



Thermodynamic extended phase space and $P - V$ criticality of black holes at Pure Lovelock gravity

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Abstract In this work the *chemistry* of asymptotically AdS black hole, for the charged and uncharged solutions of Pure Lovelock gravity, is discussed. The charged case behaves as a Van der Waals fluid and whose first order phase transitions, between small stable/large stable black holes, are analogous to the liquid/gas phase transitions as in AdS black hole for Einstein Hilbert theory. However, the thermodynamics behavior differs from the generic Lovelock theory, because there is a unique critical point, unlike the generic case where there may be more than one critical point. Also, it is shown that the thermodynamics behavior of the Pure Lovelock black holes (in the extended phase space) can be represented by variables that are analytic functions of n and d , where n corresponds to the highest power of the Riemann tensor in the Lagrangian and d corresponds to the number of dimensions. This allows to obtain several results. For instance, the critical compressibility factor Z is a function of n and d that satisfies $Z < 1$ strictly, matching the behaviour of a real gas, but the new values computed differ from the $3/8$ value of a Van der Waals gas except for $d = 4$ and $n = 1$. New versions of the Smarr formula and equation of state and its behavior near the critical points are computed, which are also functions of n , d and Z . For all the cases the critical exponent are similar to those of the Van der Waals fluid. The first law of thermodynamics, in the extended space, is deduced by the variation of parameters of the Pure Lovelock solution. The entropy, volume and electric potential are consistent with the previously known results in the literature.

1 Introduction

Certainly the existence of black holes was one of the most interesting predictions of General Relativity. In this regard, the discovery that these objects, due to quantum fluctuations, emit as black bodies with temperatures dictated by the surface gravity [1–4], shows that they are scenarios where geometry and thermodynamics intertwine.

The first law of the black hole thermodynamics (see for instance Ref. [5]) is given by

$$dM = TdS + \Omega dJ + \phi dQ, \quad (1)$$

and represents the balance of energy through the modification of the macroscopic parameters of the black hole. Here M corresponds to the mass parameter and was considered the internal energy of the system, namely the (ADM) mass. Finally, in Eq. (1) T is the temperature computed as the $(4\pi)^{-1}\kappa$ with κ is the surface gravity of the black hole horizon, S is the entropy, Ω is the angular velocity, J is the angular momentum, ϕ is the electrostatic potential and Q is the electric charge.

By comparing Eq. (1) with the first law of thermodynamics one can notice the absence of the pressure/volume term, namely $-pdV$, which would stand for the *macroscopic work* done by the system. In a matter of speaking, this was due to the lack of the concepts of volume and pressure for a black hole in the original derivation, based on accretion processes. Several works have studied this issue in the last 20 years [6–9]. To address this problem let us consider the first law of thermodynamics (see for example [9]):

$$dU = TdS - pdV + \Omega dJ + \phi dQ, \quad (2)$$

where U stands for the internal energy. To extend Eq. (1) to match Eq. (2), one can think of assigning a volume to

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the black hole by considering the *volume* defined its radius, for example $V = \frac{4}{3}\pi r_+^3$ for $d = 4$ case. Unfortunately, since entropy is a function of the horizon radius as well, then Eq. (2) would be inconsistent due to dS and dV would not be independent directions. To address this problem, in reference [10], was proposed to reinterpret the mass parameter as the Enthalpy of the black hole, instead of internal energy U . Moreover, the cosmological constant was connected with the thermodynamic pressure. The law obtained is called the first law of (black hole) thermodynamics in the *extended phase space*. In Ref. [11], on the other hand, the promotion of the mass parameter to the enthalpy is based on the fact that to form a black hole would require to cut off a region of the space, and therefore an initial energy equal to $E_0 = -\rho V$, with ρ the energy density of the system, is needed. In four dimensions the thermodynamics volume corresponds to $V = \frac{4}{3}\pi r_+^3$. Moreover, the presence of the cosmological constant defines $\rho = -P$ and therefore the mass parameter can be considered equivalent to

$$M = U - \rho V = U + pV, \tag{3}$$

Here M is to be recognized as the enthalpy H of the system. With this in mind, the first law, in this extended phase space, yields

$$dM = dH = TdS + Vdp + \phi dQ + \Omega dJ. \tag{4}$$

This extended first law (4) also was derived in reference [12] by using Hamiltonian formalism. It is worth to mention that the definition of the extended phase space have allowed to construct a *heat engine* in terms of a black hole, see some examples in references [11, 13–17], adding a new layer to our understanding of the black hole thermodynamics. Another interesting applications is the Joule Thompson expansion for black holes studied in [18–21].

1.1 Phase transitions

The study of phase transitions in black hole physics has called a renewed attention in the last years due to the AdS/CFT conjecture. For instance, it is well known that the *Hawking-Page phase transition* [22], in the context of the AdS/CFT correspondence, has been re-interpreted as the plasma gluon confinement/deconfinement phase transition in the would-be dual (conformal) field theory. Similarly, the AdS Reissner Nordström’s transitions in the $(\phi - q)$ diagram have been interpreted as liquid/gas phase transitions of Van der Waals fluids [23, 24].

Recently the analysis of the $p - V$ critical behaviors (in the extended phase space) have been under studied extensively [25–41]. For instance, in [42] was studied in the context of the charged $4D$ AdS black holes how the phase transitions

between small/large black hole are analogous to liquid/gas transitions in a Van der Waals fluid. Moreover, it was also shown that the critical exponents, near the critical points, recovers those of Van der Waals fluid with the same *compressibility factor* $Z = 3/8$. In reference [12] was introduced a *new interpretation* of the Hawking Page phase transition [22] mentioned above, but in the context of $p - V$ critical behavior.

1.2 Higher dimensions, Lovelock and thermodynamics

During the last years 50 years several branches of theoretical physics have noticed that considering higher dimensions is plausible. Now, considering higher dimension in gravity opens up a range of new possibilities that retain the core of the Einstein gravity in four dimensions. Lovelock is a one of these possibilities as, although includes higher powers of curvature corrections, its equations of motion are of second order and thus causality is still insured. Generic Lovelock theory is the sum of the Euler densities L_n in lower dimensions ($2n \leq d$) multiplied by coupling constants α_n ¹. The action is

$$S = \int d^d x \sqrt{-g} L, \tag{5}$$

where d corresponds to the number of dimensions. The Lagrangian is

$$L = \sum_{n=0}^{\lfloor d/2 \rfloor} \alpha_n L_n, \tag{6}$$

where $L_n = \frac{1}{2^n} \delta^{\mu_1 \dots \mu_{2n}}_{\nu_1 \dots \nu_{2n}} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \dots R^{\nu_{2n-1} \nu_{2n}}_{\mu_{2n-1} \mu_{2n}}$. One can notice that n corresponds to the power of the Riemann tensor in the Euler density. In this way, the $L_0 = 1$ term is related by the cosmological constant, $L_1 = R$ is related by the Ricci scalar, $L_2 = R^{\alpha\beta}_{\mu\nu} R^{\mu\nu}_{\alpha\beta} - 4R^\alpha_\beta R^\beta_\alpha + R^2$ is the Gauss Bonnet density. The higher powers in this series are increasing cumbersome to express in terms of the Riemann and Ricci tensors, and Ricci scalar. For instance

$$\begin{aligned} L_3 = & R^3 - 12RR_{\mu\nu}R^{\mu\nu} + 16R_{\mu\nu}R^\mu_\rho R^{\nu\rho} + 3RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & + 24R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} - 24R_{\mu\nu}R^\mu_{\rho\sigma\kappa}R^{\nu\rho\sigma\kappa} \\ & + 4R_{\mu\nu\rho\sigma}R^{\mu\nu\eta\xi}R^{\rho\sigma}_{\eta\xi} \\ & - 8R_{\mu\rho\nu\sigma}R^\mu_{\eta}{}^\nu{}_\xi R^{\rho\eta\sigma\xi} \end{aligned}$$

is the third-order term in the Lovelock series, see for instance [43].

¹ In even dimensions the maximum order, $n = d/2$, is in turn the corresponding Euler density of the dimension and therefore does not contribute to the equations of motion,

The EOM are:

$$\sum_{n=0}^{[d/2]} \alpha_n \mathcal{G}_{\mu\nu}^{(n)} = T_{\mu\nu}, \tag{7}$$

where T_{ν}^{μ} corresponds to the energy momentum tensor and

$$(\mathcal{G}^{(n)})_{\nu}^{\mu} = -\frac{1}{2^{n+1}} \delta_{\nu\mu_1\dots\nu_{2n}}^{\mu\nu_1\dots\nu_{2n}} R_{\nu_1\nu_2}^{\mu_1\mu_2} \dots R_{\nu_{2n-1}\nu_{2n}}^{\mu_{2n-1}\mu_{2n}}, \tag{8}$$

is the (n order) generalization of the Einstein tensor. This satisfies the identity $\nabla_{\mu}(\mathcal{G}^{(n)})^{\mu\nu} \equiv 0$.

One potential drawback of a generic Lovelock gravity is the existence of more than single ground state, namely more than a single constant curvature spaces solution, or equivalently more than a single potential *effective cosmological constants* [44]. In fact, for general $\{\alpha_n\}$'s the potential effective cosmological constants can be complex numbers. However, there are two families of Lovelock gravities that have indeed a single ground state. The first family has been originally studied in [45] and has a unique k -fold degenerated ground state.

The second case with a single effective cosmological constant is called *Pure Lovelock gravity*. In this case the Lagrangian is a just the n -single term of Lagrangian plus a cosmological constant, *i.e.*, $L = \alpha_n L_n + \alpha_0 L_0$. Now, the interest in Pure Lovelock is due to the fact that of the all families of solutions of Lovelock gravity, Pure Lovelock is the only case that has a single AdS ground state, and thus solutions with a single asymptotically AdS region, by dynamical reasons instead of purely kinematic. This is discussed in more details in [46] where it is displayed that there is a single real *negative effective cosmological constant*, being the rest strictly complex numbers with non vanishing imaginary part. Indeed, we would like to stress that the analysis of the generic Lovelock gravity must be done carefully in the sense that the limit of several physical quantities cannot be taken smoothly on the real numbers. In practice, it is far simpler to do the computations directly on Pure Lovelock theory than to considering the general case applied to Pure Lovelock gravity. Finally, the study of solutions in Pure Lovelock theory has called the attention in recent years. For instance, studies on vacuum black hole solutions can be found in references [47–52], on regular black holes in references [46,53] and on stellar distributions in references [54–56]. See other applications in references [57–60]. The action for Pure Lovelock theory is:

$$\int d^d x \sqrt{-g} (\alpha_n L_n + \alpha_0 L_0). \tag{9}$$

The action has been written following definition of cosmological constant Λ in reference [52], but by a numeri-

cal factor. For pure Lovelock the gravitational term, meaning L_n , has different units than the Ricci scalar (except for $n = 1$ obviously) and therefore the corresponding gravitational constant, roughly speaking $1/\alpha_n$, must have different units to accommodate the units of L_n .

Given that the action principle must be dimensionless, the units of the different elements are constrained. First one can noticed that $[d^d x \sqrt{-g}] = \ell_p^d$ and $[L_n] = \ell_p^{-2n}$, where ℓ_p corresponds to a unit of length such as the Planck length. This implies that the coupling constants must satisfy $[\alpha_n] = \ell_p^{2n-d}$ [61]. In this case the cosmological constant is defined by the usual relation between the cosmological constant and the gravitational constant,

$$\alpha_0 = -2\alpha_n \Lambda. \tag{10}$$

In this way the cosmological constant must have units $[\Lambda] = \ell_p^{-2n}$ as in reference [52]. The reason for this definition is to avoid introducing an unnecessary addition constant parameter in the action principle. From this it is direct that $[\Lambda] = \ell_p^{-2n}$. Moreover, the cosmological constant can be expressed as

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2}, \tag{11}$$

where l^2 the square of the radius of the ground state (geometry) solution [60]. The equation of motion are:

$$(\mathcal{G}^{(n)})_{\mu}^{\nu} + \delta_{\nu}^{\mu} \Lambda = T_{\nu}^{\mu}. \tag{12}$$

For Pure Lovelock there is not constraints on the value of the coupling constants α_n , because these do not determine the form of the solution (unlike, for example in n fold degenerated theory [45]). Thus, the value of α_n can be arbitrary and, for simplicity, we have set the coupling constants to unity as in references [54,57]

In generic Lovelock theory the $p - V$ criticality have been widely studied in literature for different particular cases. See for instance [62–66]. For instance, in [62] were computed the equations of state for the Einstein Gauss Bonnet and the 3rd-order Lovelock case. In both cases, there may be more than one critical point.

Due to the differences mentioned between Pure Lovelock and generic Lovelock theories is of physical interest to study the $p - V$ critical behavior of the Pure Lovelock solutions. Below will be tested if thermodynamic behavior of the solutions is analogue to a (generalized) Van der Waals fluid.

In this work the thermodynamic of the Pure Lovelock solutions will be carried out in the extended phase space. For this, the cosmological constant will be promoted to the extensive thermal pressure of the system. It will be computed new versions of the equations of state for the charged and uncharged

cases and will be analysed the phase transitions. Finally, the critical coefficient near the critical points will be computed. On the other hand, by means of the variation of parameters of the Pure Lovelock solution in the extended phase space, will be tested if the values computed of the entropy, volume and electric potential coincide with the values previously known and thus, if the variation of parameters of the Pure Lovelock black hole follows a first law of thermodynamics in the extended phase space.

1.2.1 Vacuum black hole in pure lovelock gravity: uncharged asymptotically AdS case

Let us consider the static spherically symmetric geometry described by the line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2. \tag{13}$$

For the line element (13) the (t, t) and (r, r) components of the equations of motion (12) are the same, see [49], and are given by:

$$\frac{d}{dr} \left(r^{d-2n-1} (1 - f(r))^n + \frac{r^{d-1}}{l^{2n}} \right) = 0, \tag{14}$$

whereas the angular equations are satisfied identically by the solution to the component (t, t) or (r, r) of $\mathcal{G}^{\mu\nu} = 0$. Solutions of these equations have been studied in references [47,49]. In terms of Eq. (13) the solution is defined by

$$f(r) = 1 - \left(\frac{2M}{r^{d-2n-1}} - \frac{r^{2n}}{l^{2n}} \right)^{1/n}. \tag{15}$$

The thermodynamic pressure can be read off this definition as

$$p = -\frac{\Lambda}{(d-2)\Omega_{d-2}}, \tag{16}$$

where Ω_{d-2} is the unitary area of a $d-2$ sphere. It is worth to stress that our definition of pressure for Pure Lovelock gravity, see Eqs. (11, 16), differ from the standard definitions for generic Lovelock theories, where the pressure has the same value independent of the power of n [62,65]. Our definition only coincide for $n = 1$ with the standard definitions of Refs. [12,42].

It must be stressed that, given that the cosmological constant must be negative to have a positive pressure, see equation (16), $f(r)$ is bound to take complex values, for ranges of r , for even n . This forbids the existence of a proper asymptotic region, namely for $r \rightarrow \infty$. Because of this, as in reference [46], in this work will be considered only the case of odd n , neglecting even n . This, for example, removes the

pure Gauss Bonnet case ($n = 2$) from the discussion. Now, replacing Eqs. (11, 16) into Eq. (15) yields

$$f(r) = 1 - \left(\frac{2M}{r^{d-2n-1}} - \frac{2\Omega_{d-2}r^{2n}}{d-1} p \right)^{1/n}, \tag{17}$$

where the dependence of $f(r)$ on p, M have been made explicit.

1.2.2 Charged pure Lovelock solution

The charged case is slightly difference since it is necessary to solve the Maxwell equations and to include an energy momentum tensor into the gravitational equations. Let us start by defining $A_\mu = A_t(r)\delta^t_\mu$, which defines only the non-vanishing component of the Maxwell tensor

$$F_{tr} = -\partial_r A_t(r). \tag{18}$$

For the charged case, the equations of motion correspond to $\mathcal{G}^{\mu\nu} = T^{\mu\nu}$ in conjunction with the Maxwell equations:

$$\nabla_\mu F^{\mu\nu} = 0. \tag{19}$$

For the line element (13), the (t, t) and (r, r) components of the gravitational equations lead to

$$\frac{d}{dr} \left(r^{d-2n-1} (1 - f(r))^n + \frac{r^{d-1}}{l^{2n}} \right) = (F_{tr})^2 r^{d-2}, \tag{20}$$

where, again, the remaining angular equations are identically satisfied. The only non vanishing component of the Maxwell equations is given by

$$\nabla_\mu F^{\mu\nu} = 0 \rightarrow \frac{d}{dr} \left(r^{d-2} F_{tr} \right) = 0. \tag{21}$$

One can notice that it is straightforward to integrate Eqs. (21, 20). This yields, see [52],

$$f(r) = 1 - \left(\frac{2M}{r^{d-2n-1}} - \frac{r^{2n}}{l^{2n}} - \frac{Q^2}{(d-3)r^{2d-2n-4}} \right)^{1/n}, \tag{22}$$

and

$$F_{tr} = \frac{Q}{r^{d-2}} \rightarrow A_t(r) = \phi_\infty + \frac{Q}{(d-3)r^{d-3}}, \tag{23}$$

$\phi_\infty = 0$ is fixed such that $\lim_{r \rightarrow \infty} A_t(r) = 0$.

Now, by direct observation, one can notice that for even n the existence of certain ranges of r where $f(r)$ can take complex values. To avoid this only odd n will be considered from now on.

As done previously, by replacing Eqs. (11, 16) into Eq. (22), $f(r)$ can be written in terms of the thermodynamics variables as

$$f(r) = 1 - \left(\frac{2M}{r^{d-2n-1}} - \frac{2\Omega d-2r^{2n}}{d-1} p - \frac{Q^2}{(d-3)r^{2d-2n-4}} \right)^{1/n}. \tag{24}$$

2 Extended phase space in vacuum pure Lovelock gravity

In this section will be found the entropy, thermodynamic volume and electric potential, based in the variation of the function $f(r_+)$ respect to its parameters M, p and Q in the extended phase space. Let us start by noticing that the first law of the thermodynamics, Eq. (4), for the non rotating case takes the form

$$dH = dM = TdS + Vdp + \phi dQ. \tag{25}$$

Now, in order to construct a thermodynamic interpretation one must notice that under any transformation of the parameters the function $f(r_+, M, Q, l)$ must still vanishes, otherwise the transformation would not be mapping black holes into black holes in the space of solutions. Indeed, $\delta f(r_+, M, Q, l) = 0$ and $f(r_+, M, Q, l) = 0$ are to be understood as constraints on the evolution along the space of parameters. However, there is another approach by recalling that the mass parameter, M , is also to be understood as a function of the parameters $M(r_+, l, Q)$ as well.

Since the thermodynamic parameters are S, p and Q , therefore it is convenient to reshape $M = M(S, p, Q)$ in order to explicitly obtain

$$dM = \left(\frac{\partial M}{\partial S} \right)_{p,Q} dS + \left(\frac{\partial M}{\partial p} \right)_{S,Q} dp + \left(\frac{\partial M}{\partial Q} \right)_{p,S} dQ. \tag{26}$$

This corresponds to the definitions of the component of the tangent vector in the space of parameters, but also correspond to the definitions of the temperature, thermodynamic volume and electric potential in the form of

$$T = \left(\frac{\partial M}{\partial S} \right)_{p,Q}, \tag{27}$$

$$V = \left(\frac{\partial M}{\partial p} \right)_{S,Q} \text{ and} \tag{28}$$

$$\phi = \left(\frac{\partial M}{\partial Q} \right)_{S,p}. \tag{29}$$

On the other hand, the variation along the space of parameters of the condition defined by $f(r_+, M, p, Q) = 0$,

$$df(r_+, M, p, Q) = 0 = \frac{\partial f}{\partial r_+} dr_+ + \frac{\partial f}{\partial M} dM + \frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial Q} dQ, \tag{30}$$

yields a second expression for dM given by

$$dM = \left(\frac{1}{4\pi} \frac{\partial f}{\partial r_+} \right) \left(-\frac{1}{4\pi} \frac{\partial f}{\partial M} \right)^{-1} dr_+ + \left(-\frac{\partial f}{\partial M} \right)^{-1} \left(\frac{\partial f}{\partial p} \right) dp + \left(-\frac{\partial f}{\partial M} \right)^{-1} \left(\frac{\partial f}{\partial Q} \right) dQ, \tag{31}$$

which must coincide with equation (26). In Eq. (31) one can recognize presence of the temperature, which geometrically is defined as

$$T = \frac{1}{4\pi} \frac{\partial f}{\partial r_+}, \tag{32}$$

which is a very known result, yielding

$$\left(-\frac{1}{4\pi} \frac{\partial f}{\partial M} \right)^{-1} dr_+ = dS. \tag{33}$$

It is worth to stress that this expression can be also derived using the Wald’s formalism, roughly speaking $\delta S = \delta \int \frac{\partial L}{\partial R}$ provided $df = 0$ is satisfied. See Eq. (30). This will become manifest below in Eq. (38).

Now, by the same token, the thermodynamic volume and electric potential are given by

$$V = \left(\frac{\partial M}{\partial p} \right)_{S,Q} = \left(-\frac{\partial f}{\partial M} \right)^{-1} \left(\frac{\partial f}{\partial p} \right) \tag{34}$$

and

$$\phi = \left(\frac{\partial M}{\partial Q} \right)_{S,p} = \left(-\frac{\partial f}{\partial M} \right)^{-1} \left(\frac{\partial f}{\partial Q} \right), \tag{35}$$

respectively. In this way, it is possible to compute the entropy, thermodynamic volume and electric potential by mean of the variation of the function $f(r)$ respect to its parameters M, p and Q . These expressions will be discussed in the next sections to test if the values computed coincide with the values previously known for Pure Lovelock theory in the literature.

2.1 New version of the Smarr expression

Considering Plank units, one can notice that p , the pressure, has units of $[p] = \ell^{-2n}$. See Eqs. (11, 16, 41). Likewise, one can check that $[M] = \ell^{d-2n-1}$, $[Q] = \ell^{d-n-2}$ and $[S] = \ell^{d-2n}$.

Following *Euler's theorem* [67], with $M(S, p, Q)$, one can construct the *Smarr formula* for Pure Lovelock gravity given by

$$\frac{d-2n-1}{d-2n}M = TS + \frac{d-n-2}{d-2n}\phi Q - \frac{2n}{d-2n}VP. \tag{36}$$

It must mention that due to the structure of the generic Lovelock theories is not possible in general to write down a Smarr formula as a function of the different powers presented in the Lovelock Lagrangian. However, for Pure this can be done swiftly. This expression coincides with the definitions discussed in [12,42,67] for $n = 1$. Derivations of the Smarr formula for generic Lovelock theory are discussed in [68,69].

2.2 Uncharged asymptotically AdS

Replacing the solution of Eq. (17) into Eq. (33) yields

$$\left(-\frac{1}{4\pi} \frac{\partial f}{\partial M}\right)^{-1} dr_+ = 2\pi n r_+^{d-2n-1} = d \left(\frac{2}{d-2n} n\pi r_+^{d-2n}\right) = dS, \tag{37}$$

or equivalently

$$S = \frac{2}{d-2n} n\pi r_+^{d-2n}, \tag{38}$$

which coincides with reference [47]. Although it is not obvious, it is straightforward to check that equation (38) can be obtained from Wald's prescription [70] in this case. It is worth to state that in this case the entropy differs completely from the area law for $n > 1$. This in the sense that, in our case, the entropy is not a mere correction to the area law but scales with different power of the horizon radius or area as in the generic Lovelock case, see reference [71]. This is due to the Lagrangian is the (single) n -term in generic Lovelock Lagrangian, and therefore only contains n - powers of the Riemann tensor. Equivalently, this can be obtained by using the expression in [72] for Lovelock gravity with the all, but one, vanishing coefficients. Equation (34), on the other hand, yields

$$\left(-\frac{\partial f}{\partial M}\right)^{-1} \left(\frac{\partial f}{\partial p}\right) = V = \frac{\Omega_{d-2}}{d-1} r_+^{d-1}, \tag{39}$$

which corresponds to the volume of a $(d-2)$ sphere of radius r_+ . This coincides with the definition in Refs. [12,42]. In this way, it has been shown that, by means of the variation of the function $f(r)$ respect to its parameters M and p in Pure Lovelock gravity, the values of entropy and thermodynamics volume coincide with the values previously known.

2.2.1 New version of the fluid equation of state

It is worth to notice at this point that the temperature, see Ref. [46], can be expressed as

$$4\pi n T = \frac{d-2n-1}{r_+} + (d-1) \frac{r_+^{2n-1}}{l^{2n}}, \tag{40}$$

where, the temperature has units of ℓ^{-1} . One can make explicit the dependence on the pressure, by inserting Eqs. (11, 16) into Eq. (40). Therefore,

$$2\Omega_{d-2} p = 4\pi n \frac{T}{r_+^{2n-1}} - \frac{d-2n-1}{r_+^{2n}}. \tag{41}$$

It is worth to mention that this state equation differs from the previously known state equation found in the literature for Lovelock theories. As for example, for $n = 3$ the Pure cubic state equation has only two terms whereas the state equation in 3rd-order generic Lovelock gravity has six terms taken the uncharged case [62].

2.2.2 Physical pressure

Before to proceed a digression is necessary. As mentioned above $[p] = \ell^{-2n}$, however the physical pressure, p_G , must be satisfied $[Force/Area] = \ell_p^{-d}$, since the area has units of $[area] = \ell_p^{d-2}$. In the literature p_g is called the geometrical pressure. Since p and p_G must be connected by $p_G \sim \alpha_n p$, namely $[p_G] = [\alpha_n p] = \ell_p^{-d}$, therefore

- the *coupling constants* must satisfy $[\alpha_n] = \ell^{2n-d}$ [61]. For $n = 1$ this coincides with the inverse of the higher dimensional Newton constant G_d^{-1} [73] and
- there is still room for a dimensionless constant which can be used to adjust the definition.

With this in mind, p_G can be taken as

$$p_G = 2\Omega_{d-2} \ell_p^{2n-d} p = 4\pi n \frac{T}{\ell_p^{d-2n} r_+^{2n-1}} + \dots, \tag{42}$$

which, in turns, defines the *specific volume* of the system as

$$v = \frac{\Omega_{d-2}}{2\pi} \ell_p^{d-2n} r_+^{2n-1}. \tag{43}$$

where the magnitude of $\ell_p = 1$ [74]. Notice that v indeed has units of volume, namely $[v] = \ell_p^{d-1}$. Finally, after some replacements,

$$p = \frac{nT}{v} - \frac{1}{2\Omega_{d-2}(2\pi/\Omega_{d-2})^{2n/(2n-1)} v^{2n/(2n-1)}}, \tag{44}$$

which can be recognized as the Van der Walls equation for $n = 1$, namely $P = T/(v - b) - a/v^2$, with $b = 0$ [12].

2.2.3 Hawking-page phase transition

In reference [22] was analyzed the thermodynamic behavior of Schwarzschild AdS space, which differs from the Schwarzschild case due to the presence of the AdS gravitational potential, namely the presence of $\sim r^2/l^2$ in $f(r)$ [75]. In this case there is a phase transition between black hole and AdS radiation at a critical temperature T_{HP} where the Gibbs free energy, $G = M - TS$ vanishes [42, 75, 76].

Figure 1 displays the numerical behavior of the Gibbs free energy v/s temperature for $n = 3$ and $d = 10$ for Pure Lovelock. The upper curve represents the unstable small black hole (namely with negative heat capacity) and the lower curve represents the stable large black hole. It is direct to show that this behavior is similar for any other set of values of n and d . The intersection between the two curves defines a temperature T_{min} , whereas the intersection between the stable large BH curve and the horizontal axis defines a temperature T_{HP} .

One can notice that for $[T_{min}, T_{HP}]$ the large stable black hole has positive Gibbs free energy, therefore, the preferred state corresponds to the thermal AdS radiation. On the other hand, for $T > T_{HP}$, the preferred state is the large stable black hole whose Gibbs free energy is negative. This hints the existence of a Hawking Page phase transition between radiation and the large black hole states at $T = T_{HP}$.

Finally, it is worth to notice, from Fig. 1, that the value of T_{HP} increases as the pressure increases. This behavior is similar to the HP phase transition for the Schwarzschild AdS black holes [12, 77] or the polarized AdS black holes [78].

2.3 Charged pure Lovelock solution

By replacing Eq. (24) into Eqs. (33, 34) the expressions for the entropy (38) and the volume (39) are obtained. On the other hand, replacing Eq. (24) into Eq. (35) yields

$$\phi = \frac{Q}{(d-3)r^{d-3}}. \tag{45}$$

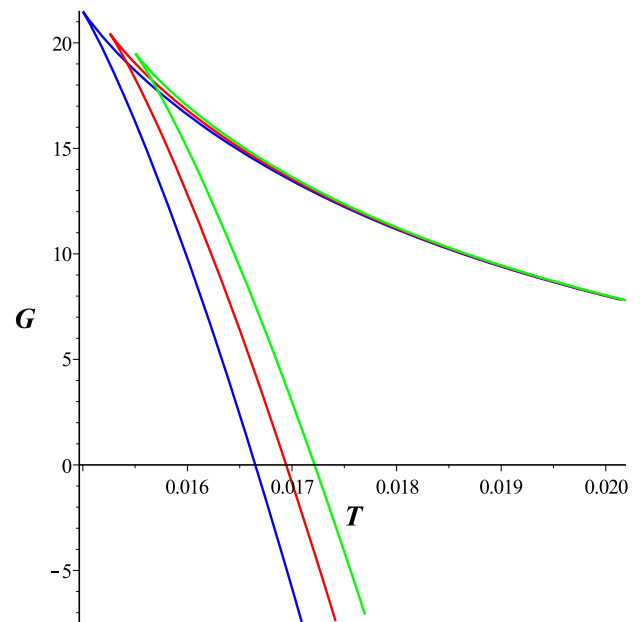


Fig. 1 HP phase transitions for $n = 3, d = 10$ and $Q = 1, p = 0.0000045$ (blue), $p = 0.0000050$ (red), $p = 0.0000055$ (green). T (horizontal axis) v/s G (vertical axis)

The values of the entropy, thermodynamic volume and the electric potential, obtained by means of the variation of the function $f(r)$ with respect to its parameters M, p and Q in Pure Lovelock gravity, coincide with the values previously known.

2.3.1 New version of the fluid equation of state

In this case the temperature can be written as

$$4\pi nT = \frac{d-2n-1}{r_+} + (d-1) \frac{r_+^{2n-1}}{l^{2n}} - \frac{Q^2}{r_+^{2d-2n-3}}. \tag{46}$$

By inserting equations (11, 16) into equation (46) is obtained

$$2\Omega_{d-2}p = 4\pi n \frac{T}{r_+^{2n-1}} - \frac{d-2n-1}{r_+^{2n}} + \frac{Q^2}{r_+^{2d-4}}. \tag{47}$$

Now, by using the definition of the specific volume, defined in Eq. (43) with $\ell_p = 1$,

$$p = \frac{nT}{v} - \frac{1}{2\Omega_{d-2}(2\pi/\Omega_{d-2})^{2n/(2n-1)} v^{2n/(2n-1)}} + \frac{1}{2\Omega_{d-2}(2\pi/\Omega_{d-2})^{(2d-4)/(2n-1)} v^{(2d-4)/(2n-1)}} \frac{Q^2}{v^{(2d-4)/(2n-1)}}. \tag{48}$$

It must be stressed that state equation found, for Pure Lovelock theory, differs from the state equations found in liter-

Table 1 Critical values and compressibility factor (Z)

n	d	v_c	T_c	p_c	$Z = \frac{p_c v_c}{T_c}$
1	4	$\Omega_{d-2}/(2\pi)Q\sqrt{6}$	$1/(18\pi Q)\sqrt{6}$	$1/(24\Omega_{d-2}Q^2)$	3/8
1	5	$\Omega_{d-2}/(4\pi)(120Q^2)^{1/4}$	$4/(75\pi)(6750/Q^2)^{1/4}$	$2/(45\Omega_{d-2}Q)\sqrt{30}$	5/12
1	6	$\Omega_{d-2}/(6\pi)(6804Q^2)^{1/6}$	$9/(98\pi)(201.684/Q^2)^{1/6}$	$9/(112\Omega_{d-2})(294/Q^2)^{1/3}$	7/16
3	8	$\Omega_{d-2}/(2\pi)(14Q^2)^{5/6}$	$3/(490\pi)(14^5/Q^2)^{1/6}$	$1/(280\Omega_{d-2}Q^2)$	7/24
3	9	$\Omega_{d-2}/(4\pi)(21^5 \cdot 2^3 Q^{10})^{1/8}$	$8/(945\pi)(21^7 \cdot 2/Q^2)^{1/8}$	$4/(735\Omega_{d-2})(21 \cdot 2^3/Q^6)^{1/4}$	9/28
3	10	$\Omega_{d-2}/(18\pi)(792)^{1/2}Q$	$3/(242\pi)(2^7 \cdot 11^9 \cdot 9/Q^2)^{1/10}$	$9/(704\Omega_{d-2})(726/Q^6)^{1/5}$	11/32
5	12	$\Omega_{d-2}/(2\pi)(22Q^2)^{9/10}$	$5/(2178\pi)(22^9/Q^2)^{1/10}$	$1/(792\Omega_{d-2}Q^2)$	11/40

ature for other Lovelock theories. See for instance [34,64]. For example, for $n = 3$ the Pure cubic state equation has only three terms whereas the state equation in 3rd-order generic Lovelock gravity has seven terms [62].

2.3.2 Critical points and compressibility factor

To compare Eq. (48) with the behavior of a Van der Waals fluid it is necessary to determine the critical points of the system. The second order critical points are defined by the conditions,

$$\frac{\partial p}{\partial v} = 0 \quad \text{and} \quad \frac{\partial^2 p}{\partial v^2} = 0. \tag{49}$$

These determine the critical values

$$v_c = \frac{\Omega_{d-2}}{2\pi} \left(\frac{2d^2 - 2dn - 7d + 4n + 6}{n(d - 2n - 1)} Q^2 \right)^{(2n-1)/(2d-2n-4)}, \tag{50}$$

and

$$T_c = \frac{n(d - 2n - 1)}{2\pi n(2n - 1)(2\pi/\Omega_{d-2})^{1/(2n-1)}v^{1/(2n-1)} - \frac{(d - 2)Q^2(2\pi/\Omega_{d-2})^{(2n-2d+3)/(2n-1)}v^{(2n-2d+3)/(2n-1)}}{2\pi n(2n - 1)}} \tag{51}$$

Notice that $p_c = p(v_c, T_c)$ can be determined from equation (48) by evaluation on the critical values T_c and v_c . Thus, it can be noted that the critical values of v , T and p are analytic functions of n and d for a fixed value of Q . Indeed, there is a single critical point (v_c, T_c, p_c) for a fixed value of Q , unlike generic Lovelock theories where there may be a much larger number. For instance, up to three critical points in Ref. [62].

In the Table 1 new critical values v_c , T_c and p_c and the compressibility factor $Z = p_c v_c / T_c$ are displayed for different values of n and d . For $n = 1$ and $d = 4$ the compressibility

factor has the exact value $Z = 3/8$ which coincides with the value of the compressibility for a Van der Waals fluid. However, for $n > 1$ and $d > 4$ the expression is different and given by

$$Z = \frac{2d - 2n - 3}{4(d - 2)}, \tag{52}$$

It must be stressed that $Z < 1$ strictly, implying that in general this can be interpreted as a real gas.

2.3.3 $P - v$ curve

In Fig. 2 is displayed for $n = 3$ and $d = 10$ the behavior of the curve $p - v$ defined by equation (48). Although this is an example still this behavior is generic for any values of n and d .

For values of temperature $T > T_c$, the second and third factors of Eq. (48) are negligible in comparison with the first one, and therefore the curve approximates the form $p \cdot v \propto T$ and thus mimicking the behavior of an ideal gas. Conversely, for $T < T_c$ as v increases the p , which diverges for $v = 0$, decreases until reach a local minimum, located at $v = v_{\min}$. Next, p increases until reaching a local maximum, located at $v = v_{\max}$. Finally p decreases asymptotically until reaching $p = 0$. Thus for $T < T_c$ (or for $p < p_c$ due that $T \propto p$) the behavior is analogue to the Van der Waals fluid.

In standard vapor-liquid theory is well established that an increase in pressure must be correlated with a decrease in volume and vice versa. Therefore, see Fig. 2, one can notice the existence of a range of the specific volume, says $]v_{\min}, v_{\max}[$, which must be considered nonphysical due to both pressure and volume increase simultaneously. On the other hand, it can be also noticed that for a single value of the specific pressure, there might exist up to three possible values of specific volume v with one of them always within the nonphysical region. Therefore, for analysis one must only consider the two proper solutions that satisfy either $v_1 < v_{\min}$ or $v_2 > v_{\max}$ with $p(v_1) = p(v_2)$. In fluid theory these two

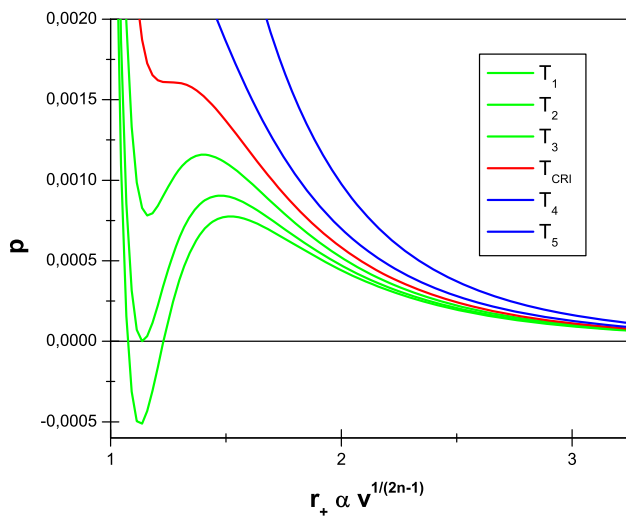


Fig. 2 $p - v$ curve for $T_1 < T_2 < T_3 < T_c < T_4 < T_5$

solutions are known as the *Van der Waals loop* and physically this corresponds to a *vapor liquid equilibrium*, highlighting that phase transitions take place.

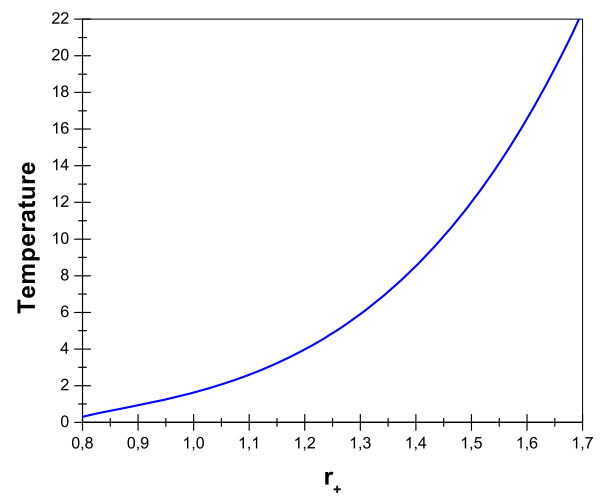
2.3.4 Temperature

The behavior of the temperature is displayed in Fig. 3a for $p > p_c$, in Fig. 3b for $p = p_c$ and in Fig. 3c for $p < p_c$. Although for these figures $n = 3$ and $d = 10$ it is straightforward to show that this enfolds the generic behavior for any n and d . One can notice the presence of an extreme black hole case for small r_+ where the temperature vanishes. For $p > p_c$ the temperature is an increasing function of r_+ . For $p = p_c$ the temperature has one inflexion point at $r_+ = r_{infl}$. More relevant for this discussion is the case for $p < p_c$, where the fluid is analogous to the Van der Waals, and where the temperature has a local minimum and a local maximum at $r_+ = r_{min}$ and $r_+ = r_{max}$, respectively.

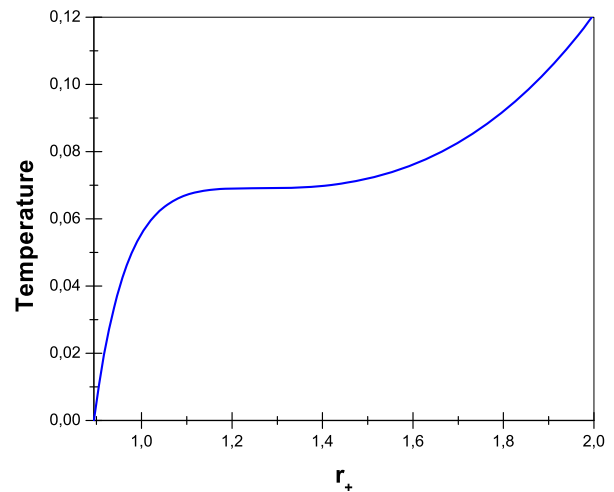
2.3.5 Heat capacity

The heat capacity is displayed in Fig. 4a for $p > p_c$, in Fig. 4b for $p = p_c$ and in Fig. 5 for $p < p_c$. As previously, although $n = 3$ and $d = 10$ it is straightforward to show that Fig. 4 enfolds the generic behavior for any n and d . One can notice that

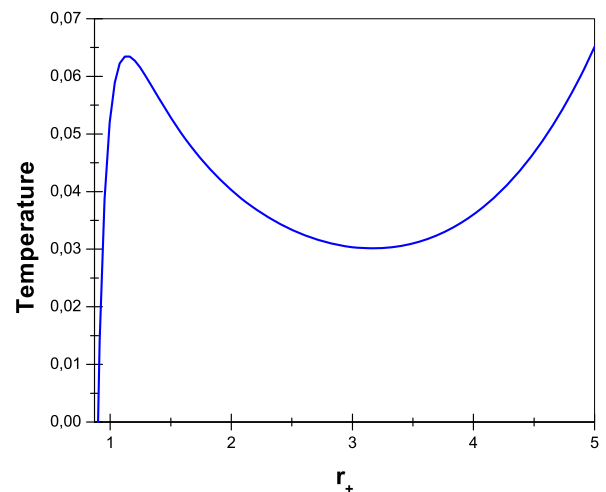
- For $p > p_c$ the heat capacity is a positive increase function of r_+ , and there is no phase transition, thus black hole is always stable.
- For $p = p_c$ small and large black hole coexist at the inflexion point $r_+ = r_{infl}$, where the heat capacity $C \rightarrow \infty$.



(a) Temperature for $p = 1 > p_c$.

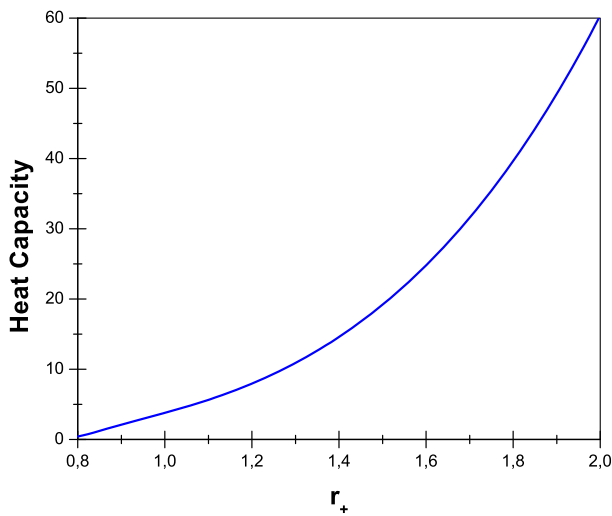


(b) Temperature for $p = p_c \approx 0,001608$

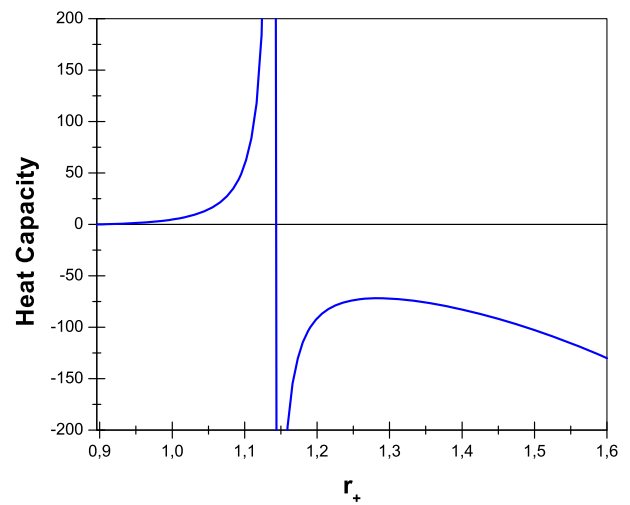


(c) Temperature for $p = 0,00001 < p_c$

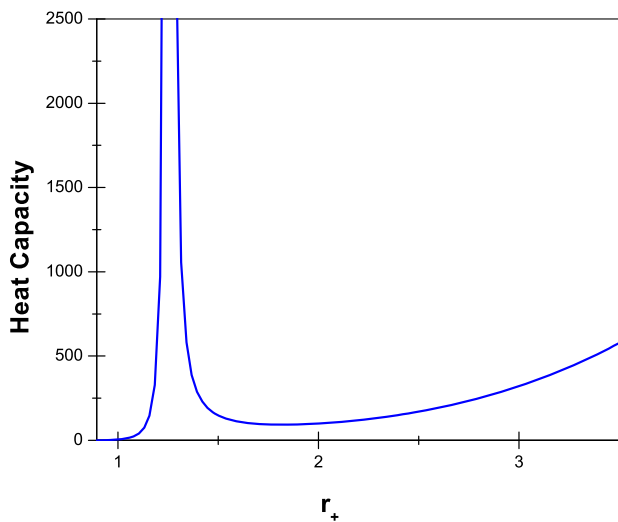
Fig. 3 Temperature behavior for $n = 3$ and $d = 10$ with $Q = 1$



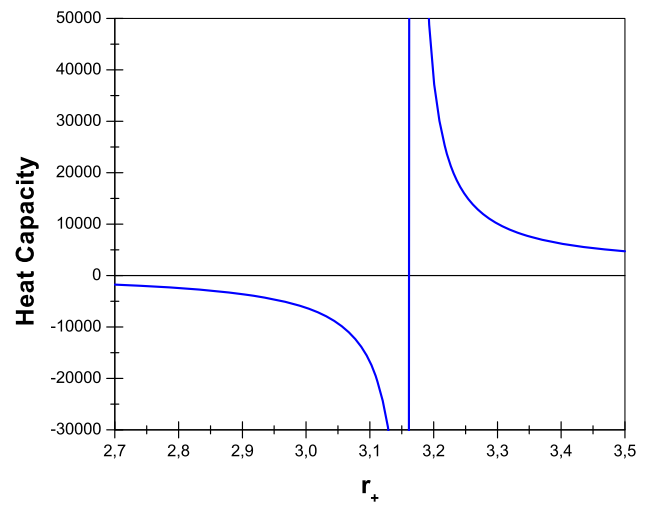
(a) Heat capacity for $p = 1 > p_c$.



(a) Heat capacity for $r_+ \in (0.896, 1.6)$.



(b) Heat capacity for $p = p_c \approx 0,001608$



(b) Heat capacity for $r_+ \in (2.7, 3.5)$

Fig. 4 Behavior of heat capacity for $n = 3$ and $d = 10$ with $Q = 1$

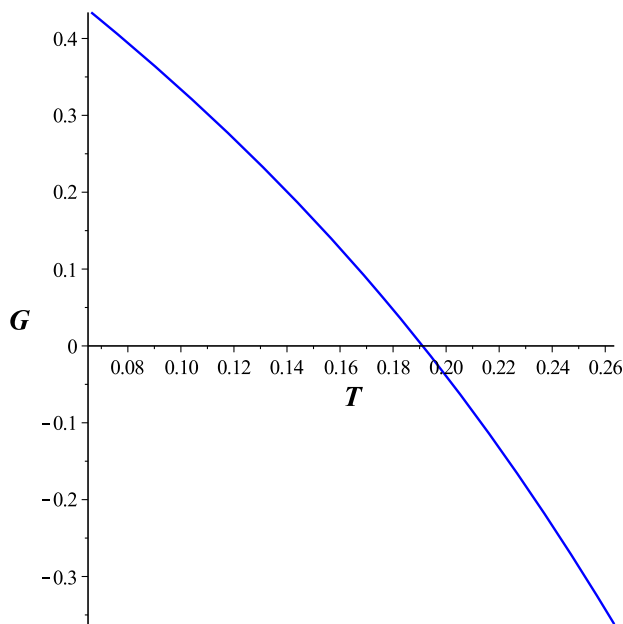
Fig. 5 Behavior of heat capacity for $n = 3$ and $d = 10$ with $Q = 1$, $p = 0,00001 < p_c$

- For the $p < p_c$ case, the derivative (dT/dr_+) can vanish for two values of $r_+ = r_{\min}$ and $r_+ = r_{\max}$, as observed in the Fig. 3c. This implies, due to $C = (dS/dr_+)/ (dT/dr_+)|_{p,Q}$, that the heat capacity becomes ill-defined at those values of r_+ . In this way r_{\min} and r_{\max} define three regions. For $r_+ < r_{\max}$ one can notice that $C > 0$ defining a small stable black hole. Next, there is small unstable ($C < 0$) region for $r \in]r_{\max}, r_{\min}[$. Finally there is a third region $r_+ > r_{\min}$ where the system is a large stable black hole ($C > 0$). This hints the existence of phase transitions but, by means of the following analysis of the Gibbs free energy, one can check that only the *small stable bh/large stable bh* transition is allowed.

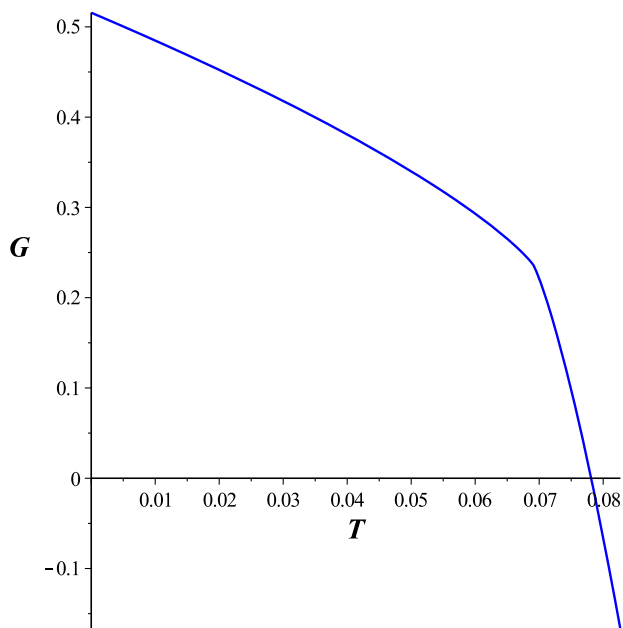
2.3.6 Gibbs free energy

The Gibbs free energy, defined as $G = M - TS$ [12,74–76], is displayed for $n = 3$ and $d = 10$ in Fig. 6a for $p > p_c$, in Fig. 6b for $p = p_c$ and in 7 for $p < p_c$. It is direct to check that this behavior is similar for other values of n and d . As well known [25] the Gibbs free energy describes the global stability of the system. It is worth to recall that the global minimum represents the most likely state, while the preferred state at fixed temperature corresponds to the minimal value of the Gibbs free energy.

- For pressure larger than the critical pressure, the Gibbs free energy is a single valued function.



(a) Gibbs for $p = 1 > p_c$.



(b) Gibbs for $p = p_c \approx 0,001608$

Fig. 6 Gibbs free energy (vertical axis) v/s T (horizontal axis) for $n = 3$ and $d = 10$ with $Q = 1$

– For $p = p_c$ there is a cusp at $T = T_c$, which coincides with the radius $r_+ = r_{infl}$, thus, since at this point $C = TdS/dT = -T(\partial^2 G/\partial T^2) \rightarrow \infty$, the discontinuity on the second derivative of Gibbs function implies the presence of second order phase transition between small stable/large stable black holes.

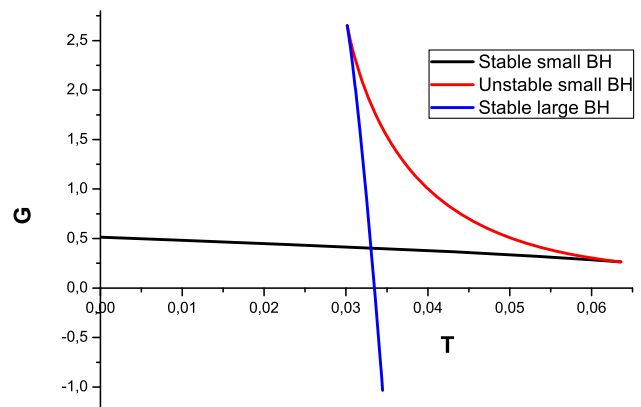


Fig. 7 Gibbs for $p = 0.00001 < p_c$

– The behavior of the Gibbs free energy for $p < p_c$ is displayed in Fig. 7. We see three possible black hole states: small stable, small unstable and large stable. The intersection between the stable small and the stable large curves defines a temperature T_0 and the intersection between the unstable small and the stable small curves defines a temperature T_{max} . The preferred state is such that the Gibbs free energy has the minimum value. Thus, for $]T_0, T_{max}]$ the preferred state correspond to stable large black hole. However, for $T < T_0$ the preferred state corresponds to the small stable black hole. Thus, at $T = T_0$ there is a first order phase transition between large/small stable black hole.

2.3.7 Behavior near critical points

Let's define the following dimensionless variables

$$\omega = \left(\frac{V}{V_c}\right)^{2n-1} - 1 = \left(\frac{v}{v_c}\right)^{d-1} - 1 \tag{53}$$

and

$$t = \frac{T}{T_c} - 1, \tag{54}$$

to analyze the behavior nearby the critical points. It is direct to check that near the critical points, *i.e.* $V \rightarrow V_c$ (or $v \rightarrow v_c$) and $T \rightarrow T_c$, the variables $\omega \rightarrow 0$, and $t \rightarrow 0$, respectively. The pressure in Eq. (48) is displayed in Table 2 at $O(t\omega^2, \omega^4)$. The truncation will be justified below.

In general it is possible to approximate the pressure as

$$p \approx 1 + \frac{n}{Z}t - \frac{nt}{(d-1)Z}\omega - \frac{(2d-2n-3)n}{6Z(d-1)^3(2n-1)^2}\omega^3. \tag{55}$$

Table 2 Behavior of pressure near critical points

n	d	p
1	4	$p \approx 1 + 8/3t - 8/9t\omega - 4/81\omega^3$
1	5	$p \approx 1 + 12/5t - 3/5t\omega - 1/32\omega^3$
1	6	$p \approx 1 + 16/7t - 16/35t\omega - 8/375\omega^3$
3	8	$p \approx 1 + 72/7t - 72/49t\omega - 12/8575\omega^3$
3	9	$p \approx 1 + 28/3t - 7/6t\omega - 7/6400\omega^3$
3	10	$p \approx 1 + 96/11t - 32/33t\omega - 16/18225\omega^3$
5	12	$p \approx 1 + 200/11t - 200/121t\omega - 100/323433\omega^3$

It must be noticed that the behavior of the equation of state near the critical points is a analytic function of n , d and Z . On the other hand,

$$dP = -\frac{P_c n}{Z(d-1)} \left(t + \frac{(2d-2n-3)}{2(d-1)^2(2n-1)^2} \omega^2 \right) d\omega. \tag{56}$$

To compute the *critical exponents* one can follow [42].

- The α exponent describes the behavior of the heat capacity at constant volume defined as

$$C_v = T \frac{\partial S}{\partial T} \propto |t|^{-\alpha}. \tag{57}$$

In the case at hand, since entropy and volume are both functions of horizon radius, see Eqs. (38, 39), then a constant volume implies a constant entropy as well. Therefore it is satisfied that $C_v = 0$ which implies that there is no dependence on $|t|$. In turn this implies that

$$\alpha = 0. \tag{58}$$

- The exponent β describes the behavior of the *order parameter* defined as

$$\eta = V_l - V_s \propto |t|^\beta. \tag{59}$$

To compute this order parameter one can use the *Maxwell's area law*

$$\oint V dP = 0, \tag{60}$$

where, the volume (53) is approximated such that $V dP$ is truncated to $O(t\omega^3, \omega^5)$, in other words,

$$V \approx V_c \left(1 + \frac{1}{2n-1} \omega \right). \tag{61}$$

Therefore, closed integral (60) becomes

$$V_c \oint dP + \frac{V_c}{2n-1} \oint \omega dP = 0. \tag{62}$$

The second integral of the left side of (62) yields

$$\int_{\omega_l}^{\omega_s} \omega dP = \int_{\omega_l}^{\omega_s} \omega \left(t + \frac{(2d-2n-3)}{2(d-1)^2(2n-1)^2} \omega^2 \right) d\omega = 0. \tag{63}$$

which has the non trivial solution given by

$$\omega_s = -\omega_l. \tag{64}$$

On the other hand, the first integral of the left side of (62) yields

$$\begin{aligned} \int_{\omega_l}^{\omega_s} dP &= 0 \\ 1 + \frac{n}{Z} t - \frac{nt}{(d-1)Z} \omega_l - \frac{(2d-2n-3)n}{6Z(d-1)^3(2n-1)^2} \omega_l^3 \\ &= 1 + \frac{n}{Z} t - \frac{nt}{(d-1)Z} \omega_s - \frac{(2d-2n-3)n}{6Z(d-1)^3(2n-1)^2} \omega_s^3, \end{aligned} \tag{65}$$

This, by condition (64), yields

$$\omega_l = (d-1)(2n-1) \sqrt{-\frac{6}{2d-2n-3} t}, \tag{66}$$

with $t < 0$. Replacing Eqs. (61) and (66) into Eq. (59) one can obtain

$$\eta = \frac{2V_c}{2n-1} \omega_l = 2(d-1)V_c \sqrt{-\frac{6}{2d-2n-3} t}, \tag{67}$$

Comparing this result with Eq. (59) one can uncover that

$$\beta = \frac{1}{2}. \tag{68}$$

- Now one can compute the *exponent* γ which describes the behavior under *isothermal compressibility*, κ_T , defined by

$$\kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \Big|_T \propto |t|^{-\gamma}. \tag{69}$$

By using Eqs. (55, 61) one can prove that

$$\frac{\partial P}{\partial V} = P_c \frac{\partial p}{\partial \omega} \frac{\partial \omega}{\partial V} \propto -(2n - 1) \frac{P_c}{V_c} \frac{n}{(d - 1)Z} t, \tag{70}$$

and therefore

$$\kappa_T \propto \frac{(d - 1)Z}{P_c n (2n - 1)t}, \tag{71}$$

from which is direct to read that

$$\gamma = 1. \tag{72}$$

- Finally, one can compute *exponent* δ which describes the behavior on the critical isotherm $T = T_c$, and therefore for $t = 0$. In this case this is defined by

$$|P - P_c| \propto |V - V_c|^\delta. \tag{73}$$

From Eq. (55), at $t = 0$, it is possible to notice that

$$p - 1 \approx -\frac{(2d - 2n - 3)n}{6Z(d - 1)^3(2n - 1)^2} \omega^3. \tag{74}$$

Using the approximation of Eq. (61) one can show that

$$\frac{P - P_c}{P_c} \approx -\frac{(2d - 2n - 3)n}{6Z(d - 1)^3(2n - 1)^2} \left((2n - 1) \frac{V - V_c}{V_c} \right)^3, \tag{75}$$

and therefore

$$\delta = 3. \tag{76}$$

This critical exponents just computed are similar to those of Van der Waals gas. Although the presence of extra dimensions and the value of n modify the value of the compressibility factor respect to the well known value $Z = 3/8$, they do not affect the value of the critical exponents, and thus, the behavior is still similar to that of Van der Waals fluid near the critical exponents.

3 Conclusion and discussion

In this article it has been analyzed the thermodynamics of the Pure Lovelock solutions in d dimensions, in an extended phase space including the introduction of pressure and volume as dual thermodynamic variables and the mass parameter standing for the enthalphy of the system. A linear relation

between the cosmological constant and the thermodynamics pressure, valid for all value of n (odd) and d , has been established.

The first law of thermodynamics, in the extended phase space for Pure Lovelock gravity, is constructed through the variation of the (lapse) function $f(r_+) = 0$ with respect to its parameters M , p and Q . The entropy deduced coincides with the value computed *a la Wald* [46,47]. The electric potential matches the usual known definition and the volume corresponds to a geometric volume of a $(d - 2)$ sphere of radius r_+ .

It is shown that the thermodynamics behavior of the Pure Lovelock black holes (in the extended phase space) can be represented by variables that are analytic functions of n and d . Similarly, a new version of the Smarr formula that corresponds to a function of n is provided. For the charged case, it was found that the compressibility factor, Z , is a generic function of d and n given by Eq. (52), thus, the new values computed differ from the Van der Waals value $Z = 3/8$ for $n > 1$ and $d > 4$. It is shown that $Z < 1$ strictly and therefore the behavior always corresponds to a real gas. This is novel result to our knowledge. On the other hand, the state equation and its behavior near the critical points are also generic functions of n , d and Z , where for all the cases the critical exponent are similar to those of the Van der Waals fluid.

In reference [34] is conjectured that in generic Lovelock theories there might be *n-tuple critical points*. In reference [64] was showed that in third order Lovelock black holes there are two critical points in dimensions $d = 8, 9, 10, 11$. In [62] was shown that there are up to three critical points in Gauss-Bonnet and 3rd-order Lovelock gravities. For Pure Lovelock the critical values of v , p and T are also functions of n and d for Q to be determined by Eqs. (50, 51). However, unlike the generic case, there is a unique critical point (v_c, T_c, p_c) for a fixed value of Q . Moreover, for Pure Lovelock gravity the $p - V$ critical behavior is similar for all d and n odd. This differs from the result for generic theories where the number of critical point depends on the value of d and the coupling constants of the different powers of Riemann tensor presented in the Lovelock Lagrangian.

New versions of the state equation for charged and uncharged Pure Lovelock gravity were computed as generic functions of n . These differ from the state equation for the generic case. As for example, for $n = 3$ the Pure cubic state equation has only two (three) terms in the uncharged (charged) case, whereas the state equation in 3rd-order generic Lovelock gravity has six (seven for the charged case) terms [62].

It has been shown that for the uncharged case, the state equation leads to a Hawking-Page-Like phase transitions between thermal radiation and large stable black hole. On the other hand, the thermodynamics behavior of the charged case is also analogous to the Van der Waals fluid as in generic

Lovelock theories despite their different ground state structure. It was found the existence of a critical temperature, T_c , where phase transitions occur. The mapping of the $p - v$ curves indicates that for Pure Lovelock gravity the behavior is similar to an ideal gas for $T > T_c$. For $T < T_c$ the behavior is analogous to a Van der Waals fluid. Furthermore, there are a first order phase transition between small stable/large stable black hole, which are analogous to liquid/gas phase transitions.

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