  $a_{\pm}$  should be properly tuned in order to make the Poisson bracket of  $L_+$  and  $L_-$  vanish. It can be shown that (6.5) are a combination of the first-class constraints, corresponding to the local 3d Lorentz transformations and diffeomorphisms [1, 49]. Using the general formula (5.3) one finds that for the  $AdS_3$  solution (6.1) the Poisson bracket of  $L_+$  and  $L_-$  reduces to

$$\begin{aligned} \{L_+[\chi], L_-[\eta]\} &= \varphi_\omega[[\chi, \eta]] \left( a_+ a_- + 2m^2 C(1 + \alpha\sigma) \right. \\ &\quad \left. + m\beta\sigma + m^3 C^2 \rho\sigma \right) \\ &\quad + (\varphi_e[[\chi, \eta]] + mC\varphi_h[[\chi, \eta]]) (a_+ + a_- + 2\alpha mC \\ &\quad + \beta + m^2 C^2 \rho). \end{aligned} \tag{6.6}$$

<sup>9</sup> A more general class of vacuum solutions in MMG including those with a positive cosmological constant were considered e.g in [40–47], in particular at a specific point called “merger point”. The merger point is a point in the space of the parameters of the theory at which for all values of  $C$  defined by the Eq. (6.2) the cosmological constant  $\Lambda$  (6.4) has a unique value. It would be of interest to study a similar class of vacuum solutions also in the HMCSG context.

Note that on the  $AdS_3$  solution (6.1) the second term in (5.3) vanishes. Also the boundary contribution [the last term in (5.3)] vanishes. To see this, one should take into account the linear redefinition which relates  $\Omega$  with  $\omega$  (3.15), the vacuum value of the  $\Omega$  spin connection (6.1), and the corresponding  $AdS_3$  asymptotic symmetry parameters  $\chi$  and  $\eta$  (see [49] for details). Requiring that the Poisson bracket (6.6) vanishes, we find that the parameters  $a_{\pm}$  should have the following values

$$a_{\pm} = \pm \frac{1}{l} - \alpha m C - \frac{\beta}{2} - \frac{C^2 m^2 \rho}{2}, \tag{6.7}$$

where  $l$  is the radius of the  $AdS_3$  background defined in (6.4). Using the general expression (5.3) once again, one also finds

$$\begin{aligned} \{L_{\pm}[\chi], L_{\pm}[\eta]\} &= \pm \frac{2}{l} L_{\pm}[[\chi, \eta]] \\ &\pm \frac{2}{l} \left( \sigma \pm \frac{1}{ml} + \alpha C + \frac{\beta}{2m} + \frac{m\rho C^2}{2} \right) \int_{\partial\Sigma} d\phi \chi \\ &\cdot \left( \partial_{\phi} \eta + \bar{\Omega}_{\phi} \times \eta \pm \frac{1}{l} \bar{e}_{\phi} \times \eta \right), \end{aligned} \tag{6.8}$$

where in order to get the boundary term expressed via the  $AdS_3$  spin connection  $\bar{\Omega}$ , we made use of (3.15) and (6.1). After expanding the asymptotic symmetry parameters  $\eta$  and  $\chi$  in Fourier modes, the commutation relations above represent two copies of the Virasoro algebra with central charges

$$c_{\pm} = \frac{3l}{2G} \left( \pm \frac{1}{ml} + \sigma + \frac{\beta}{2m} + \alpha C + \frac{m\rho C^2}{2} \right), \tag{6.9}$$

where to be in agreement with the Brown–Henneaux central charge expression [48] we have included the Newton’s constant by restoring  $1/16\pi G$  in the action (3.10).

For the boundary CFT associated with (6.8) to be unitary both central charges should be positive, which implies

$$\sigma + \frac{\beta}{2m} + \alpha C + \frac{m\rho C^2}{2} - \frac{1}{|ml|} > 0. \tag{6.10}$$

For certain choices of the parameters  $\alpha, \beta, \rho$  and  $\sigma = \pm 1$ , the above expressions reduce to those of pure GR [48], TMG [49] and MMG [1].

### 7 Linearized theory around an $AdS_3$ background

We shall now study, following [1, 53], the conditions on the parameters of our model for which the propagating mode is neither a tachyon nor a ghost. To this end let us consider perturbations around the  $AdS_3$  vacuum solution which are convenient to take as follows

$$\begin{aligned} e &= \bar{e} + k, & \Omega &= \bar{\Omega} - \frac{\rho m^2 C^2}{2} (\bar{e} + k) - mC\rho p + v, \\ h &= mC(\bar{e} + k) + p, \end{aligned} \tag{7.1}$$

where  $k, v$  and  $p$  denote infinitesimal excitations of the fields. Then, using the relation (6.2) and the definition (6.4) of  $l^{-2}$  we get the linearized equations for (3.16) as

$$\begin{aligned} \bar{\nabla} v + l^{-2} \bar{e} \times k + \bar{e} \times p \left( \beta\alpha + m(1 + \alpha\sigma)^2 - \rho(\Lambda_0 + m\beta) \right) &= 0, \\ \bar{\nabla} p + M \bar{e} \times p &= 0, \\ \bar{\nabla} k + \bar{e} \times v &= 0, \end{aligned} \tag{7.2}$$

where <sup>10</sup>

$$M = \frac{1}{2} \left( \beta + 2m\sigma(1 + \alpha\sigma) + 2mC(m\rho - \alpha) - 3m^2 C^2 \rho \right). \tag{7.3}$$

The integrability condition (3.19) for the above equations reduces to

$$\bar{e} \cdot p = 0.$$

Making the redefinition (assuming that  $|\ell M| \neq 1$ )

$$f_{\pm} = \pm l^{-1} k + \frac{\beta\alpha + m(1 + \alpha\sigma)^2 - \rho(\Lambda_0 + m\beta)}{(\pm l^{-1} - M)} p + v, \tag{7.4}$$

one diagonalizes two of the Eq. (7.2) and gets

$$\begin{aligned} \bar{\nabla} f_{\pm} \pm l^{-1} \bar{e} \times f_{\pm} &= 0, \\ \bar{\nabla} p + M \bar{e} \times p &= 0. \end{aligned} \tag{7.5}$$

The first two equations in (7.5) describe the linearized 3d Einstein gravity with a cosmological constant and the third equation describes the propagation of the spin-2 mode  $p$  with the mass  $\mathcal{M}$  given by

$$\mathcal{M}^2 = M^2 - l^{-2}.$$

In accordance with the general Hamiltonian analysis we thus see that the HMCSG model has exactly the same field content as the MMG. The no-tachyon condition is [1]

$$M^2 - l^{-2} > 0. \tag{7.6}$$

Let us now find the form of the action (3.10) up to the second order in perturbations. Upon taking into account the form of the transformation (3.15), the excitations (7.1) and the linear redefinition (7.4) one gets

$$\begin{aligned} S_2 &= \int_{\mathcal{M}_3} \lambda_+ \left( f_+ \bar{\nabla} f_+ + l^{-1} \bar{e} \cdot f_+ \times f_+ \right) \\ &+ \lambda_- \left( f_- \bar{\nabla} f_- - l^{-1} \bar{e} \cdot f_- \times f_- \right) \\ &+ \int_{\mathcal{M}_3} \frac{1}{m(1 - 2C)} \left( p \bar{\nabla} p + M \bar{e} \cdot p \times p \right), \end{aligned} \tag{7.7}$$

<sup>10</sup> Note that in the MMG case (i.e. when  $\beta = \rho = 0$ ) the value  $M = 0$  defines the merger point [40] for the values of the cosmological constant. This, however, is not the case anymore for  $\rho \neq 0$ .

where

$$\lambda_{\pm} = \frac{1}{2m} \mp \frac{l}{4m} (2m\sigma + \beta + 2mC\alpha + m^2C^2\rho). \tag{7.8}$$

The first two terms are two linearized  $SL(2, R)$  Chern–Simons terms. Comparing (6.9) with (7.8) we see that  $c_{\pm} = \pm 3\lambda_{\mp}/G$ .

The product of  $\lambda_+$  and  $\lambda_-$  is

$$\lambda_+\lambda_- = -\frac{l^2}{4}(1 - 2C). \tag{7.9}$$

If the product is negative, the first two terms describe the linearized pure GR as the difference of two  $SL(2, R)$  Chern–Simons terms. In the general case, however, the product may also have the positive sign, then the resulting theory can be interpreted as a kind of “exotic” GR with additional terms.<sup>11</sup> However,  $-\lambda_+$  and  $\lambda_-$  (7.8) are proportional to the central charges  $c_+$  and  $c_-$  (6.9) in the asymptotic algebra and if we require both central charges to be positive (6.10), then the product of  $\lambda_+$  and  $\lambda_-$  (7.9) must be negative and hence  $(1 - 2C) > 0$ . Note that at the chiral point of the theory, at which one of the boundary central charges vanishes,  $1 - 2C = 0$  and Eq. (7.7) becomes singular.

The last term in (7.7) describes the propagating massive spin-2 mode. The no-ghost condition implies (see [1] for details)

$$(1 - 2C)mM < 0. \tag{7.10}$$

We shall now consider in more detail three particular cases in which the values of the parameters differ from the original MMG.

### 7.1 $C = 0$

In this case the Eq. (6.2) reduces to the following relation

$$\Lambda_0 + m\beta = 0 \Rightarrow \Lambda_0 = -m\beta, \tag{7.11}$$

while (6.4) and (7.3) respectively simplify to

$$l^{-2} = \frac{1}{4}(\beta^2 + 4m\beta\sigma) = m^2 \left( \frac{\beta}{2m} + \sigma \right)^2 - m^2 > 0, \tag{7.12}$$

and

$$M = m \left( \frac{\beta}{2m} + \sigma + \alpha \right). \tag{7.13}$$

<sup>11</sup> Note that the CS action for GR corresponds to the  $SO(2, 2)$  bilinear form  $\langle J^a, P^a \rangle = \langle J_+^a, J_+^a \rangle - \langle J_-^a, J_-^a \rangle = \eta^{ab}$ , where  $J_{\pm}^a$  are two copies of  $SO(1, 2)$  generators, related to that of  $SO(2, 2)$  as  $J^a = J_+^a + J_-^a$  and  $P^a = J_+^a - J_-^a$ . One can use the additional bilinear form of the  $SO(2, 2)$  algebra given by  $\langle J^a, J^a \rangle = \langle P^a, P^a \rangle = c\eta^{ab}$  with a constant  $c$  to extend the GR action by the topological and torsion terms  $c(\omega d\omega + \frac{1}{3}\omega^3) + ce\nabla e$ . At the linearized level the sign of the product (7.9) depends on the value of the constant parameter  $c$ .

From (7.12) we have

$$\frac{\beta}{2m} + \sigma > 1 \quad \text{or} \quad \frac{\beta}{2m} + \sigma < -1. \tag{7.14}$$

The no-tachyon condition (7.6) takes the form

$$2\alpha \left( \frac{\beta}{2m} + \sigma \right) + 1 + \alpha^2 > 0. \tag{7.15}$$

In the action (7.7) we now have  $\lambda_+\lambda_- = -l^2/4 < 0$ . Hence, the first two terms are the difference of two linearized  $SL(2, R)$  Chern–Simons terms describing the linearized 3d Einstein gravity. The last term describes the propagating massive spin-2 mode whose no-ghost condition (7.10) requires

$$mM < 0 \Rightarrow \frac{\beta}{2m} + \sigma + \alpha < 0. \tag{7.16}$$

The positive central charge condition in the case  $C = 0$  is

$$\sigma + \frac{\beta}{2m} - \frac{1}{|ml|} > 0. \tag{7.17}$$

Now we would like to analyze consequences of the conditions (7.14)–(7.17). From (7.17) we see that  $\sigma + \frac{\beta}{2m} > 0$  which is compatible with the first choice in (7.14). Then (7.16) and (7.15) require that

$$\alpha < -1, \quad \left( \alpha + \frac{\beta}{2m} + \sigma \right)^2 > \left( \frac{\beta}{2m} + \sigma \right)^2 - 1. \tag{7.18}$$

So finally, the range of the parameters which satisfies the conditions (7.14)–(7.17) is

$$\begin{aligned} \frac{\beta}{2m} + \sigma > 1, \quad \alpha < -\sqrt{\left( \frac{\beta}{2m} + \sigma \right)^2 - 1} - \left( \frac{\beta}{2m} + \sigma \right), \\ \Lambda_0 = -m\beta, \end{aligned} \tag{7.19}$$

and  $\rho$  is arbitrary.

### 7.2 $\rho = 0$

In this case we have

$$\begin{aligned} \alpha m^2 C^2 - (\beta + 2m\sigma(1 + \alpha\sigma))mC - (\Lambda_0 + m\beta) &= 0, \\ l^{-2} = 2mC(\alpha\beta + m(1 + \alpha\sigma)^2) + \left( \frac{\beta^2}{4} + \alpha\Lambda_0 + m\beta\sigma(1 + \alpha\sigma) \right), \\ M = \frac{\beta}{2} + m\sigma(1 + \alpha\sigma) - m\alpha C. \end{aligned} \tag{7.20}$$

The solution for  $C$  is (assuming  $\alpha \neq 0$ )

$$C = \frac{\beta + 2m\sigma(1 + \alpha\sigma)}{2m\alpha} \mp \sqrt{\frac{\Lambda_0 + m\beta}{m^2\alpha} + \frac{(\beta + 2m\sigma(1 + \alpha\sigma))^2}{4m^2\alpha^2}}, \tag{7.21}$$

so

$$M = \pm m\alpha \sqrt{\frac{\Lambda_0 + m\beta}{m^2\alpha} + \frac{(\beta + 2m\sigma(1 + \alpha\sigma))^2}{4m^2\alpha^2}}. \tag{7.22}$$

and

$$l^{-2} = -m^2(1 - 2C) \left( \frac{\alpha\beta}{m} + (1 + \alpha\sigma)^2 \right) + M^2 > 0,$$

The no-tachyon condition is

$$M^2 - l^{-2} = m^2(1 - 2C) \left( \frac{\alpha\beta}{m} + (1 + \alpha\sigma)^2 \right) > 0, \tag{7.23}$$

and the no-ghost condition is as in (7.10). Note that  $C$  is real iff  $M^2 \geq 0$ . Hence, the no-tachyon condition also guarantees  $C$  to be real. For  $M = 0$ , the  $C$  equation (7.20) has a double root, but this case is un-physical since the no-tachyon condition is violated. Also note that unlike the original MMG, for which  $\beta = 0$ , the no-tachyon condition does not in general imply  $(1 - 2C) > 0$  which is, however, required by the positivity of the asymptotic central charges.

Collecting the positive central charge, the no-tachyon and the no-ghost conditions together we have

$$\begin{aligned} \alpha C + \sigma + \frac{\beta}{2m} - \frac{1}{|ml|} > 0 &\Rightarrow 1 - 2C > 0; \\ \frac{\alpha\beta}{m} + (1 + \alpha\sigma)^2 > 0, \quad mM < 0. \end{aligned} \tag{7.24}$$

The positivity of  $l^{-2}$  in (7.23) also requires that

$$0 < 1 - 2C < \frac{M^2}{m^2 \left( \frac{\alpha\beta}{m} + (1 + \alpha\sigma)^2 \right)}.$$

Due to the definition of  $M$  in (7.20), the condition  $mM < 0$  is the same as

$$\alpha(C - 1) - \frac{\beta}{2m} - \sigma > 0, \tag{7.25}$$

which when summed up with the first condition in (7.24) gives

$$\alpha(1 - 2C) + \frac{1}{|ml|} < 0 \Rightarrow \alpha < -\frac{1}{|ml|(1 - 2C)} < 0.$$

From (7.25) and the fact that  $\alpha$  is negative it follows, that in the solution (7.21) we should choose the minus sign in front of the square root and plus one in (7.22).

We have thus identified a range of the parameters compatible with the conditions (7.24). One can proceed further with the analysis and specify this range in more detail. Namely, from the third condition in (7.24), as in the case  $C = 0$ , it also follows that

$$\left( \frac{\beta}{2m} + \sigma + \alpha \right)^2 > \left( \frac{\beta}{2m} + \sigma \right)^2 - 1. \tag{7.26}$$

- Iff  $|\frac{\beta}{2m} + \sigma| \geq 1$ , we have

$$\alpha < -\sqrt{\left( \frac{\beta}{2m} + \sigma \right)^2 - 1} - \left( \frac{\beta}{2m} + \sigma \right) \quad \text{or}$$

$$\alpha > \sqrt{\left( \frac{\beta}{2m} + \sigma \right)^2 - 1} - \left( \frac{\beta}{2m} + \sigma \right).$$

These are compatible with  $\alpha < 0$  iff  $\frac{\beta}{2m} + \sigma \geq 1$ .

- Another brunch of (7.26) is

$$-1 < \frac{\beta}{2m} + \sigma < 1, \quad -1 < \alpha < 0,$$

for which a particularly simple case is  $\frac{\beta}{2m} + \sigma = 0$ .

- One more simple case is  $\Lambda_0 = -m\beta$  for which either  $C = 0$  and we are back to Sect. 7.1, or  $\alpha C = 2 \left( \frac{\beta}{2m} + \sigma + \alpha \right)$  and hence, due to (7.25) and (7.23),

$$\frac{\beta}{2m} + \sigma + \alpha > 0, \quad -1 < \alpha < 0.$$

Let us now consider the case in which  $\alpha = 0$  (but  $\beta \neq 0$ ). Then we have

$$\begin{aligned} C &= -\frac{\Lambda_0 + m\beta}{m(\beta + 2m\sigma)}, \quad M = \frac{\beta}{2} + m\sigma, \\ \ell^{-2} &= -\frac{2m(\Lambda_0 + m\beta)}{\beta + 2m\sigma} + \frac{\beta^2}{4} + m\sigma\beta. \end{aligned} \tag{7.27}$$

So, the no-tachyon condition is

$$m^2 + \frac{2m(\Lambda_0 + m\beta)}{\beta + 2m\sigma} > 0, \tag{7.28}$$

and the no-ghost condition is

$$\left( m + \frac{2(\Lambda_0 + m\beta)}{\beta + 2m\sigma} \right) (\beta + 2m\sigma) < 0. \tag{7.29}$$

From (7.28) and (7.29) we see that

$$\frac{\beta}{2m} + \sigma < 0. \tag{7.30}$$

Note that if  $\beta = 0$ , the model under consideration (3.10) is TMG [34,35] for which the above no-ghost condition requires  $\sigma = -1$ .

We will now show that also when  $\beta \neq 0$ , the model with  $\alpha = \rho = 0$  is equivalent to the TMG [34,35]. Indeed, upon making the redefinition of the connection (3.15) in the action (3.10) with  $\alpha = \rho = 0$ , we get

$$\begin{aligned} S &= \frac{1}{2} \int_{\mathcal{M}_3} \left( -\frac{2m\sigma + \beta}{m} e^a R_a + \frac{1}{m} (\Omega^a d\Omega_a \right. \\ &\quad \left. + \frac{1}{3} \varepsilon_{abc} \Omega^a \Omega^b \Omega^c) \right) \\ &\quad + \frac{1}{6} \int_{\mathcal{M}_3} \left( \Lambda_0 - \frac{\beta^3}{8m} \right) \varepsilon^{abc} e^a e^b e^c \\ &\quad + \frac{1}{2} \int_{\mathcal{M}_3} \left( 2h + \frac{\beta^2 + 4m\beta\sigma}{4m} e^a \right) \nabla e_a. \end{aligned} \tag{7.31}$$

This coincides with the first-order action for the topological massive gravity upon the redefinition  $\tilde{h}^a = h^a + \frac{1}{2} \left( \frac{\beta^2}{4m} + \beta\sigma \right) e^a$  and appropriate rescalings of  $e^a$  and  $\tilde{h}^a$ .

$$7.3 \quad \tilde{\alpha} = \tilde{\beta} = \tilde{\rho} = 0$$

Let us now consider the case in which in the action (3.1) we have  $\tilde{\alpha} = \tilde{\beta} = \tilde{\rho} = 0$ . This is the situation in which the spontaneous breaking of the Hietarinta/Maxwell symmetry occurs only due to the contribution associated with the classical lower-derivative Volokov–Akulov-like Goldstone (“cosmological”) term (2.15). In this case the parameters (3.11) in the MMG-like action (3.10) are

$$\beta = -\frac{\Lambda_0}{m}, \quad \alpha = \frac{\Lambda_0}{m^2} - \sigma, \quad \rho = -\frac{\Lambda_0 + m^2\sigma}{m^3}. \tag{7.32}$$

Hence, similar to the case of Sect. 7.1 we have  $\Lambda_0 + m\beta = 0$  but with particular expressions for  $\alpha$  and  $\rho$  in terms of  $\Lambda_0$  and  $m$ . Then the Eq. (6.2) for  $C$  reduces to

$$C \left[ (\Lambda_0 + m^2\sigma)C^2 - 2\Lambda_0C + \Lambda_0 \right] = 0. \tag{7.33}$$

For the solution  $C = 0$  of this equation from (6.4) and (7.3) we get

$$\ell^{-2} = \frac{\Lambda_0^2}{4m^2} - \Lambda_0\sigma, \quad M = \frac{\Lambda_0}{2m}.$$

We see that to satisfy the no-tachyon (7.6) and the no-ghost (7.10) conditions together with the requirement  $\ell^{-2} > 0$  we need  $\sigma = -1$  and  $\Lambda_0 < -4m^2$  which are in agreement with (7.19). Note that if  $\Lambda_0 = 0$ , then  $C = 0$  and hence this possibility is ruled out by the last inequality. Actually, in this case the background is flat not  $AdS$ , while the model reduces to the Chern–Simons theory with the unbroken Hietarinta (2.8) or Maxwell symmetry (2.9).

When  $\rho = 0$ , i.e.  $\Lambda_0 = -m^2\sigma$ , then from (7.33) we see that either  $C = 0$  or  $C = 1/2$ . In the both cases the no-tachyon condition (7.6) is not satisfied.

Finally, let us consider the case in which  $C \neq 0$ ,  $\Lambda_0 \neq 0$  and  $\rho \neq 0$ . Then from (7.33) we get

$$C = \frac{\Lambda_0 \pm \sqrt{-\Lambda_0\sigma m^2}}{\Lambda_0 + m^2\sigma} \tag{7.34}$$

So the existence of the real solutions (associated with  $AdS_3$  vacua) requires that

$$\Lambda_0\sigma < 0. \tag{7.35}$$

In this case from (6.4) and (7.3) we find

$$\ell^{-2} = m^2C^2, \quad M = \frac{\Lambda_0(C - 1)}{m}.$$

Note that  $C = 1$  is not a solution of (7.33). The no-tachyon condition (7.6) becomes

$$-(\Lambda_0\sigma + m^2)C^2 > 0 \Rightarrow \Lambda_0\sigma < -m^2,$$

and the no-ghost condition (7.10) using (7.33) implies

$$(\Lambda_0 + m^2\sigma)(1 - C) < 0.$$

Combining the last two inequalities we see

$$\sigma(1 - C) > 0, \tag{7.36}$$

which shows that we should take the plus sign in the solution of  $C$  in (7.34).

From (7.9) the positivity of the central charges implies  $1 - 2C > 0$  and since  $-\Lambda_0\sigma > 0$ , the no-ghost condition takes the form

$$\Lambda_0(C - 1) < 0 \Rightarrow \sigma(-\Lambda_0\sigma)(1 - C) < 0 \Rightarrow \sigma(1 - C) < 0,$$

which is incompatible with (7.36). Therefore, for the choice of the parameters considered in this Subsection the no-ghost and no-tachyon conditions, and the positivity of central charges are satisfied only for  $C = 0$ .

### 8 Conclusion

In this paper we have shown that both the TMG [34,35] and the MMG [1] can be treated as spontaneously broken phases of the Chern–Simons theory based on the Hietarinta/Maxwell algebra. In general, the spontaneous symmetry breaking in the HMCSG theory leads to a more general class of minimal massive gravities propagating a single massive spin-2 mode and having two more coupling parameters with respect to the MMG. For a certain range of the parameters these models have neither tachyons nor ghosts and their asymptotic algebra has positive central charges thus giving rise to unitary boundary CFTs. A further more detailed analysis of these models in the AdS/CFT context might be of interest.

As a generalization of the results of this paper, it would be interesting to identify the group-theoretical structure of Chern–Simons theories whose symmetry breaking gives rise e.g. to “New”, “General” [54,55], “Zwei-Dreibein” [51,56] and “Exotic” Massive Gravities [36,57,58], for more references see [4]. And of course the most challenging issue is to find an Englert–Brout–Higgs–Guralnik–Hagen–Kibble mechanism which might lead to such a symmetry breaking.

Another interesting direction is to look for a relation of the HMCSG to a “simple” theory of 3d massive gravity constructed and studied in [59,60]. The simplicity of this model is due to the fact that it contains only two one-form fields, a dreibein and a would-be Lorentz spin connection, but the local Lorentz symmetry in this model is broken. For a certain choice of the parameters in the letter its field equations



reproduce those of the MMG. A question is whether for a more general range of the parameters the simple massive gravity may also reproduce the equations of motion of the HMCSG theory constructed in this paper [upon solving for  $h^a$  in (3.16)].

It might also be of interest to consider supersymmetric and higher-spin extensions of these models elaborating on the constructions obtained e.g. in [27–30, 38, 61–64].

Regarding supersymmetric generalizations, let us make the following final remark. The simplest extension of the Maxwell algebra (1.2) by a two-component Majorana spinor generator  $Q_\alpha$  [65] is such that  $[Q_\alpha, P_a] = [Q_\alpha, Z_a] = 0$  and  $\{Q_\alpha, Q_\beta\} = 2\gamma_{\alpha\beta}^a Z_a$ , i.e. the anti-commutator of  $Q$  can only close on  $Z$  due to Jacobi identities. Hence, this simplest super-Maxwell algebra is not an extension of the conventional  $N = 1, D = 3$  super-Poincaré algebra. On the contrary, the similar supersymmetric extension of the Hietarinta algebra (1.1) is the extension of the simple  $3d$  super-Poincaré algebra since in this case the Jacobi identities allow  $\{Q_\alpha, Q_\beta\} = 2\gamma_{\alpha\beta}^a P_a$ . This gives one more evidence to the fact that the physical models based on the two versions of the same algebra are a priori different.

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