



# Scalar field and deflection of light under the effects of topologically charged Ellis–Bronnikov-type wormhole spacetime

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**Abstract** We have analyzed the effects of a simple wormhole, known as the Ellis–Bronnikov-type wormhole, on a scalar field, where, analytically, we determine solutions of bound states and show that the relativistic energy profile of this scalar field is drastically influenced by the topology of space-time characterized by the presence of a global monopole. Before this analysis, we investigated the effects of this background on the deflection of light, which is influenced by the parameters associated with the wormhole throat and the topological defect.

## 1 Introduction

Based on condensed matter systems, it is expected that, due to the symmetry break associated with the decoupling of fundamental interactions, in the phase transition process of the primordial Universe, to observe cosmological objects known as topological defects [1], although, until now, these objects have not been observed. The topological defects more known in the literature are domain walls [2], cosmic string [3–5] and the global monopole [6]. The latter two are the strongest candidates to be observed [1]. In particular, the global monopole (GM) has been extensively investigated and applied in several branches of physics [7–13]. In the context of non-relativistic quantum mechanics, the GM has been studied on the harmonic oscillator [14, 15], on a particle subjected to the a self-interaction potential [7], in a particle interacting with a Kratzer potential [16] and on a charged particle-magnetic monopole scattering [17]. In relativistic quantum mechanics the GM has been investigated on the hydrogen atom and

the pionic atom [18], in the exact solutions of the Klein–Gordon equation with presence of a dyon, magnetic flux and scalar potential [19], in exact solutions of scalar bosons with the Aharonov–Bohm effect and the Coulomb potential [20–22] and on the Dirac and Klein–Gordon oscillators [23]. Recently, in the context of modified gravity, one of us investigated how the energy-momentum tensor associated with the region outside the GM core [6] can give rise to wormhole solutions [24, 25].

Wormhole are solutions of the General Theory of Relativity (GTR) that describe a shortcut between different regions of spacetime or even between different universes [26, 27]. The simplest wormhole solution is the Ellis–Bronnikov wormhole [28, 29], the Raychaudhuri equation for a congruence of null radial geodesics in that spacetime is given by<sup>1</sup>

$$\frac{d\Theta}{d\lambda} = -R_{\mu\nu}k^\mu k^\nu - \frac{\theta^2}{2}, \quad (1)$$

where  $\Theta$  is the congruence expansion,  $\lambda$  is the affine parameter associated with geodesics,  $R_{\mu\nu}$  is the Ricci tensor and  $k^\mu$  is a null tangent vector to the geodesic. In the wormhole throat, the expansion vanish ( $\Theta = 0$ ), but  $\frac{d\Theta}{d\lambda} \geq 0$ , which imply  $R_{\mu\nu}k^\mu k^\nu \leq 0$ . The defocusing of the congruence in the wormhole throat, therefore, violates null convergence condition (NCC), defined as  $R_{\mu\nu}k^\mu k^\nu \geq 0$  in [30]. Then, by admitting the validity of the GTR, the violation of the CCN also implies the violation of the Null Energy Condition (NEC).<sup>2</sup> In the GTR, NCC is used to demonstrate the attractive character of gravitational fields generated by conventional sources of matter; as in the the wormhole case it is violated in the throat, this implies that to support worm-

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<sup>1</sup> See Sect. 12.4 of the reference [27] for more details.

<sup>2</sup> The NEC states that, for a null vector  $k^\mu$ ,  $T_{\mu\nu}k^\mu k^\nu \geq 0$ ; where  $T_{\mu\nu}$  is the energy-momentum tensor.

hole solutions in the GTR, the presence of exotic matter is necessary.

However, related research to the cosmology, such as that involving increasingly accurate measurements of temperature variations in cosmic microwave background, suggests that the universe is expanding rapidly and GTR does not explain this phenomenon [31–35]. In addition, almost all classical GTR solutions lead to geodesically incomplete spacetimes, that is, singular spacetimes.

In order to avoid singularities and exotic components, such as dark matter and energy, for example, alternative theories of gravity are sought where, in the appropriate regime, these problems can be avoided. Among the many proposals for modifications, we want to highlight the well-called Eddington-inspired Born-Infeld gravity (EiBI) [36] that has aroused a lot of interest in recent years due to its ability to avoid geodesic singularities without the need for exotic matter and without quantum effects, among other interesting properties presented in Ref. [37]. Among the various solutions in EiBI gravity [24, 38–41], the simplest one so far describes a static and spherically symmetrical Ellis–Bronnikov-type wormhole with the topological charge of the GM [24, 42]. Such a solution was obtained by coupling to the spacetime geometry the energy-momentum tensor associated with the region outside the GM core [6], i.e.,  $T_\nu^\mu = \text{diag}\left(-\frac{\eta^2}{r^2}, -\frac{\eta^2}{r^2}, 0, 0\right)$ . It is important to highlight that the solution found, being a wormhole, has no core. Instead, it is supported on topological grounds, such a solution can be interpreted as a geon, in the same sense introduced by Wheeler [43]. The line element of such a solution can be written as [24]

$$ds^2 = -dt^2 + \frac{dx^2}{\alpha^2} + (x^2 + a^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where  $-\infty < x < \infty$ ,  $0 < \alpha = 1 - 8\pi^2 G\eta_0^2 < 1$ , with  $G$  being the universal constant of gravitation and  $\eta_0$  the dimensionless volumetric mass density of the pointlike GM [18], and  $a = \text{const.}$  is the radius of the wormhole throat. It is interesting to note that in this context of EiBI the source of matter that supports the wormhole is due to the energy-momentum tensor referring to the region outside the GM core and, therefore, it does not violate the energy conditions, even though it violates the CCN, as is expected [44]. In this work we will explore some aspects related to the metric (2), such as the deflection of light and the scalar field in this background.

The structure of this paper is as follows: in Sect. (2) we have shown that the topological charge presence in an Ellis–Bronnikov-type wormhole increases the angular deflection of the light in relation to the classic case without topological charge; in Sect. (2) we investigate a scalar field immersed in an Ellis–Bronnikov-type wormhole, where, in the search for solutions of bound states, we determine the relativistic

energy profile of the relativistic quantum system; in Sect. (4) we present our conclusions.

## 2 On the deflection of light in the topologically charged Ellis–Bronnikov-type wormhole spacetime

In this section we calculate the deflection angle of light in the spacetime of the topologically charged Ellis–Bronnikov-type wormhole to assess how the topological charge influences the result. As MG, the Ellis–Bronnikov wormhole has no mass, but even so it causes a deflection in the trajectories of light rays. In the case of an Ellis–Bronnikov wormhole, Nakajima and Asada obtained the expression for the deflection of light in terms of the ratio between the radius of the throat of the wormhole and the impact parameter, the result obtained by them corrects previously obtained results in higher orders [45]. Our solution (2) is symmetrical in relation to the throat, but in [46] the authors studied several aspects of gravitational lensing by spherically symmetric wormholes which are asymmetrical with respect to the throat.

To calculate the deflection of light, it is sufficient to consider only one region of the wormhole, which without loss of generality we choose  $x > 0$ . The respective Lagrangian for radial geodesic in the equatorial plane  $\theta = \frac{\pi}{2}$  is given by

$$\mathcal{L} = -\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{\alpha^2}\left(\frac{dx}{d\lambda}\right)^2 + (x^2 + a^2)\left(\frac{d\phi}{d\lambda}\right)^2, \quad (3)$$

where  $\lambda$  is the parameter associated to the geodesics. From the Euler-Lagrange equation for the coordinates  $t$  and  $\phi$ , we show that the conserved quantities are the following  $L = (x^2 + a^2)\frac{d\phi}{d\lambda}$  and  $E = \frac{dt}{d\lambda}$ , which, in the case of null geodesics, are the angular momentum and the energy, respectively. In terms of these conserved quantities and remembering that for null geodesics  $\mathcal{L} = 0$ , Eq. (3) becomes

$$\frac{1}{\alpha^2}\left(\frac{dx}{d\lambda}\right)^2 = E^2 - \frac{L^2}{x^2 + a^2}. \quad (4)$$

By defining the parameter  $\beta = \frac{L}{E}$  into Eq. (4), we have

$$\frac{d\phi}{dx} = \frac{\beta}{\alpha\sqrt{(x^2 + a^2)^2 - \beta^2(x^2 + a^2)}}. \quad (5)$$

The angular deflection  $\delta\phi$  is given by  $\delta\phi = \Delta\phi - \pi$ , where

$$\Delta\phi = 2 \int_{x_0}^{\infty} \frac{\beta dx}{\alpha\sqrt{(x^2 + a^2)^2 - \beta^2(x^2 + a^2)}}, \quad (6)$$

where  $x_0$  is the closest approximation radius where the light orbit will have a turning point. This radius is obtained when the right side of Eq. (4) cancels, that is,  $x_0 = \sqrt{\beta^2 - a^2}$ . Based Ref. [45], we make the following change of variable

$r^2 = x^2 + a^2$  such that we obtain

$$\Delta\phi = 2 \int_{\beta}^{\infty} \frac{\beta dr}{\sqrt{\alpha^2(r^2 - a^2)(r^2 - \beta^2)}}. \tag{7}$$

Now, let us consider the new variable  $w = \frac{\beta}{r}$  and the constant  $g = \frac{a}{\beta}$  into Eq. (7). In this case, we have

$$\Delta\phi = \frac{2K(g)}{\alpha}, \tag{8}$$

where

$$K(g) = \int_0^1 \frac{dw}{\sqrt{(1-w^2)(1-g^2w^2)}}, \tag{9}$$

is a complete elliptic integral of first kind. With that, we finally find the deflection  $\delta\phi$

$$\delta\phi = \frac{2K(g)}{\alpha} - \pi. \tag{10}$$

We can note that by taking  $\alpha = 1$ , Eq. (10) corresponds to the deflection of light in the spacetime of an Ellis–Bronnikov-type wormhole [45], but in general  $\alpha < 1$ , thus, in the Ellis–Bronnikov-type spacetime with charge topological the deflection of light is greater. In addition, for the weak field approximation, where  $a \ll \beta$  and  $g \ll 1$ , the deflection of light is given by:

$$\delta\phi \approx \frac{\alpha^2\pi}{2} + \frac{\pi}{4} \left(\frac{a}{\beta}\right)^2 + \frac{9\pi}{64} \left(\frac{a}{\beta}\right)^4 + \frac{\pi\alpha^2}{8} \left(\frac{a}{\beta}\right)^2 + \mathcal{O}\left(\frac{a}{\beta}\right)^6. \tag{11}$$

The first term in the expression above (11) corresponds to the deflection in the case of pure MG. The second and third terms correspond to the deflection in an Ellis–Bronnikov-type wormhole, and the fourth term it is a joint contribution that takes into account both the wormhole throat and the topological charge of the MG. Therefore, in the weak field regime, we see more clearly how the deflection of light in the wormhole spacetime with topological charge increases compared to the case without topological charge.

### 3 On a scalar field in the topologically charged Ellis–Bronnikov-type wormhole spacetime

In this section, we investigate quantum dynamics of a scalar field in an Ellis–Bronnikov-type wormhole. The Klein–Gordon equation in this background can be written as

$$\frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}g^{\mu\nu})\partial_{\nu}\Phi - m^2\Phi = 0, \tag{12}$$

where  $g = \det(g_{\mu\nu})$ ,  $g^{\mu\nu}$  is the inverse metric tensor and  $m$  is the rest mass of the scalar field. In this way, from Eq. (2),

Eq. (12) becomes

$$-\frac{\partial^2\Phi}{\partial t^2} + \alpha^2 \frac{\partial^2\Phi}{\partial x^2} + \frac{2\alpha^2 x}{(x^2 + a^2)} \frac{\partial\Phi}{\partial x} + \frac{1}{(x^2 + a^2) \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\Phi}{\partial\theta} \right) + \frac{1}{(x^2 + a^2) \sin^2\theta} \frac{\partial^2\Phi}{\partial\varphi^2} - m^2\Phi = 0. \tag{13}$$

The solution to Eq. (13) can be written in the form

$$\Phi(r, \theta, \varphi, t) = f(x)Y_{l,m}(\theta, \varphi)e^{-i\mathcal{E}t}, \tag{14}$$

where  $f(x)$  is a radial wave function and  $Y_{l,m}(\theta, \varphi)$  are the spherical harmonics. Then, by substituting Eq. (14) into Eq. (13), we have

$$\frac{\alpha^2}{f} \frac{d^2 f}{dx^2} + \frac{2\alpha^2 x}{(x^2 + a^2)f} \frac{df}{dx} + \mathcal{E}^2 - m^2 - \frac{1}{(x^2 + a^2)} \left[ -\frac{1}{\sin\theta Y_{l,m}} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Y_{l,m}}{\partial\theta} \right) - \frac{1}{\sin^2\theta Y_{l,m}} \frac{\partial^2 Y_{l,m}}{\partial\varphi^2} \right] = 0, \tag{15}$$

or

$$\frac{d^2 f}{dx^2} + \frac{2x}{(x^2 + a^2)} \frac{df}{dx} - \frac{l(l+1)}{\alpha^2(x^2 + a^2)} f + \frac{(\mathcal{E}^2 - m^2)}{\alpha^2} f = 0, \tag{16}$$

where we use the definition

$$\left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{l,m}(\theta, \varphi) = -l(l+1)Y_{l,m}(\theta, \varphi), \tag{17}$$

with  $l = 0, 1, 2, \dots$

We can rewrite Eq. (17) as follows

$$\frac{d^2 f}{dx^2} + \frac{2x}{(x^2 + a^2)} \frac{df}{dx} - \frac{l^2}{(x^2 + a^2)} f + b^2 f = 0, \tag{18}$$

where we define the new parameters

$$l^2 = \frac{l(l+1)}{\alpha^2}; \quad b^2 = \frac{\mathcal{E}^2 - m^2}{\alpha^2}. \tag{19}$$

Let us define  $u = -\frac{x^2}{a^2}$ , then Eq. (18) becomes

$$\frac{d^2 f}{du^2} + \left[ \frac{1}{2u} + \frac{1}{(u-1)} \right] \frac{df}{du} + \left[ \frac{l^2 - a^2 b^2}{4u} - \frac{l^2}{4(u-1)} \right] f = 0. \tag{20}$$

Eq. (20) is the confluent Heun equation [47,48] and  $f(u)$  is the confluent Heun function:

$$f(u) = H_c \left( 0, -\frac{1}{2}, 0, -\frac{a^2 b^2}{4}, \frac{1}{4} - \frac{l^2}{4} + \frac{a^2 b^2}{4}; u \right). \tag{21}$$

Since we wish to find the bound state solutions, then, let us write [48]

$$f(u) = \sum_{j=0}^{\infty} c_j u^j. \tag{22}$$

Then, by substituting Eq. (22) into Eq. (20), we obtain the recurrence relation

$$c_{j+2} = \frac{[2(j+1)(2j+3) - l^2 + a^2 b^2]c_{j+1} - a^2 b^2 c_j}{2(j+2)(2j+3)}, \tag{23}$$

with the coefficient

$$c_1 = \frac{(a^2 b^2 - l^2)}{2} c_0. \tag{24}$$

Solutions of bound states can be determined by imposing that the recurrence relation (24) is truncated, of which allow us to obtain a polynomial solution to the confluent Heun series (23). Thus, let us consider  $j = n - 1$  into Eq. (23) such that  $c_{n+1} = 0$ , which results in the following condition:

$$c_n = \frac{a^2 b^2}{[2n(2n+1) - l^2 + a^2 b^2]} c_{n-1}, \tag{25}$$

where  $n = 1, 2, 3 \dots$  is radial mode. We can note that, in order to continue the solutions of bound states, we must analyze Eq. (25) for each radial mode separately. Then, let us discuss, here, the particular case of the radial mode  $n = 1$  which, from the physical point of view, represents the lowest energy state of the system. Then, for  $n = 1$ , Eq. (25) becomes

$$c_1 = \frac{a^2 b^2}{(a^2 b^2 - l^2 + 6)} c_0. \tag{26}$$

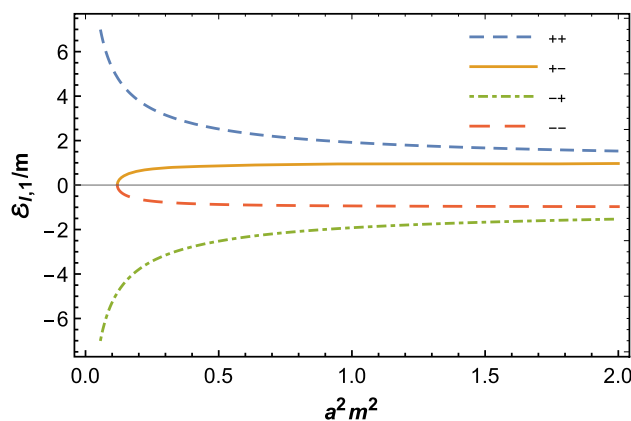
By substituting Eq. (24) into Eq. (26), we have the following fourth degree algebraic equation for  $\mathcal{E}$ :

$$\mathcal{E}_{l,1}^4 - 2 \left[ m^2 - \frac{\alpha^2(2-l^2)}{a^2} \right] \mathcal{E}_{l,1}^2 + m^4 - \frac{2\alpha^2(2-l^2)}{a^2} m^2 + \frac{l^2(l^2-6)\alpha^4}{a^4} = 0, \tag{27}$$

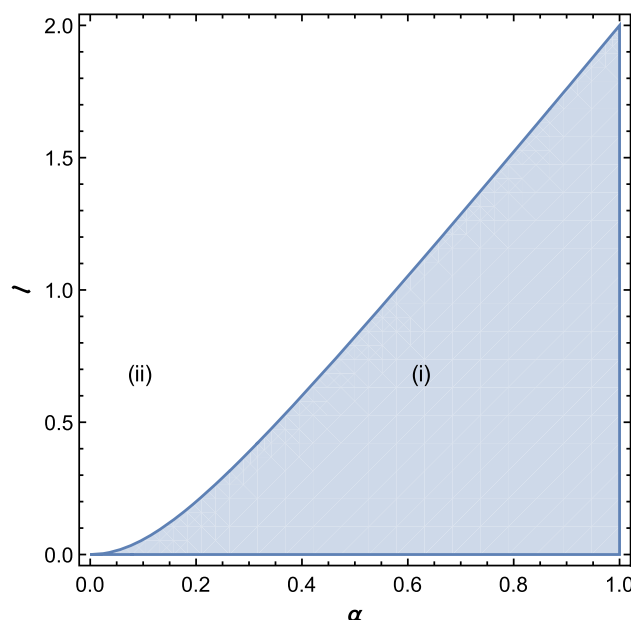
where we have labelled  $\mathcal{E} = \mathcal{E}_{l,1}$  in order to emphasize that the possible values of this parameter depend on the quantum number of the relativistic system  $\{l, n\}$ . Thereby, the four permitted values of the lowest energy state of the system, are

$$\mathcal{E}_{l,1} = \pm \sqrt{m^2 + \frac{l(l+1) - 2\alpha^2}{a^2}} \pm \frac{1}{a^2} \sqrt{2\alpha^2[l(l+1) + 2\alpha^2]}. \tag{28}$$

In the Fig. (1), we plot  $\frac{\mathcal{E}_{l,1}}{m}$  as a function of  $a^2 m^2$ . Let's look at the two possible cases regarding the signs ( $\pm$ ) inside the square root in Eq. (28). When the signal is positive, the behavior of  $\left| \frac{\mathcal{E}_{l,1}}{m} \right|$  is always the same, decreasing. On the other



**Fig. 1** Behavior of the four possibilities for lowest energy system. We choose  $l = 1$  and  $\alpha = 0.6$ . In the legend, the first sign refers to the sign outside the square root, while the second sign, inside the square root



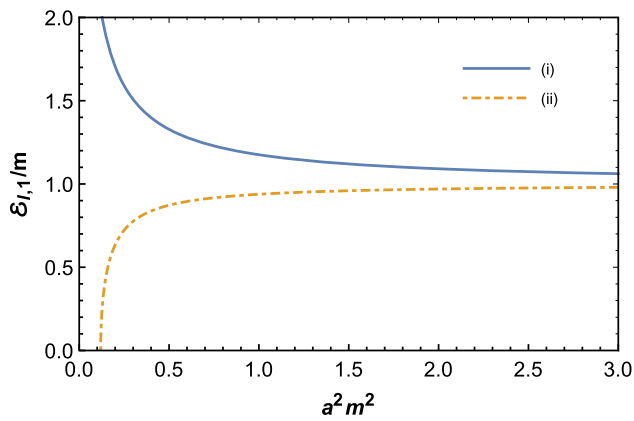
**Fig. 2** Allowed values of  $\alpha$  e  $l$  regarding conditions (i) and (ii)

hand, when the sign is negative, we have two possibilities for the behavior of  $\left| \frac{\mathcal{E}_{l,1}}{m} \right|$ :

- (i) if  $\sqrt{2\alpha^2(l(l+1) + 2\alpha^2)} > l(l+1) - 2\alpha^2$ ,  $\left| \frac{\mathcal{E}_{l,1}}{m} \right|$  is decreasing;
- (ii) if  $\sqrt{2\alpha^2(l(l+1) + 2\alpha^2)} < l(l+1) - 2\alpha^2$ ,  $\left| \frac{\mathcal{E}_{l,1}}{m} \right|$  is increasing.

We illustrate in Fig. (2) the possible values of  $l$  and  $\alpha$  so that conditions (i) and (ii) are satisfied and in Fig. (3). We show the behavior of  $\frac{\mathcal{E}_{l,1}}{m}$  for these two cases.

We can observe that the energy profile of a scalar field in the spacetime of a topologically charged Ellis–Bronnikov–



**Fig. 3** Behavior of  $\left| \frac{E_{l,1}}{m} \right|$  when the signal inside the square root is negative. For this illustration we take  $\alpha = 0.5$  in the case (i) and  $\alpha = 0.6$  in the case (ii), in both cases we take  $l = 1$

type wormhole is influenced by the background, as we can see through the presence of the parameters associated with the topological charge,  $\alpha$ , and the wormhole throat,  $a$ , in the equation (28). We can also note that the energy of the scalar field is not free and continuous energy, but determined by the quantum numbers of the system, that is, the energy of the scalar field in an Ellis–Bronnikov-type wormhole is quantized, even though there is no explicit interaction. In addition, the allowed energy values for the lowest energy state of the system depend on the parameters associated with the wormhole throat and the topological defect. By making  $\alpha = 1$  we recover the allowed energy values for the lowest energy state of the system in the spacetime of an Ellis–Bronnikov-type wormhole.

#### 4 Conclusion

In this work we analyze some phenomena related to topologically charged Ellis–Bronnikov-type wormhole spacetime. This spacetime, which is an exact solution of the modified field equations in EiBI gravity, is supported by the energy-momentum tensor associated with the region outside the GM core. First, we calculated the angular deflection of light in this spacetime and found that it increases compared to ordinary Ellis–Bronnikov wormhole due to the topological charge, as shown in the weak field approach. In addition to our analysis, we analyzed the relativistic quantum dynamics of a scalar field in this background characterized by the topologically charged Ellis–Bronnikov-type wormhole. In the search for solutions of bound states, we analytically determine the allowed values of relativistic energy for the lowest energy state of the system, which are determined by the quantum numbers of the system and the parameters associated with topologically charged wormhole, that is, the parameter

related to the radius of the throat of the wormhole and GM. In addition, we show that, taking  $\alpha = 1$ , we recover the values of relativistic energy for the lowest energy state of a scalar field in the Ellis–Bronnikov-type wormhole spacetime.

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