Mixed state entanglement for holographic axion model

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Received: 9 January 2020 / Accepted: 13 April 2020 / Published online: 15 May 2020
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Abstract We study the mixed state entanglement in a holographic axion model. We find that the holographic entanglement entropy (HEE), mutual information (MI) and entanglement of purification (EoP) exhibit very distinct behaviors with system parameters. The HEE exhibits universal monotonic behavior with system parameters, while the behaviors of MI and EoP relate to the specific system parameters and configurations. We find that MI and EoP can characterize mixed state entanglement better than HEE since they are less affected by thermal effects. Specifically, the MI partly cancels out the thermal entropy contribution, while the holographic EoP is not dictated by the thermal entropy in any situation. Moreover, we argue that EoP is more suitable for describing mixed state entanglement than MI. Because the MI of large configurations are still dictated by the thermal entropy, while the EoP will never be controlled only by the thermal effects.

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1 Introduction

Quantum entanglement is one of the most striking features of quantum systems that distinguish them from classical systems. Recently, quantum entanglement has been extensively studied in quantum information, condensed matter theory, quantum gravity, and become the core of interdisciplinary of these fields. Quantum entanglement can characterize the quantum phase transition of strong correlation systems and the topological quantum phase transitions, and plays a key role in the emergence of spacetime [1–8].

There are many measures of quantum entanglement, such as entanglement entropy (EE), Rényi entanglement entropy, negativity, mutual information (MI), and so on. Each metric can characterize different aspects of quantum entanglement. EE is a good measure for pure state entanglement, but is unsuitable for mixed state entanglement. In fact, mixed state are more common than pure states. For example, any system with finite temperature corresponds to a mixed state, while only zero temperature system can be a pure state. Therefore, many new entanglement metrics have been proposed to characterize mixed state entanglement, such as the non-negativity, entanglement of purification and the entanglement of formation [9, 10]. However, entanglement measures are notoriously difficult to calculate, because the Hilbert space of quantum systems is often extremely large.

Gauge/gravity duality is a powerful tool for studying strongly correlated systems, and it also brings geometric prescriptions for entanglement related physical quantities. The

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holographic entanglement entropy (HEE) associates the EE of a subregion with the area of the minimum surface in the dual gravity system [5]. HEE has many important applications, such as characterizing quantum phase transitions and thermodynamic phase transitions [11–15]. The HEE proposal opened the door for exploring the information-related properties in holographic theories. For instance, as a more general measure of entanglement, Rényi entropy is proposed to be proportional to the minimal area of cosmic branes [16]. Moreover, the butterfly effect as a typical phenomenon of quantum chaos has been extensively studied in holographic theory recently [17–26]. In addition, holographic duality of quantum complexity has also been proposed [27–33]. Recently, the purification of purification (EoP) was associated with the area of the minimum cross-section of the entanglement wedge [34,35]. The geometric dual of EoP provides a novel tool for studying the mixed state entanglement.

At present, HEE has been widely studied, but the research on mixed state entanglement—MI and EoP, is still to be enhanced. In particular, the difference between MI and EoP, and their effectiveness in characterizing mixed state entanglement, are still unclear. For this purpose, we study the properties of HEE, MI and EoP, in a holographic axion model. We choose axion model because it is simple enough to have analytical solutions, meanwhile it has some important properties such as momentum dissipation.

We organize this paper as follows: we introduce the holographic axion model in Sect. 2.1, entanglement measures (HEE, MI, EoP) and their holographic duality in Sect. 2.2. In Sect. 3, we discuss the properties of these three information related physical quantities in axion model. Finally, we summarize in Sect. 4.

2 The holographic axion model and information related quantities

In this section, we introduce the holographic axion model, as well as the concepts and calculations of HEE, MI and EoP.

2.1 Holographic axion model

We consider the following action [36–38],

$$S = \int d^4x \sqrt{-g} \left[R + 6 - V(X) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where $V(X)$ is the kinematic term for the axion fields. The ansatz and the corresponding solutions are,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\delta_{ij}dx^i dx^j$$

$$A_t = \mu - \frac{\rho}{r}, \quad \psi^I = k\delta_i^I x^i, \quad (2)$$

$$X = \frac{1}{2} g^{ab} \delta_{IJ} \partial_a \psi^I \partial_b \psi^J, \quad \text{where } i, j, I, J = 1, 2,$$

with

$$f(r) = \frac{\rho^2}{4r} \left(\frac{1}{r} - \frac{1}{r_h} \right) + \frac{1}{2r} \int_{r_h}^r \left[6u^2 - V \left(\frac{k^2}{u^2} \right) u^2 \right] du. \quad (3)$$

The ψ^I represents the linear axion field with a constant linear factor k . The A is the Maxwell field, and μ , ρ represent the chemical potential and the charge density of the dual field theory. The regularity of the Maxwell field on the horizon requires that $A_t|_{r_h} = 0$, i.e., $\rho = \mu r_h$. The linear axion fields break the translational symmetry, so the system has finite DC conductivity, which reads [38],

$$\sigma_{DC} = 1 + \frac{\mu^2}{k^2 \tilde{V}(r_h)}, \quad (4)$$

with $\tilde{V}(r) \equiv \sum_{n=1}^{\infty} 2n V'(X^n/2)^{n-1}$. The limit $k \rightarrow \infty$ in here is the incoherent limit, that plays a crucial role in explaining the universal strange metals behaviors [22,39–42].

For clarity, we focus on the $V(X) = X^2$ case, which goes back to [43]. We have also examined the cases for $V(X) = X, X^3, X^4, X^5$, numerically the phenomena we revealed in this paper are the same as that of $V(X) = X^2$. Moreover, we have also implemented analytical treatments for general $V(X) = X^N$ in appendix A.

The system (2) is invariant under the following rescaling,

$$(t, x, y) \rightarrow \alpha(t, x, y), \quad (r, k, \mu) \rightarrow (r, k, \mu)/\alpha, \quad (f(r), \rho) \rightarrow (f(r), \rho)/\alpha^2. \quad (5)$$

We focus on scaling-invariant physical quantities, so we adopt μ as the scaling unit by setting $\mu = 1$. Hawking temperature is given by,

$$T = \frac{1}{4\pi} \frac{df}{dr} \Big|_{r=r_h} = -\frac{r_h^2 + 2k^4 - 12r_h^4}{16\pi r_h^3}. \quad (6)$$

2.2 The holographic information-related quantities

The HEE is identified as the area of the minimum surface stretching into the bulk. It is more convenient to use $z \equiv \frac{r_h}{r}$ coordinate for numerical calculation, where

$$ds^2 = \frac{r_h^2}{z^2} \left[-g(z)dt^2 + \frac{dz^2}{g(z)r_h^2} + dx^2 + dy^2 \right], \quad (7)$$

Fig. 1 The left plot: the cartoon of a minimum surface. The right plot: The cartoon of the minimum cross-section (green surface) of the entanglement wedge

with

$$g(z) = \frac{(z-1)(z^3 r_h^2 - 4(z^2 + z + 1)r_h^4 + 2k^4 z^3)}{4r_h^4}. \quad (8)$$

For simplicity, we consider the infinite strip configuration along y -axis (see the left plot of Fig. 1). The minimum surface is invariant along y -axis, and hence can be described by $z(x)$. Therefore, solving the minimum surface only involves in ordinary differential equations, instead of partial differential equations. The HEE S of the minimum surface and the corresponding width w of the strip are,

$$\begin{aligned} S(z_*) &= \int_{\epsilon}^{z_*} \frac{4z_*^2 r_h^3}{z^2} T^{-1/2} dz, \\ w(z_*) &= \int_0^{z_*} 4z^2 r_h T^{-1/2} dz. \end{aligned} \quad (9)$$

where $T \equiv (1-z)(z^4 - z_*^4)(z^3 r_h^2 - 4(z^2 + z + 1)r_h^4 + 2k^4 z^3)$, and z_* represents the top of the minimum surface. The asymptotic AdS boundary leads to a divergent HEE, therefore we introduce a cutoff ϵ . Subtracting a cutoff-dependent quantity, the regularized HEE is given by,

$$S(z_*) = \int_0^{z_*} \left(\frac{4z_*^2 r_h^3}{z^2} T^{-1/2} - \frac{2r_h}{z^2} \right) dz - \frac{2r_h}{z_*}. \quad (10)$$

When the size of the subregion is small (where the size can be represented by the width w of the infinite strips), the minimum surface will approach the horizon of the black brane, so that the HEE will be mainly contributed by the thermal entropy. The HEE will behaves as

$$S \simeq s w, \quad (11)$$

where s denotes the thermal entropy density. A qualified mixed state entanglement measure must be able to get rid of the influence of thermal entropy, therefore HEE is not a suitable measure of entanglement of mixed states.

The mutual information between two separate subsystems A and C (with separation B) is defined by the following formula,

$$I(A; C) = S(A) + S(C) - S(A \cup C). \quad (12)$$

Apparently, a non-trivial MI requires $S(A \cup C) = S(B) + S(A \cup B \cup C)$ (see Fig. 2 for cartoon). MI measures the entanglement between subsystems. Unlike the holographic entanglement entropy, this definition cancels out the divergence from the AdS. Moreover, MI also partly cancels out the thermal entropy contribution [44]. For convenience, let's consider parallel infinite stripes A and C with separation B , whose widths are a , b and c respectively. Therefore we can label a configuration with (a, b, c) (see the right plot of Fig. 1).

The entanglement of purification, defined as the minimum entanglement entropy among all possible purification, is a distinct entanglement measure from the HEE and MI [45]. Recently, it is proposed that the EoP is proportional to the minimal cross-section for connected configuration of MI (see the right plot of Fig. 1) [34],

$$E_W(\rho_{AC}) = \min_{\Sigma_{AC}} \left(\frac{\text{Area}(\Sigma_{AC})}{4G_N} \right). \quad (13)$$

EoP can measure mixed state entanglement for separate subsystems, and satisfies several important inequalities [34, 45]. Again for simplicity, we study the EoP for symmetric configurations, where the minimum cross-section can be easily identified as the vertical line connecting tops of the minimum surfaces (see Fig. 2). Because the vertical line does not lie in the near horizon region, EoP is not determined by thermal entropy. Instead, the EoP is more affected by the boundary region than by the horizon region due to the factor $1/z^2$ from the asymptotic AdS geometry.

Fig. 2 A cartoon of the mutual information and EoP for symmetric configurations. The black horizontal line on the bottom represents the AdS boundary. A non-trivial MI equals the area of the blue surfaces (connected configuration) minus the area of the red curves (disconnected configuration). The vertical green line denotes the minimal cross section, that connects the tops (z_{*1} and z_{*2}) of C_b and $C_{a,b,c}$

Next, we explore the HEE, MI and EoP on the axion model.

3 The entanglement measures on the axion model

3.1 Holographic entanglement entropy

The Hawking temperature (6) suggests that r_h is a function of (k, T) , which we plot in Fig. 3. At first, r_h hardly changes with k for small k . With the increase of k , r_h gradually increases with k , and finally exhibits a linear relationship with an constant slope. This fact can be deduced from (6). At a fixed temperature T , we obtain

$$\left. \frac{\partial r_h}{\partial k} \right|_T = \frac{8k^3 r_h}{12r_h^4 + r_h^2 + 6k^4}. \quad (14)$$

Solving k from (6) and inserting it into (14) we obtain that,

$$\lim_{r_h \rightarrow \infty} \left. \frac{\partial r_h}{\partial k} \right|_T = \lim_{r_h \rightarrow \infty} \frac{2\sqrt{2}\sqrt{r_h}(-16\pi T r_h + 12r_h^2 - 1)^{3/4}}{-24\pi T r_h + 24r_h^2 - 1} = 6^{-1/4}. \quad (15)$$

Therefore, the slope of r_h vs k at any temperature approaches $6^{-1/4}$ for large k . For $V = X^N$ we obtain that $\lim_{r_h \rightarrow \infty} \left. \frac{\partial r_h}{\partial k} \right|_T = 6^{-\frac{1}{2N}}$ (see Appendix A for more detailed analysis). The linear relationship between r_h and k in large k limit plays an important role in the behavior of information related physical quantities.

Next, we show the HEE behavior with k and T at width $w = 2$ respectively in Fig. 4. Apparently, the HEE monotonically increases with k and T , regardless of the values of k , T and the subregion size. So far, we are not able to prove the

Fig. 3 The r_h vs k

monotonically increasing behavior of HEE for arbitrary values of k , T (see the Appendix A for detailed expressions). However, we can understand the monotonically increasing behavior of the HEE with k by the small k expansion of the $\frac{\partial S}{\partial k}$,

$$\begin{aligned} \frac{\partial S}{\partial k} &= \int_0^w \frac{16k^3}{(12r_h^2 + 1) \sqrt{1 - \frac{4z'(x)^2}{4(z^3 - 1)r_h^2 - z^4 + z^3}}} \\ &\times \left[\frac{1}{z^2} + \frac{p(z)z'(x)^2}{1 - z} \right] dx > 0 + \mathcal{O}(k^4), \end{aligned} \quad (16)$$

where $p(z) \equiv \frac{4(3z^3 + 2z^2 + 2z + 2)r_h^2 - 3z^3}{(z^4 - 4z(z^2 + z + 1)r_h^2)^2} > 0$.

For general $V(X) = X^N$, we find $S \simeq S_{(0)} + S_{(1)}k^{2N} + \mathcal{O}(k^{2N+1})$ in small k limit. Therefore we can derive that, the S will approach a constant $S_{(0)}$, and the first order correction is of order k^{2N} in small k limit. The MI will behave as $I \simeq I_{(0)} + I_{(1)}k^{2N} + \mathcal{O}(k^{2N+1})$ because MI is defined with HEE. Therefore, we can conclude that the HEE and the MI will be flat in small k region because $\frac{\partial S}{\partial k} \sim k^{2N-1}$, $\frac{\partial I}{\partial k} \sim k^{2N-1}$.

For large values of the k , r_h will also be large and approach the boundary, hence any minimum surface will approach the horizon. As a result, the HEE will be dominated by the thermal entropy,

$$\lim_{k \rightarrow \infty} S \rightarrow wr_h^2 = 6^{-1/N} wk^2. \quad (17)$$

For small T , the system approaches a zero temperature black brane system, which has regular background geometry. Therefore, the HEE and the MI are all finite, and exhibits no scaling relation with T . However, for large T limit we will again have $r_h \rightarrow \infty$. Like the limit $k \rightarrow \infty$, we will again have that $S \rightarrow wr_h^2$.

Next, we study two different kinds of mixed state entanglement: MI and EoP.

3.2 Mutual information

Compared with the monotonic dependence of HEE on system parameters, we find that MI presents more abundant phenom-

Fig. 4 HEE at width $w = 2$ vs k (the upper plot) and T (the lower plot). For other values of w we have qualitatively similar behaviors

ena. Moreover, the dependence of MI with k is different from that of MI with T . Thus, we explore MI vs k and MI vs T respectively.

3.2.1 Mutual information vs k

The MI of small configuration is different from that of large configuration, and the mechanisms behind these phenomena are also different. Therefore, we discuss the behavior of MI in small and large configurations respectively.

1. Small configurations

For small configurations, HEE and MI are mainly determined by the asymptotic AdS geometry. We may analytically explore the MI vs k .

We find that MI decreases monotonically with k (see Fig. 5). According to the MI definition (18), the monotonic behavior of I corresponds to $\partial_k I = \partial_k S(A) + \partial_k S(B) - \partial_k S(B) - \partial_k S(A \cup B \cup C) < 0$. In fact, the $\partial_k I$ mainly depends on the $\partial_k S(A \cup B \cup C)$ because minimum surface of $A \cup B \cup C$ probes deeper into the bulk, and hence are more affected than the other three quantities by the deviation from the AdS geometry. Therefore, we can conclude that $\partial_k I < 0$ following $\partial_k S > 0$ from (16).

Another significant phenomenon is that MI almost does not change with k when k is small, which can be understood by $\partial_k S \sim k^3$ from (16). For general $V(X) = X^N$, we will have $\partial_k I \sim k^{2N-1}$.

2. Large configuration

The minimum surfaces of a large configuration will

Fig. 5 The plot of I vs k at different temperatures specified by the plot legends. The small configuration is $(a, b, c) = (0.1, 0.05, 0.1)$

Fig. 6 The plot of I vs k at different configurations specified by the plot legends, at $b = 0.05$, $T = 0.1818$. The behavior of MI at other temperatures is qualitatively the same

stretch deep into the bulk, so it can not be understood analytically by the geometry deviation near the boundary. We use numerical method to explore MI vs k for large configurations.

When the configuration increases, we find that MI monotonically decreases with k at the beginning. But when a, c increases, we find that the MI becomes non-monotonic with k . No matter how large a, c becomes, MI always monotonically decreases with k in the small k region. These phenomena are shown in Fig. 6.

The universal monotonic behavior for small k can also be understood from (16). For large configurations, the MI will be dominated by $S(A)$, $S(C)$, $S(A \cup B \cup C)$ since they approach the horizon [see the second term of (16)] These three quantities are mainly contributed by the thermal entropy s . Therefore, the $\partial_k I$ will be determined by $-\partial_k s(B) < 0$. As a result, we have $\partial_k I < 0$ again.

Fig. 7 The plot of I vs T at different temperatures specified by the plot legends. The small configuration is $(a, b, c) = (0.1, 0.05, 0.1)$

Fig. 8 The plot of I vs T at different temperatures specified by the plot legends. The large configuration is $(a, b, c) = (10, 0.05, 10)$

In large k limit, we proved that the HEE is dictated by the thermal entropy (11). Therefore, the MI will vanish because

$$I(A; C) = S(A) + S(C) - S(A \cup C) = s(w_A + w_C - (w_A + w_C)) = 0. \quad (18)$$

Next we explore the relationship between MI and temperature.

3.2.2 Mutual information vs T

The relationship between MI and temperature also relate to specific system parameters and configurations. Comparing Figs. 7 and 8, we see that when configuration increases, the behavior of MI with temperature becomes more abundant. Specifically, we find that $\partial_T I < 0$ when k is small and $\partial_T I > 0$ when k is relatively large, regardless of the configurations.

Fig. 9 The plot of $m(z)$

First, we discuss the small k limit. Taking the limit $k \rightarrow 0$, $T \rightarrow 0$, we find,

$$\frac{\partial S}{\partial T} = \int_0^w \frac{1}{2z^2 \sqrt{\frac{12z'(x)^2}{(3z-4)z^3+1} + 1}} \times \left[1 - \frac{6m(z)z'(x)^2}{(1-z)^3(z(3z+2)+1)^2} \right] dx \quad (19)$$

where $m(z) \equiv z(2z-1)(3z+1)-1$. The sign of $\partial_T S$ depends on $m(z)$. From the plot of $m(z)$ in Fig. 9, we can deduce that $\partial_T S > 0$ for small configuration since the minimum surface resides only in the small- z region. Following the arguments for the MI behavior for small configurations in the previous section, we see that $\partial_T I < 0$ for small configurations.

For large configurations, the minimum surface approaches the horizon and goes to the region with positive $m(z)$. At this time, the sign of $\partial_T S$ is not transparent from (19). However, the temperature behavior of the HEE as well as the MI, depends on the thermal entropy density since the HEE will be mainly contributed by the thermal entropy s . Therefore, we still have $\partial_T S > 0$. Following the arguments for large configurations in the previous subsection, we have $\partial_T I < 0$ again.

Next, the study the large k limit. The small k -limit analysis technique used in the previous section does not apply to the analysis of large k -limit. Because the radius of the horizon r_h becomes large for large k . As a result, the minimum surface of any finite subregion approaches the horizon, leading to a vanishing MI. As can also be seen in Figs. 7 and 8, MI in the low-temperature region tends to vanish when k is large enough. Therefore, the monotonous increase of MI with temperature in low temperature region occurs when MI is about to vanishes. At this time, the minimum surface is neither near the boundary nor near the black hole horizon. Therefore, the analysis techniques of large and small configuration limits are also not applicable. At this stage, we can only numerically address this phenomenon. We plan to explore an analytical understanding in the near future.

Fig. 10 The left and the right plot is the critical k and T for disentangling phase transitions

Fig. 11 The first plot: EoP vs a, c . The temperature is fixed as $t = 0.1818$, and the separation b is fixed as 0.249 . Different curves in the diagram represent different k values specified by the plot legends. The

second plot: $E_W - I/2$ vs a . The second plot shows that E_W is always greater than half of the mutual information $I/2$

Another important quantity is the critical parameter of the disentangling phase transitions, where two disjoint subregions disentangle. We plot the critical values of k and T with respect to T and k respectively in Fig. 10. Apparently, k_c decreases with T and T_c decreases with k as well. The reason is that, r_h increases with both k and T , which means that the horizon will approach the boundary for larger k or T . As a result, the HEE will become more dependent on the thermal entropy, and thus makes the subregions disentangle more easily. Therefore, we can understand that k_c decreases with T and T_c decreases with k .

Next we examine the behavior of the EoP.

3.3 Entanglement of purification

The EoP for a symmetric configuration equals the area of the vertical line connecting the tops of the minimum surfaces (see Fig. 2). For simplicity, we focus on the EoP of symmetric configurations in this paper. Most of the previous studies on

EoP only considered symmetric configurations [34,35,46–53]. For asymmetric configurations ($a \neq c$), one needs to search a two-dimensional parameter space to find the EoP, which is a hard task [54].

We show the relationship of EoP with a, c in Fig. 11, and EoP with b in Fig. 12. EoP increases with the increase of a, c and decreases with the increase of b . These phenomena verify the inequality $E_W(\rho_{A(BC)}) \geq E_W(\rho_{AB})$ satisfied by EoP [34]. In addition, from the right plots in Figs. 11 and 12 we see that EoP is always larger than $I/2$. This verifies another important inequality $E_W(\rho_{AC}) \geq \frac{I(A;C)}{2}$ that EoP satisfies [34].

Next, we study EoP vs k and EoP vs T .

3.3.1 Entanglement of purification vs k

First, EoP monotonically increases with k , regardless of the specific system parameters and configurations. When k reaches the critical k_c , EoP vanishes because MI vanishes. We

Fig. 12 The first plot: EoP vs b at $k = 0.909$ and $a = c = 4$. The second plot: $E_W - I/2$ vs b . The second plot shows that E_W is always greater than half of the mutual information $I/2$. Each curve in both plots corresponds to k specified by the plot legends

Fig. 13 The relation between EoP and k . The reason for the sudden drop of EoP to 0 is that the MI under this configuration is actually 0. That is to say, this point corresponds to the critical point of disentangling transition

show these phenomena in Fig. 13. The monotonic behavior can be seen from,

$$\frac{\partial E_W}{\partial k} = \int_{z_{*1}}^{z_{*2}} \frac{8k^3 r_h (n(z) + 2\pi T z^3) \sqrt{\frac{r_h}{(1-z)n(z)}}}{z^2 n(z) (12r_h^4 + r_h^2 + 6k^4)} dz > 0, \quad (20)$$

where $n(z) \equiv (-3z^3 + z^2 + z + 1)r_h + 4\pi T z^3$, the z_{*1} and z_{*2} are the tops of the minimum surfaces (see Fig. 2). Therefore, we conclude that E_W monotonically increases with k . This fact shows that the bipartite entanglement of the dual field theory of the axion model always increases with the k . Analytically, we also find $\frac{\partial E_W}{\partial k} > 0$ for $V = X$. However, for $V = X^3$, we cannot prove this monotonic behaviors analytically (see Appendix A for detailed expressions). For general $V(X) = X^N$ we have $E_W \simeq E_{W(0)} + E_{W(1)}k^{2N} + \mathcal{O}(k^{2N+1})$.

3.3.2 Entanglement of purification vs T

For small configurations, we find that EoP increases with T , regardless of the values of k , T . We find that this can be proved analytically for any parameter k , T and for general $V(X) = X^N$. In the Appendix A we proved that $\frac{\partial E_W}{\partial T} < 0$. We have also implemented the numerical treatment on the relation between the EoP and the temperature. We plot E_W vs T in Fig. 14, in which the E_W decreases with T before it vanishes. Meanwhile, we notice that EoP will vanish as MI vanishes when T is large. This is because r_h becomes large for high temperature, and the minimum surface of any finite subregion approaches the horizon. This situation of large T limit is the same as that of the large k limit, where we have vanishing MI. The vanishing E_W at large temperatures means that the thermal effects will break the entanglement between separate subregions.

We also checked the $V = X, X^3, X^4, X^5$ with numerics, and we find the behaviors of EoP with k and T are qualitatively the same as that of $V = X^2$.

3.4 Comparison of three entanglement measures

We have studied the behavior of HEE, MI and EoP on axion model, and found that they exhibit very distinct behaviors. First, HEE increases with k and T monotonically, which can be understood analytically in certain limits. However, the monotonic behavior of HEE does not mean that the entanglement of the system increases with k , T , because the thermal entropy can bury the quantum entanglement [44]. When k or T is large, r_h will be large for any finite temperature (see Fig. 3). Therefore the horizon tends to approach the AdS boundary, and the HEE of course will be determined by the thermal entropy.

Fig. 14 The relation between EoP and T . The reason for the sudden drop of EoP to 0 is that the MI under this configuration is actually 0. That is to say, this point corresponds to the critical point of disentangling transition

The MI typically exhibits non-monotonic behaviors with k and T . Nevertheless, the MI decreases with k monotonically when k is small (see Figs. 5 and 6). This fact can be justified analytically for small configurations and large configurations (see Sect. 3.2.1). For the relationship between MI and the temperature, we find that for low temperatures, the MI always decreases with T for small k ; while for large enough k , we observed a universal increasing behavior with T . Compared with the monotonic behaviors of HEE with k and T , we can conclude that these phenomena suggests that MI captures distinct entanglement structure from the HEE. However, although exhibiting distinct behaviors from the HEE, MI may still be dictated by the thermal entropy due to its dependence on HEE, especially for large configurations. For large configurations, the MI behavior with k and T originates from the HEE that is dictated by thermal entropy, following the analysis given in Sect. 3.2.2, we see that the MI may still be determined by thermal effects. Therefore, MI is not an ideal mixed state entanglement measure in the sense that the effect of thermal entropy cannot be completely removed.

The EoP monotonically increases with k before it vanishes, which can be proved analytically in certain cases (See Sect. 3.3.1, Fig. 13 and related discussion). This unusual monotonic behavior implies that the EoP captures very different entanglement structures from the MI, since MI does not show any universal monotonic behavior. Moreover, we find that the EoP monotonically decreases with T (see Fig. 14), this is another distinct property compared with MI, remind that the MI increases with T for large enough k . The monotonically decreasing behavior of the EoP with the temperature can be proved analytically (see Sect. 3.3.1). More importantly, the EoP does not show dependence on the near horizon geometry, even for large configurations. The

reason is that the EoP involves the bulk degrees of freedom even in the large configurations limit. Remind that the EoP is proposed as proportional to the area of the minimum cross-section living in the entanglement wedge, whose value actually are more affected by the area away from the horizon region because of the factor $1/z^2$ in asymptotic AdS geometry. That is to say, the EoP will never be dictated by the thermal effects. Therefore, the EoP can be a better mixed entanglement measure than the MI in the sense that it is not under control of thermal entropy in any situation.

4 Discussion

In this paper, we have studied HEE, MI and EoP on holographic axion model, and found that they exhibit very different behaviors. Combined with numerical and analytical analysis, we found and proved their monotonic behavior in certain limits. Their differences show that they depict different aspects of the entanglement properties. Specifically, MI and EoP can cancel out the thermal effect compared with HEE, and hence exhibits more diverse phenomena. Moreover, EoP can be a better mixed state entanglement measure than the MI due to its independence from the thermal entropy. Next, we point out several topics worthy of further study.

First, the techniques in this paper can be directly applied to holographic models with analytical solutions, such as Gubser–Rocha model [55], massive gravity theory [56], Gauss–Bonnet gravity theory [57], and so on. For numerical background solutions, in certain limits, the techniques in this paper may still be applicable. In addition, the EoP of asymmetric configuration in this paper is worth studying, it would be interesting to test whether the monotonic behavior of E_W with k is universal. Finally, it is also desirable to study the behavior of HEE, MI and EoP during phase transitions.

Acknowledgements Peng Liu would like to thank Yun-Ha Zha for her kind encouragement during this work. This work is supported by the Natural Science Foundation of China under Grant No. 11805083, 11847055, 11905083, 11947067.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.]

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Funded by SCOAP³.

Appendix A: Analytical treatment for general $V = X^N$

In the appendix we provide the analytical treatment for general $V = X^N$. The Hawking temperature expression for general N reads,

$$\frac{\partial S}{\partial k} = \int_0^w \frac{k^{2N-1} \left(8N(z-1)r_h^5 - 8N(2N-3)z^2v_1(z)r_h^{2N+3}z'(x)^2 \right)}{(z-1)z^{3/2} \left(2(2N-1)r_h^2k^{2N} + (12r_h^2+1)r_h^{2N} \right) \sqrt{zr_h^2 + \frac{4z'(x)^2}{q(z)}}} dx, \quad (\text{A5})$$

1. The holographic entanglement entropy

The expression of the HEE reads,

$$S = \int_0^w z^{-5/2} r_h \sqrt{\frac{4z'(x)^2}{q(z)} + zr_h^2} dx, \quad (\text{A3})$$

where

$$q(z) \equiv (1-z) \left(\frac{2k^{2N}(z^{2N}-z^3)r_h^{2-2N}}{(2N-3)z^3} + 4 \left(\frac{1}{z^3} - 1 \right) r_h^2 + z - 1 \right) \quad (\text{A4})$$

We compute the $\frac{\partial S}{\partial k}$,

$$v_1(z) \equiv \frac{2r_h^2k^{2N}(z^{2N}-z^3) - r_h^{2N}((12r_h^2+1)z^{2N} + z^3(-12r_h^2+4N-7) + (6-4N)z^4)}{(2r_h^2k^{2N}(z^{2N}-z^3) + (2N-3)(z-1)z^3r_h^{2N} - 4(2N-3)(z^3-1)r_h^{2N+2})^2}. \quad (\text{A6})$$

$$T = \frac{-2k^{2N}r_h^{2-2N} + 12r_h^2 - 1}{16\pi r_h}. \quad (\text{A1})$$

From (A5) we see that the integrand of (A5) can be either positive or negative, depending on specific values of N , k , T and z .

Now we compute the $\frac{\partial S}{\partial T}$,

$$\frac{\partial S}{\partial T} = \int_0^w \frac{32\pi(z-1)r_h^{2N+4} - 64\pi(2N-3)z^2v_2(z)r_h^{4N+2}z'(x)^2}{(z-1)z^{3/2} \left(2(2N-1)r_h^2k^{2N} + (12r_h^2+1)r_h^{2N} \right) \sqrt{zr_h^2 + \frac{4z'(x)^2}{q(z)}}} dx, \quad (\text{A7})$$

where

$$v_2(z) \equiv \frac{2Nr_h^2k^{2N}(z^{2N}-z^3) + (2N-3)(z-1)z^3r_h^{2N}}{(2r_h^2k^{2N}(z^{2N}-z^3) + (2N-3)(z-1)z^3r_h^{2N} - 4(2N-3)(z^3-1)r_h^{2N+2})^2}. \quad (\text{A8})$$

At a finite temperature, we have $\lim_{k \rightarrow \infty} r_h \rightarrow \infty$. Implementing the procedure of (14) and (15), we obtain,

$$\lim_{k \rightarrow \infty} \frac{\partial r_h}{\partial k} \rightarrow 6^{-\frac{1}{2N}}. \quad (\text{A2})$$

And hence we have that $r_h \simeq 6^{-\frac{1}{2N}} k$ in large k limit.

From (A7) we see that the integrand of (A7) can be either positive or negative, depending on specific values of N , k , T and z .

The above equations mean that for finite values of k , T , we can not determine the monotonic behavior of HEE directly from these analytical treatments. However, we stress that our numerical results for several different cases of N all show that the HEE increases with k and T .

Variate the temperature expression (A2) and the EoP expression (A9) we obtain that,

$$\frac{\partial E_W}{\partial T} = \int_{z_{*1}}^{z_{*2}} n_1(z)n_2(z)dz, \quad (\text{A10})$$

where

$$\begin{aligned} n_1(z) &\equiv \frac{32\pi r_h^{4N+4} \sqrt{zq(z)}}{(2r_h^2 k^{2N} (z^{2N} - z^3) + (2N-3)(z-1)r_h^{2N} (z^3 - 4(z^2 + z + 1)r_h^2))^2} > 0, \\ n_2(z) &\equiv -\frac{(2N-3)(2Nr_h^2 k^{2N} (z^3 - z^{2N}) + (2N-3)(1-z)z^3 r_h^{2N})}{(1-z)(2(2N-1)r_h^2 k^{2N} + (12r_h^2 + 1)r_h^{2N})} < 0. \end{aligned} \quad (\text{A11})$$

For small k limit and small T limit, we can also expand the (A5) and (A7) with respect to k and T . We find that we can only prove for several cases of N , such as $N = 2, 3$, that the HEE increases with k . A general proof of these results even at certain limits, such as $k \rightarrow 0$, is still in need.

Therefore we obtain that $\frac{\partial E_W}{\partial T} < 0$ for any N .

For the relationship between the EoP and k , we carry out the same procedure,

$$\frac{\partial E_W}{\partial k} = \int_{z_{*1}}^{z_{*2}} m_1(z)m_2(z)dz, \quad (\text{A12})$$

where

$$\begin{aligned} m_1(z) &\equiv \frac{Nk^{2N-1}r_h^{1-2N}}{8\pi} n_1(z) > 0, \\ m_2(z) &\equiv -\frac{(2N-3)(z^{2N}(8\pi Tr_h + 1) - z^3(8\pi Tr_h - 2N + 4) - (2N-3)z^4)}{(1-z)(N(-16\pi Tr_h + 12r_h^2 - 1) + 8\pi Tr_h + 1)}. \end{aligned} \quad (\text{A13})$$

2. The entanglement of purification

For general N we can deduce that the symmetric EoP reads,

$$E_W = \int_{z_{*1}}^{z_{*2}} \frac{2r_h z^{-5/2}}{\sqrt{q(z)}} dz. \quad (\text{A9})$$

The sign of $m_2(z)$ depends on specific values of N . Taking $N = 2$ we will obtain (20). However, taking other values of N , for example, $N = 3$, we find that $m_2(z)$ can be either positive or negative, depending on the values of z . That is to say, we can not determine the sign of $\frac{\partial E_W}{\partial k}$ by the analytical expression (A12).

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