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Left-right entanglement entropy for a Dp-brane with dynamics and background fields

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Abstract We investigate the left-right entanglement entropy of a boundary state, corresponding to a dynamical Dp-brane with the internal and background fields. We assume that the brane has a tangential linear motion and a rotation, and is dressed with an internal $U(1)$ gauge potential and the Kalb–Ramond tensor field $B_{\mu\nu}$. We derive the entanglement entropy via the Rényi entropy by applying the replica trick. Our calculations will be in the context of the bosonic string theory.

1 Introduction

In a composite quantum system, which consists of subsystems, entanglement relates the different parts of the system. The subsystems can become entangled if the quantum state of each subsystem cannot be described independent of the states of the other subsystems. In fact, the quantum systems are capable to become entangled through the various types of processes such as interactions, particles creation and etc. For instance, in the decay of the subatomic particles, because of the conservation laws, the measured quantum labels for the daughter particles are highly correlated. Traditionally, for quantifying entanglement, geometric setups have been intensively studied in the literature [1–6]. Entanglement entropy is a favorable quantity for measuring the entanglement between the subsystems. Also, this quantity has been drastically studied in the context of the AdS/CFT [7,8].

At first consider a bipartite system with the subsystems A and B. The division can occur in the Hilbert space (instead of the configuration space), i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $|a_i\rangle$ and $|b_j\rangle$ be the eigen-bases which span $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Thus, $|a_i\rangle \otimes |b_j\rangle$ forms an eigen-basis for $\mathcal{H}$. Hence, a generic state $|\psi\rangle$ in $\mathcal{H}$ possesses the expansion

$$|\psi\rangle = \sum_{i,j} c_{ij} |a_i\rangle \otimes |b_j\rangle. \tag{1.1}$$

If the correlation coefficients c_{ij} can be decomposed, e.g., as $c_{ij} = \alpha_i \beta_j$ we acquire the product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. In this case the subsystems A and B are not entangled. For the case $|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ we have an entangled system.

In our system the left- and right-moving oscillating modes of closed strings are the bases of the two subsystems, hence, the Hilbert space possesses the factorized form $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. The Schmidt decomposition of the boundary state with respect to the left- and right-moving modes can be written as [9,10],

$$|B\rangle = \mathcal{N} \sum_{\vec{m}} |\vec{m}\rangle \otimes |U\tilde{\vec{m}}\rangle, \tag{1.2}$$

where the states $|\vec{m}\rangle$ and $|\tilde{\vec{m}}\rangle$ are complete orthonormal bases for $\mathcal{H}_L$ and $\mathcal{H}_R$, and U is an anti-unitary operator which acts on $\mathcal{H}_R$. Both states $|\vec{m}\rangle$ and $|\tilde{\vec{m}}\rangle$ depend on a set of the integer numbers $\{m_1, m_2, \ldots\}$. Now consider Eq. (1.1) for the maximally entangled case, i.e. $c_{ij} = c\delta_{ij}$, and compare it with Eq. (1.2). This comparing clarifies that the decomposition (1.2) represents the boundary state $|B\rangle$ as a maximally entangled state of the left- and right-moving modes. Thus, we can choose the boundary state as our composite system and the left- and right-moving modes of closed strings as its subsystems.

On the other hand, the D-branes as dynamical objects are essential for studying different areas of string theory. We shall investigate one of the attractive characteristics of a D-brane, i.e. the so called left-right entanglement entropy (LREE) [11–14]. The left-right entanglement is a non-geometrical version of the entanglement. Since the boundary state accurately encodes all properties of a D-brane [15–32], it is a useful tool for investigating the LREE corresponding to the D-brane.

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Zayas and Quiroz previously worked out the LREE for a one-dimensional boundary state in a free bosonic 2D CFT with the Dirichlet or the Neumann boundary condition [11]. Besides, they derived the LREE for the bare-static D-branes [13]. By making use of their approach, in this paper we shall obtain the LREE for a bosonic Dp-brane which is dressed by the Kalb–Ramond background field $B_{\mu\nu}$ and an internal $U(1)$ gauge potential A_α which lives in the brane worldvolume. In addition, we impose a tangential dynamics to the brane, which includes linear motion and rotation. We shall observe that the LREE of our setup may be interpreted as a thermodynamical entropy.

In fact, the entanglement entropy of the D-branes potentially has relation with the black holes entropies [6,33]. Therefore, we are motivated to investigate the LREE of a special Dp-brane. Precisely, a brane configuration with the background and internal fields can be corresponded to a charged black hole. Besides, a dynamical brane, especially those with internal rotations, may be associated with a rotating black hole. Ultimately, the LREE of our brane configuration may find a connection with the entropy of the charged-rotating black holes.
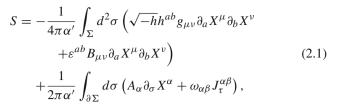
Since the extracted quantities of the bosonic string theory are similar to their counterparts in the NS–NS sector of the superstring theory we shall begin our calculations for the foregoing bosonic D-brane. Beside, the bosonic computations are more simple than the superstring calculations. Hopefully, in the subsequent works we shall extend our calculations to the supersymmetric version.

The paper is organized as follows. In Sect. 2, we shall introduce the boundary state, corresponding to the Dp-brane, then, the interaction amplitude between two parallel and identical Dp-branes will be introduced. This amplitude is required for calculating the Rényi entropy. In Sect. 3, we shall compute the LREE for a bare-static Dp-brane and for a dressed-dynamical one. We shall terminate this section with a thermodynamical interpretation of the LREE of our system. In Sect. 4, some simple examples will be presented to clarify the parametric dependence of the setup. Section 5 is devoted to the conclusions.

2 The dressed-dynamical Dp-branes: boundary state and interaction

2.1 The boundary state

In the beginning we introduce the boundary state, associated with a Dp-brane with tangential dynamics, in the presence of the antisymmetric tensor $B_{\mu\nu}$ and the internal gauge field A_α. Thus, we apply the following closed string action

$$
\begin{aligned}
S = &-\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(\sqrt{-h}\, h^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right. \\
&\left. + \varepsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \\
&+ \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} d\sigma \left(A_\alpha \partial_\sigma X^\alpha + \omega_{\alpha\beta} J_\tau^{\alpha\beta} \right),
\end{aligned}
\tag{2.1}
$$

where the indices $a, b \in \{0, 1\}$ are devoted to the string worldsheet and $\alpha, \beta \in \{0, 1, \ldots, p\}$ belong to the Dp-brane worldvolume. Let the spacetime be flat, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. In addition, the string worldsheet will be flat. The tensors $\omega_{\alpha\beta}$ and $J_\tau^{\alpha\beta} = X^\alpha \partial_\tau X^\beta - X^\beta \partial_\tau X^\alpha$ indicate the tangential angular velocity and the angular momentum density, respectively. The angular velocity $\omega_{\alpha\beta}$, the Kalb–Ramond field $B_{\mu\nu}$ and the field strength of the gauge potential, i.e. $F_{\alpha\beta}$, are taken to be constant, hence, we utilize the gauge $A_\alpha = -\frac{1}{2} F_{\alpha\beta} X^\beta$. Because of the presence of the fields on the brane worldvolume the Lorentz symmetry breaks down, thus, the tangential dynamics along the worldvolume directions obviously is sensible. In the rest of the paper we take $\alpha' = 2$.

The boundary state equations can be obtained by vanishing of the variation of the action with respect to X^μ,

$$
\begin{aligned}
&\left[\left(\eta_{\alpha\beta} + 4\omega_{\alpha\beta} \right) \partial_\tau X^\beta + \mathcal{F}_{\alpha\beta} \partial_\sigma X^\beta \right. \\
&\left. + B_{\alpha i} \partial_\sigma X^i \right]_{\tau=0} |B_x\rangle = 0,
\end{aligned}
\tag{2.2}
$$

$$
\left(X^i - y^i \right)_{\tau=0} |B_x\rangle = 0,
\tag{2.3}
$$

where $\mathcal{F}_{\alpha\beta} \equiv B_{\alpha\beta} - F_{\alpha\beta}$. The Dirichlet directions are shown by $\{x^i | i = p + 1, \ldots, d - 1\}$ and the parameters y^i specify the brane position. One can use the mode expansion of X^μ to rewrite the above equations in terms of the closed string oscillators

$$
\begin{aligned}
&\left[\left(\eta_{\alpha\beta} + 4\omega_{\alpha\beta} - \mathcal{F}_{\alpha\beta} \right) \alpha_m^\beta \right. \\
&\left. + \left(\eta_{\alpha\beta} + 4\omega_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \right) \tilde{\alpha}_{-m}^\beta \right] |B_{\text{osc}}\rangle = 0, \\
&\left(\eta_{\alpha\beta} + 4\omega_{\alpha\beta} \right) p^\beta |B\rangle^{(0)} = 0,
\end{aligned}
\tag{2.4}
$$

for the tangential directions, and

$$
\begin{aligned}
&(\alpha_m^i - \tilde{\alpha}_{-m}^i)|B_{\text{osc}}\rangle = 0, \\
&(x^i - y^i)|B\rangle^{(0)} = 0,
\end{aligned}
\tag{2.5}
$$

for the perpendicular directions to the worldvolume. The following decomposition was also applied $|B_x\rangle = |B_{\text{osc}}\rangle \otimes |B\rangle^{(0)}$.

The second equation of Eq. (2.4) eventuates to $p^\alpha \det(\eta + 4\omega) = 0$. Thus, there are two possibilities depending on whether $\left(\eta_{\alpha\beta} + 4\omega_{\alpha\beta} \right)$ is invertible or not. We consider the invertible case which leads to the vanishing tangential momentum $p^\alpha = 0$. Hence, by applying the commutation relations and the coherent state formalism we find the zero-mode part and oscillatory sector of the boundary state as follows

$$|B\rangle^{(0)} = \frac{T_p}{2} \prod_{i=p+1}^{d-1} \delta\left(x^i - y^i\right) |p^i = 0\rangle \prod_{\alpha=0}^{p} |p^\alpha = 0\rangle,$$

(2.6)

$$|B_{\text{osc}}\rangle = \sqrt{-\det M} \exp\left[-\sum_{m=1}^{\infty} \left(\frac{1}{m} \alpha_{-m}^\mu S_{\mu\nu} \tilde{\alpha}_{-m}^\nu \right) \right] |0\rangle_\alpha$$

$$\otimes |0\rangle_{\tilde{\alpha}},$$

(2.7)

where T_p is the brane tension, and the matrix $S_{\mu\nu}$ is defined by

$$S_{\mu\nu} = \left(Q_{\alpha\beta} \equiv (M^{-1}N)_{\alpha\beta}, -\delta_{ij} \right),$$
$$M_{\alpha\beta} = \eta_{\alpha\beta} + 4\omega_{\alpha\beta} - \mathcal{F}_{\alpha\beta},$$
$$N_{\alpha\beta} = \eta_{\alpha\beta} + 4\omega_{\alpha\beta} + \mathcal{F}_{\alpha\beta}.$$

(2.8)

The prefactor in the oscillating part comes from the normalization of the disk partition function. For prefactors of the stationary setups see, e.g., Refs. [34,35]. One may define the combination $\mathcal{T}_p = T_p \sqrt{-\det M}$ as an effective tension for the dynamical brane in the presence of the internal and background fields.

In fact, the coherent state method enabled us to acquire the boundary state (2.7) under the condition $SS^T = 1$. This condition reduces the number of the total parameters from $3p(p+1)/2$ to $p^2 - 1$ independent parameters.

In addition to the foregoing sectors of the boundary state, there also exists a contribution from the conformal ghosts too

$$|B_{\text{gh}}\rangle = \exp\left[\sum_{m=1}^{\infty} (c_{-m}\tilde{b}_{-m} - b_{-m}\tilde{c}_{-m}) \right]$$

$$\frac{c_0 + \tilde{c}_0}{2} |q = 1\rangle |\tilde{q} = 1\rangle.$$

(2.9)

Therefore, the total bosonic boundary state, corresponding to the Dp-brane, is given by

$$|B\rangle = |B_{\text{osc}}\rangle \otimes |B\rangle^{(0)} \otimes |B_{\text{gh}}\rangle.$$

(2.10)

2.2 The amplitude of interaction

The interaction amplitude of two parallel Dp-branes enables us to extract the partition function, which will be required for computing the LREE. For calculating the interaction amplitude we can look at the one-loop diagram of an open string, stretched between the branes, or equivalently study the tree-level diagram of the exchanged closed string. This equivalence is a consequence of the conformal invariance of string theory.

Here, we apply the second approach in which the interaction amplitude is given by the overlap of the two boundary states, corresponding to the two dressed-dynamical Dp-

branes, via the closed string propagator D,

$$\mathcal{A} = \langle B_1 | D | B_2 \rangle,$$
$$D = 4 \int_0^\infty dt \, e^{-tH},$$

(2.11)

where H is the closed string Hamiltonian. Accordingly, the interaction amplitude finds the feature

$$\mathcal{A} = \frac{T_p^2 \, V_{p+1}}{8(2\pi)^{d-p-1}} \sqrt{\det(M_1^T M_2)} \int_0^\infty dt \left[e^{(d-2)\pi t/6} \right.$$

$$\times \left(\sqrt{\frac{1}{2t}} \right)^{d-p-1} \exp\left(-\frac{1}{8\pi t} \sum_{i=p+1}^{d-1} \left(y_1^i - y_2^i \right)^2 \right)$$

$$\left. \times \prod_{n=1}^{\infty} \left(\det[1 - Q_1^T Q_2 e^{-4n\pi t}]^{-1} \left(1 - e^{-4n\pi t} \right)^{p-d+3} \right) \right],$$

(2.12)

where V_{p+1} is the brane worldvolume. The first exponential comes from the zero-point energy, the next factor of it originates from the zero-modes of the Dirichlet directions, and the second exponential specifies the dependence on the distance of the branes. Furthermore, the factor $\prod_{n=1}^{\infty}(1 - e^{-4n\pi t})^{p-d+3}$ is due to the oscillators of the Dirichlet directions and the conformal ghosts, while the second determinant originates from the oscillators of the Neumann directions. We observe that the interaction amplitude is exponentially damped by the square distance of the branes. Note that analogous analysis in the presence of an additional background field (i.e. the tachyon field) has been worked out in Ref. [36]. Beside, similar results for a setup without rotation have been found in Ref. [37]. For more investigation also see Refs. [15–32].

3 LREE corresponding to a Dp-brane

3.1 Entanglement entropy of a bipartite system

Let $|\psi\rangle$ denote the pure state of the whole composite system, including the subsystems A and B. The density operator which is associated to this state is specified by $\rho = |\psi\rangle\langle\psi|$. It satisfies the probability conservation condition $\text{Tr}\rho = 1$. Moreover, the reduced density matrix for the subsystem A is defined by taking the partial trace over the subsystem B as $\rho_A = \text{Tr}_B\rho$.

Among the various quantities for measuring entanglement, the entanglement entropy and the Rényi entropy are more interesting and attractive. The entanglement entropy is given by the von Neumann formula $S = -\text{Tr}(\rho_A \ln \rho_A)$ [38] and the Rényi entropy is defined as $S_n = \frac{1}{1-n} \ln \text{Tr}\rho_A^n$ with $n \geq 0$, $n \neq 1$ [39]. Note that the limit $n \to 1$ of the Rényi entropy gives the entanglement entropy.

3.2 The density operator of the setup

By expanding the exponential part of the state (2.7) we receive a series which elaborates an entanglement between the left- and right-moving parts of the Hilbert space. Since in our configuration all elements of the matrix $S_{\mu\nu}$ are nonzero we have an extremely non-trivial composite system with the left-right entanglement.

For a given boundary state $|B\rangle$, associated with a Dp-brane, we may immediately take the density matrix as $\rho = |B\rangle\langle B|$. Since the inner product $\langle B|B\rangle$ is divergent, see Eq. (3.2) in the limit $\epsilon \to 0$, this choice does not satisfy the condition $\mathrm{Tr}\rho = 1$. Thus, a finite correlation length ϵ is introduced and the density matrix is defined by [40,41],

$$\rho = \frac{e^{-\epsilon H}|B\rangle\langle B|e^{-\epsilon H}}{Z(2\epsilon)}, \tag{3.1}$$

where $Z(2\epsilon)$ is fixed by $\mathrm{Tr}\rho = 1$. Therefore, the amplitude (2.12) conveniently enables us to extract $Z(2\epsilon)$ as in the following

$$\begin{aligned}
Z(2\epsilon) &= \langle B|e^{-2\epsilon H}|B\rangle \\
&= \frac{T_p^2\, V_{p+1}}{8(2\pi)^{d-p-1}}|\det M|\left[e^{(d-2)\pi\epsilon/3}\left(\sqrt{\frac{1}{4\epsilon}}\right)^{d-p-1} \right.\\
&\quad \times \prod_{n=1}^{\infty}\left(\det[\mathbf{1} - Q^{\mathrm{T}}Qe^{-8n\pi\epsilon}]^{-1} \right.\\
&\quad \left.\left. \left(1 - e^{-8n\pi\epsilon}\right)^{p-d+3} \right) \right].
\end{aligned} \tag{3.2}$$

Note that the two interacting boundary states exactly are alike, and their corresponding branes have been located at the same position. Consequently, the y-dependent exponential disappeared and also the indices 1 and 2 were omitted. Hence, $Z(2\epsilon)$ can be manifestly interpreted as the tree-level amplitude which a closed string propagates for the time 2ϵ between the very near Dp-branes.

At first, we shall construct the LREE corresponding to a bare-static brane as a simple system, and then LREE will be computed for a rotating-moving brane in the presence of the Kalb–Ramond field and $U(1)$ gauge potential.

3.3 LREE corresponding to a bare-static brane

For this setup, quench the internal and background fields, and also stop the rotation and linear motion of the brane. Therefore, the partition function (3.2) is simplified with $\det M = -1$ and $Q^{\mathrm{T}}Q = \mathbf{1}$. In this case we call it $Z_{(0)}(2\epsilon)$. For deriving the Rényi entropy we need to compute $\mathrm{Tr}\rho_{\mathrm{L}}^n$ for the real number n, where the subsystem "L" is the left-moving part of the Hilbert space. The replica trick enables

us to accurately calculate $\mathrm{Tr}\rho_{\mathrm{L}}^n$, which yields

$$\mathrm{Tr}\rho_{\mathrm{L}}^n \sim \frac{Z_{(0)}(2n\epsilon)}{Z_{(0)}^n(2\epsilon)} \equiv \frac{Z_{(0)n}(\mathrm{L})}{Z_{(0)}^n}, \tag{3.3}$$

where $Z_{(0)n}(\mathrm{L})$ is called "replicated partition function". By defining $q = e^{-4\pi\epsilon}$, the last relation can be expressed in terms of the Dedekind η-function

$$\eta(q) = q^{1/12}\prod_{m=1}^{\infty}\left(1 - q^{2m}\right).$$

Since in the limit $\epsilon \to 0$ the variable q does not vanish, the open/closed worldsheet duality is employed to go to the open string channel. Using the transformation $4\epsilon \to 1/4\epsilon$ we obtain the new variable $\tilde{q} = \exp\left(-\frac{\pi}{4\epsilon}\right)$ which vanishes at the limit $\epsilon \to 0$. Hence, by expanding the Dedekind η-function for small $\tilde{q}$, we acquire

$$\begin{aligned}
\frac{Z_{(0)n}(\mathrm{L})}{Z_{(0)}^n} &\approx K_0^{1-n}\left(\left(2\sqrt{\epsilon}\right)^{1-n}\sqrt{n}\right)^{d-p-1} \\
&\quad \exp\left[\frac{(d-2)\pi}{48\epsilon}\left(\frac{1}{n} - n\right)\right] \\
&\quad \times \prod_{m=1}^{\infty}\left\{1 + (d-2)\left[-n\, e^{-m\pi/2\epsilon} + e^{-m\pi/2\epsilon n}\right.\right. \\
&\quad \left. - n(d-2)\, e^{-(1+1/n)m\pi/2\epsilon} + \frac{d-1}{2}\, e^{-m\pi/\epsilon n}\right. \\
&\quad \left.\left. + \frac{n^2}{2}\left(d - 2 - \frac{1}{n}\right)\, e^{-m\pi/\epsilon}\right]\right\},
\end{aligned} \tag{3.4}$$

where $K_0 = T_p^2 V_{p+1}/8(2\pi)^{d-p-1}$.

Finally, by taking the limit $n \to 1$ of the Rényi entropy we receive the entanglement entropy as in the following

$$\begin{aligned}
S_{(0)\mathrm{LREE}} &= \lim_{n\to 1}\left[\frac{1}{1-n}\ln\frac{Z_{(0)n}(\mathrm{L})}{Z_{(0)}^n}\right] \\
&\approx \ln K_0 + \frac{d-p-1}{2}\,(2\ln 2 + \ln\epsilon - 1) \\
&\quad +(d-2)\left[\frac{\pi}{24\epsilon} + \left(1 - \frac{\pi}{2\epsilon}\right)e^{-\pi/2\epsilon}\right. \\
&\quad \left. + \frac{3}{2}\left(1 - \frac{\pi}{\epsilon}\right)e^{-\pi/\epsilon}\right],
\end{aligned} \tag{3.5}$$

up to the order $\mathcal{O}\left(\exp\left(-3\pi/2\epsilon\right)\right)$. The first term, i.e. $\ln K_0$, depends on the tension and the worldvolume of the brane. It is related to the boundary entropy of the brane. In Refs. [42,43] similar relations concerning the boundary entropy have been found. However, the second factor denotes the zero-mode contribution which originates from the Dirichlet directions. The other terms are due to the oscillators. The factor -2 in $(d-2)$ comes from the conformal ghosts. The divergence $(d-2)\pi/24\epsilon$ can be justified by the sum over all oscillating

modes $\tilde{\alpha}_n$ which become more and more energetic [11]. For the special case $d = 3$ and $p = 1$, the leading terms (the terms without exponential factors) of Eq. (3.5) are exactly compatible with the result of the Ref. [11].

3.4 LREE corresponding to a dressed-dynamical brane

Now we calculate the LREE regarding a generalized configuration. Therefore, our Dp-brane possesses a tangential dynamics and is dressed by the background field $B_{\mu\nu}$ and the gauge potential A_α. In the previous section we obtained the corresponding boundary state and the associated partition function, i.e. Eq. (3.2). Writing the ratio Z_n/Z^n in terms of the Dedekind η-function, and applying the transformation $4\epsilon \to 1/4\epsilon$ for receiving the open string channel, and finally expanding the η-function for small $\tilde{q}$, give rise to the equation

$$
\frac{Z_n}{Z^n} \approx K^{1-n} \left((2\sqrt{\epsilon})^{1-n} \sqrt{n} \right)^{d-p-1}
$$

$$
\times \exp\left[\frac{(d-2)\pi}{48\epsilon} \left(\frac{1}{n} - n \right) \right]
$$

$$
\times \prod_{m=1}^{\infty} \left\{ 1 + (d-p-3)\left[-ne^{-m\pi/2\epsilon} \right.\right.
$$

$$
+ \frac{n^2}{2}\left(d-p-3-\frac{1}{n} \right) e^{-m\pi/\epsilon}
$$

$$
+ e^{-m\pi/2\epsilon n} - n(d-p-3)e^{-(1+1/n)m\pi/2\epsilon}
$$

$$
\left.\left. + \frac{(d-p-2)}{2}\, e^{-m\pi/\epsilon n} \right] \right\}
$$

$$
\times \prod_{m=1}^{\infty} \left\{ 1 - n\,\mathrm{Tr}(Q^{\mathrm{T}}Q)\, e^{-m\pi/2\epsilon} \right.
$$

$$
- n\,[\mathrm{Tr}(Q^{\mathrm{T}}Q)]^2\, e^{-m\pi(1/\epsilon+1/n\epsilon)}
$$

$$
+ \mathrm{Tr}(Q^{\mathrm{T}}Q)\, e^{-m\pi/2n\epsilon}
$$

$$
+ \frac{n}{2}\left[-\mathrm{Tr}(Q^{\mathrm{T}}Q)^2 + n[\mathrm{Tr}(Q^{\mathrm{T}}Q)]^2 \right]e^{-m\pi/\epsilon}
$$

$$
\left. + \frac{1}{2}\left[\mathrm{Tr}(Q^{\mathrm{T}}Q)^2 + [\mathrm{Tr}(Q^{\mathrm{T}}Q)]^2 \right]e^{-m\pi/n\epsilon} \right\}, \tag{3.6}
$$

where $K = T_p^2 V_{p+1}|\det M|/8(2\pi)^{d-p-1}$. The exponential in the first line and the first infinite product are consequences of the expansion of the η-function. Besides, the second infinite product is due to the expansion of the determinants in the partition function and the replicated one.

The entanglement entropy of this generalized configuration finds the feature

$$
S_{\mathrm{LREE}} = \lim_{n \to 1} \left[\frac{1}{1-n} \ln \frac{Z_n}{Z^n} \right]
$$

$$
\approx \ln K + \frac{(d-p-1)}{2}\, (2\ln 2 + \ln \epsilon - 1)
$$

$$
+ \frac{(d-2)\pi}{24\epsilon} + \left(1 - \frac{\pi}{2\epsilon} \right)[d-p-3
$$

$$
+ \mathrm{Tr}(Q^{\mathrm{T}}Q)\Big]e^{-\pi/2\epsilon}
$$

$$
+ \left(1 - \frac{\pi}{\epsilon} \right)\left[\frac{3}{2}(d-p-3) + \mathrm{Tr}(Q^{\mathrm{T}}Q) \right.
$$

$$
\left. + \frac{1}{2}\mathrm{Tr}(Q^{\mathrm{T}}Q)^2 \right]e^{-\pi/\epsilon}, \tag{3.7}
$$

up to the order $\mathcal{O}\left(\exp(-3\pi/2\epsilon)\right)$. As it can be seen, the second and third phrases are the same as for the bare-static Dp-brane. In addition, the effects of the background and internal fields and the brane dynamics have been prominently accumulated in the Q-dependent terms and $\ln K$. However, by turning off the fields and stopping the brane, Eq. (3.7) is reduced to the entanglement entropy of the bare-static brane, as expected.

3.5 Comparison with a thermal entropy

We can associate the LREE of the dressed-dynamical brane, i.e. Eq. (3.7), to the thermodynamics. This resemblance can be done by defining a temperature which is proportional to the inverse of the infinitesimal parameter ϵ. From this point of view, the limit $\epsilon \to 0$ is equivalent to the high temperature limit of the thermal system. According to the partition function (3.2), the thermodynamical entropy of the system in the limit $\beta = 2\epsilon \to 0$ takes the form

$$
S_{\mathrm{th}} = \beta^2 \frac{\partial}{\partial \beta}\left(-\frac{1}{\beta}\ln Z \right)
$$

$$
\approx \ln K + \frac{(d-p-1)}{2}\left[2\ln 2 + \ln \frac{\beta}{2} - 1 \right] + \frac{(d-2)\pi}{12\beta}
$$

$$
+ \left(1 - \frac{\pi}{\beta} \right)\left[d-p-3 + \mathrm{Tr}(Q^{\mathrm{T}}Q) \right]e^{-\pi/\beta}
$$

$$
+ \left(1 - \frac{2\pi}{\beta} \right)\left[\frac{3}{2}(d-p-3) + \mathrm{Tr}(Q^{\mathrm{T}}Q) \right.
$$

$$
\left. + \frac{1}{2}\mathrm{Tr}(Q^{\mathrm{T}}Q)^2 \right]e^{-2\pi/\beta}, \tag{3.8}
$$

up to the order $\mathcal{O}\left(e^{-3\pi/\beta}\right)$. We observe that this thermal entropy exactly is equal to the LREE which was specified by Eq. (3.7). In fact, these two entropies basically are different quantities. This desirable connection may reveal a close relation between the entanglement entropy and thermodynamic entropy. There are also other works which illustrate such connections. For instance, the Refs. [44–46] provide a relation similar to the first law of thermodynamics via the entanglement entropy.

4 Some simple configurations with $p = 2$

For clarifying our results, we reduce the general complicated case to the D2-brane. Let the brane sit on the x^1x^2-plane.

The matrices for the D2-brane are given by

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & \Omega \\ -v_2 & -\Omega & 0 \end{pmatrix}, \quad \mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & \mathcal{E}_1 & \mathcal{E}_2 \\ -\mathcal{E}_1 & 0 & \mathcal{B} \\ -\mathcal{E}_2 & -\mathcal{B} & 0 \end{pmatrix}, \quad (4.1)$$

where the parameters exhibit the following quantities,

v_1 and v_2: the components of the linear velocity of the brane,

Ω: the angular velocity of the brane,

$\mathcal{E}_1$ and $\mathcal{E}_2$: the components of the total electric field inside the brane,

$\mathcal{B}$: total magnetic field, in which $\mathcal{E}_1 \equiv \mathcal{F}_{01} = F_{01} - B_{01}$ (similarly for $\mathcal{E}_2$), $\Omega = \omega_{12}$ and $\mathcal{B} \equiv \mathcal{F}_{12} = F_{12} - B_{12}$. For more illustration we shall decompose this setup into the following special configurations.

A dressed-boosted D2-brane At first let us only turn on v_1 and $\mathcal{E}_2$. In fact, to have a sensible tangential velocity v_1, presence of the electric field $\mathcal{E}_2$ is inevitable, otherwise, the Lorentz invariance is restored and consequently there is no preferable direction. For this case we obtain

$$\text{Tr}\left(Q^{\text{T}}Q\right) = 3 + \frac{4v_1^2\mathcal{E}_2^2}{1 - v_1^2 - \mathcal{E}_2^2}. \quad (4.2)$$

Hence, the LREE (up to the order $e^{-\pi/\epsilon}$) becomes

$$S_{(1)\text{LREE}} \approx S_0 + \frac{4v_1^2\mathcal{E}_2^2}{1 - v_1^2 - \mathcal{E}_2^2}\left(1 - \frac{\pi}{2\epsilon}\right)e^{-\pi/2\epsilon}, \quad (4.3)$$

where S_0 denotes the LREE for a bare-static D2-brane. It is given by Eq. (3.5) with $p = 2$.

The prefactor of Eq. (2.7) gives rise to the condition $\det M < 0$. On the basis of this, the denominator of Eq. (4.3) for a moving D2-brane does not vanish, i.e. $1 - v_1^2 - \mathcal{E}_2^2 > 15v_1^2$. Thus, the entropy $S_{(1)\text{LREE}}$ for any finite value of $\mathcal{E}_2$ satisfactorily remains finite.

As the second special case, we consider v_1 and $\mathcal{B}$ to be nonzero. Hence, the trace factor is given by
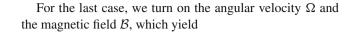
$$\text{Tr}\left(Q^{\text{T}}Q\right) = 3 - \frac{4v_1^2\mathcal{B}^2}{1 - v_1^2 + \mathcal{B}^2}. \quad (4.4)$$

Therefore, the LREE takes the form

$$S_{(2)\text{LREE}} \approx S_0 - \frac{4v_1^2\mathcal{B}^2}{1 - v_1^2 + \mathcal{B}^2}\left(1 - \frac{\pi}{2\epsilon}\right)e^{-\pi/2\epsilon}. \quad (4.5)$$

A dressed-rotating D2-brane Another profitable option is illustrated by turning on the fields and the brane rotation. Again note that the fields are indeed necessary for sensibility of the tangential rotation. At first, consider a rotating D2-brane with the angular velocity Ω which is dressed with $\mathcal{E}_1$. Accordingly, we receive the following LREE

$$S_{(3)\text{LREE}} \approx S_0 - \frac{4\Omega^2\mathcal{E}_1^2}{1 + \Omega^2 - \mathcal{E}_1^2}\left(1 - \frac{\pi}{2\epsilon}\right)e^{-\pi/2\epsilon}. \quad (4.6)$$

For the last case, we turn on the angular velocity Ω and the magnetic field $\mathcal{B}$, which yield

$$S_{(4)\text{LREE}} \approx S_0 + \frac{8\Omega\mathcal{B}}{1 + (\Omega - \mathcal{B})^2}\left(1 - \frac{\pi}{2\epsilon}\right)e^{-\pi/2\epsilon}. \quad (4.7)$$

Note that similar to the finiteness of $S_{(1)\text{LREE}}$, again the condition $\det M < 0$ eventuates to the finiteness of the other three foregoing entropies.

5 Conclusions

At first, we acquired the LREE of a bare-static Dp-brane. Then, the LREE of a rotating-moving Dp-brane in the presence of the Kalb–Ramond background field and an internal $U(1)$ gauge potential was computed. For this purpose, we utilized the boundary state, associated with the Dp-brane, and the interaction amplitude between the two identical and parallel Dp-branes. For the dressed-dynamical brane presence of the various parameters in the setup dedicated a generalized feature to the LREE. By varying the parameters the value of the LREE can be accurately adjusted to any desirable value.

The partition function enabled us to conveniently calculate a reliable thermodynamic entropy. The LREE of the dressed-dynamical Dp-brane was compared with this entropy. We observed that, by redefinition of the temperature, the two entropies exactly are the same. This connection may be useful for the future works. For example, by deriving the LREE for those supersymmetric configurations which represent the black holes, one may find the Bekenstein–Hawking entropy.

Finally, for explicit appearance of the various parameters, we reduced the general case to the D2-brane with either a linear velocity or an angular velocity in the presence of the total electric field $\mathcal{E}$ or the total magnetic field $\mathcal{B}$.

References

1. P. Calabrese, J. Cardy, Int. J. Quantum Inf. **4**, 429 (2006)
2. P. Calabrese, J. Cardy, J. Phys. A **42**, 504005 (2009)
3. V. Balasubramanian, O. Parrikar, Phys. Rev. D **97**, 066025 (2018)
4. M. Rangamani, T. Takayanagi, *Holographic Entanglement Entropy* (Springer International Publishing, Berlin, 2017)
5. S.A. Hartnoll, E. Mazenc, Phys. Rev. Lett. **115**, 121602 (2015)
6. M. Srednicki, Phys. Rev. Lett. **71**, 666–669 (1993)
7. S. Ryu, T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006)
8. S. Ryu, T. Takayanagi, JHEP **0608**, 045 (2006)
9. N. Ishibashi, Mod. Phys. Lett. A **4**, 251 (1989)
10. R. Blumenhagen, E. Plauschinn, Lect. Notes Phys. **779**, 1–256 (2009)
11. L.A. Pando Zayas, N. Quiroz, JHEP **1501**, 110 (2015)
12. M.B. Cantcheff, Phys. Rev. D **80**, 046001 (2009)
13. L.A. Pando Zayas, N. Quiroz, JHEP **1611**, 023 (2016)
14. D. Das, S. Datta, Phys. Rev. Lett. **115**(13), 131602 (2015)
15. P. Di Vecchia, A. Liccardo, N.A.T.O. Adv, Study Inst. Ser. C. Math. Phys. Sci. **556**, 1–59 (2000)
16. P. Di Vecchia, A. Liccardo, YITP Proceedings Series No. 4 (1999)
17. M. Billo, P. Di Vecchia, D. Cangemi, Phys. Lett. B **400**, 63–70 (1997)
18. P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, R. Russo, Nucl. Phys. B **507**, 259–276 (1997)
19. H. Arfaei, M.M. Sheikh Jabbari, Phys. Lett. B **394**, 288–296 (1997)
20. H. Arfaei, D. Kamani, Phys. Lett. B **452**, 54 (1999)
21. H. Arfaei, D. Kamani, Nucl. Phys. B **561**, 57–76 (1999)
22. H. Arfaei, D. Kamani, Phys. Lett. B **475**, 39–45 (2000)
23. D. Kamani, Phys. Lett. B **487**, 187–191 (2000)
24. D. Kamani, Europhys. Lett. **57**, 672–676 (2002)
25. D. Kamani, Phys. Lett. B **580**, 257–264 (2004)
26. D. Kamani, Eur. Phys. J. C **26**, 285–291 (2002)
27. D. Kamani, Mod. Phys. Lett. A **15**, 1655–1664 (2000)
28. D. Kamani, Ann. Phys. **354**, 394–400 (2015)
29. D. Kamani, Nucl. Phys. B **601**, 149–168 (2001)
30. F. Safarzadeh-Maleki, D. Kamani, Phys. Rev. D **90**, 107902 (2014)
31. E. Maghsoodi, D. Kamani, Nucl. Phys. B **922**, 280 (2017)
32. E. Maghsoodi, D. Kamani, Nucl. Phys. B **942**, 381 (2019)
33. L. Bombelli, R.K. Koul, J. Lee, R.D. Sorkin, Phys. Rev. D **34**, 373–383 (1986)
34. C.G. Callan, C. Lovelace, C.R. Nappi, S.A. Yost, Nucl. Phys. B **308**, 221–284 (1988)
35. E. Bergshoeff, E. Sezgin, C.N. Pope, P.K. Townsend, Phys. Lett. B **188**, 70 (1987)
36. F. Safarzadeh-Maleki, D. Kamani, Phys. Rev. D **89**, 026006 (2014)
37. M. Saidy-Sarjoubi, D. Kamani, Phys. Rev. D **92**, 046003 (2015)
38. J. von Neumann, Gott. Nachr. **273** (1927)
39. A. Renyi, On measures of entropy and information, in *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability* vol. 1, pp. 547–561 (1961)
40. T. Takayanagi, T. Ugajin, JHEP **1011**, 054 (2010)
41. J. Cardy, Theory. Phys. Rev. Lett. **112**, 220401 (2014)
42. J.A. Harvey, S. Kachru, G.W. Moore, E. Silverstein, JHEP **03**, 001 (2000)
43. I. Affleck, A.W.W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991)
44. D.D. Blanco, H. Casini, L.Y. Hung, R.C. Myers, JHEP **1308**, 060 (2013)
45. G. Wong, I. Klich, L.A. Pando Zayas, D. Vaman, JHEP **1312**, 020 (2013)
46. J. Bhattacharya, M. Nozaki, T. Takayanagi, T. Ugajin, Phys. Rev. Lett. **110**, 091602 (2013)