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Comment

Comment on "Dirac fermions in Som–Raychaudhuri space-time with scalar and vector potential and the energy momentum distributions [Eur. Phys. J. C (2019) 79:541]"

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In a recent paper in this Journal, Sedaghatnia et al. [1] have studied the Dirac equation in the presence of scalar and vector potentials in a class of flat Gödel-type space-times called Som–Raychaudhuri space-times by using the methods quasi-exactly solvable (QES) differential equations and the Nikiforov Uvarov (NU) form. To achieve their goal, the authors have mapped the system into second-order differential equation (Schrödinger-like problem). It is worthwhile to mention that the expressions obtained [Eqs. (2.8)–(2.17)] in Ref. [1] are correct. On the other hand, the second-order differential equation in Ref. [1] is wrong, probably due to erroneous calculations in the manipulation of the two coupled first-order differential equations. This fact jeopardizes the results of [1]. The purpose of this comment is to calculate the correct differential equation and following the appropriate procedure to obtain the solution for this problem.

The Gödel-type solution with torsion and a topological defect can be written in cylindrical coordinates by the line element

$$ds^2 = -(dt + \alpha\Omega r^2 d\varphi)^2 + dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2\,. \tag{1}$$

The Dirac equation for a free Fermi field Ψ of mass M in a Som–Raychaudhuri space-time with scalar and vector potentials is given by [1]

$$\left[i\gamma^{\mu}\left(\nabla_{\mu} + ieA_{\mu}\right) - (M + S(r))\right]\Psi(r,t) = 0 \tag{2}$$

where $A_{\mu} = (V(r), 0, 0, 0)$, $\nabla_{\mu} = (\partial_{\mu} + \Gamma_{\mu})$ and Γ_{μ} is the affine connection. Now, using the correct results of Ref. [1] and considering the solution in the form

$$\Psi(t,r,\varphi,z) = e^{-iEt} e^{i(m\varphi + kz)} \begin{pmatrix} \bar{\psi}(r) \\ \bar{\chi}(r) \end{pmatrix}, \tag{3}$$

the Dirac equation (2) (with $k = 0$) becomes

$$\hat{O}\bar{\chi}(r) = \left[(E - V(r)) - \frac{\Omega}{2}\sigma^3 - (M + S(r))\right]\bar{\psi}(r), \tag{4}$$

$$\hat{O}\bar{\psi}(r) = \left[(E - V(r)) - \frac{\Omega}{2}\sigma^3 + (M + S(r))\right]\bar{\chi}(r). \tag{5}$$

Here $\hat{O} = \Omega r(E - V(r))\sigma^2 - i\sigma^1\partial_r + \frac{m_\alpha}{r}\sigma^2 - \frac{1}{2r}\sigma^2\sigma^3$ with $m_\alpha = m/\alpha$. Using the expression for $\bar{\chi}(r)$ obtained from (5) with $V(r) = S(r)$, redefining the spinor as $\bar{\psi}(r) = \frac{\psi(r)}{\sqrt{r}}$ and inserting it in (4) we obtain

$$\begin{aligned}\psi''(r) + \Bigg[&(E^2 - M^2) - 2V(r)(E + M) \\ &- 2\Omega(E - V(r))\sigma^3 + \frac{\Omega^2}{4} - \Omega^2 r^2 (E - V(r))^2 \\ &- 2m_\alpha\Omega(E - V(r)) + \Omega r\left(\frac{dV(r)}{dr}\right)\sigma^3 \\ &- \frac{\left(m_\alpha - \frac{\sigma^3}{2}\right)^2 - \frac{1}{4}}{r^2}\Bigg]\psi(r) = 0. \end{aligned}\tag{6}$$

Equation (6) is effectively a Schrödinger-type equation. The second-order differential equation obtained in [1] [Eq. (2.18)] is not similar to our result (6) probably due to erroneous calculations in the manipulation of the two coupled first-order differential Eqs. (4) and (5).

As in Ref. [1], firstly we concentrate our efforts on $V(r) = 0$. Using $\psi(r) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, $\sigma^3\psi_s(r) = s\psi_s(r)$ with $s = \pm 1$ and $V(r) = 0$, (6) reduces to

$$\psi_s''(r) + \left(\lambda_3 - \lambda_1 r^2 - \frac{\lambda_2}{r^2}\right)\psi_s(r) = 0, \tag{7}$$

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where

$$\lambda_1 = E^2 \Omega^2 , \tag{8}$$

$$\lambda_2 = \left(m_\alpha - \frac{s}{2}\right)^2 - \frac{1}{4} , \tag{9}$$

$$\lambda_3 = E^2 - M^2 - 2E\Omega s + \frac{\Omega^2}{4} - 2m_\alpha \Omega E . \tag{10}$$

The equation of motion (7) describes the quantum dynamics of a Dirac particle in a Som–Raychaudhuri space-time. The expression for λ_2 obtained in Ref. [1] [Eq. (2.21)] is wrong. The solution for (7) with λ_1 and λ_3 real is precisely the well-known solution of the Schrödinger equation for the harmonic oscillator. The solution for all r can be expressed as

$$\psi_s(r) = N_n r^{|m_\alpha - \frac{s}{2}| + \frac{1}{2}} e^{-\sqrt{\lambda_1} r^2/2} L_n^{|m_\alpha - \frac{s}{2}|}(\sqrt{\lambda_1} r^2) , \tag{11}$$

where N_n is a normalization constant. Moreover, the spectrum is expressed as (for $E\Omega > 0$)

$$E = \nu_{n,m} + \sqrt{\nu_{n,m}^2 + M^2 - \frac{\Omega^2}{4}} \tag{12}$$

with $\nu_{n,m} = \left(2n + 1 + \left|m_\alpha - \frac{s}{2}\right| + m_\alpha + s\right)\Omega$. The eigenvalue of energy (2.25) obtained in Ref. [1] is not similar to our result (12) due to it having been obtained from a wrong differential equation.

As a second example, let us consider an attractive Coulomb potential $V(r) = -\frac{a}{r}$. By introducing the Coulomb potential into Eq. (6), and using $\psi(r) = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$ and $\sigma^3 \psi_s(r) = s\psi_s(r)$ with $s = \pm 1$, we get

$$\frac{d^2\psi_s}{dr^2} + \left[\mathcal{E}^2 + \frac{A}{r} - Br - Cr^2 - \frac{\left(m_\alpha - \frac{s}{2}\right)^2 - \frac{1}{4}}{r^2}\right] \psi_s = 0 , \tag{13}$$

where

$$\mathcal{E}^2 = E^2 - M^2 - 2E\Omega s + \frac{\Omega^2}{4} - 2m_\alpha E\Omega - \Omega^2 a^2 , \tag{14}$$

$$A = 2(E + M)a - \Omega a s - 2m_\alpha \Omega a , \tag{15}$$

$$B = 2E\Omega^2 a , \tag{16}$$

$$C = E^2 \Omega^2 . \tag{17}$$

The solution for (13), with C necessarily real and positive, is the solution of the Schrödinger equation for the three-dimensional harmonic oscillator plus a Cornell potential [3–6]. By setting

$$\psi_s = r^{\frac{1}{2} + |m_\alpha - \frac{s}{2}|} \exp\left(-\frac{\sqrt{C}}{2} r^2 - \frac{B}{2\sqrt{C}} r\right) \phi_s(r) \tag{18}$$

and by introducing the new variable and parameters

$$x = \sqrt[4]{C}\, r , \tag{19}$$

$$\omega = 2\left|m_\alpha - \frac{s}{2}\right| , \tag{20}$$

$$\rho = \frac{B}{\sqrt[4]{C^3}} , \tag{21}$$

$$\tau = \frac{B^2 + 4C\varepsilon^2}{4\sqrt{C^3}} , \tag{22}$$

one finds that the solution for all r can be expressed as a solution of the biconfluent Heun differential equation [5–11]

$$x\frac{d^2\phi_s}{dx^2} + (\omega + 1 - \rho x - 2x^2)\frac{d\phi_s}{dx} + [(\tau - \omega - 2)x - \Theta]\phi_s = 0 , \tag{23}$$

with $\Theta = \frac{1}{2}[\delta + \rho(\omega + 1)]$ and

$$\delta = -\frac{2A}{\sqrt[4]{C}} . \tag{24}$$

It is well known that the biconfluent Heun equation has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$ [4]. The regular solution at the origin is

$$H_b(\omega, \rho, \tau, \delta; x) = \sum_{j=0}^{\infty} \frac{\Gamma(1+\omega)}{\Gamma(1+\omega+j)} \frac{A_j}{j!} x^j , \tag{25}$$

where $\Gamma(z)$ is the gamma function, $A_0 = 1$, $A_1 = \Theta$ and the remaining coefficients for $\rho \neq 0$ satisfy the recurrence relation

$$A_{j+2} = [(j+1)\rho + \Theta]A_{j+1} - (j+1)(j+\omega+1)(\Delta - 2j)A_j , \tag{26}$$

where $\Delta = \tau - \omega - 2$. The series is convergent and tends to $\exp\left(x^2 + \rho x\right) = \exp\left(\sqrt{C} r^2 + \frac{B}{\sqrt{C}} r\right)$ as $x \to \infty$. This asymptotic behavior perverts the normalizability of the solution (18), because $\psi(r) \propto \exp\left(\frac{\sqrt{C}}{2} r^2 + \frac{B}{2\sqrt{C}} r\right)$ as $r \to \infty$. This impasse can be surpassed by considering a polynomial solution for H_b. From the recurrence (26), H_b becomes a polynomial of degree n if and only if two conditions are satisfied:

$$\Delta = 2n \quad (n = 0, 1, 2, \ldots) \tag{27}$$

and

$$A_{n+1} = 0 . \tag{28}$$

The condition (28) furnishes a polynomial of degree $n + 1$ in δ; there are at most $n + 1$ suitable values of δ. At this stage, it is worth to mention that the energy of the system is obtained using both conditions (27) and (28).

From the condition (27), we obtain (for $E\Omega > 0$)

$$E_{n,m}^2 - M^2 + \frac{\Omega_{n,m}^2}{4} - 2\Omega_{n,m}\xi_{n,m}E = 0\,, \tag{29}$$

where

$$\xi_{n,m} = s + m_\alpha + \left|m_\alpha - \frac{s}{2}\right| + n + 1\,. \tag{30}$$

The problem does not end here; it is necessary to analyze the condition (28). For $n = 0$, the condition (28) becomes $A_1 = \Theta = 0$ and results in an algebraic equation of degree one in δ,

$$\delta + \rho(\omega + 1) = 0\,. \tag{31}$$

Substituting (20), (21) and (24) into (31), we obtain

$$\Omega_{0,m} = \frac{2\,(E + M)}{\epsilon_m}\,, \tag{32}$$

where $\epsilon_m = 2\left(m_\alpha + \frac{s}{2}\right) + 2\left|m_\alpha - \frac{s}{2}\right| + 1$. Substituting (32) into (29) for $n = 0$, we have

$$E_{0,m} = M\frac{\epsilon_m^2 - 1}{1 - \epsilon_m^2 - 2(s+1)\epsilon_m}\,. \tag{33}$$

Equation (33) represents the energy eigenvalue for $n = 0$. For $n = 1$, the condition (28) becomes $A_2 = 0$ and results in an algebraic equation of degree two in δ. For $n \geq 2$ the algebraic equations are cumbersome. In this comment, we will only consider the solution for $n = 0$ for simplicity.

In summary, we studied the Dirac equation in the presence of scalar and vector potentials in a class of flat Gödel-type space-times called Som–Raychaudhuri space-times. We calculated the correct second-order differential equation for this system. As in Ref. [1], we have considered two cases: (1) $V(r) = 0$ and (2) the Coulomb potential. For the first case, $V(r) = 0$, the problem was mapped into a Schrödinger-like equation with the harmonic oscillator potential. The correct energy spectrum for this case was obtained. For the second case, we considered an attractive Coulomb potential. In this case, the problem was mapped into a biconfluent Heun differential equation and appropriately using the quantization conditions (27) and (28), we found the correct energy spectrum for $n = 0$. Finally, we showed that the results obtained in Ref. [1] are incorrect, due to them having been obtained from a wrong differential equation.

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