On dynamic aspects of the Proca field screening by a black hole

Can the existence of an arbitrarily small photon mass be observable?

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Received: 2 April 2020 / Accepted: 8 May 2020 / Published online: 2 June 2020

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Abstract Classical black holes are known for almost half a century to nullify exterior classical massive vector field associated with a charge crossing the event horizon. This implies that, from the point of view of an external observer, the Proca field of the charge is screened with the strength gradually increasing as the charge adiabatically approaches the event horizon. In this paper we reject the adiabaticity constraint and calculate analytically the field evolution with respect to a distant observer in the frame of the simplest model of a contracting charged spherical shell concentrically surrounding a Schwarzschild black hole. We show that a time scale of the screening is determined by a mass of the black hole and, moreover, loss of Proca hair during the collapse of charged matter has the same temporal character. Due to existence of the event horizon, there is discontinuous jump between massive and massless electrodynamics. This means that presence of an arbitrarily small mass of the photon induces observable effects, which include generation of electric asymmetry of the Universe and galactic magnetic fields.

1 Introduction

The absence of massive vector fields associated with black holes (BH) is a particular conclusion of no-hair theorems proved by Bekenstein and Teitelboim. This statement was proved for both spherically symmetric [1,2] and stationary [3] BHs. Moreover, an asymptotically flat spherically symmetric solution of the Einstein–Proca equations was shown by numerical calculations of Obukhov and Vlachynsky [4] to be inevitably horizonless. The absence of vector hair in the Abelian Higgs model for spherically symmetric BHs was demonstrated by Adler and Pearson [5].

An exact static solution of the Proca equation on the Schwarzschild background was obtained in one form or

another in early works [6–8]. An analysis [8,9] demonstrated a non-trivial dependence of the Proca static field created by a uniformly charged spherical shell surrounding the BH on the distance between the shell surface and the horizon. The measured field from the point of view of an observer fixed outside the charged sphere gradually disappears as the surface of the sphere adiabatically approaches the horizon. This suppression of the field can be considered as effective screening of the charge.

A similar smooth behavior of the static field occurs for an arbitrarily small mass value of the gauge boson. In other words, the transition to Maxwell electrodynamics is discontinuous *in the static case*. As shown in [2,8–10] this discontinuity is explained by the difference in the boundary conditions for the field at the horizon.

However, it is argued sometimes that the solution of a fully dynamical problem will manifest the continuous connection between the behavior of massive and massless fields in the vicinity of the BH. As stated without proof in the paper of Coleman, Preskill and Wilczek [11], when the charge crosses the horizon, the Schwarzschild time scale for the field of the charge to vanish is consistent with the Compton wavelength of the gauge boson. Thus, as the mass of the boson gets smaller, the time scale for the field to decay gets longer and longer with the Maxwell case realized in the limit of the mass going to zero.

An attempt to prove the statement of [11] was made in [12]. The author of this paper concluded that the monopole massive vector field created by spherically symmetric distribution of charged matter collapsing into a Schwarzschild BH does have the decay time scale governed by the Compton wavelength of the corresponding gauge boson. Thus, according to the results of [12], the prediction of [11] is correct at least in the monopole case.

In this paper we show that the time scale of the screening and the Proca hair loss is equal in order of magnitude to the gravitational radius of the BH and, therefore, the result of

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[12] is incorrect. We analytically calculate the evolution of the field of the uniformly charged spherical shell surrounding the BH and contracting according to some undefined law. Direct calculation provides a strong proof of the discontinuity between the Proca and Maxwell physics in the limit of vanishing field mass and demonstrates unique behavior of the field in this limit. The results of this paper are valid for any classical Abelian massive vector field.

Study of the massive vector field dynamics in the vicinity of the BH, of course, is important for the phenomenology of quantum gravity but, may be, more fruitful for cosmological consequences. An arbitrarily small but nonzero photon mass causes the effect of electrogenesis via BHs [13] with the rate of this process determined by the dynamics of the field created by the accreting charge. This process violates the electric symmetry of the Universe and probably makes a significant contribution to the generation of galactic magnetic fields.

The photon mass can be generated efficiently through the Stückelberg [14] mechanism or spontaneous symmetry breaking via the Higgs mechanism. This implies the existence of a new elementary scalar field. However, the existence of the additional scalar field to produce the photon mass is not necessary. The photon mass, for example, can be generated topologically [15, 16] by a four-dimensional analogue of the mechanism first described by Schwinger for two dimensions [17, 18].

If the photon is the massive particle, the most general solution of the Einstein equation describing BHs is the Kerr metric. If the electromagnetic field really obeys the Proca equation, the superradiant instability of massive vector fields around rotating BHs disappearing in the massless limit allows one to set an upper bound on the photon mass [19]. At the instability threshold, BHs with nonzero angular momentum were found in [20] to be able to possess massive vector hair, but that has a harmonic temporal dependence and is not continuously connected with the Kerr-Newman solution. Consequently, this hairy behavior is not directly related to the problem of electrogenesis. It is natural to expect in this context that the qualitative picture of the field evolution in the limit of a small photon mass coincides with that for the Schwarzschild case, because axisymmetric BHs do not have massive vector hair that respect space-time symmetry.

The article is organized as follows. In Sect. 2 we give a brief derivation of the static solution of the Proca equation for the fixed charged shell and discuss some related issues. The resolution of the dynamical problem is based on the static solution and described in detail in Sect. 3. At the end of this section, we write formulas to calculate the field evolution and define bounds of their applicability. The results of our calculations are demonstrated and discussed in Sect. 4. We conclude the article by summary of the results noting open areas for further research.

2 Static solution of the Proca equation

Like any massive vector field generated by current density j^μ , massive electromagnetic field with the photon mass m satisfies the Proca equation

$$\nabla_\mu F^{\mu\nu} + m^2 A^\nu = 4\pi j^\nu, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and ∇ is the covariant derivative in the background gravitational field. Strictly speaking, the Eq. (1) is valid if the Proca field and its source does not change space-time geometry. We will indeed assume throughout the paper that the stress-energy tensor of the Proca field and the mass of charged matter are negligible in this sense. Therefore, the gravitational field of the BH will be described by the unperturbed Schwarzschild metric

$$ds^2 = h(r) dt^2 - dr^2/h(r) - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2, \quad (2)$$

$$h(r) \equiv 1 - r_g/r, \quad (3)$$

where r_g is the gravitational radius of the BH. The latter is assumed to be surrounded by a concentric uniformly charged spherical shell with a charge Q and a fixed radius r_s . The corresponding current density has only a temporal component

$$j^t = \frac{Q\delta(r - r_s)}{4\pi r_s^2}. \quad (4)$$

It is natural to expect that the field inherits the symmetry of the metric and the source. Therefore A_ϕ and A_θ are equal to zero and the Eq. (1) reduces to a single equation for the electrostatic potential

$$\frac{1}{r^2} \partial_r (r^2 \partial_r A_t) - \frac{m^2}{h(r)} A_t = 4\pi j^t, \quad (5)$$

which can be solved by conversion to a form of the Whittaker equation [21]. The corresponding general homogeneous solution is a linear combination of functions

$$A_1 = h(r) e^{-m(r-r_g)} \Phi(mr_g/2 + 1, 2; 2m(r - r_g)), \quad (6)$$

$$A_2 = h(r) e^{-m(r-r_g)} \Psi(mr_g/2 + 1, 2; 2m(r - r_g)), \quad (7)$$

where Φ and Ψ denote confluent hypergeometric functions [21] of the first and second kind, respectively.

From the no-hair theorem we know that any non-trivial homogeneous solution of the equation (5) cannot simultaneously satisfy physically justified boundary conditions at the surface of BH and spatial infinity. The physically acceptable boundary conditions require the finiteness of observed invariants of the field

$$I_1 \equiv F_{\alpha\beta} F^{\alpha\beta} \quad \text{and} \quad I_2 \equiv A_\alpha A^\alpha, \quad (8)$$

included in the Proca field energy-momentum tensor that has the components

$$T^{\mu\nu} = \frac{1}{16\pi} [4F_{\beta\alpha} g^{\beta\mu} F^{\nu\alpha} + I_1 g^{\mu\nu}]$$

$$-\frac{m^2}{8\pi}[2A^\mu A^\nu - I_2 g^{\mu\nu}]. \quad (9)$$

Contributions of the solution (6) into the field strength $E \equiv \partial_r A_t = -\sqrt{I_1}$ and the invariant I_2 are finite at the horizon, but grow without bound at $r \rightarrow \infty$. On the other hand, as we approach the gravitational radius, the contribution of the solution (7) to the field strength diverges logarithmically and the invariant I_2 grows infinitely according to the law $1/h(r)$. Considered at spatial infinity this solution vanishes quasi exponentially. Consequently, only trivial static solution is possible if the radius of the charged shell approaches gravitational one.

Solutions inside and outside the shell are represented by the functions A_1 and A_2 respectively as follows from the behavior of these functions described above. From junction conditions of these solutions, it follows that the radial component of the field strength has the form

$$E_0(r) = -2Qm \Gamma(1 + mr_g/2) [\theta(r - r_s) A_1(r_s) \partial_r A_2(r) + \theta(r_s - r) A_2(r_s) \partial_r A_1(r)], \quad (10)$$

where Γ denotes the Euler gamma function [21] and θ is the Heaviside step function. We omitted here the details of pretty standard calculations. The solution (10) was obtained in one form or another in the early works [8, 9].

Now let the mass of the gauge boson be small so that we can neglect the product mr_s and, of course, mr_g . Then the field strength being measured by a distant observer outside the shell will be determined by a simple expression

$$E_0(r) = \frac{Q}{r^2} \frac{r_s - r_g}{r_s} e^{-mr} (1 + mr). \quad (11)$$

Thus, if the photon has an arbitrarily small but nonzero mass, the effective charge and the field disappear as the shell adiabatically approaches the BH horizon. However, it is well known that in the case when the photon mass is strictly zero, the screening of the charge does not occur and we would simply have $E_0(r) = Q/r^2$ everywhere outside the shell regardless of its radius.

The nature of this discontinuity was in a varying degree studied in the papers [2, 8–10]. The massive vector field acts as a source of space-time curvature, that is expressed by the second term in the stress-energy tensor (9). Since the Proca equation is not gauge-invariant, the vector potential A^μ becomes physically observable. Therefore, the boundary conditions at the horizon for the massive electromagnetic field do not coincide with the conditions in Maxwell's theory. For any nonzero photon mass, we must additionally require the invariant I_2 to be finite. It is worth noting that this discontinuity in the behavior of the effective charge takes place only for BH, i.e. if the source of spherically symmetric gravitational field has the event horizon [8].

Fig. 1 The strength $E = \partial_r A_t$ (in Schwarzschild coordinates) depending on the mass of the photon (a) and the position of the charged shell for the photon with a negligible mass (b). The additional parameters are also shown in the figure. Black dashed curves correspond to the Maxwell's case

Shown in Fig. 1 is the behavior of the electrostatic field strength of a shell with a unit charge for various values of the photon mass and the shell radius in comparison with the massless case.

3 Analytical calculation of the field evolution

Formulation of the problem and its resolution technique We suppose now the charged shell surrounding the BH begins to radially contract at some moment of the global time. We calculate the evolution of the Proca field far outside the shell, without specifying the driving force of the contraction and assuming that the stress-energy tensor of the source of this force is negligibly small. The dimensionless parameter mr_g is also assumed to be small throughout this section.

Due to spherical symmetry, the field strength is connected to the vector potential by the relation $E = \partial_t A_r - \partial_r A_t$. The Proca equation (1) for this value has the form

$$\frac{\partial_t^2 E}{h(r)} - \partial_r \left[\frac{h(r)}{r^2} \partial_r (r^2 E) \right] + m^2 E = 4\pi (\partial_t j_r - \partial_r j_t). \quad (12)$$

Keeping the notation introduced in the previous Sect. 2 the temporal and spatial components of the current density are determined as

$$j_t = \frac{Qh(r)\delta(r-r_s)}{4\pi r_s^2} \quad \text{and} \quad j_r = -\frac{Q\partial_t r_s \delta(r-r_s)}{4\pi h(r)r_s^2}. \quad (13)$$

An analytical approach to calculating field behavior in this paper is based on the well-known method of retarded Green function. We apply this approach for the equation (12) previously translated into a frequency region by the Fourier transform, defined as follows:

$$\mathcal{E} = \int dt e^{-i\omega t} E(t, r) \quad E = \frac{1}{2\pi} \int d\omega e^{i\omega t} \mathcal{E}(\omega, r).$$

The Green function $G = G(r, x, \omega)$ for the Fourier transform of the equation (12) by definition satisfies the equation

$$-\partial_r \left[\frac{h(r)}{r^2} \partial_r (r^2 G) \right] - \frac{\omega^2}{h(r)} G + m^2 G = \delta(r-x) \quad (14)$$

and the correct boundary conditions at the horizon which include the finiteness of the response function G . The homogeneous equation corresponding to (14) has the form

$$\partial_r^2 \mathcal{E} + p(r) \partial_r \mathcal{E} + q(\omega, r) \mathcal{E} = 0, \quad (15)$$

where we denote

$$p(r) \equiv \frac{1}{r} + \frac{1}{r-r_g},$$

$$q(\omega, r) \equiv \frac{1}{r-r_g} \left[\frac{\omega^2 r^2}{r-r_g} - \frac{2}{r} \left(1 - \frac{2r_g}{r} \right) - m^2 r \right],$$

and $\mathcal{E}$ is an unknown function to be determined.

The law of the shell contraction is required to be described by a smooth function $r_s(t)$ decreasing monotonically. If r_i and r_f denote the initial and final radii of the shell, the Fourier image of the function in the right-hand side of the equation (12) equals to zero in the regions $r > r_i$ and $r < r_f$. This function is represented as

$$F(\omega, r) = Q [F_T(\omega, r) + F_R(\omega, r)], \quad (16)$$

where F_T and F_R denote contributions from the temporal and spatial components of the current density, namely

$$F_T(\omega, r) = \partial_r \left[\partial_r t_s(r) e^{-i\omega t_s(r)} h(r)/r^2 \right], \quad (17)$$

$$F_R(\omega, r) = \frac{i\omega e^{-i\omega t_s(r)}}{h(r)r^2}. \quad (18)$$

Here $t_s(r)$ is the function inverse to $r_s(t)$.

Fig. 2 The effective potential (22) as a function of r_* for $r_g = 1$. The potential is plotted for three values of m indicated in the figure

We use the equation (12) because its homogeneous solution for $\omega = 0$ has a simple form obtained by taking the derivative of a linear combination of the functions (6) and (7) with respect to r . Solutions for the static problem in the regions $r < x$ and $r > x$ are respectively expressed as

$$E_1(r) = \partial_r A_1(r), \quad (19)$$

$$E_2(r) = \partial_r A_2(r). \quad (20)$$

However, it is also useful to represent a homogeneous equation corresponding (12) by the standard replacement $E = \Psi/r$ and introducing the Regge-Wheeler coordinate

$$r_* = r + r_g \ln(r/r_g - 1) + c,$$

where c is some constant. In this notation, the equation of the field evolution takes the form of the Schrödinger equation

$$\partial_t^2 \Psi - \partial_{r_*}^2 \Psi + V(r_*) \Psi = 0, \quad (21)$$

with an effective potential $V(r_*)$:

$$V(r_*) = \left(1 - \frac{r_g}{r} \right) \left[\frac{2}{r^2} + m^2 - \frac{3r_g}{r^3} \right]. \quad (22)$$

Frequency dependence of the transmission coefficient of the barrier formed by the effective potential V determines the evolution of the field. The maximum of the barrier is located near $r = 1.75 r_g$ for sufficiently small m (Fig. 2).

We also require smallness of the dimensionless parameter ωr_g , which is not a significant constraint of generality, as will become clear in what follows. We will start to solve the equation (15) by computing the solution in the region with $r < x$. Analytical continuing that outside of this domain allows us to find the solution in the region with $r > x$, which thereby unambiguously determines the Green function and the solution of the problem.

Determination of the solution for $r < x$ Being taken at the point $r = r_g$ the indicial equation $\rho(\rho - 1) + \rho + r_g\omega = 0$ has the roots $\rho = \pm ir_g\omega$. Since the difference of these roots does not equal to an integer real number for any nonzero product ωr_g , both linearly independent solutions in a sufficiently small neighborhood of the gravitational radius can be found in the form of a series [22]. Unlike the static problem, we cannot choose here the only physically acceptable combination of the linearly independent solutions, relying only on their finiteness in a neighborhood of the gravitational radius.

To determine a desired combination of the solutions, it will be useful to study its behavior in the limit $r_g \rightarrow 0$. It is natural to require that the solution in this limit becomes the same as for the case without the BH. Strictly speaking, this follows from the fact that in the case of flat space-time, due to spherical symmetry and vector nature of the field, the latter vanishes in the center of the charged sphere for any law of its contraction. Thus, the emergence of a sufficiently small BH in the center of the shell will not have any significant impact on behavior of the field everywhere outside the BH, since the interaction between the field of the shell and gravity of the BH in its exterior is negligible. Introducing

$$\gamma = \sqrt{m^2 - \omega^2}$$

we can write the general homogeneous solution of the equation (15) in flat space-time in the form

$$\mathcal{E}_0 = \frac{m^2}{2\gamma^3 r^2} [\alpha e^{-\gamma r} (1 + \gamma r) + \beta e^{\gamma r} (\gamma r - 1)], \quad (23)$$

where α and β are some constants. The only combination being regular in the center of the spherical coordinate system is obtained when these constants are equal that is physically explained by vector character of the field and energy balance between converging and diverging waves. We will consider that $\alpha = \beta = 1$ in the limit of vanishing curvature.

How will the solution change when the BH appears? It is natural to expect that smallness of the parameters mr_g and ωr_g implies a high quality approximation of the homogeneous solution by a superposition (23) in the limit of large r . But how will the balance between the coefficients α and β change? An answer can in fact be easily obtain directly from properties of the expression (23) which we rewrite as

$$\mathcal{E}_0 = \frac{\alpha + \beta}{2} \frac{m^2}{\gamma^2} \partial_r \frac{\sinh \gamma r}{\gamma r} + \frac{\beta - \alpha}{2} \frac{m^2}{\gamma^2} \partial_r \frac{\cosh \gamma r}{\gamma r}. \quad (24)$$

Because $\partial_t g_{\mu\nu} = 0$ in the Schwarzschild geometry, the energy is conserved that requires $|\alpha| = |\beta|$. Hence, any violation of this equality means that the field changes the metric. In the specific case, this change is explained as the wave absorption by the BH. Note that for small γr

$$\partial_r \frac{\sinh \gamma r}{\gamma r} = \frac{\gamma^2 r}{3} + \mathcal{O}(\gamma^4 r^3), \quad (25)$$

$$\partial_r \frac{\cosh \gamma r}{\gamma r} = -\frac{1}{\gamma r^2} + \mathcal{O}(\gamma). \quad (26)$$

An additional condition $\chi \equiv r/r_g \gg 1$ ensures validity of the approximations (25) and (26) in the case of $r_g \neq 0$. Fixing the ratio χ in this way, one finds that the balance condition $\alpha = \beta$ in the limit $r_g \rightarrow 0$ gives

$$\lim_{r_g \rightarrow 0} \frac{\alpha - \beta}{\gamma r^2} \Big|_{\chi} = \lim_{r_g \rightarrow 0} \frac{\alpha - \beta}{\gamma \chi^2 r_g^2} = 0.$$

In other words, the difference $\alpha - \beta$ is more than the second order of smallness and we can neglect the second term in (24). The influence of the absorption on the field dynamics is negligible, since characteristic wavelengths, as we require, significantly exceed the gravitational radius. The potential barrier determined by (22) is impenetrable for ingoing waves with this property. Thus, we can require that the amplitudes of the ingoing and outgoing waves coincide in absolute value and, moreover, $\alpha = \beta$.

Consider the dependence of the effective potential (22) on the radial coordinate. Influence of the mass term on behavior of the desired homogeneous solution can be neglected near the BH, while the terms with r_g can be neglected for sufficiently large r , where solution behavior significantly depends on m . A coordinate of the border between these regions can be defined by the equality in absolute value of the last two terms in square brackets, which gives

$$r_b \equiv \sqrt[3]{\frac{3r_g}{m^2}}. \quad (27)$$

The regions with $r < r_b$ and $r > r_b$ will be referred to below as regions of dominant curvature and mass, respectively. As expected, this distinction is also justified by the behavior of the static solution (19), since its expansion has the form

$$E_1(r) = \frac{r_g}{r^2} + \frac{1}{3} m^2 r + o(m^2 r) \quad (28)$$

with the same terms in the same proportion. We will require the desired solution $\mathcal{E}$ to go into the static one E_1 in the limit $\omega \rightarrow 0$.

Now we can simply solve our problem with stated accuracy if the region with $r < r_b$ is sufficiently wide that corresponds to the inequality $r_b \gg r_g$. The function describing field behavior in the mass dominated region far enough from its border is the flat space-time solution that we determine as the function (23) with $\alpha = \beta = 1$:

$$\begin{aligned} \mathcal{E}_0 &\sim \frac{m^2}{2\gamma^3 r^2} [e^{-\gamma r} (1 + \gamma r) + e^{\gamma r} (\gamma r - 1)] \\ &= \frac{m^2}{\gamma^2} \partial_r \frac{\sinh \gamma r}{\gamma r}. \end{aligned} \quad (29)$$

Recall that the Eqs. (15) and (21) are equivalent and r_* is asymptotically equals to r for large r/r_g . To compute the

corresponding approximation to the solution in the curvature dominated region, we first assume that $r_g \ll r \ll r_b$. The effective potential V is well reproduced by the centrifugal term $2/r^2$ in this spherical layer and thus we can neglect both the mass and the gravitational radius in the equation (15) to find a function approximating the solution in the aforementioned layer. Since the function (29) behaves as $m^2 r/3$ for small values of the product mr and in the additional limit $\omega \rightarrow 0$, the desired form of the approximate solution for said range of r in the low frequency limit goes to r_g/r^2 , as seen from the expansion (28). This property is inherent in only one combination of these functions that respects energy balance between converging and diverging waves. This combination has the form

$$\begin{aligned} \mathcal{E}_< &\sim \frac{r_g}{2r^2} \left[e^{-i\omega r} (i\omega r + 1) - e^{i\omega r} (i\omega r - 1) \right] \\ &= -r_g \partial_r \frac{\cos \omega r}{r}. \end{aligned} \quad (30)$$

In fact, this expression can be considered with good accuracy applicable up to the horizon. Indeed, the solutions of the equation (21) in the frequency region and in the limit of large ωr_g are represented by a combination

$$\Psi = A_\omega \exp(i\omega r_\star) + B_\omega \exp(-i\omega r_\star) + \mathcal{O}(\omega r_g)^{-1}, \quad (31)$$

where A_ω and B_ω are coefficients. However this expression is valid for any frequency ω in a close neighborhood of the horizon. Since we require the parameter ωr_g to be small, ωr with radii r of the order of r_g is also a small value. Hence, the constant component of the solution dominates near the BH except for a small neighborhood of the horizon, where it is necessary to take into account logarithmic divergent terms contained by (31). The latter also means that the condition $\omega r_g \ln(r/r_g - 1) \ll 1$ is a more precise applicability restriction. Therefore, we have determined and joined the approximate solutions in the regions of dominant curvature and mass far enough from the border between these regions.

Determination of the solution for $r > x$ We can describe the solution in the region with $r > x$ by its asymptotic form, since the observer measuring the field is assumed to be far from the BH. This function denoted by $\mathcal{E}_>$ describes a diverging wave in both the curvature and mass dominated regions. It follows that we can consider the form of the function $\mathcal{E}_>$ does not depend on the region. Therefore, we can fix the observer and study the limit of the Proca mass going to zero, when the region of dominant curvature expands unlimitedly.

The solutions $\mathcal{E}_<$ and $\mathcal{E}_>$ are well known to be connected by the integral relation

$$\mathcal{E}_>(\omega, r) = \mathcal{E}_<(\omega, r) \int_r^\infty \frac{d\xi}{\xi(\xi - r_g) \mathcal{E}_<^2(\omega, \xi)}, \quad (32)$$

with the integration constant determined for the solution $\mathcal{E}_>$ to vanish at spatial infinity. Given the asymptotic form of $\mathcal{E}_<$, the relation (32) allows us to determine the asymptotic form of $\mathcal{E}_>$. To find this form in the frame of the assumptions made, it is enough to set $r_g = 0$ in this integral relation and use the expression (29) as the asymptotic form of $\mathcal{E}_<$.

Green function Dynamics of the field created by the shell located in the region of dominant curvature is of the greatest interest in the context of this work. Therefore, we impose the condition $r_i \ll r_b$. Since the same constraint holds for each point belonging to the support of the function (16), the Green function can be written in the form

$$G(r, x, \omega) = \frac{\mathcal{E}_<(\omega, x) \mathcal{E}_>(\omega, r)}{h(x) W(\mathcal{E}_>(\omega, x), \mathcal{E}_<(\omega, x))}, \quad (33)$$

where the function $\mathcal{E}_<$ is represented by the expression (29), while $W(\mathcal{E}_>, \mathcal{E}_<)$ is the Wronskian of the functions $\mathcal{E}_<$ and $\mathcal{E}_>$ with the latter defined by (32).

Substituting (32) into the formula (33) we finally obtain a rather simple expression for the Green function:

$$G = -x^2 \mathcal{E}_>(\omega, r) \mathcal{E}_<(\omega, x) = -\partial_r \frac{e^{-\gamma r}}{m^2 r} x^2 \mathcal{E}_<(\omega, x). \quad (34)$$

Convolution The strength of the Proca field $E = E(\tau, r)$ outside the shell at a large distance from the BH will be described by the expression

$$\begin{aligned} E &= \int \frac{d\omega}{2\pi} e^{i\omega\tau} \int_{r_f}^{r_i} d\xi G(r, \xi, \omega) F(\omega, \xi) \\ &= -\partial_r \int \frac{d\omega}{2\pi m^2 r} e^{i\omega\tau - \gamma r} \int_{r_f}^{r_i} \xi^2 \mathcal{E}_<(\omega, \xi) F(\omega, \xi) d\xi, \end{aligned} \quad (35)$$

where τ is the Schwarzschild time coordinate of the observer. Recall that the function $\mathcal{E}_<$ is entire for any finite $\xi > r_g$ in the frame of our approximation. Therefore, the Green function has only branch points corresponding $\gamma = 0$. To satisfy retarded boundary conditions, the contour of the integration goes around these singularities at $\text{Im} \omega < 0$.

Of course, the expression (17) is by no means a regular function of ω in a neighborhood of zero frequency. It is advisable to explicitly recover the corresponding singularity to take the inverse Fourier transform in the expression (35). In order to avoid using cumbersome expressions, we first introduce a notation

$$\mathcal{F}(\omega, r) \equiv \partial_r \left\{ \partial_r \left[r^2 \mathcal{E}_<(\omega, r) \right] h(r)/r^2 \right\}. \quad (36)$$

To regularize the integral over ξ , let also r_0 be the shell radius at some point in the time denoted by t_0 . Integrating now twice in parts the relevant function of ξ in the expression (35) from r_f to r_0 , we obtain

$$I_1 = I_1(r_f, r_0) \equiv \int_{r_f}^{r_0} d\xi \xi^2 \mathcal{E}_<(\omega, \xi) F_T(\omega, \xi) =$$

$$\begin{aligned}
&= e^{-ipt_0} h(r_0) \mathcal{E}_<(\omega, r_0) \partial_{r_0} t_s(r_0) \\
&+ e^{-ipt_0} \frac{h(r_0)}{ir_0^2 p} \partial_{r_0} \left[r_0^2 \mathcal{E}_<(\omega, r_0) \right] \\
&- \frac{1}{p} \int_{r_f}^{r_0} d\xi e^{-ipt_s(\xi)} \mathcal{F}(\omega, \xi), \quad (37)
\end{aligned}$$

where $p = \omega - i\epsilon$ and ϵ is an infinitesimal positive parameter of regularization. Behavior of the field in the light cone of the future with the vertex at the point (r_0, t_0) is not completely determined by this expression, in contrast to the case of a massless field. Since the propagation speed of a massive field wave is less than the speed of light and depends on its frequency, behavior of the field in the space-time region mentioned above is determined, strictly speaking, by entire history of the change in the state of the shell at the time $t < t_0$. However, we can easily make the expression (37) to work taking t_0 to the distant past and requiring the product ϵt_0 to be small in this limit, wherein the influence of the history of the shell until the time t_0 becomes negligibly small and this expression can be greatly simplified. Namely, in the subsequent integration over ω due to the infinitely fast oscillating behavior of the nonintegral terms, the contribution of the first term to the Fourier integral is zero, while the contribution of the second term can be calculated through a residue. Thus, we can make an equivalent replacement

$$\begin{aligned}
I_1(r_f, r_i) &\rightarrow 2\pi \delta(\omega) \partial_{r_i} \left[r_i^2 \mathcal{E}_<(\omega, r_i) \right] h(r_i) / r_i^2 \\
&- \frac{1}{i\omega} \int_{r_f}^{r_i} d\xi e^{-i\omega t_s(\xi)} \mathcal{F}(\omega, \xi), \quad (38)
\end{aligned}$$

where we imply that the pole in the second term belongs to the upper halfplane. As it should be, the contribution of this term into the integral over ω is equal to zero if the condition $\tau - t(\xi) < r$ is satisfied. Indeed, the integrand is an analytic function in the lower halfplane and decays exponentially on the lower semicircle.

To conveniently represent the contribution of the first term in the expression (38) to the field strength, we should use the fact that the static homogeneous Proca equation corresponding to the equation (5) gives

$$\partial_r \left[r^2 E_1(r) \right] = \frac{m^2 r^2}{h(r)} A_1(r). \quad (39)$$

It is easy to see now that the result of the inverse Fourier transform of the first term in the approximation of a small mr_i has the form

$$E_0(r_i, r) = \frac{Q}{r^2} h(r_i) e^{-mr} (1 + mr), \quad (40)$$

already known to be the solution of the static problem for the case of the shell with the radius r_i . Denoting

$$\Delta\tau = \Delta\tau(\xi) \equiv \tau - t_s(\xi)$$

we can write the expression (35) as

$$\begin{aligned}
E &= E_0(r_i, r) + \frac{Q}{2\pi} \int_{r_f}^{r_i} d\xi \partial_r \int d\omega e^{i\omega\Delta\tau - \gamma r} \frac{\mathcal{F}(\omega, \xi)}{i\omega m^2 r} \\
&- \frac{Q}{2\pi} \int_{r_f}^{r_i} \frac{d\xi}{h(\xi)} \partial_r \partial_\tau^2 \int d\omega e^{i\omega\Delta\tau - \gamma r} \frac{\mathcal{E}_<(\omega, \xi)}{i\omega m^2 r}. \quad (41)
\end{aligned}$$

This expression is dramatically simplified, because using the equation (15), the function $\mathcal{F}$ defined by the formula (36) can easily be converted to the form

$$\mathcal{F}(\omega, r) = m^2 \mathcal{E}_<(\omega, r) - \frac{\omega^2}{h(r)} \mathcal{E}_<(\omega, r). \quad (42)$$

Being applied in the expression (41) this relation generates a term that diverges in the limit $m \rightarrow 0$. But it is easy to see that the contribution from the last term in the right-hand side of the relation (42) completely reduces the last term in this expression. This reduction saves the only term that are finite in the limit of m going to zero. The expression (41), up to negligible corrections, takes the form

$$\begin{aligned}
E &= E_0(r_i, r) + \frac{Q}{2\pi} \int_{r_f}^{r_i} d\xi \partial_r \int d\omega e^{i\omega\Delta\tau - \gamma r} \frac{\mathcal{E}_<(\omega, \xi)}{i\omega r} \\
&= E_0(r_i, r) + \frac{Q r_g}{2\pi} \int_{r_f}^{r_i} \frac{d\xi}{\xi^2} \partial_r \int \frac{d\omega}{i\omega r} e^{i\omega\Delta\tau - \gamma r} \cos \omega \xi \\
&- \frac{i Q r_g}{2\pi} \int_{r_f}^{r_i} \frac{d\xi}{\xi} \partial_r \int \frac{d\omega}{r} e^{i\omega\Delta\tau - \gamma r} \sin \omega \xi. \quad (43)
\end{aligned}$$

Inverse Fourier transform To proceed from the frequency plane into the time region, we need to calculate the integral

$$\begin{aligned}
I_2(\alpha, \beta) &\equiv \frac{1}{2\pi} \int \frac{d\xi}{i\xi} \exp(i\alpha\xi - \beta\sqrt{1-\xi^2}) \\
&= \int_{-\infty}^{\alpha} \frac{d\xi}{2\pi} \int d\eta \exp\left(i\xi\eta - \beta\sqrt{1-\eta^2}\right), \quad (44)
\end{aligned}$$

which depends on the parameters α and β . The choice of the lower limit in the right-hand side of this expression dictated by the fact that the initial integral is equal to zero if $\alpha = -\infty$ or, more precisely, if $\alpha < \beta$ since the singularities of the integrand are located above the real axis that corresponds to retarded boundary conditions.

The inner integral in the right-hand side of the expression (44) is connected by a simple relation with the retarded Green function of the Klein–Gordon equation. This function is known to be written as

$$D_{ret}(t, r) = \frac{1}{2\pi} \theta(t) \left[\delta(\lambda^2) - \theta(\lambda^2) \frac{m}{2\lambda} J_1(m\lambda) \right], \quad (45)$$

where $\lambda^2 = t^2 - r^2$ and J_1 is the Bessel function of the first kind. Indeed

$$D_{ret} = -\frac{1}{(2\pi)^4} \int \frac{d\omega d^3k \exp[i\omega t - i\mathbf{k}\mathbf{x}]}{\omega^2 - \mathbf{k}^2 - m^2 + i\epsilon\omega}$$

$$= \frac{1}{8\pi^2 r} \int d\omega \exp(i\omega t - r\sqrt{\gamma^2 + i\epsilon\omega}), \quad (46)$$

that can be easily proved by integration over the components of the vector $\mathbf{k}$ in the four-dimensional integral. Taking into account the formulas (45) and (46) we obtain

$$I_2 = \beta \int_{-\infty}^{\alpha} d\xi \theta(\xi) \left[2\delta(\lambda_\xi^2) - \theta(\lambda_\xi^2) J_1(\lambda_\xi)/\lambda_\xi \right], \quad (47)$$

where we denote $\lambda_\xi = \sqrt{\xi^2 - \beta^2}$. Hence, we can write

$$\partial_\beta [I_2(\alpha, \beta)/\beta] = \partial_\beta [\mathcal{D}(\alpha, \beta) + e^{-\beta}/\beta] \theta(\lambda_\alpha^2) - 2\theta(\alpha)\delta(\lambda_\alpha^2), \quad (48)$$

having defined a new function

$$\mathcal{D}(\alpha, \beta) \equiv \int_{\alpha}^{\infty} d\xi J_1(\lambda_\xi)/\lambda_\xi. \quad (49)$$

Resulting formulas and their applicability Combining the expressions (43) and (48) and getting rid of the generalized functions we eventually find the final formulas describing the Proca field strength measured by a distant observer:

$$E(\tau, r) = \frac{1}{2} [E_+(\tau, r) + E_-(\tau, r)], \quad (50)$$

$$E_\pm(\tau, r) = E_0(r_{s\pm}, r) + R_\pm(\tau, r), \quad (51)$$

with slightly cumbersome functions $R_\pm$ defined as

$$\begin{aligned} R_\pm \equiv & \frac{Qr_g \dot{r}_{s\pm}}{rr_{s\pm}^2} \pm \frac{Qr_g \dot{r}_{s\pm}}{rr_{s\pm}} \left[\frac{\dot{r}_{s\pm}}{r_{s\pm}} - \frac{\ddot{r}_{s\pm}}{\dot{r}_{s\pm}} - \frac{1}{r} \right] \\ & + Qmr_g \int_{r_{s\pm}}^{r_i} \frac{d\xi}{\xi^2} \partial_r \mathcal{D}(m\Delta\tau \pm m\xi, mr) \\ & \mp Qm^2 r_g \partial_r \int_{r_{s\pm}}^{r_i} \frac{d\xi}{\xi} \frac{J_1(m\sqrt{[\Delta\tau \pm \xi]^2 - r^2})}{m\sqrt{[\Delta\tau \pm \xi]^2 - r^2}}. \end{aligned} \quad (52)$$

Here $r_{s\pm}$ is the root of the equation

$$\tau - t_s(\xi) \pm \xi = r \quad (53)$$

with respect to ξ and the point denotes derivative by time τ .

Let us briefly recall and clarify the limits of applicability for the derived expressions. Despite the fact that the source of the field was assumed during the derivation to be in the region of dominant curvature, this constraint can be omitted in the case of a small mr_i because this condition provides the trivial dynamics in the limit $r_g \rightarrow 0$. In addition to the requirement that mr_i and r_g/r are small parameters, we also implied the smallness of the parameter $\omega r_g \ln(r_f/r_g - 1)$ that imposes an insignificant constraint on the class of possible laws of the shell contraction. Obviously, this neglect is justified if the characteristic time scale of the contraction is substantially larger than r_g and r_f is not too close to r_g . It is worth noting that the formulas obtained can be significantly simplified in the case of a nonrelativistic contraction of the shell. Under

this assumption we can neglect the highest powers of the product $\omega\xi$ in (43) and find

$$\begin{aligned} E(\tau, r) = & E_0(r_s, r) + \frac{Qr_g \dot{r}_s}{r_s^2 r} \\ & + Qmr_g \int_{r_s}^{r_i} \frac{d\xi}{\xi^2} \partial_r \mathcal{D}(m\Delta\tau, mr), \end{aligned} \quad (54)$$

where r_s this time is the a single root of the simple equation $\tau - t_s(\xi) = r$. On the other hand, the solution describing the field outside the shell is also approximated up to terms of the order of $\omega r_g \ln(r/r_g - 1)$, as can be seen from the asymptotics (31). Therefore, our result seem to potentially lose accuracy in the region of very large, but noninteresting, radial coordinates.

4 A revealed picture of field evolution and time scale for loss of massive vector hair

Now analyze behavior of the various terms in the resulting formulas (50) and (54). The function E_0 determines the usual Yukawa contribution suppressed by the factor h . The nonintegral terms reproduce the impulse response of the field to an instantaneous perturbation, with this response propagating at the speed of light. Stretching behind the latter is a low-frequency tail described by the integral terms. Because the amplitude of this tail is suppressed by a factor m^2 , the field evolution does not depend on the mass of the field if this mass is very small. Of course, the time scale of setting the field configuration that corresponds to the final radius r_f depends on the characteristic rapidity of the shell contraction and on the radius of the shell. But the transient duration in the limit $m \rightarrow 0$ is of the order of r_g that is clear already from an elementary dimensional analysis.

In order to demonstrate typical behavior of the field, we accept the law of the shell contraction to have the form

$$r_s(t) = \frac{r_i + r_f e^{tv}}{1 + e^{tv}}, \quad (55)$$

where v is the parameter setting the rapidity of the contraction. In accordance with the conditions of applicability, the parameter vr_g must be a small value. Shown in Fig. 3 are temporal dependencies of the Proca field strength computed using the formula (54) for the law (55) with different parameters v and m .

The dynamics of the monopole field of the charge approaching a BH is very unusual – the resulting impulse has the polarity, which is opposite to that expected from the Lenz rule in the Maxwell's electrodynamics.

Strictly speaking, the results obtained above are valid for the eternal BH and now we will discuss the influence of replacing this idealized source of gravity by a more realistic

Fig. 3 The evolution of the Proca field strength of the shell contracted according to the law (55) depending on the rapidity (a) and the Proca mass (b). The additional parameters are also shown in the figure. Thin dashed curves correspond to the first term in the formula (54)

gravitating object formed as a result of the collapse. According to the work [12], if a charged star collapses into a BH, the exterior massive vector field associated with the charge of the star decays over time of the order of $1/m$. However, the results of our calculations show that this is not the case.

The existence of the event horizon, as follows from our calculations, is a sufficient condition for the discontinuity of the Proca field behavior in the limit $m \rightarrow 0$. On the other hand, the event horizon is well known to form even before the end of the collapse [23]. Namely, the region of space-time enclosed under the horizon is expanding with the speed of light until the moment when the stellar surface crosses the boundary of this region. A characteristic time scale of this process is consistent with r_g . Consequently, the exterior field created by the charge, which is located in the volume of the star, also decreases quickly as the charge crosses the

expanding horizon, since the collapse is much faster process than the contraction of the shell considered in our problem.

Hence, the time scale for loss of massive vector hair on a collapsing star coincides in order of magnitude with r_g . The connection between the massive case and the massless one is discontinuous, contrary to the statements of [11, 12].

5 Summary and conclusion

As was shown in this paper, the time scale of the screening of the Proca field associated with a source approaching the event horizon is determined by the mass of the BH. Being a conceptually identical process, the massive vector hair loss during the collapse of a star has the same temporal behavior. Considered in the immediate vicinity of the event horizon, the physics of a classical massive vector field in the limit of the Proca mass going to zero does not coincide with Maxwell's physics.

The incorrect estimate of the time scale for loss of the massive vector hair in the paper mentioned above led authors of [13] to a significantly low estimate of the electrical asymmetry of the Universe, which is generated by superheavy BHs in the centers of large galaxies. The difference in the mobilities of electrons and protons in the ionized plasma surrounding such BHs leads to an uncompensated radial current. If the photon has a mass, due to the radial redistribution of the charge, the positive charge of the protons is screened more strongly than the negative charge of the electrons, which leads to an excess of the negative charge in the Universe. A study of the generated asymmetry requires a careful approach and will be done in the future work.

A sharp difference in behavior of low-frequency modes of electromagnetic radiation in the vicinity of a BH for a massive photon and massless one provides a unique experimental opportunity to establish whether the photon has a mass or is it a massless particle [24]. A similar conclusion can be also made by comparing the theoretical and experimental [25] data on the electrical asymmetry of the Universe.

Acknowledgements I wish to thank A. D. Dolgov for suggesting this problem. I grateful to A. A. Pomeransky and V. M. Khatsymovsky for the discussion and careful reading of the manuscript. I also wish to thank A. S. Rudenko and A. V. Reznichenko for the helpful discussions. The work is supported by the RSF Grant No. 19-42-02004.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Being completely theoretical, this work has no associated data.]

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Funded by SCOAP³.

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