Scalar 1-loop Feynman integrals as meromorphic functions in space-time dimension d , II : special kinematics

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Abstract Based on the method developed in Phan and Riemann (Phys Lett B 791:257, 2019), detailed analytic results for scalar one-loop two-, three-, four-point integrals in general d -dimension are presented in this paper. The calculations are considered all external kinematic configurations and internal mass assignments. Analytic formulas are expressed in terms of generalized hypergeometric series such as Gauss ${}_2F_1$, Appell F_1 and Lauricella F_S functions.

1 Introduction

Scalar one-loop integrals in general d are important for several reasons. In general framework for computing two-loop and higher-loop corrections, higher-terms in the ϵ -expansion ($\epsilon = 2 - d/2$) for one-loop integrals are necessary for building blocks. For example, they are used for building counterterms. Furthermore, in the evaluations for multi-loop Feynman integrals, we may combine several methods in [1, 2] to optimize the master integrals. As a result, the resulting integrals may include of one-loop functions in arbitrary space-time dimensions. Last but not least, scalar one-loop functions at $d = 4 + 2n \pm 2\epsilon$ with $n \in \mathbb{N}$ may be taken into account in the reduction for tensor one-loop Feynman integrals [3].

One-loop functions in space-time dimensions d have been performed in [4–9]. However, not all of the calculations have covered at general configurations of external momenta and internal masses. Recently, scalar one-loop three-point functions have been performed by applying multiple unitarity cuts for Feynman diagrams [10]. In Ref. [10], analytic results have been presented in terms of hypergeometric functions in special cases of external invariants and internal masses. In more general cases, the results have only presented ϵ^0 -terms in space-time $d = 4 - 2\epsilon$. The algebraic structure of cut Feynman integrals, the diagrammatic coaction and its appli-

cations have proposed in [11–13]. However, the detailed analytic results for one-loop Feynman integrals have not been shown in the mentioned papers. More recently, detailed analytic results for one-loop three-point functions which are expressed in terms of Appell F_1 hypergeometric functions have been reported in [15].

A recurrence relation in d for Feynman loop integrals has proposed by Tarasov [2, 14, 16]. By solving the differential equation in d , analytic results for scalar one-loop integrals up to four-points have been expressed in terms of generalized hypergeometric series such as Gauss ${}_2F_1$, Appell F_1 , Lauricella–Saran F_S functions. In [16], boundary terms have obtained by applying asymptotic theory of complex Laplace-type integrals. These terms are only valid in sub-domain of external momentum and mass configurations which the theory are applicable. Hence, the general solutions for one-loop integrals in arbitrary kinematics have not been found [16], as pointed out in [17, 18]. General solutions for this problem have been derived in [19] which proposed a different kind of recursion relation for one-loop integrals in comparison with [2, 14, 16]. In the scope of this paper, based on the method in Ref. [19], detailed analytic results for scalar one-loop two-, three- and four-point functions in general d -dimensions are presented. The calculations are considered all external kinematic configurations and internal mass assignments. Thus, we go beyond the material presented in [19].

The layout of the paper is as follows: In Sect. 2, we discuss briefly the method for evaluating one-loop integrals. We then apply this method for computing scalar one-loop two-, three- and four-point functions in Sects. 3, 4, and 5. Conclusions and plans for future work are presented in Sect. 6.

2 Methods

In this section, we describe briefly the method for evaluating scalar one-loop N -point Feynman integrals. Detailed

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Fig. 1 Generic Feynman diagrams at one-loop with N external momenta

description for this method can be found in Ref. [19]. A general recursion relation between scalar one-loop N -point and $(N-1)$ -point Feynman integrals is shown in this section. From the representation, analytic formulas for scalar one-loop N -point functions can be constructed from basic integrals which are scalar one-point integrals. For illustrating, analytic expressions for scalar one-loop two-, three-, four-point functions are derived in detail in next sections.

The scalar one-loop N -point Feynman integrals are defined:

$$J_N(d; \{p_i p_j\}, \{m_i^2\}) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{P_1 P_2 \dots P_N}. \quad (1)$$

The inverse Feynman propagators are given:

$$P_j = (k + q_j)^2 - m_j^2 + i\rho, \text{ for } j = 1, 2, \dots, N. \quad (2)$$

where p_j (m_j) for $j = 1, 2, \dots, N$ are external momenta (internal masses) respectively. The momenta q_j are given: $q_1 = p_1, q_2 = p_1 + p_2, \dots, q_j = \sum_{i=1}^j p_i$ and $q_N = \sum_{j=1}^N p_j = 0$ thanks to momentum conservation. They are inward as described in Fig. 1. The term $i\rho$ is Feynman's prescription and d is space-time dimension. Several cases of physical interests for d are $d = 4 + 2n \pm 2\epsilon$ with $n \in \mathbb{N}$. In the complex-mass scheme [20], the internal masses take the form of $m_j^2 = m_{0j}^2 - im_{0j}\Gamma_j$ where $\Gamma_j \geq 0$ are decay widths of unstable particles.

The Cayley and Gram determinants [16] related to one-loop Feynman N -point topologies are defined as follows:

$$Y_N \equiv Y_{12\dots N} = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1N} \\ Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix},$$

$$G_{N-1} \equiv G_{12\dots N} = -2^N \begin{vmatrix} q_1^2 & q_1 q_2 & \dots & q_1 q_{N-1} \\ q_1 q_2 & q_2^2 & \dots & q_2 q_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_1 q_{N-1} & q_2 q_{N-1} & \dots & q_{N-1}^2 \end{vmatrix} \quad (3)$$

with $Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2$.

In this report, analytic solutions for one-loop integrals are expressed in terms of generalized hypergeometric with arguments given by ratios of the above determinants. Hence, it is worth to introduce the following kinematic variables

$$R_N \equiv R_{12\dots N} = -\frac{Y_N}{G_{N-1}} \text{ for } G_{N-1} \neq 0. \quad (4)$$

The kinematics R_N play a role of the squared internal masses. In fact, when we shift $m_j^2 \rightarrow m_j^2 - i\rho$, one verifies easily that $R_N \rightarrow R_N - i\rho$ [16].

The recursion relation for J_N [19] is given (master equation):

$$J_N(d; \{p_i p_j\}, \{m_i^2\}) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \times \frac{\Gamma(-s) \Gamma\left(\frac{d-N+1}{2} + s\right) \Gamma(s+1)}{2\Gamma\left(\frac{d-N+1}{2}\right)} \left(\frac{1}{R_N}\right)^s \times \sum_{k=1}^N \left(\frac{\partial_k R_N}{R_N}\right) \mathbf{k}^- J_N(d+2s; \{p_i p_j\}, \{m_i^2\}), \quad (5)$$

for $i, j = 1, 2, \dots, N$ and $\partial_k = \partial/\partial m_k^2$. Here the operator $\mathbf{k}^-$ is defined as [19]

$$\mathbf{k}^- J_N(d; \{p_i p_j\}, \{m_i^2\}) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{P_1 P_2 \dots P_{k-1} P_{k+1} \dots P_{N-1} P_N} \quad (6)$$

The relation (5) indicates that the integral J_N can be constructed by taking one-fold Mellin–Barnes (MB) integration over J_{N-1} in $d+2s$. This representation has several advantages. First, analytic formulas for J_N can be derived from basic functions which are scalar one-loop one-point functions. Second, J_N is expressed as functions of kinematic variables such as m_j^2 for $j = 1, 2, \dots, N$ and R_N . As a consequence of this fact, analytic expressions for J_N reflect the symmetry as well as threshold behavior of the corresponding Feynman topologies. Two special cases of (5) are also mentioned as follows:

1. $Y_N \rightarrow 0$ and $G_{N-1} \neq 0$: In this case, $R_N \rightarrow 0$ and we have [16] (deriving this equation for $N = 4$ is shown in the Appendix D)

$$J_N(d; \{p_i p_j\}, \{m_i^2\}) = \frac{1}{d - N - 1}$$

$$\times \sum_{k=1}^N \left(\frac{\partial_k Y_N}{G_{N-1}} \right) \mathbf{k}^- J_N(d-2; \{p_i p_j\}, \{m_i^2\}). \quad (7)$$

2. $G_{N-1} \rightarrow 0$ and $Y_N \neq 0$: In this case, $R_N \rightarrow \infty$. We close the integration contour in (5) to the right. Taking residue contributions from poles of $\Gamma(\cdots - s)$. In the limit $R_N \rightarrow \infty$, we find only the term with $s = 0$ is non-zero. The result then reads

$$J_N(d; \{p_i p_j\}, \{m_i^2\}) = -\frac{1}{2} \sum_{k=1}^N \left(\frac{\partial_k Y_N}{Y_N} \right) \mathbf{k}^- J_N(d; \{p_i p_j\}, \{m_i^2\}). \quad (8)$$

This equation is equivalent to (65) in [21] and (3) in [16].

We turn our attention to apply the method for evaluating scalar one-loop Feynman integrals. The detailed evaluations for scalar one-loop two-, three- and four-point functions are presented in next sections. As we pointed out in this section, the prescription $i\rho$ always follows with R_N as $R_N - i\rho$. In order to simplify the notation, we omit $i\rho$ in R_N in the next calculations. This term puts back into the final results when it is necessary.

3 One-loop two-point functions

The master equation for J_2 is obtained by setting $N = 2$ in (5). MB representation for J_2 then reads

$$J_2 \equiv J_2(d; p^2, m_1^2, m_2^2) = \frac{1}{2\pi i} \times \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s)\Gamma(\frac{2-d}{2}-s)\Gamma(\frac{d-1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-1}{2})} \times \left(\frac{1}{R_2} \right)^s \left\{ \left(\frac{\partial_2 R_2}{R_2} \right) (m_1^2)^{\frac{d-2}{2}+s} + (1 \leftrightarrow 2) \right\}. \quad (9)$$

Note that we used the analytic formula for J_1 in [22] with d shifted to $d \rightarrow d + 2s$. We write J_1 in $d + 2s$ explicitly as follows:

$$J_1(d + 2s; m^2) = -\Gamma\left(\frac{2-d}{2} - s\right) (m)^{\frac{d-2}{2}+s}. \quad (10)$$

In order to evaluate the MB integrals in (9), we close the integration contour to the right. The residue contributions to J_2 at the sequence poles of $\Gamma(-s)$ and $\Gamma(\frac{2-d}{2} - s)$ are taken into account.

First, we calculate the residue at the poles of $\Gamma(-s)$. In this case, $s = m$ for $m = 0, 1, \dots, \mathbb{N}$. Subsequently, we can apply the reflect formula for gamma functions (103) in the

Appendix B. In detail, it is implied that

$$\Gamma\left(\frac{2-d}{2} - s\right) \Gamma\left(\frac{d}{2} + s\right) = -(-1)^s \Gamma\left(\frac{4-d}{2}\right) \Gamma\left(\frac{d-2}{2}\right). \quad (11)$$

With the help of (11), the MB representation in (9) is casted into the form of

$$\frac{J_2}{\Gamma(\frac{4-d}{2})} \Big|_{s=m} = -\frac{\Gamma(\frac{d-2}{2})}{2\Gamma(\frac{d}{2})} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \times \frac{\Gamma(-s)\Gamma(\frac{d-1}{2}+s)\Gamma(s+1)}{\Gamma(\frac{d}{2}+s)} \times \left\{ \left(\frac{\partial_2 R_2}{R_2} \right) (m_1^2)^{\frac{d-2}{2}} \times \left(-\frac{m_1^2}{R_2} \right)^s + (1 \leftrightarrow 2) \right\}. \quad (12)$$

Using (1.6.1.6) in Ref. [25], these MB integrals are expressed in terms of Gauss hypergeometric series as follows:

$$\frac{J_2}{\Gamma(\frac{4-d}{2})} \Big|_{s=m} = -\frac{\Gamma(\frac{d-2}{2})}{2\Gamma(\frac{d}{2})} \times \left\{ \left(\frac{\partial_2 R_2}{R_2} \right) (m_1^2)^{\frac{d-2}{2}} {}_2F_1 \left[1, \frac{d-1}{2}; \frac{m_1^2}{R_2} \right] + (1 \leftrightarrow 2) \right\}, \quad (13)$$

provided that $|m_1^2/R_2| < 1$, $|m_2^2/R_2| < 1$ and $\text{Re}(d-2) > 0$. Using (1.3.15) in [25], we arrive at another representation for (13)

$$\frac{J_2}{\Gamma(\frac{4-d}{2})} \Big|_{s=m} = -\frac{\Gamma(\frac{d-2}{2})}{2\Gamma(\frac{d}{2})} \left\{ \left(\frac{\partial_2 R_2}{R_2} \right) \frac{(m_1^2)^{\frac{d-2}{2}}}{\sqrt{1-m_1^2/R_2}} \times {}_2F_1 \left[\frac{d-2}{2}, \frac{1}{2}; \frac{m_1^2}{R_2} \right] + (1 \leftrightarrow 2) \right\}, \quad (14)$$

provided that $|m_1^2/R_2| < 1$, $|m_2^2/R_2| < 1$ and $\text{Re}(d-2) > 0$.

We next consider the residue at the second sequence poles of $\Gamma(\frac{2-d}{2} - s)$. In this case, $s = \frac{2-d}{2} + m$ for $m \in \mathbb{N}$. These contributions read

$$J_2 \Big|_{s=\frac{2-d}{2}+m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(\frac{d-2}{2}-m)\Gamma(\frac{4-d}{2}+m)\Gamma(m+\frac{1}{2})}{2\Gamma(\frac{d-1}{2})} \times (R_2)^{\frac{d-2}{2}} \left[\left(\frac{\partial_2 R_2}{R_2} \right) \left(\frac{m_1^2}{R_2} \right)^m \right]$$

$$+ \left(\frac{\partial_1 R_2}{R_2} \right) \left(\frac{m_2^2}{R_2} \right)^m \Bigg] \quad (15)$$

$$= \frac{\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{4-d}{2}\right)}{2\Gamma\left(\frac{d-1}{2}\right)} (R_2)^{\frac{d-2}{2}}$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m+1)} \left[\left(\frac{\partial_2 R_2}{R_2} \right) \left(\frac{m_1^2}{R_2} \right)^m + \left(\frac{\partial_1 R_2}{R_2} \right) \left(\frac{m_2^2}{R_2} \right)^m \right] \quad (16)$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{4-d}{2}\right) \Gamma\left(\frac{d-2}{2}\right)}{2\Gamma\left(\frac{d-1}{2}\right)} (R_2)^{\frac{d-4}{2}}$$

$$\times \left[\frac{\partial_2 R_2}{\sqrt{1 - m_1^2/R_2}} + \frac{\partial_1 R_2}{\sqrt{1 - m_2^2/R_2}} \right]. \quad (17)$$

Noting that from (15) to (16), we have already applied the reflect formulas for gamma functions [see (103) in Appendix A for more detail]. Summing all the above contributions in Eqs. (14, 17), we finally get

$$\frac{J_2}{\Gamma\left(\frac{4-d}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}$$

$$\times (R_2)^{\frac{d-4}{2}} \left[\frac{\partial_2 R_2}{\sqrt{1 - m_1^2/R_2}} + \frac{\partial_1 R_2}{\sqrt{1 - m_2^2/R_2}} \right]$$

$$- \frac{\Gamma\left(\frac{d-2}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)} \left\{ \left(\frac{\partial_2 R_2}{R_2} \right) \frac{(m_1^2)^{\frac{d-2}{2}}}{\sqrt{1 - m_1^2/R_2}} \right.$$

$$\left. \times {}_2F_1\left[\frac{d-2}{2}, \frac{1}{2}; \frac{m_1^2}{R_2}\right] + (1 \leftrightarrow 2) \right\}, \quad (18)$$

provided that $|m_1^2/R_2| < 1$, $|m_2^2/R_2| < 1$ and $\text{Re}(d-2) > 0$. Equation (18) is unchanged with exchanging $m_1^2 \leftrightarrow m_2^2$. This reflects the symmetry of the scalar one-loop two-point Feynman diagrams. The result in (18) has shown in [18] and gives fully agreement with [16]. It is an important to remark that the solution for J_2 in (18) is also valid when $d \rightarrow d+2n$ for $n \in \mathbb{N}$. We would like to stress that one can perform the analytic continuation for J_2 in (18) to extend the kinematic regions for one-loop two-point functions. The analytic continuation formulas for Gauss hypergeometric functions are given from (109) to (112) in the Appendix B. As an example, using (111), the result reads

$$\frac{J_2}{\Gamma\left(\frac{4-d}{2}\right)} = \left(\frac{\partial_2 R_2}{R_2} \right)$$

$$\times (m_1^2)^{\frac{d-2}{2}} {}_2F_1\left[1, \frac{d-1}{2}; \frac{m_1^2}{R_2}\right]$$

$$+ (1 \leftrightarrow 2). \quad (19)$$

With (108), we arrive at

$$\frac{J_2}{\Gamma\left(\frac{4-d}{2}\right)} = \left(\frac{\partial_2 R_2}{R_2} \right)$$

$$\times R_2^{\frac{d-2}{2}} {}_2F_1\left[\frac{4-d}{2}, \frac{1}{2}; 1 - \frac{m_1^2}{R_2}\right]$$

$$+ (1 \leftrightarrow 2). \quad (20)$$

We are going to consider special cases for scalar one-loop two-point Feynman integrals.

3.1 $G_1 \neq 0$ and $R_2 = 0$

If $R_2 = 0$, we verify that $p^2 = (m_1 - m_2)^2$, or $(m_1 + m_2)^2$. Applying (7) for $N = 2$ the result reads

$$\frac{J_2}{\Gamma\left(\frac{4-d}{2}\right)} = \frac{1}{d-3} \left[\frac{(m_1^2)^{\frac{d-3}{2}}}{(m_1 \pm m_2)^3} + (1 \leftrightarrow 2) \right]. \quad (21)$$

In the limit of $m_1 \rightarrow m_2 = m$ and $p^2 = 4m^2$, the result reads

$$J_2 = \frac{\Gamma\left(\frac{4-d}{2}\right)}{2(d-3)} (m^2)^{\frac{d-6}{2}}. \quad (22)$$

3.2 $G_1 = 0$

Following (8) for $N = 2$, we arrive at

$$J_2 = \Gamma\left(\frac{4-d}{2}\right) (m_2^2)^{\frac{d-4}{2}} {}_2F_1\left[\frac{4-d}{2}, 1; 1 - \frac{m_1^2}{m_2^2}\right]$$

$$= \Gamma\left(\frac{2-d}{2}\right) \frac{(m_2^2)^{\frac{d-2}{2}} - (m_1^2)^{\frac{d-2}{2}}}{m_1^2 - m_2^2}. \quad (23)$$

If $m_1^2 = m_2^2$, one presents J_2 as

$$J_2 = \Gamma\left(\frac{4-d}{2}\right) (m^2)^{\frac{d-4}{2}}. \quad (24)$$

3.3 $R_2 = m_1^2$ or $R_2 = m_2^2$

For the case of $R_2 = m_1^2$ or $R_2 = m_2^2$, one relies on (20). If $R_2 = m_1^2 = m_2^2 = m^2$, the result in (20) simplifies to

$$J_2 = \Gamma\left(\frac{4-d}{2}\right) (m^2)^{\frac{d-4}{2}}. \quad (25)$$

Under the condition $\text{Re}(d-4) > 0$, when $m^2 \rightarrow 0$, one then gets $J_2 = 0$.

$$3.4 \quad m_1^2 = m_2^2 = 0$$

If one of $m_i^2 = 0$, for $i = 1, 2$, we rely on (18). In the case of $m_1^2 = m_2^2 = 0$, from (18) we get

$$J_2 = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{4-d}{2}\right) \Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(-\frac{p^2}{4}\right)^{\frac{d-4}{2}}, \quad (26)$$

provided that $\text{Re}(d-2) > 0$. We note that p^2 means $p^2 + i\rho$. Therefore, if $p^2 > 0$ the term $(-p^2/4)^{\frac{d-4}{2}}$ is well-defined.

4 One-loop three-point functions

Setting $N = 3$ in (5), master equation for J_3 reads

$$J_3 \equiv J_3(d; \{p_i^2\}, \{m_i^2\}) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \times \frac{\Gamma(-s) \Gamma\left(\frac{d-2}{2} + s\right) \Gamma(s+1)}{2\Gamma\left(\frac{d-2}{2}\right)} \left(\frac{1}{R_3}\right)^s \times \sum_{k=1}^3 \left(\frac{\partial_k R_3}{R_3}\right) \mathbf{k}^- J_3(d+2s; \{p_i^2\}, \{m_i^2\}), \quad (27)$$

for $i = 1, 2, 3$. The term $\mathbf{k}^- J_3(d+2s; \{p_i p_j\}, \{m_i^2\})$ becomes scalar one-loop two-point functions by shrinking an propagator k -th in the integrand of J_3 . In the next steps, we take the contour integrals in (27). In order to understand how to take the contour integrals, we chose the term with $k = 3$ in (27) for illustrating. This term is written explicitly as follows:

$$J_{3,(123)} = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(s+1) \Gamma\left(\frac{d-2}{2} + s\right)}{2\Gamma\left(\frac{d-2}{2}\right)} \times \left(\frac{1}{R_3}\right)^s \left(\frac{\partial_3 R_3}{R_3}\right) J_2(d+2s; p_1^2, m_1^2, m_2^2) \quad (28)$$

$$= -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \times \frac{\sqrt{\pi} \Gamma(-s) \Gamma(s+1) \Gamma\left(\frac{d-2}{2} + s\right) \Gamma\left(\frac{4-d}{2} - s\right) \Gamma\left(\frac{d-2}{2} + s\right)}{4\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d-1}{2} + s\right)} \times \left(\frac{\partial_3 R_3}{R_3}\right) \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1-m_2^2/R_{12}}} \right] \times (R_{12})^{\frac{d-4}{2}} \left(-\frac{R_{12}}{R_3}\right)^s + \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \times \frac{\Gamma(-s) \Gamma(s+1) \Gamma\left(\frac{d-2}{2} + s\right) \Gamma\left(\frac{4-d}{2} - s\right) \Gamma\left(\frac{d-2}{2} + s\right)}{4\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d}{2} + s\right)}$$

$$\times \left(\frac{\partial_3 R_3}{R_3}\right) \left\{ \left(\frac{\partial_2 R_{12}}{R_{12}}\right) \times \frac{(m_1^2)^{\frac{d-2}{2}}}{\sqrt{1-m_1^2/R_{12}}} {}_2F_1\left[\frac{d-2}{2} + s, \frac{1}{2}; \frac{m_1^2}{R_{12}}\right] \times \left(-\frac{m_1^2}{R_3}\right)^s + (1 \leftrightarrow 2) \right\}. \quad (29)$$

For taking the MB integrals in (29), one closes the integration contour to the right. The residue contributions at the poles of the $\Gamma(\dots - s)$ are taken into account.

First, the contributions of residua at the poles $s = m$ with $m \in \mathbb{N}$. In this case, one first applies the reflect formula (103) for gamma function:

$$\Gamma\left(\frac{4-d}{2} - s\right) \Gamma\left(\frac{d-2}{2} + s\right) = (-1)^s \Gamma\left(\frac{4-d}{2}\right) \Gamma\left(\frac{d-2}{2}\right). \quad (30)$$

Using this identity, the first MB integration reads

$$\frac{J_{3,(123)}^{1\text{-term}}}{\Gamma\left(\frac{4-d}{2}\right)} \Big|_{s=m} = -\frac{\sqrt{\pi}}{4} \left(\frac{\partial_3 R_3}{R_3}\right) \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1-m_2^2/R_{12}}} \right] (R_{12})^{\frac{d-4}{2}} \times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(s+1) \Gamma\left(\frac{d-2}{2} + s\right)}{\Gamma\left(\frac{d-1}{2} + s\right)} \times \left(-\frac{R_{12}}{R_3}\right)^s. \quad (31)$$

This MB integral is then expressed in terms of hypergeometric ${}_2F_1$ as follows:

$$\frac{J_{3,(123)}^{1\text{-term}}}{\Gamma\left(\frac{4-d}{2}\right)} \Big|_{s=m} = -\frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \times \left(\frac{\partial_3 R_3}{R_3}\right) \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + (1 \leftrightarrow 2) \right] \times (R_{12})^{\frac{d-4}{2}} {}_2F_1\left[\frac{d-2}{2}, 1; \frac{R_{12}}{R_3}\right], \quad (32)$$

provided that $|R_{12}/R_3| < 1$ and $\text{Re}(d-3) > 0$. The second MB integral reads

$$\frac{J_{3,(123)}^{2\text{-term}}}{\Gamma\left(\frac{4-d}{2}\right)} \Big|_{s=m} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(s+1) \Gamma\left(\frac{d-2}{2} + s\right)}{\Gamma\left(\frac{d}{2} + s\right)} \times \left(\frac{\partial_3 R_3}{4R_3}\right) \quad (33)$$

$$\begin{aligned}
& \times \left\{ \left(\frac{\partial_2 R_{12}}{R_{12}} \right) \frac{(m_1^2)^{\frac{d-2}{2}}}{\sqrt{1-m_1^2/R_{12}}} {}_2F_1 \left[\frac{d-2}{2} + s, \frac{1}{2}; \frac{m_1^2}{R_{12}} \right] \right. \\
& \quad \left. + (1 \leftrightarrow 2) \right\} \\
& = \frac{\Gamma(\frac{d-2}{2})}{4\Gamma(\frac{d}{2})} \\
& \quad \times \left(\frac{\partial_3 R_3}{R_3} \right) \left[\left(\frac{\partial_2 R_{12}}{R_{12}} \right) \frac{(m_1^2)^{\frac{d-2}{2}}}{\sqrt{1-m_1^2/R_{12}}} \right. \\
& \quad \left. \times F_1 \left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \right], \quad (34)
\end{aligned}$$

provided that $|m_{(1,2)}^2/R_3|, |m_{(1,2)}^2/R_{12}| < 1$ and $\text{Re}(d-2) > 0$.

In the next steps, the residue contributions at the second sequence poles $s = \frac{4-d}{2} + m$ for $m \in \mathbb{N}$ are taken into account. The next MB integrations are considered as follows:

$$\begin{aligned}
& J_{3,(123)}^{1-\text{term}} \Big|_{s=\frac{4-d+2m}{2}} \\
& = -\frac{\sqrt{\pi}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(s+1) \Gamma^2\left(\frac{d-2}{2} + s\right) \Gamma\left(\frac{4-d}{2} - s\right)}{4 \Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d-1}{2} + s\right)}
\end{aligned} \quad (35)$$

$$\begin{aligned}
& \times \left(\frac{\partial_3 R_3}{R_3} \right) \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1-m_2^2/R_{12}}} \right] \\
& \quad \times (R_{12})^{\frac{d-4}{2}} \left(-\frac{R_{12}}{R_3} \right)^s \\
& = \left(\frac{\partial_3 R_3}{2 R_3} \right) \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1-m_2^2/R_{12}}} \right] (R_3)^{\frac{d-4}{2}} \\
& \quad \times {}_2F_1 \left[\frac{1}{2}, 1; \frac{R_{12}}{R_3} \right], \quad (36)
\end{aligned}$$

$$\begin{aligned}
& \frac{J_{3,(123)}^{2-\text{term}}}{\Gamma\left(\frac{4-d}{2}\right)} \Big|_{s=\frac{4-d}{2}+m} = - \left(\frac{\partial_3 R_3}{2 R_3} \right) \left[\left(\frac{\partial_2 R_{12}}{2 R_{12}} \right) \frac{m_1^2 (R_3)^{\frac{d-4}{2}}}{\sqrt{1-m_1^2/R_{12}}} \right. \\
& \quad \left. \times F_1 \left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \right]
\end{aligned} \quad (37)$$

provided that $|m_{(1,2)}^2/R_3|, |m_{(1,2)}^2/R_{12}| < 1$ and $|R_{12}/R_3| < 1$.

Summing all the above contributions, the final result for J_3 is written as a compact form

$$\begin{aligned}
& \frac{J_3}{\Gamma\left(\frac{4-d}{2}\right)} = -J_{123}^{(d=4)} (R_3)^{\frac{d-4}{2}} + J_{123}^{(d)} \\
& \quad + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \\
& \quad + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}. \quad (38)
\end{aligned}$$

where $J_{123}^{(d)}$ is obtained from (32) and (33). It is given by

$$\begin{aligned}
J_{123}^{(d)} & = -\frac{\sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right)}{4\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{\partial_3 R_3}{R_3} \right) \\
& \quad \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + (1 \leftrightarrow 2) \right] \\
& \quad \times (R_{12})^{\frac{d-4}{2}} {}_2F_1 \left[\frac{d-2}{2}, 1; \frac{R_{12}}{R_3} \right] \\
& \quad + \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\Gamma\left(\frac{d}{2}\right)} \left(\frac{\partial_3 R_3}{R_3} \right) \left[\left(\frac{\partial_2 R_{12}}{R_{12}} \right) \frac{(m_1^2)^{\frac{d-2}{2}}}{\sqrt{1-m_1^2/R_{12}}} \right. \\
& \quad \left. \times F_1 \left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \right], \quad (39)
\end{aligned}$$

provided that $|m_i^2/R_{ij}| < 1$, $|R_{ij}/R_{ijk}| < 1$ for $i, j, k = 1, 2, 3$ and $\text{Re}(d-2) > 0$. The latter condition always meets when $d > 2$. The kinematic variables R_{ijk} , R_{ij} and m_i for $i, j, k = 1, 2, 3$, etc., may not satisfy the former conditions. If the absolute value of the arguments of ${}_2F_1$ and the Appell functions F_1 in (39) are larger than one, we have to perform analytic continuations for these functions as in [25, 31]. The result for J_3 has been shown in [18, 19]. The term $J_{123}^{(d=4)}$ is obtained from (36, 37) or taking $d \rightarrow 4$ of (39). This term is given

$$\begin{aligned}
J_{123}^{(d=4)} & = - \left(\frac{\partial_3 R_3}{2 R_3} \right) \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} \right. \\
& \quad \left. + (1 \leftrightarrow 2) \right] {}_2F_1 \left[\frac{1}{2}, 1; \frac{R_{12}}{R_3} \right] \\
& \quad + \left(\frac{\partial_3 R_3}{2 R_3} \right) \left[\left(\frac{\partial_2 R_{12}}{2 R_{12}} \right) \frac{m_1^2}{\sqrt{1-m_1^2/R_{12}}} \right. \\
& \quad \left. \times F_1 \left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \right]. \quad (40)
\end{aligned}$$

We emphasize that the solution (38) for J_3 with (39) is equivalent to (74) in Ref. [16]. But the terms $J_{123}^{(d=4)}, \dots$ in our solution cover the condition (73) in Ref. [16]. Since the boundary term given in (74) of Ref. [16] was obtained by asymptotic theory of complex Laplace-type integrals. This term is only valid in a kinematic sub-domain in which the asymptotic theory of Laplace-type can be applied. The analytic continuation for the boundary term in [16] has not been discussed. We provide a complete analytic solution for J_3 in comparison with (74) in Ref. [16]. We refer to our previous work [18] in which the numerical studies for this problem have discussed.

One-fold integral and all transformations for F_1 can be found in Appendix B. Applying the relation (122) for F_1 in Appendix C, we arrive at another representation for (39, 40):

$$J_{123}^{(d)} = \frac{(\partial_3 R_3)(\partial_2 R_{12})}{2(m_1^2 - R_3)} (m_1^2)^{\frac{d-4}{2}} \times F_1 \left(1; \frac{4-d}{2}, 1; \frac{3}{2}; 1 - \frac{R_{12}}{m_1^2}, \frac{R_{12} - m_1^2}{R_3 - m_1^2} \right) + (1 \leftrightarrow 2), \quad (41)$$

$$J_{123}^{(d=4)} = \frac{(\partial_3 R_3)(\partial_2 R_{12})}{2(m_1^2 - R_3)} \times {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{R_{12} - m_1^2}{R_3 - m_1^2} \right] + (1 \leftrightarrow 2), \quad (42)$$

provided that the absolute value of the arguments of ${}_2F_1$ and the Appell functions F_1 in this presentation are less than 1.

4.1 Massless internal lines

For the massless case, under the condition $\text{Re}(d-2) > 0$, all terms related to Appell F_1 functions in (38) vanish, the result then reads

$$\frac{J_3}{\Gamma\left(\frac{4-d}{2}\right)} = - (R_3)^{\frac{d-4}{2}} \left(\frac{\partial_3 R_3}{2R_3} \right) \Big|_{m_i^2 \rightarrow 0} {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{R_{12}}{R_3} \right] + \frac{\sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right)}{4 \Gamma\left(\frac{d-1}{2}\right)} \left(\frac{\partial_3 R_3}{R_3} \right) \Big|_{m_i^2 \rightarrow 0} \times (R_{12})^{\frac{d-4}{2}} {}_2F_1 \left[\begin{matrix} \frac{d-2}{2}, 1; \\ \frac{d-1}{2}; \end{matrix} \frac{R_{12}}{R_3} \right] + \left\{ (1, 2, 3) \rightarrow (2, 3, 1) \right\} + \left\{ (1, 2, 3) \rightarrow (3, 1, 2) \right\}. \quad (43)$$

In order to cross check with the result in [6], we write J_3 as a function of p_1^2, p_2^2, p_3^2 explicitly

$$\frac{J_3}{\Gamma\left(\frac{4-d}{2}\right)} = - \left(\frac{p_2^2 + p_3^2 - p_1^2}{2 p_2^2 p_3^2} \right) \left(\frac{p_1^2 p_2^2 p_3^2}{\lambda(p_1^2, p_2^2, p_3^2)} \right)^{\frac{d-4}{2}} \times {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} - \frac{\lambda(p_1^2, p_2^2, p_3^2)}{4 p_2^2 p_3^2} \right] + \frac{\sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right)}{4 \Gamma\left(\frac{d-1}{2}\right)} \left(\frac{p_2^2 + p_3^2 - p_1^2}{p_2^2 p_3^2} \right) \left(-\frac{p_1^2}{4} \right)^{\frac{d-4}{2}} \times {}_2F_1 \left[\begin{matrix} 1, \frac{d-2}{2}; \\ \frac{d-1}{2}; \end{matrix} - \frac{\lambda(p_1^2, p_2^2, p_3^2)}{4 p_2^2 p_3^2} \right] + \left\{ (1, 2, 3) \rightarrow (2, 3, 1) \right\} + \left\{ (1, 2, 3) \rightarrow (3, 1, 2) \right\}. \quad (44)$$

Here $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is the Källén function. We remark that $p_i^2 \rightarrow p_i^2 + i\rho$ in this formula. With applying (1.3.3.5) in Ref. [25], one can present J_3 as

$$\frac{J_3}{\Gamma\left(\frac{4-d}{2}\right)} = \frac{2}{(p_1^2 - p_2^2 - p_3^2)} \left(\frac{p_1^2 p_2^2 p_3^2}{\lambda(p_1^2, p_2^2, p_3^2)} \right)^{\frac{d-4}{2}} \times {}_2F_1 \left[\begin{matrix} 1, \frac{1}{2}; \\ \frac{3}{2}; \end{matrix} \frac{\lambda(p_1^2, p_2^2, p_3^2)}{(p_2^2 + p_3^2 - p_1^2)^2} \right] + \frac{\sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \times \frac{(-p_1^2/4)^{\frac{d-4}{2}}}{p_2^2 + p_3^2 - p_1^2} {}_2F_1 \left[\begin{matrix} 1, \frac{1}{2}; \\ \frac{d-1}{2}; \end{matrix} \frac{\lambda(p_1^2, p_2^2, p_3^2)}{(p_2^2 + p_3^2 - p_1^2)^2} \right] + \left\{ (1, 2, 3) \rightarrow (2, 3, 1) \right\} + \left\{ (1, 2, 3) \rightarrow (3, 1, 2) \right\}, \quad (45)$$

provided that $\left| \frac{\lambda(p_1^2, p_2^2, p_3^2)}{(p_2^2 + p_3^2 - p_1^2)^2} \right| < 1$ and $\text{Re}(d-2) > 0$.

This equation is equivalent to (10) in [6]. We note that we can arrive to this result by inserting J_2 at $d+2s$ in (26) into (27) and taking the corresponding MB integrals.

4.2 $R_{ij} = 0$

We consider the terms in J_3 with $R_{12} = 0$ as an example. In this case, the terms $J_{231}^{(d)}$ and $J_{312}^{(d)}$ are given in the same form of (39) or (41). While the term $J_{123}^{(d)}$ is obtained by performing analytic continuation the result in (41). In detail, one takes the limit of $R_{12} \rightarrow 0$ in (41), we arrive at

$$J_{123}^{(d)} = \frac{(\partial_3 R_3)(\partial_2 R_{12})}{2(m_1^2 - R_3)} (m_1^2)^{\frac{d-4}{2}} \times F_1 \left(1; \frac{4-d}{2}, 1; \frac{3}{2}; 1, \frac{m_1^2}{m_1^2 - R_3} \right) + (1 \leftrightarrow 2). \quad (46)$$

Using (36) in Ref. [30], the term $J_{123}^{(d)}$ simplifies to

$$J_{123}^{(d)} = - \frac{\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{\partial_3 R_3}{4R_3} \right) \times \left\{ (\partial_2 R_{12}) (m_1^2)^{\frac{d-4}{2}} {}_2F_1 \left[\begin{matrix} 1, \frac{d-2}{2}; \\ \frac{d-1}{2}; \end{matrix} \frac{m_1^2}{R_3} \right] + (1 \leftrightarrow 2) \right\}, \quad (47)$$

provided that $\text{Re}(d-2) > 0$ and $|m_i^2/R_3| < 1$ for $i = 1, 2$. Taking $d \rightarrow 4$, we have

$$J_{123}^{(d=4)} = - \left(\frac{\partial_3 R_3}{2R_3} \right) \left\{ (\partial_2 R_{12}) {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{m_1^2}{R_3} \right] + (1 \leftrightarrow 2) \right\}. \quad (48)$$

4.3 $R_{ij} = m_{i(j)}^2$ for $i, j = 1, 2, 3$

Next we consider $R_{12} = m_1^2$ as an example. In this case, the terms $J_{231}^{(d)}$, $J_{312}^{(d)}$ are given by (39). Beside that, one verifies

$$\partial_2 R_{12} = 0. \quad (49)$$

As a result, we obtain

$$J_{123}^{(d=4)} = \frac{(\partial_3 R_3)(\partial_1 R_{12})}{2(m_2^2 - R_3)} {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{m_1^2 - m_2^2}{R_3 - m_2^2} \right], \quad (50)$$

$$J_{123}^{(d)} = \frac{(\partial_3 R_3)(\partial_1 R_{12})}{2(m_2^2 - R_3)} (m_2^2)^{\frac{d-4}{2}} \times F_1 \left(1; \frac{4-d}{2}, 1; \frac{3}{2}; 1 - \frac{m_1^2}{m_2^2}, \frac{m_1^2 - m_2^2}{R_3 - m_2^2} \right), \quad (51)$$

provided that the amplitude of arguments of hypergeometric functions appearing in this formula are less than 1. For $R_{12} = m_1^2 = m_2^2$, the function F_1 in (50) is equal 1. The result reads

$$J_{123}^{(d)} = \frac{(\partial_3 R_3)}{2(m^2 - R_3)} (m^2)^{\frac{d-4}{2}}. \quad (52)$$

4.4 $R_3 = m_k^2$ for $k = 1, 2, 3$

As an example, consider the terms of J_3 in (38) with $R_3 = m_1^2$. One verifies that

$$\partial_1 R_3 = 1, \quad \partial_i R_3 = 0, \quad \text{for } i = 2, 3. \quad (53)$$

As a result, J_3 is casted into the form of

$$\frac{J_3}{\Gamma\left(\frac{4-d}{2}\right)} = -J_{231}^{(d=4)} (R_3)^{\frac{d-4}{2}} + J_{231}^{(d)}, \quad (54)$$

with $J_{231}^{(d)}$ taking the same form of (39) or (41). We take (41) as example for $J_{231}^{(d)}$. In detail, it takes

$$J_{231}^{(d)} = \frac{(\partial_1 R_3)(\partial_3 R_{23})}{2(m_2^2 - R_3)} (m_2^2)^{\frac{d-4}{2}} \times F_1 \left(1; \frac{4-d}{2}, 1; \frac{3}{2}; 1 - \frac{R_{23}}{m_2^2}, \frac{R_{23} - m_2^2}{R_3 - m_2^2} \right) + (2 \leftrightarrow 3). \quad (55)$$

Taking $d \rightarrow 4$, the result reads

$$J_{231}^{(d=4)} = \frac{(\partial_1 R_3)(\partial_3 R_{23})}{2(m_2^2 - R_3)} \times {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{R_{23} - m_2^2}{R_3 - m_2^2} \right] + (2 \leftrightarrow 3). \quad (56)$$

4.5 $G_2 \neq 0$ and $R_3 = 0$

By setting $N = 3$ in (7), one obtains

$$J_3 = \frac{1}{(d-4)} \sum_{k=1}^3 \left(\frac{\partial_k Y_3}{G_2} \right) \mathbf{k}^- J_3(d-2; \{p_i^2\}, \{m_i^2\}). \quad (57)$$

The resulting of $\mathbf{k}^- J_3(d-2; \{p_i^2\}, \{m_i^2\})$ is J_2 in (18) or in (20) with $d \rightarrow d-2$. As an example, we take J_2 in (20) at $d-2$. The result reads

$$\begin{aligned} \frac{J_3}{\Gamma\left(\frac{6-d}{2}\right)} &= \frac{1}{(4-d)} \left(\frac{\partial_3 Y_3}{G_2} \right) \left\{ \left(\frac{\partial_2 R_{12}}{R_{12}} \right) (R_{12})^{\frac{d-4}{2}} \right. \\ &\quad \times {}_2F_1 \left[\begin{matrix} \frac{6-d}{2}, \frac{1}{2}; \\ \frac{3}{2}; \end{matrix} 1 - \frac{m_1^2}{R_{12}} \right] + (1 \leftrightarrow 2) \Big\} \\ &\quad + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \\ &\quad + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}. \end{aligned} \quad (58)$$

We can confirm (57) again by using analytic continuation result of J_3 in (41). In fact, when $R_3 \rightarrow 0$ the Eq. (41) becomes

$$J_{123}^{(d)} = -\frac{1}{2} \left(\frac{\partial_3 Y_3}{G_2} \right) \left[(\partial_2 R_{12})(m_1^2)^{\frac{d-4}{2}} \times F_1 \left(1, \frac{4-d}{2}, 1, \frac{3}{2}, 1 - \frac{R_{12}}{m_1^2}, 1 - \frac{R_{12}}{m_1^2} \right) + (1 \leftrightarrow 2) \right] \quad (59)$$

$$= -\frac{1}{2} \left(\frac{\partial_3 Y_3}{G_2} \right) \left\{ (\partial_2 R_{12})(m_1^2)^{\frac{d-4}{2}} \times {}_2F_1 \left[\begin{matrix} \frac{6-d}{2}, 1; \\ \frac{3}{2}; \end{matrix} 1 - \frac{R_{12}}{m_1^2} \right] + (1 \leftrightarrow 2) \right\} \quad (60)$$

$$= -\frac{1}{2} \left(\frac{\partial_3 Y_3}{G_2} \right) \left\{ \left(\frac{\partial_2 R_{12}}{R_{12}} \right) (R_{12})^{\frac{d-6}{2}} \times {}_2F_1 \left[\begin{matrix} \frac{6-d}{2}, 1; \\ \frac{3}{2}; \end{matrix} 1 - \frac{m_1^2}{R_{12}} \right] + (1 \leftrightarrow 2) \right\}. \quad (61)$$

Plugging (61) into (38), we arrive at (58).

4.6 $R_3 = R_{ij}$ for $i, j = 1, 2, 3$

We consider the case of $R_3 = R_{12}$ as an example. In this case, one verifies that

$$\partial_3 R_3 = 0. \quad (62)$$

As a result, J_3 is

$$\begin{aligned} \frac{J_3}{\Gamma\left(\frac{4-d}{2}\right)} &= -J_{231}^{(d=4)} (R_3)^{\frac{d-4}{2}} + J_{231}^{(d)} \\ &\quad + \{(2, 3, 1) \leftrightarrow (3, 1, 2)\}. \end{aligned} \quad (63)$$

Here $J_{231}^{(d)}$ takes the same form as (39) or (41). We take (41) as example for $J_{231}^{(d)}$. In detail, it takes

$$J_{231}^{(d)} = \frac{(\partial_1 R_3)(\partial_3 R_{23})}{2(m_2^2 - R_3)} (m_2^2)^{\frac{d-4}{2}} \times F_1 \left(1; \frac{4-d}{2}, 1; \frac{3}{2}; 1 - \frac{R_{23}}{m_2^2}, \frac{R_{23} - m_2^2}{R_3 - m_2^2} \right) + (2 \leftrightarrow 3), \quad (64)$$

$$J_{231}^{(d=4)} = \frac{(\partial_1 R_3)(\partial_3 R_{23})}{2(m_2^2 - R_3)} \times {}_2F_1 \left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{R_{23} - m_2^2}{R_3 - m_2^2} \right] + (2 \leftrightarrow 3). \quad (65)$$

4.7 $G_2 = 0$

Setting $N = 3$ in (8), the result reads

$$J_3 = -\frac{1}{2} \sum_{k=1}^3 \left(\frac{\partial_k Y_3}{Y_3} \right) \mathbf{k}^- J_3(d; \{p_i^2\}, \{m_i^2\}). \quad (66)$$

This equation is equivalent to (46) in Ref. [21]. The term $\mathbf{k}^- J_3(d; \{p_i^2\}, \{m_i^2\})$ corresponds to J_2 in (20) as an example. We obtain

$$\begin{aligned} \frac{J_3}{\Gamma(\frac{4-d}{2})} &= - \left(\frac{\partial_3 Y_3}{2Y_3} \right) \left\{ \left(\frac{\partial_2 Y_{12}}{Y_{12}} \right) (m_1^2)^{\frac{d-2}{2}} \right. \\ &\quad \times {}_2F_1 \left[\begin{matrix} \frac{d-1}{2}, 1; \\ \frac{3}{2}; \end{matrix} 1 - \frac{m_1^2}{R_{12}} \right] + (1 \leftrightarrow 2) \Big\} \\ &\quad + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \\ &\quad + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}. \end{aligned} \quad (67)$$

4.8 $G_{1(ij)} = 0$ for $i, j = 1, 2, 3$

$G_{1(ij)}$ are the Gram determinants of two-point functions which are obtained by shrinking a propagator $k \neq i, j$ in the three-point ones. Taking $G_{1(12)} = 0$ as an example, the term $J_{123}^{(d)}$ is evaluated as follows. We put J_2 in (23) into (27). Taking the corresponding MB integrations, the results read as form of (38) with

$$J_{123}^{(d=4)} = - \left(\frac{\partial_3 R_3}{R_3} \right) \frac{m_1^2}{m_1^2 - m_2^2} {}_2F_1 \left[\begin{matrix} 1, 1; \\ 2; \end{matrix} \frac{m_1^2}{R_3} \right] + (1 \leftrightarrow 2), \quad (68)$$

$$\begin{aligned} J_{123}^{(d)} &= - \frac{\Gamma(\frac{d-2}{2})}{2\Gamma(\frac{d}{2})} \left(\frac{\partial_3 R_3}{R_3} \right) \frac{(m_1^2)^{\frac{d-2}{2}}}{m_1^2 - m_2^2} \\ &\quad \times {}_2F_1 \left[\begin{matrix} \frac{d-2}{2}, 1; \\ \frac{d}{2}; \end{matrix} \frac{m_1^2}{R_3} \right] + (1 \leftrightarrow 2). \end{aligned} \quad (69)$$

It is valid under the conditions that the arguments of the hypergeometric functions appearing in this formula are less than 1 and $\text{Re}(d-2) > 0$.

4.9 Cross check with other papers

We consider $p_2^2 = p_3^2 = 0$; $p_1^2 \neq 0$, $m_1^2 = m_3^2 = 0$ and $m_2^2 \neq 0$ as an example [10, 15]. We confirm that

$$\begin{aligned} R_3 &= \frac{m_2^2(p_1^2 + m_2^2)}{p_1^2}, \quad \frac{\partial_1 R_3}{R_3} = \frac{\partial_3 R_3}{R_3} \\ &= - \frac{1}{p_1^2 + m_2^2}, \quad \frac{\partial_2 R_3}{R_3} \\ &= \frac{2m_2^2 + p_1^2}{m_2^2(m_2^2 + p_1^2)}. \end{aligned} \quad (70)$$

The J_3 in (27) becomes

$$\begin{aligned} J_3 &= -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(s+1) \Gamma(\frac{d-2}{2} + s) \Gamma(\frac{4-d}{2} - s)}{2 \Gamma(\frac{d-2}{2})} \\ &\quad \times \left(\frac{1}{R_3} \right)^s \left\{ \frac{\sqrt{\pi}}{2} \frac{2m_2^2 + p_1^2}{m_2^2(m_2^2 + p_1^2)} \frac{\Gamma(\frac{d-2}{2} + s)}{\Gamma(\frac{d-1}{2} + s)} \right. \\ &\quad \times \left(\frac{-p_1^2}{4} \right)^{\frac{d-4}{2} + s} - \frac{2}{p_1^2 + m_2^2} \frac{\Gamma(\frac{d-2}{2} + s)}{\Gamma(\frac{d}{2} + s)} (m_2^2)^{\frac{d-4}{2} + s} \Big\}. \end{aligned} \quad (71)$$

We note that the first term in curly bracket of (71) is J_2 in the case of (26) with d shifted to $d + 2s$. While the second term in curly bracket of (71) is corresponding to J_2 in (23) at $d + 2s$ (and with a massless internal line). In the following we perform the contour integration of (71) starting with the second contour integral:

$$\begin{aligned} J_3^{(1)} &= -\frac{(m_2^2)^{\frac{d}{2}-2}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \\ &\quad \times \frac{\Gamma(-s) \Gamma(s+1) \Gamma(\frac{d-2}{2} + s)^2 \Gamma(\frac{4-d}{2} - s)}{2 \Gamma(\frac{d-2}{2}) \Gamma(\frac{d}{2} + s)} \left(\frac{m_2^2}{R_3} \right)^s. \end{aligned} \quad (72)$$

By closing the integration contour to the right, the residue contributions at the poles of $\Gamma(-s)$ and $\Gamma(\frac{4-d}{2} - s)$ are calculated. For the first sequence poles, the result reads

$$\begin{aligned} J_3^{(1,a)} &= - \frac{\Gamma(2 - \frac{d}{2}) \Gamma(\frac{d}{2} - 1)}{2\Gamma(\frac{d}{2})} (m_2^2)^{\frac{d}{2}-2} \\ &\quad \times {}_2F_1 \left[\begin{matrix} 1, \frac{d-2}{2}; \\ \frac{d}{2}; \end{matrix} \frac{p_1^2}{p_1^2 + m_2^2} \right] \\ &= - \frac{\Gamma(2 - \frac{d}{2}) \Gamma(\frac{d}{2} - 1)}{2\Gamma(\frac{d}{2})} \end{aligned} \quad (73)$$

$$\times \frac{p_1^2 + m_2^2}{m_2^2} (m_2^2)^{\frac{d}{2}-2} {}_2F_1\left[\begin{matrix} 1, 1; \\ \frac{d}{2}; \end{matrix} -\frac{p_1^2}{m_2^2}\right]. \quad (74)$$

For the second sequence poles, we arrive at

$$J_3^{(1,b)} = -\frac{\Gamma(\frac{d}{2}-2)\Gamma(3-\frac{d}{2})}{2\Gamma(\frac{d}{2}-1)} (R_3)^{\frac{d}{2}-2} \times {}_2F_1\left[\begin{matrix} 1, 1; \\ 2; \end{matrix} \frac{p_1^2}{p_1^2 + m_2^2}\right] \quad (75)$$

$$= -\frac{\Gamma(\frac{d}{2}-2)\Gamma(3-\frac{d}{2})}{2\Gamma(\frac{d}{2}-1)} (R_3)^{\frac{d}{2}-2} \times \frac{p_1^2 + m_2^2}{m_2^2} {}_2F_1\left[\begin{matrix} 1, 1; \\ 2; \end{matrix} -\frac{p_1^2}{m_2^2}\right]. \quad (76)$$

Second type of MB integral is considered

$$J_3^{(2)} = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\sqrt{\pi}}{2} \frac{\Gamma(-s)\Gamma(s+1)\Gamma(\frac{d-2}{2}+s)^2\Gamma(\frac{4-d}{2}-s)}{2\Gamma(\frac{d-2}{2})\Gamma(\frac{d-1}{2}+s)} \times \left(\frac{1}{R_3}\right)^s \left(\frac{-p_1^2}{4}\right)^{\frac{d}{2}-2+s}. \quad (77)$$

For the first sequence poles of $\Gamma(-s)$, the result is

$$J_3^{(2,a)} = -\frac{\sqrt{\pi}\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{4\Gamma(\frac{d-1}{2})} \left(\frac{-p_1^2}{4}\right)^{\frac{d}{2}-2} \times {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}; \\ \frac{d-1}{2}; \end{matrix} \frac{-(p_1^2)^2}{4m_2^2(p_1^2 + m_2^2)}\right].$$

Applying transforms for Gauss hypergeometric function which are (see (1.8.10) in [25] for the first relations and page 49, [26] for the later case)

$${}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = (1-z)^{c-b-a} {}_2F_1\left[\begin{matrix} c-a, c-b; \\ c; \end{matrix} z\right], \quad (78)$$

$${}_2F_1\left[\begin{matrix} a, b; \\ 2b; \end{matrix} z\right] = (1-z)^{-a/2} {}_2F_1\left[\begin{matrix} \frac{a}{2}, b-\frac{a}{2}; \\ b+\frac{1}{2}; \end{matrix} \frac{z^2}{4(z-1)}\right] \quad (79)$$

one obtains

$$J_3^{(2,a)} = -\frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d-1}{2})} \left(\frac{-p_1^2}{4}\right)^{\frac{d}{2}-2} \times \frac{(p_1^2 + m_2^2)}{(p_1^2 + 2m_2^2)} {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}; \\ d-2; \end{matrix} -\frac{p_1^2}{m_2^2}\right]. \quad (80)$$

Taking into account the residue at the poles $\Gamma(\frac{4-d}{2}-s)$, we get

$$J_3^{(2,b)} = -\frac{\Gamma(\frac{d}{2}-2)\Gamma(3-\frac{d}{2})}{2\Gamma(\frac{d}{2}-1)} \left[\frac{4m_2^2(p_1^2 + m_2^2)}{(2m_2^2 + p_1^2)^2}\right] \times R_3^{\frac{d}{2}-2} {}_2F_1\left[\begin{matrix} 1, \frac{1}{2}; \\ \frac{3}{2}; \end{matrix} \frac{(p_1^2)^2}{(2m_2^2 + p_1^2)^2}\right]. \quad (81)$$

Using the relation (see Eq. (3.1.7) in [27])

$${}_2F_1\left[\begin{matrix} a, b; \\ 2b; \end{matrix} z\right] = \left(1-\frac{z}{2}\right)^{-a} \times {}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2}; \\ b+\frac{1}{2}; \end{matrix} \left(\frac{z}{2-z}\right)^2\right], \quad (82)$$

one gets

$$J_3^{(2,b)} = -\frac{\Gamma(\frac{d}{2}-2)\Gamma(3-\frac{d}{2})}{\Gamma(\frac{d}{2}-1)} \frac{(p_1^2 + m_2^2)}{2m_2^2 + p_1^2} R_3^{\frac{d}{2}-2} \times {}_2F_1\left[\begin{matrix} 1, 1; \\ 2; \end{matrix} -\frac{p_1^2}{m_2^2}\right]. \quad (83)$$

Combining all the terms, J_3 reads

$$J_3 = \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d}{2})} (m_2^2)^{\frac{d}{2}-3} {}_2F_1\left[\begin{matrix} 1, 1; \\ \frac{d}{2}; \end{matrix} -\frac{p_1^2}{m_2^2}\right] - \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d-1}{2})m_2^2} \left(\frac{-p_1^2}{4}\right)^{\frac{d}{2}-2} \times {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}; \\ d-2; \end{matrix} -\frac{p_1^2}{m_2^2}\right] \quad (84)$$

provided that $|p_1^2/m_2^2| < 1$ and $\text{Re}(d-2) > 0$. It agrees with Eq. (B2) in [10] and (22) in [15].

5 One-loop four-point functions

The master equation for J_4 is obtained from (5) with $N = 4$,

$$J_4 \equiv J_4(d; \{p_i^2, s, t\}, \{m_i^2\}) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \times \frac{\Gamma(-s)\Gamma(\frac{d-3}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-3}{2})} \left(\frac{1}{R_4}\right)^s \times \sum_{k=1}^4 \left(\frac{\partial_k R_4}{R_4}\right) \mathbf{k}^- J_4(d+2s; \{p_i^2, s, t\}, \{m_i^2\}). \quad (85)$$

We substitute the analytic solution for $J_3(d+2s; \{p_i^2\}, \{m_i^2\})$ in (38) into (85) and take the contour integrals in (85). With the help of MB integrations in (131, 134) in Appendix C, a compact expression for J_4 can be derived and expressed as follows:

$$\frac{J_4}{\Gamma(\frac{4-d}{2})} = -J_{1234}^{(d=4)} (R_4)^{\frac{d-4}{2}} + J_{1234}^{(d)} + \left\{ (1, 2, 3, 4) \leftrightarrow (2, 3, 4, 1) \right\} + \left\{ (1, 2, 3, 4) \leftrightarrow (3, 4, 1, 2) \right\}$$

$$+ \left\{ (1, 2, 3, 4) \leftrightarrow (4, 1, 2, 3) \right\} \quad (86)$$

with

$$\begin{aligned} J_{1234}^{(d)} = & - \left(\frac{\partial_4 R_4}{2R_4} \right) J_{123}^{(d=4)} (R_{123})^{\frac{d-4}{2}} {}_2F_1 \left[\frac{d-3}{2}, 1; \frac{R_{123}}{R_4} \right] \\ & + \frac{\sqrt{\pi} \Gamma(\frac{d-2}{2})}{\Gamma(\frac{d-1}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left(\frac{\partial_3 R_{123}}{R_{123}} \right) \\ & \times \left[\frac{\partial_2 R_{12}}{\sqrt{1 - m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1 - m_2^2/R_{12}}} \right] \\ & \times \frac{(R_{12})^{\frac{d-4}{2}}}{\sqrt{1 - R_{12}/R_{123}}} F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{R_{12}}{R_4}, \frac{R_{12}}{R_{123}} \right) \\ & - \frac{\Gamma(\frac{d-2}{2})}{8 \Gamma(\frac{d}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left[\frac{\partial_3 R_{123}}{(R_{123} - m_1^2)} \frac{\partial_2 R_{12}}{(R_{12} - m_1^2)} (m_1^2)^{\frac{d-2}{2}} \right. \\ & \times F_S \left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \right. \\ & \left. \left. \frac{m_1^2}{R_4}, \frac{m_1^2}{m_1^2 - R_{123}}, \frac{m_1^2}{m_1^2 - R_{12}} \right) + (1 \leftrightarrow 2) \right] \\ & + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \\ & + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}. \quad (87) \end{aligned}$$

where $J_{123}^{(d=4)}, \dots$ are given by (40). It is important that this representation is valid under the conditions that $\text{Re}(d-3) > 0$ and the absolute values of arguments of hypergeometric functions are smaller than one. If the absolute value of these arguments are larger than one, we have to perform analytic continuations for the Gauss hypergeometric and Appell F_1 functions, cf. [25,31]. Further, the Saran function F_S may be expressed by a Mellin–Barnes representation, or Euler integrals in this case. The result for J_4 has been shown in [19]. There are two important points we would like to emphasize in this paper as follows. (1) Ref. [16] have not shown conditions for the boundary term in (100). (2) J_4 is constructed from J_3 for arbitrary kinematics. However, the boundary term for J_3 for general kinematics have not been provided in [16], as mentioned in the previous section and in [18]. Subsequently, the first term in (99) of [16] is only valid in special kinematic regions. Therefore, the solution in (98) of Ref. [16] may not be considered as a complete solution for J_4 . In contrast to [16], we provide a complete solution for J_4 in this article.

5.1 Massless internal lines

We are going to take $m_i^2 \rightarrow 0$ for $i = 1, 2, 3, 4$. The terms related to F_S vanish. Therefore, in the massless case the result reads

$$J_{1234}^{(d)} = \left(\frac{\partial_4 R_4}{2R_4} \right) \Big|_{m_i^2 \rightarrow 0} \left(\frac{\partial_3 R_{123}}{2R_{123}} \right) \Big|_{m_i^2 \rightarrow 0}$$

$$\begin{aligned} & \times {}_2F_1 \left[\frac{1}{2}, 1; \frac{R_{12}}{R_{123}} \right] (R_{123})^{\frac{d-4}{2}} {}_2F_1 \left[\frac{d-3}{2}, 1; \frac{R_{123}}{R_4} \right] \\ & + \frac{\sqrt{\pi} \Gamma(\frac{d-2}{2})}{2 \Gamma(\frac{d-1}{2})} \left(\frac{\partial_4 R_4}{2R_4} \right) \Big|_{m_i^2 \rightarrow 0} \left(\frac{\partial_3 R_{123}}{2R_{123}} \right) \Big|_{m_i^2 \rightarrow 0} \\ & \times \frac{(R_{12})^{\frac{d-4}{2}}}{\sqrt{1 - R_{12}/R_{123}}} \\ & F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{R_{12}}{R_4}, \frac{R_{12}}{R_{123}} \right) \\ & + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \\ & + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}, \quad (88) \end{aligned}$$

provided that $\text{Re}(d-3) > 0$ and that the absolute values of arguments of the hypergeometric functions are smaller than one. Taking $d \rightarrow 4$, we have

$$\begin{aligned} J_{1234}^{(d=4)} = & \left(\frac{\partial_4 R_4}{2R_4} \right) \Big|_{m_i^2 \rightarrow 0} \left(\frac{\partial_3 R_{123}}{2R_{123}} \right) \Big|_{m_i^2 \rightarrow 0} \\ & \times {}_2F_1 \left[\frac{1}{2}, 1; \frac{R_{12}}{R_{123}} \right] {}_2F_1 \left[\frac{1}{2}, 1; \frac{R_{123}}{R_4} \right] \\ & + \left(\frac{\partial_4 R_4}{2R_4} \right) \Big|_{m_i^2 \rightarrow 0} \left(\frac{\partial_3 R_{123}}{2R_{123}} \right) \Big|_{m_i^2 \rightarrow 0} \\ & \times \left(\frac{R_{123}}{R_{123} - R_{12}} \right) {}_2F_1 \left[\frac{1}{2}, 1; \frac{R_{12}(R_{123} - R_{12})}{R_4(R_{123} - R_{12})} \right] \\ & + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \\ & + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}, \quad (89) \end{aligned}$$

This is a new result for J_4 in the massless case at general d . We are going to consider the special cases for J_4 in the following subsections.

5.2 $R_4 = 0$

From (7), we set $N = 4$ and get

$$J_4 = \frac{1}{d-5} \sum_{k=1}^4 \left(\frac{\partial_k Y_4}{G_3} \right) \mathbf{k}^- J_4(d-2; \{p_i^2, s, t\}, \{m_i^2\}). \quad (90)$$

The term $\mathbf{k}^- J_4(d-2; \{p_i^2, s, t\}, \{m_i^2\})$ is given by J_3 in (38) with $d \rightarrow d-2$.

5.3 $R_4 = R_{ijk}$ for $i, j, k = 1, 2, 3, 4$

As an example, we consider the case $R_4 = R_{123}$. In this case, we verify that

$$(\partial_4 R_4) = 0. \quad (91)$$

As a result, the terms $J_{1234}^{(d)}$ vanish, other terms in (86) are of the same form in (87).

5.4 $R_4 = R_{ij}$ for $i, j = 1, 2, 3, 4$

For example, the terms of J_4 in (86) meet the condition $R_4 = R_{12}$. Because that R_2 depends only the internal masses m_1^2, m_2^2 , one has

$$\partial_i R_4 = \partial_i R_2 = 0, \quad \text{for } i = 3, 4. \quad (92)$$

As a matter of this fact, only two terms (2, 3, 4, 1) and (3, 4, 1, 2) in (86) contribute to J_4 .

5.5 $R_4 = m_k^2$ for $k = 1, 2, 3, 4$

For example, one considers the terms of J_4 in (86) having $R_4 = m_1^2$. One verifies that

$$\partial_i R_4 = 0 \quad \text{for } i = 2, 3, 4. \quad (93)$$

As a result, only the term (2, 3, 4, 1) in (86) contributes to J_4 .

5.6 $R_{ijk} = 0$ for $i, j, k = 1, 2, 3, 4$

We assume that J_4 in (86) contains $R_{123} = 0$ as an example. The term (1, 2, 3, 4) in (87) with $R_{123} = 0$ is evaluated by applying the same previous procedure. The result reads

$$\begin{aligned} J_{1234}^{(d)} = & -\frac{\sqrt{\pi}}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-2}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_3 Y_{123}}{G_{12}}\right) \\ & \times \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1-m_2^2/R_{12}}} \right] \\ & \times (R_{12})^{\frac{d-6}{2}} {}_2F_1\left[1, \frac{d-4}{2}; \frac{R_{12}}{R_4}\right] \\ & + \frac{1}{2\Gamma^2\left(\frac{d-2}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_3 Y_{123}}{G_{12}}\right) \\ & \times \left[\left(\frac{\partial_2 R_{12}}{R_{12}}\right) \frac{(m_1^2)^{\frac{d-4}{2}}}{\sqrt{1-m_1^2/R_{12}}} \right. \\ & \times F_{1;1;0}^{1;2;1}\left(\frac{d-4}{2}; \frac{d-3}{2}, 1; \frac{1}{2}; \frac{m_1^2}{R_4}, \frac{m_1^2}{R_{12}}\right) \\ & + (1 \leftrightarrow 2)] \\ & + \{(1, 2, 3) \leftrightarrow (2, 3, 1)\} \\ & + \{(1, 2, 3) \leftrightarrow (3, 1, 2)\}. \end{aligned} \quad (94)$$

Where $F_{1;1;0}^{1;2;1}$ is Kampé de Fériet [33] (see Appendix B for more detail). We also refer to [34] which analytic continuations for a class of the Kampé de Fériet functions have been studied. This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and

$\text{Re}(d-4) > 0$. In the massless case, one has

$$\begin{aligned} J_{1234}^{(d)} = & \frac{\sqrt{\pi}}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-2}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_3 Y_{123}}{G_{12}}\right) (R_{12})^{\frac{d-6}{2}} \\ & \times {}_2F_1\left[1, \frac{d-4}{2}; \frac{R_{12}}{R_4}\right] + \{(1, 2, 3) \leftrightarrow (2, 3, 1)\} \\ & + \{(1, 2, 3) \leftrightarrow (3, 1, 2)\}. \end{aligned} \quad (95)$$

5.7 $R_{ijk} = R_{ij}$ for $i, j, k = 1, 2, 3, 4$

We examine the terms of J_4 in (86) having $R_{123} = R_2$. We check that

$$\partial_3 R_{123} = 0. \quad (96)$$

As a result, the terms (2, 3, 4, 1), (3, 4, 1, 2) and (4, 1, 2, 3) of J_4 get the same formula for $J_{1234}^{(d)}$ provided in (87). The terms (1, 2, 3) of $J_{1234}^{(d)}$ in (87) are vanished.

5.8 $R_{ijk} = m_{i,(j,k)}^2$ for $i, j, k = 1, 2, 3, 4$

Assuming the terms of J_4 in (86) with $R_{123} = m_1^2$, we check that

$$\partial_1 R_{123} = 1, \quad \text{and} \quad \partial_i R_{123} = 0, \quad \text{for } i = 2, 3. \quad (97)$$

The terms $J_{2341}^{(d)}$, $J_{3412}^{(d)}$ and $J_{4123}^{(d)}$ get the same formula as (87). The term (1, 2, 3) and (3, 1, 2) of $J_{1234}^{(d)}$ in (87) vanish, only the (2, 3, 1)-term contribute to $J_{1234}^{(d)}$.

5.9 $R_{ij} = 0$ for $i, j = 1, 2, 3, 4$

We consider that J_4 in (86) with $R_{12} = 0$. In this case, the terms $J_{2341}^{(d)}$ and $J_{3412}^{(d)}$ are unchanged. The terms $J_{1234}^{(d)}$ and $J_{4123}^{(d)}$ with $R_{12} = 0$ are evaluated again by applying same previous procedure. Taking $J_{1234}^{(d)}$ as an example. The result reads

$$\begin{aligned} J_{1234}^{(d)} = & -\left(\frac{\partial_4 R_4}{R_4}\right) J_{123}^{(d=4)} (R_{123})^{\frac{d-4}{2}} {}_2F_1\left[1, \frac{d-3}{2}; \frac{R_{123}}{R_4}\right] \\ & + \frac{\Gamma\left(\frac{d-3}{2}\right)}{4\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_3 R_{123}}{R_{123}}\right) \\ & \times \left[\left(\frac{\partial_2 Y_{12}}{G_{12}}\right) \frac{(m_1^2)^{\frac{d-4}{2}}}{\sqrt{1-m_1^2/R_{123}}} \right. \\ & \times F_{1;1;0}^{1;2;1}\left(\frac{d-3}{2}; \frac{d-1}{2}, \frac{d-2}{2}, 1; \frac{1}{2}; \frac{m_1^2}{R_4}, \frac{m_1^2}{R_{123}}\right) \\ & + (1 \leftrightarrow 2)] + \left\{ -\left(\frac{\partial_4 R_4}{2R_4}\right) J_{231}^{(d=4)} (R_{231})^{\frac{d-4}{2}} \right. \\ & \times {}_2F_1\left[\frac{d-3}{2}, 1; \frac{R_{231}}{R_4}\right] + \frac{\sqrt{\pi}\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_1 R_{231}}{R_{231}}\right) \\ & \times \left[\frac{\partial_3 R_{23}}{\sqrt{1-m_2^2/R_{23}}} + \frac{\partial_1 R_{23}}{\sqrt{1-m_2^2/R_{23}}} \right] \end{aligned}$$

$$\begin{aligned}
& \times \frac{(R_{23})^{\frac{d-4}{2}}}{\sqrt{1-R_{23}/R_{231}}} F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{R_{23}}{R_4}, \frac{R_{23}}{R_{231}} \right) \\
& - \frac{\Gamma(\frac{d-2}{2})}{8 \Gamma(\frac{d}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left[\frac{\partial_1 R_{231}}{(R_{231}-m_2^2)} \frac{\partial_3 R_{23}}{(R_{23}-m_2^2)} (m_2^2)^{\frac{d-2}{2}} \right. \\
& \times F_S \left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \right. \\
& \left. \left. \frac{m_2^2}{R_4}, \frac{m_2^2}{m_2^2-R_{231}}, \frac{m_2^2}{m_2^2-R_{23}} \right) + (1 \leftrightarrow 2) \right] \\
& + \left\{ (2, 3, 1) \leftrightarrow (3, 1, 2) \right\}. \quad (98)
\end{aligned}$$

This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and $\text{Re}(d-3) > 0$.

5.10 $R_{ij} = m_i^2$ or m_j^2 for $i, j = 1, 2, 3, 4$

Taking $R_{12} = m_1^2$ as an example, one confirms that

$$\partial_2 R_{12} = 0. \quad (99)$$

As a result, the terms $(1, 2, 3)$ of $J_{1234}^{(d)}$ that are multiplied by $\frac{\partial R_2}{\partial m_2^2}$ vanish. Other terms of $J_{1234}^{(d)}$ are given by (87). The terms $J_{2341}^{(d)}$, $J_{3412}^{(d)}$ and $J_{4123}^{(d)}$ of J_4 are given by (87).

5.11 $G_3 = 0$

In this case, one has

$$J_4 = -\frac{1}{2} \sum_{k=1}^4 \left(\frac{\partial_k Y_4}{Y_4} \right) \mathbf{k}^- J_4(d; \{p_i^2, s, t\}, \{m_i^2\}). \quad (100)$$

This equation is equivalent with (65) in Ref. [21]. The term $\mathbf{k}^- J_4(d; \{p_i^2, s, t\}, \{m_i^2\})$ is given by J_3 in (38).

5.12 $G_{2(ijk)} = 0$ for $i, j, l = 1, 2, 3, 4$

In the same notation, $G_{2(ijk)}$ are the Gram determinants of J_3 that are obtained by shrinking k th propagator in J_4 . We take $|G_{2(123)}| = 0$ as an example. By using (66) for $J_{123}^{(d)}$, we then evaluate $J_{1234}^{(d)}$ again, the result is

$$\begin{aligned}
J_{1234}^{(d)} &= -\frac{\sqrt{\pi} \Gamma(\frac{d-2}{2})}{8 \Gamma(\frac{d-1}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left(\frac{\partial_3 Y_{123}}{Y_{123}} \right) \\
& \times \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1-m_2^2/R_{12}}} \right] \\
& \times (R_{12})^{\frac{d-4}{2}} {}_2F_1 \left[1, \frac{d-3}{2}; \frac{d-1}{2}; \frac{R_{12}}{R_4} \right] \\
& + \frac{\Gamma(\frac{d-2}{2})}{8 \Gamma(\frac{d}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left(\frac{\partial_3 Y_{123}}{Y_{123}} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{\partial_2 R_{12}}{\sqrt{1-m_1^2/R_{12}}} (m_1^2)^{\frac{d-4}{2}} F_1 \left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \right. \right. \\
& \left. \left. \frac{m_1^2}{R_4}, \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \right] \\
& + \{(1, 2, 3) \leftrightarrow (2, 3, 1)\} \\
& + \{(1, 2, 3) \leftrightarrow (3, 1, 2)\}. \quad (101)
\end{aligned}$$

This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and $\text{Re}(d-3) > 0$.

5.13 $G_{1(ij)} = 0$ for $i, j = 1, 2, 3, 4$

One assumes that the term $J_{1234}^{(d)}$ has $G_{1(12)} = 0$. Recalculating this term, the result reads in term of Gauss and Appell F_3 functions

$$\begin{aligned}
J_{1234}^{(d)} &= -\left(\frac{\partial_4 R_4}{2 R_4} \right) J_{123}^{(d=4)} (R_{123})^{\frac{d-4}{2}} {}_2F_1 \left[1, \frac{d-3}{2}; \frac{d-2}{2}; \frac{R_{123}}{R_4} \right] \\
& + \frac{\Gamma(\frac{d-2}{2})}{4 \Gamma(\frac{d}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left(\frac{\partial_3 R_{123}}{R_{123}} \right) \\
& \times \left[\frac{R_{123}}{(R_{123}-m_1^2)} \frac{(m_1^2)^{\frac{d-2}{2}}}{(m_1^2-m_2^2)} \right. \\
& \times F_3 \left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{m_1^2}{R_4}, \frac{m_1^2}{m_1^2-R_{123}} \right) \\
& \left. + (1 \leftrightarrow 2) \right] + \left\{ -\left(\frac{\partial_4 R_4}{2 R_4} \right) J_{231}^{(d=4)} (R_{231})^{\frac{d-4}{2}} \right. \\
& \times {}_2F_1 \left[\frac{d-3}{2}, 1; \frac{R_{231}}{R_4} \right] + \frac{\sqrt{\pi} \Gamma(\frac{d-2}{2})}{\Gamma(\frac{d-1}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \left(\frac{\partial_1 R_{231}}{R_{231}} \right) \\
& \times \left[\frac{\partial_3 R_{23}}{\sqrt{1-m_2^2/R_{23}}} + \frac{\partial_1 R_{23}}{\sqrt{1-m_2^2/R_{23}}} \right] \\
& \times \frac{(R_{23})^{\frac{d-4}{2}}}{\sqrt{1-R_{23}/R_{231}}} F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \right. \\
& \left. \frac{R_{23}}{R_4}, \frac{R_{23}}{R_{231}} \right) - \frac{\Gamma(\frac{d-2}{2})}{8 \Gamma(\frac{d}{2})} \left(\frac{\partial_4 R_4}{R_4} \right) \\
& \times \left[\frac{\partial_1 R_{231}}{(R_{231}-m_2^2)} \frac{\partial_3 R_{23}}{(R_{23}-m_2^2)} (m_2^2)^{\frac{d-2}{2}} \right. \\
& \times F_S \left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \frac{m_2^2}{R_4}, \right. \\
& \left. \frac{m_2^2}{m_2^2-R_{231}}, \frac{m_2^2}{m_2^2-R_{23}} \right) + (1 \leftrightarrow 2) \left. \right] \\
& + \left\{ (2, 3, 1) \leftrightarrow (3, 1, 2) \right\}. \quad (102)
\end{aligned}$$

where the terms $J_{123}^{(d=4)}$ and $J_{231}^{(d=4)}$ are given in (40). This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and $\text{Re}(d-3) > 0$. The Appell F_3 functions are described in detail in Appendix B (see (123) in more detail).

For future prospect of this work, a package which provides a general ϵ -expansion and numerical evaluations for one-loop functions at general d is planned. To achieve this purpose, many related works are worth mentioning in this paragraph. First, automatized analytic continuation of Mellin–Barnes integrals have been presented in [38]. The construction of Mellin–Barnes representations for Feynman integrals has been performed in [39, 40]. Recent development for treating numerically Mellin–Barnes integrals in physical regions has been proposed in [41–43]. The hypergeometric functions in this work can be expressed as the multi-fold MB integrals and they may be evaluated numerically by following the above works. Furthermore, the ϵ -expansion of the hypergeometric functions appearing in our analytic results may be also performed by using the packages `Sigma`, `EvaluateMultiSums` and `Harmonic Sums` [44–50]. Numerical ϵ -expansion of hypergeometric functions may be done by using `NumEXP` [51]. Besides that, analytic ϵ -expansion for the hypergeometric functions has been carried out in [52–59]. Differential reduction of generalized hypergeometric functions has been also reported in [60–63].

In the context of dimensional recurrence relations, the tensor reductions for one-loop up to five-point functions have been worked out in [64] and for higher-point functions have been developed in [65]. In practice, one encounters integrals with denominator powers higher than one and their reduction needs to be considered, see e.g. [3] for the scalar case. IBP reduction can be combined with dimensional recurrence relations to reduce them to master integrals of higher space-time dimensions.

6 Conclusions

In this article, we have been presented the analytic results for scalar one-loop two-, three- and four-point functions in detail. The results have been expressed in terms of Gauss ${}_2F_1$, Appell F_1 and F_S hypergeometric functions. All cases of external momentum and internal mass assignments have considered in detail in this work. The higher-terms in the ϵ -expansion for one-loop integrals can be performed directly from analytic expressions in this work. These terms are necessary building blocks in computing two-loop and higher-loop corrections. Moreover, one-loop functions in arbitrary d in this work may be taken account in the evaluations for higher-loop Feynman integrals. The one-loop functions with $d \geq 4$ can also used in the reduction for tensor one-loop Feynman integrals. For future works, a package for numerical evaluations for one-loop integrals at general d and general ϵ -expansion for these integrals is planned. Additionally, the method can extend to evaluate two- and higher-loop Feynman integrals.

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Appendix A: Useful formulas

In this appendix, we show some useful formulas used in this paper. We applied the reflect formula for gamma functions [24]:

$$\Gamma(1 - z - n) = (-1)^n \frac{\Gamma(z)\Gamma(1 - z)}{\Gamma(z + n)}, \quad (103)$$

provided that $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We have mentioned the duplication formula for gamma functions [24]:

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2})}{\sqrt{\pi}}, \quad (104)$$

provided that $z \in \mathbb{C}$.

Appendix B: Generalized hypergeometric series

Generalized hypergeometric functions are presented in this appendix.

Gauss hypergeometric series

Gauss hypergeometric series are given [25]:

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (105)$$

provided that $|z| < 1$. The pochhammer symbol $(a)_n = \Gamma(a + n) / \Gamma(a)$ is used.

The integral representation for Gauss hypergeometric functions [25] reads

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 du u^{b-1} (1-u)^{c-b-1} (1-zu)^{-a}, \quad (106)$$

provided that $|z| < 1$ and $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$.

Basic linear transformation formulas for Gauss ${}_2F_1$ hypergeometric functions which are collected from Refs. [25, 28], are listed as follows:

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = {}_2F_1\left[\begin{matrix} b, a \\ c \end{matrix}; z\right] \quad (107)$$

$$= (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right] \quad (108)$$

$$= (1-z)^{-a} {}_2F_1\left[\begin{matrix} a, c-b \\ c \end{matrix}; \frac{z}{z-1}\right] \quad (109)$$

$$= (1-z)^{-b} {}_2F_1\left[\begin{matrix} b, c-a \\ c \end{matrix}; \frac{z}{z-1}\right] \quad (110)$$

$$\begin{aligned} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ &\quad \times {}_2F_1\left[\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right] \\ &\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ &\quad \times {}_2F_1\left[\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-z\right] \quad (111) \end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} \\ &\quad \times {}_2F_1\left[\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}; \frac{1}{z}\right] \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} \\ &\quad \times {}_2F_1\left[\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}; \frac{1}{z}\right]. \quad (112) \end{aligned}$$

Appell series

Appell F_1 series are defined [25, 32]:

$$F_1(a; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n} m! n!} x^m y^n, \quad (113)$$

provided that $|x| < 1$ and $|y| < 1$.

The single integral representation for F_1 reads [25]

$$F_1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 du u^{a-1} \times (1-u)^{c-a-1} (1-xu)^{-b} (1-yu)^{-b'}, \quad (114)$$

provided that $\operatorname{Re}(c) > \operatorname{Re}(a) > 0$ and $|x| < 1, |y| < 1$.

We collect all transformations for Appell F_1 functions from Refs. [25, 30]. The first relation for F_1 is mentioned,

$$F_1\left(a; b, b'; c; x, y\right) = (1-x)^{-b} (1-y)^{-b'} \times F_1\left(c-a; b, b'; c; \frac{x}{x-1}, \frac{y}{y-1}\right). \quad (115)$$

If $b' = 0$, we arrive at the well-known Pfaff–Kummer transformation for the ${}_2F_1$. Further, we have

$$F_1\left(a; b, b'; c; x, y\right) = (1-x)^{-a} F_1\left(a; -b-b'+c, b'; c; \frac{x}{x-1}, \frac{y-x}{1-x}\right). \quad (116)$$

Furthermore, if $c = b + b'$, one then obtains

$$F_1\left(a; b, b'; b+b'; x, y\right) = (1-x)^{-a} {}_2F_1\left[\begin{matrix} a, b' \\ b+b' \end{matrix}; \frac{y-x}{1-x}\right] \quad (117)$$

$$= (1-y)^{-a} {}_2F_1\left[\begin{matrix} a, b \\ b+b' \end{matrix}; \frac{x-y}{1-y}\right]. \quad (118)$$

Similarly,

$$F_1\left(a; b, b'; c; x, y\right) = (1-y)^{-a} \times F_1\left(a; b, c-b-b'; c; \frac{x-y}{1-y}, \frac{y}{y-1}\right), \quad (119)$$

and

$$F_1\left(a; b, b'; c; x, y\right) = (1-x)^{c-a-b} (1-y)^{-b'} \times F_1\left(c-a; c-b-b', b'; c; x, \frac{x-y}{1-y}\right), \quad (120)$$

$$F_1\left(a; b, b'; c; x, y\right) = (1-x)^{-b} (1-y)^{c-a-b'} \times F_1\left(c-a; b, c-b-b'; c; \frac{y-x}{1-x}, y\right). \quad (121)$$

A further relation for F_1 is given

$$\begin{aligned} &F_1\left(a+1, 1, 1/2, a+2; x, y\right) \\ &= \frac{\sqrt{\pi}\Gamma(a+2)}{\Gamma(a+3/2)} y^{-a-1} {}_2F_1\left[\begin{matrix} 1, a+1 \\ a+3/2 \end{matrix}; \frac{x}{y}\right] \\ &\quad - \frac{2\Gamma(a+2)}{\Gamma(a+1)} \frac{\sqrt{1-y}}{y(1-x)} \\ &\quad \times F_1\left(1, -a, 1, 3/2; 1-\frac{1}{y}, \frac{x(1-y)}{y(1-x)}\right). \quad (122) \end{aligned}$$

Appell F_3 series

Appell F_3 series are written [25,32]

$$F_3(a, a'; b, b'; c; x, y) = \sum_{m \geq 0} \sum_{n \geq 0} \frac{(a)_m (a')_n (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n, \quad (123)$$

provided that $\operatorname{Re}(c) > \operatorname{Re}(a) > 0$ and $|x| < 1, |y| < 1$.

J. Kampé de Fériet series

In addition, J. Kampé de Fériet series [32,33] with two variables are shown:

$$F_{r,s}^{p,q} \left(\begin{matrix} a_1, \dots, a_p : b_1, b'_1; \dots; b_q, b'_q \\ c_1, \dots, c_r : d_1, d'_1; \dots; d_s, d'_s \end{matrix} ; x, y \right) = \sum_{m \geq 0} \sum_{n \geq 0} \frac{(a_1)_{m+n} \dots (a_p)_{m+n} (b_1)_m (b'_1)_n \dots (b_q)_m (b'_q)_n}{(c_1)_{m+n} \dots (c_r)_{m+n} (d_1)_m (d'_1)_n \dots (d_s)_m (d'_s)_n} \frac{x^m y^n}{m! n!}. \quad (124)$$

Lauricella–Saran function F_S

Lauricella–Saran function F_S is defined in terms of a triple hypergeometric series [35]

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_1, \gamma_1; x, y, z) = \sum_{r,m,n=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_{m+n} (\beta_1)_r (\beta_2)_m (\beta_3)_n}{(\gamma_1)_{r+m+n} r! m! n!} x^r y^m z^n. \quad (125)$$

The integral representation of F_S is defined [35]:

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1) \Gamma(\gamma_1 - \alpha_1)} \int_0^1 dt \frac{t^{\gamma_1 - \alpha_1 - 1} (1-t)^{\alpha_1 - 1}}{(1-x+tx)^{\beta_1}} F_1(\alpha_2, \beta_2, \beta_3; \gamma_1 - \alpha_1; ty, tz), \quad (126)$$

provided that $\operatorname{Re}(\gamma_1 - \alpha_1 - \alpha_2) > 0, \operatorname{Re}(\alpha_1) > 0, \operatorname{Re}(\alpha_2) > 0$, and $|x|, |y|, |z| < 1$.

Appendix C: The contour integrations

Type 1

Mellin–Barnes relation [29] is given:

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\lambda + s)}{\Gamma(\lambda)} (z)^s = \frac{1}{(1+z)^\lambda} \quad (127)$$

provided that $|\operatorname{Arg}(z)| < \pi$. The integration contour is chosen in such a way that the poles of $\Gamma(-s)$ and $\Gamma(\lambda + s)$ are well-separated.

Type 2

The Barnes-type integral for Gauss hypergeometric [25] reads

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+s) \Gamma(b+s)}{\Gamma(c+s)} (-z)^s = \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right] \quad (128)$$

provided that $|\operatorname{Arg}(-z)| < \pi$ and $|z| < 1$.

Type 3

The next Barnes-type integral applied in this paper is

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+s) \Gamma(b+s)}{\Gamma(c+s)} (-x)^s \\ \times {}_2F_1 \left[\begin{matrix} a+s, b' \\ c+s \end{matrix} ; y \right] \\ = \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F_1(a; b, b'; c; x, y) \end{aligned} \quad (129)$$

with $|\operatorname{Arg}(-x)| < \pi, |\operatorname{Arg}(-y)| < \pi$ and $|x| < 1$ and $|y| < 1$. Under these conditions, one could close the contour of integration to the right. Subsequently, we have to take into account the residua of the sequence poles of $\Gamma(-s)$. The result is expressed as the summation of Gauss hypergeometric function. The summation is then identical as a series of Appell F_1 functions [30] as follows:

$$\begin{aligned} \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \\ \times {}_2F_1 \left[\begin{matrix} a+m, b' \\ c+m \end{matrix} ; y \right] = \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F_1(a; b, b'; c; x, y). \end{aligned} \quad (130)$$

Type 4

Furthermore, in this paper we evaluate the following integral:

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+s) \Gamma(b_1+s)}{\Gamma(c+s)} (-x)^s \\ \times F_1(a' + s; b_2, b_3; c + s; y, z) \\ = \frac{\Gamma(a) \Gamma(b_1)}{\Gamma(c)} (1-y)^{-b_2} (1-z)^{-b_3} \\ \times F_S \left(a, c-a', c-a'; b_1, b_2, b_3; c, c, c; x, \frac{y}{y-1}, \frac{z}{z-1} \right). \end{aligned} \quad (131)$$

We close the contour of integration to the right. By taking into account the residua of the sequence poles of $\Gamma(-s)$, the

result reads

$$\frac{\Gamma(a)\Gamma(b_1)}{\Gamma(c)} \sum_{m=0}^{\infty} \frac{(a)_m(b_1)_m}{(c)_m} \frac{x^m}{m!} F_1(a' + m; b_2, b_3; c + m; y, z). \quad (132)$$

One applies the relation (115) for Appell F_1 functions in (132) as follows

$$F_1(a' + m; b_2, b_3; c + m; y, z) = (1 - y)^{-b_2} (1 - z)^{-b_3} \times F_1\left(c - a'; b_2, b_3; c + m; \frac{y}{y-1}, \frac{z}{z-1}\right). \quad (133)$$

Equation (132) is then presented as a series of Lauricella functions F_S [35]

$$\begin{aligned} & \frac{\Gamma(a)\Gamma(b_1)}{\Gamma(c)} (1-y)^{-b_2} (1-z)^{-b_3} \\ & \times \sum_{m,n,l=0}^{\infty} \frac{(a)_m(b_1)_m(b_2)_n(b_3)_l(c-a')_{n+l}}{(c)_{m+n+l}} \frac{x^m}{m!} \frac{1}{n!} \\ & \times \left(\frac{y}{y-1}\right)^n \frac{1}{l!} \left(\frac{z}{z-1}\right)^l \\ & = \frac{\Gamma(a)\Gamma(b_1)}{\Gamma(c)} (1-y)^{-b_2} (1-z)^{-b_3} \\ & \times F_S\left(a, c-a', c-a'; b_1, b_2, b_3; c, c, c; x, \frac{y}{y-1}, \frac{z}{z-1}\right). \end{aligned} \quad (134)$$

provided that $|x| < 1$, $\left|\frac{y}{y-1}\right| < 1$ and $\left|\frac{z}{z-1}\right| < 1$.

The last formula is mentioned in this paper relates to a transformation of Mellin–Barnes integrals (see page 156 of Ref. [28], item 14.53 in page 290 of [29]) which is

$$\begin{aligned} & \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+s) \Gamma(b+s)}{\Gamma(c+s)} (-z)^s \\ & = \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+b-c-s) \Gamma(c-a+s) \Gamma(c-b+s)}{\Gamma(c-a) \Gamma(c-b)} \\ & \times (1-z)^{c-a-b+s}, \end{aligned} \quad (135)$$

provided that $|\text{Arg}(-z)| < 2\pi$.

Appendix D: Master equation for J_N

General relation for J_N have been proved in [18]. In this appendix, we consider $N = 4$ as an example. Performing Feynman parameterization for J_4 , one arrives at

$$\frac{J_4}{\Gamma\left(4 - \frac{d}{2}\right)} = \int dS_3 \left[ax_1^2 + bx_2^2 + cx_3^2 + 2d x_1x_2 + 2e x_1x_3 + 2f x_2x_3 \right. \\ \left. + gx_1 + hx_2 + kx_3 + j - i\rho \right]^{\frac{d}{2}-4}. \quad (136)$$

Above coefficients $a, b, c, \dots, j$ are shown

$$\begin{aligned} a &= p_1^2, & b &= p_2^2, & c &= (p_2 + p_3)^2, \\ d &= -p_1p_2, & e &= -p_1p_2 - p_1p_3, & f &= p_2^2 + p_2p_3, \\ g &= m_1^2 - m_2^2 - p_1^2, & h &= -m_2^2 + m_3^2 - p_2^2, & k &= -m_2^2 + m_4^2 - (p_2 + p_3)^2, \\ j &= m_2^2. \end{aligned}$$

We used the following notation

$$\begin{aligned} \int dS_3 &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \\ &= \int_0^1 dx_2 \int_0^{1-x_2} dx_3 \int_0^{1-x_2-x_3} dx_1 \\ &= \int_0^1 dx_3 \int_0^{1-x_3} dx_1 \int_0^{1-x_1-x_3} dx_2. \end{aligned} \quad (137)$$

The integrand of J_4 is

$$\begin{aligned} \mathcal{M}_4(x_1, x_2, x_3) &= ax_1^2 + bx_2^2 + cx_3^2 + 2d x_1x_2 \\ &+ 2e x_1x_3 + 2f x_2x_3 \\ &+ gx_1 + hx_2 + kx_3 + j. \\ &= (x_1, x_2, x_3) \mathcal{G}_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &+ 2\mathcal{H}_4^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathcal{K}_4 \end{aligned} \quad (138)$$

$$\begin{aligned} &= (x_1 - y_1, x_2 - y_2, x_3 - y_3) \\ &\mathcal{G}_3 \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \end{pmatrix} + R_4 \end{aligned} \quad (139)$$

$$= \Lambda_4(x_1, x_2, x_3) + R_4. \quad (140)$$

These matrices are given

$$\mathcal{G}_3 = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}, \quad \mathcal{H}_4 = \frac{1}{2} \begin{pmatrix} g \\ h \\ k \end{pmatrix}, \quad \mathcal{K}_4 = j.$$

The $\Lambda_4(x_1, x_2, x_3)$ reads

$$\begin{aligned} \Lambda_4(x_1, x_2, x_3) &= a(x_1 - y_1)^2 + b(x_2 - y_2)^2 \\ &+ c(x_3 - y_3)^2 + 2d(x_1 - y_1)(x_2 - y_2) \\ &+ 2e(x_1 - y_1)(x_3 - y_3) \\ &+ 2f(x_2 - y_2)(x_3 - y_3). \end{aligned} \quad (141)$$

The vector $\vec{y}$ is defined as: $\vec{y} = (y_1, y_2, y_3) = -\mathcal{G}_3^{-1} H^T$. We write explicitly y_1, y_2, y_3 and y_4 as follows:

$$\begin{aligned} y_1 &= \frac{-bch + bek + cdg - dfk - efg + f^2h}{G_3} = \frac{\partial R_4}{\partial m_1^2}, \\ y_2 &= \frac{a(fk - cg) + cdh - e(dk + fh) + e^2g}{G_3} = \frac{\partial R_4}{\partial m_3^2}, \\ y_3 &= \frac{afg + beh - d(eg + fh) - abk + d^2k}{G_3} = \frac{\partial R_4}{\partial m_4^2}, \\ y_4 &= 1 - y_1 - y_2 - y_3 = \frac{\partial R_4}{\partial m_2^2}. \end{aligned} \quad (142)$$

First, we consider $G_3 \neq 0$ and $R_4 \neq 0$. In this case, Mellin–Barnes relation is applied to decompose J_4 's integrand as follows:

$$\begin{aligned} & \frac{1}{\left[\Lambda_4(x_1, x_2, x_3) + R_4 - i\rho \right]^{4-\frac{d}{2}}} \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(4 - \frac{d}{2} + s)}{\Gamma(4 - \frac{d}{2})} \\ & \quad \times \left(\frac{1}{R_4 - i\rho} \right)^{4-\frac{d}{2}} \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4 - i\rho} \right]^s, \end{aligned} \quad (143)$$

provided that $\left| \text{Arg} \left(\frac{\Lambda_4(x_1, x_2, x_3)}{R_4 - i\rho} \right) \right| < \pi$. With the help of the Mellin–Barnes relation, this brings the Feynman parameters integration to the simpler form:

$$\mathcal{F}_4 = \int dS_3 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4 - i\rho} \right]^s. \quad (144)$$

In order to carry out this integral, we consider the following differential operator (see theorem of Bernshtein [36], or [37])

$$\begin{aligned} \hat{O}_4 &= \frac{1}{2}(x_1 - y_1) \frac{\partial}{\partial x_1} \\ &+ \frac{1}{2}(x_2 - y_2) \frac{\partial}{\partial x_2} + \frac{1}{2}(x_3 - y_3) \frac{\partial}{\partial x_3}. \end{aligned} \quad (145)$$

It is easy to check that

$$\hat{O}_4 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s = s \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s. \quad (146)$$

As a matter of this fact, we can rewrite Feynman parameter integral as

$$\begin{aligned} \mathcal{F}_4 &= \frac{1}{s} \int dS_3 \hat{O}_4 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \\ &= \frac{1}{s} \left\{ \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} dx_3 (x - y_3) \frac{\partial}{\partial x_3} \right. \\ & \quad \times \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s + \int_0^1 dx_1 \int_0^{1-x_1} dx_3 \end{aligned}$$

Table 1 The coefficients $A_i, B_i, C_i, D_i, E_i, F_i$

i	1	2	3	4
A_i	b	a	a	$a + c - 2e$
B_i	c	c	b	$a + b - 2d$
C_i	f	e	d	$a - d - e + f$
D_i	h	g	g	$-2a + 2e - g + k$
E_i	k	k	h	$-2a + 2d - g + h$
F_i	j	j	j	$a + g + j$

$$\begin{aligned} & \times \int_0^{1-x_1-x_3} dx_2 (x - y_2) \frac{\partial}{\partial x_2} \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \\ &+ \int_0^1 dx_2 \int_0^{1-x_3} dx_3 \int_0^{1-x_2-x_3} dx_1 (x - y_1) \frac{\partial}{\partial x_1} \\ & \times \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \} \end{aligned} \quad (147)$$

$$\begin{aligned} &= \frac{1}{2s} \left\{ \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \frac{\partial}{\partial x_3} \right. \\ & \times \left\{ (x_3 - y_3) \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \right\} \\ &+ \int_0^1 dx_1 \int_0^{1-x_1} dx_3 \int_0^{1-x_1-x_3} dx_2 \frac{\partial}{\partial x_2} \left\{ (x_2 - y_2) \right. \\ & \times \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \} \\ &+ \int_0^1 dx_2 \int_0^{1-x_3} dx_3 \int_0^{1-x_2-x_3} dx_1 \frac{\partial}{\partial x_1} \\ & \times \left\{ (x_1 - y_1) \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \right\} \} \\ &- \frac{3}{2s} \int dS_3 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s. \end{aligned} \quad (148)$$

The last term in this equation is proportional to $\mathcal{F}_4$. It is then combined with $\mathcal{F}_4$ on the left side of Eq. (148). As a result, Eq. (148) is then casted into the form:

$$\begin{aligned} \mathcal{F}_4 &= \frac{\Gamma(s + \frac{3}{2})}{2 \Gamma(s + \frac{5}{2})} \left\{ \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \right. \\ & \times \frac{\partial}{\partial x_3} \left\{ (x_3 - y_3) \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \right\} \\ &+ \int_0^1 dx_1 \int_0^{1-x_1} dx_3 \int_0^{1-x_1-x_3} dx_2 \frac{\partial}{\partial x_2} \end{aligned}$$

Table 2 The coefficients $A_i, B_i, C_i, D_i, E_i, F_i$ in terms of external momenta and internal masses

i	1	2	3	4
A_i	p_2^2	p_1^2	p_1^2	p_4^2
B_i	t	t	p_2^2	s
C_i	$p_2(p_2 + p_3)$	$-p_1(p_2 + p_3)$	$-p_1 p_2$	$-p_4(p_1 + p_2)$
D_i	$-(p_2^2 + m_2^2 - m_3^2)$	$-(p_1^2 + m_2^2 - m_1^2)$	$-(p_1^2 - m_1^2 + m_2^2)$	$-(p_4^2 + m_1^2 - m_4^2)$
E_i	$-(t + m_2^2 - m_4^2)$	$-(t + m_2^2 - m_4^2)$	$-(p_2^2 - m_3^2 + m_2^2)$	$-(s + m_1^2 - m_3^2)$
F_i	m_2^2	m_2^2	m_2^2	m_1^2

$$\times \left\{ (x_2 - y_2) \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \right\} \\ + \int_0^1 dx_2 \int_0^{1-x_3} dx_3 \int_0^{1-x_2-x_3} dx_1 \frac{\partial}{\partial x_1} \left\{ (x_1 - y_1) \right. \\ \left. \times \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4} \right]^s \right\} \Bigg\}. \quad (149)$$

Taking over a Feynman parameter integration in Eq. (149), the result reads

$$\mathcal{F}_4 = \frac{\Gamma(s + \frac{3}{2})}{2\Gamma(s + \frac{5}{2})} \sum_{i=1}^4 y_i \int dS_2 \\ \times \left[\frac{A_i x_1^2 + B_i x_2^2 + 2C_i x_1 x_2 + D_i x_1 + E_i x_2 + F_i}{R_4} - 1 \right]^s \\ = \frac{\Gamma(s + \frac{3}{2})}{2\Gamma(s + \frac{5}{2})} \sum_{i=1}^4 y_i \int dS_2 \\ \times \left[\frac{\mathcal{M}_3(A_i, B_i, C_i, D_i, E_i, F_i)}{R_4} - 1 \right]^s. \quad (150)$$

The coefficients $A_i, B_i, C_i, D_i, E_i, F_i$ are given in Table 1.

We then write the coefficients in terms of external momenta and internal masses which are given in Table 2.

Now we can write the Mellin–Barnes integral for J_4 as

$$J_4(d; \{p_i^2, s, t\}, \{m_i^2\}) \\ = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(4 - \frac{d}{2} + s) \Gamma(s + \frac{3}{2})}{2\Gamma(s + \frac{5}{2})} \\ \times \left(\frac{1}{R_4} \right)^{\frac{d}{2}-4} \sum_{i=1}^4 \left(\frac{\partial R_4}{\partial m_i^2} \right) \int dS_2 \\ \times \left[\frac{\mathcal{M}_3(A_i, B_i, C_i, D_i, E_i, F_i)}{R_4} - 1 \right]^s. \quad (151)$$

By applying the relation (135), one gets

$$J_4(d; \{p_i^2, s, t\}, \{m_i^2\})$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-3}{2} + s) \Gamma(s + 1)}{2\Gamma(\frac{d-3}{2})} \\ \times \left(\frac{1}{R_4} \right)^{4-\frac{d}{2}} \sum_{i=1}^4 \left(\frac{\partial R_4}{\partial m_i^2} \right) \\ \times \Gamma\left(3 - \frac{d}{2} - s\right) \int dS_2 \\ \times \left[\frac{\mathcal{M}_3(A_i, B_i, C_i, D_i, E_i, F_i)}{R_4} \right]^{\frac{d}{2}-3+s}. \quad (152)$$

This equation is then written as follows:

$$J_4(d; \{p_i^2, s, t\}, \{m_i^2\}) \\ = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-3}{2} + s) \Gamma(s + 1)}{2\Gamma(\frac{d-3}{2})} \\ \times \left(\frac{1}{R_4} \right)^s \sum_{k=1}^4 \left(\frac{1}{R_4} \frac{\partial R_4}{\partial m_k^2} \right) \\ \times \mathbf{k}^- J_4(d + 2s; \{p_i^2, s, t\}, \{m_i^2\}). \quad (153)$$

In the case of $R_4 = 0$, there is no Mellin–Barnes integral for J_4 . We only apply the ring operator $\hat{O}_4$, the result arrives

$$J_4(d; \{p_i^2, s, t\}, \{m_i^2\}) = \frac{1}{d-5} \\ \times \sum_{k=1}^4 \left(\frac{\partial_k Y_4}{G_3} \right) \mathbf{k}^- J_4(d-2; \{p_i^2, s, t\}, \{m_i^2\}). \quad (154)$$

References

1. S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000)
2. O.V. Tarasov, Phys. Rev. D **54**, 6479 (1996)
3. A.I. Davydychev, J. Math. Phys. **33**, 358 (1992)
4. E.E. Boos, A.I. Davydychev, Theor. Math. Phys. **89**, 1052 (1991), [Teor. Mat. Fiz. **89** (1991) 56]
5. A.I. Davydychev, R. Delbourgo, J. Math. Phys. **39**, 4299 (1998)
6. A.I. Davydychev, Phys. Rev. D **61**, 087701 (2000)
7. A.I. Davydychev, M.Y. Kalmykov, Nucl. Phys. B **699**, 3 (2004)
8. A.I. Davydychev, Nucl. Instrum. Meth. A **559**, 293 (2006)
9. C. Anastasiou, E.W.N. Glover, C. Oleari, Nucl. Phys. B **572**, 307 (2000)

10. S. Abreu, R. Britto, H. Grönqvist, JHEP **1507**, 111 (2015)
11. S. Abreu, R. Britto, C. Duhr, E. Gardi, JHEP **1706**, 114 (2017)
12. S. Abreu, R. Britto, C. Duhr, E. Gardi, Phys. Rev. Lett. **119**(5), 051601 (2017)
13. S. Abreu, R. Britto, C. Duhr, E. Gardi, JHEP **1712**, 090 (2017)
14. O.V. Tarasov, Nucl. Phys. Proc. Suppl. **89**, 237 (2000)
15. K.H. Phan, D.T. Tran, PTEP **2019**(6), 063B01 (2019)
16. J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. B **672**, 303 (2003)
17. J. Blümlein, K.H. Phan, T. Riemann, Nucl. Part. Phys. Proc. **270–272**, 227 (2016)
18. J. Blümlein, K.H. Phan, T. Riemann, Acta Phys. Polon. B **48**, 2313 (2017)
19. K.H. Phan, T. Riemann, Phys. Lett. B **791**, 257 (2019)
20. A. Denner, S. Dittmaier, Nucl. Phys. Proc. Suppl. **160**, 22 (2006)
21. G. Devaraj, R.G. Stuart, Nucl. Phys. B **519**, 483 (1998)
22. G't Hooft, M.J.G. Veltman, Nucl. Phys. B **153**, 365 (1979)
23. Z. Bern, L.J. Dixon, D.A. Kosower, Nucl. Phys. B **412**, 751 (1994)
24. B.C. Carlson, *Special functions of applied mathematics* (Academic Press, New York, 1977)
25. L.J. Slater, *Generalized hypergeometric functions* (Cambridge University Press, Cambridge, 1966)
26. Bruce C. Berndt, *Ramanujan's Notebooks, Part II* (Springer Press, Berlin, 1999)
27. George E. Andrews, Richard Askey, Ranjan Roy, *Special functions* (Cambridge university Press, Cambridge, 1999)
28. E. W. Barnes, Proc. London Math. Soc. **s2-6** (1), 141–177 (1908)
29. E.T. Whittaker, G.N. Watson, *A course of modern analysis* (Cambridge University Press, Cambridge, 1952)
30. M.J. Schlosser, [arXiv:1305.1966](https://arxiv.org/abs/1305.1966) [math.CA]
31. P.O.M. Olsson, J. Math. Phys. **5**, 420 (1964)
32. P. Appell, J. Kampé de Fériet, Fonctions Hypergeometriques et Hyperspheriques. Polynomes d' Hermite, Gauthier-Villars, Paris (1926)
33. Kampé de Fériet, J. La fonction hypergéométrique. Gauthier-Villars, Paris (1937)
34. B. Ananthanarayan, S. Friot, S. Ghosh, [arXiv:2003.12030](https://arxiv.org/abs/2003.12030) [hep-ph]
35. S. Saran, Acta Math. **93**, 293 (1955)
36. I.N. Bershtein, Modules over the ring of differential operators. A study of the fundamental solutions of differential equations with constant coefficients. Funk. Analiz **5**(2), 1–16 (1971)
37. V.A. Golubeva, V.Z. Énoľ'skii, Math. Notes Acad. Sci. USSR **23**, 63 (1978). <https://doi.org/10.1007/BF01104888>
38. M. Czakon, Comput. Phys. Commun. **175**, 559 (2006)
39. J. Gluza, K. Kajda, T. Riemann, Comput. Phys. Commun. **177**, 879 (2007)
40. J. Gluza, K. Kajda, T. Riemann, V. Yundin, Eur. Phys. J. C **71**, 1516 (2011)
41. I. Dubovyk, J. Gluza, T. Jelinski, T. Riemann, J. Usovitsch, Acta Phys. Polon. B **48**, 995 (2017)
42. J. Usovitsch, I. Dubovyk, T. Riemann, PoS LL **2018**, 046 (2018)
43. I. Dubovyk, J. Gluza, T. Riemann, Acta Phys. Polon. B **50**, 1993 (2019)
44. C. Schneider, Sémin. Lothar, Combin. **56** 1, (2007) article B56b; in: *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation*, eds. C. Schneider, J. Blümlein (Springer, Wien, 2013) 325 [arXiv:1304.4134](https://arxiv.org/abs/1304.4134) [cs.SC]
45. J. Ablinger, J. Blümlein, S. Klein, C. Schneider, Nucl. Phys. Proc. Suppl. **205–206**, 110 (2010)
46. J. Blümlein, A. Hasselhuhn, C. Schneider, PoS (RADCOR 2011) 032
47. C. Schneider, J. Phys. Conf. Ser. **523**, 012037 (2014)
48. J. Ablinger, [arXiv:1305.0687](https://arxiv.org/abs/1305.0687) [math-ph]. [arXiv:1011.1176](https://arxiv.org/abs/1011.1176) [math-ph]
49. J. Ablinger, J. Blümlein, C. Schneider, J. Math. Phys. **52**, 102301 (2011)
50. J. Ablinger, J. Blümlein, C. Schneider, J. Math. Phys. **54**, 082301 (2013)
51. Z.W. Huang, J. Liu, Comput. Phys. Commun. **184**, 1973 (2013)
52. T. Huber, D. Maitre, Comput. Phys. Commun. **175**, 122 (2006)
53. T. Huber, D. Maitre, Comput. Phys. Commun. **178**, 755 (2008)
54. M.Y. Kalmykov, JHEP **0604**, 056 (2006)
55. M.Y. Kalmykov, B.F.L. Ward, S. Yost, JHEP **0702**, 040 (2007)
56. S. Moch, P. Uwer, Comput. Phys. Commun. **174**, 759–770 (2006). <https://doi.org/10.1016/j.cpc.2005.12.014>
57. S. Friot, D. Greynat, J. Math. Phys. **53**, 023508 (2012)
58. D. Greynat, J. Sesma, Comput. Phys. Commun. **185**, 472 (2014)
59. D. Greynat, J. Sesma, G. Vulvert, [arXiv:1310.7700](https://arxiv.org/abs/1310.7700) [math-ph]
60. V.V. Bytev, M.Y. Kalmykov, B.A. Kniehl, Nucl. Phys. B **836**, 129 (2010)
61. V.V. Bytev, M.Y. Kalmykov, B.A. Kniehl, Comput. Phys. Commun. **184**, 2332 (2013)
62. V.V. Bytev, B.A. Kniehl, Comput. Phys. Commun. **189**, 128 (2015)
63. V.V. Bytev, M.Y. Kalmykov, S.O. Moch, Comput. Phys. Commun. **185**, 3041 (2014)
64. J. Fleischer, T. Riemann, Phys. Rev. D **83**, 073004 (2011)
65. J. Fleischer, T. Riemann, Phys. Lett. B **707**, 375–380 (2012)