Exact solutions of Einstein-æther gravity in Bianchi type V cosmology

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Abstract We present the solution space of the field equations in the Einstein-æther theory for the case of a vacuum Bianchi Type V space-time. We also find that there are portions of the initial parameters space for which no solution is admitted by the reduced equations. Whenever solutions do exist, their physical interpretation is examined through the behavior of Ricci and/or Kretschmann scalar, as well as with the identification of the effective energy momentum tensor in terms of a perfect fluid. There are cases in which no singularities appear and others where the effective fluid is isotropic.

1 Introduction

Bianchi spacetimes contain various important cosmological models that have been used to describe the anisotropies of primordial universe and the evolution to an isotropic universe as observed in the present time [1–4]. In Bianchi models the spacetime manifold is foliated along the time axis, with three dimensional homogeneous hypersurfaces [5]. The corresponding geometries are thus spatially homogeneous, which means that the physical variables depend on the time variable only, reducing the Einstein and other governing equations to coupled ordinary differential equations. In total there are nine Bianchi models which are defined by the possible nine different three-dimensional algebras constructed from the isometries of the metric tensor (see for example [5,6]).

In the context of GR, the field equations for Bianchi spacetimes admit exact solutions, for instance see [7–17] and references therein. In the case where a matter source is included

in Einstein's GR, such as electromagnetic fluid, ideal gas and scalar field, some exact solutions were previously determined in [18–27].

In this work we are interested in the exact solutions of field equations for Bianchi V spacetime in Einstein-æther gravity [28,29]. This is a Lorentz violating gravitational theory, because a preferred frame is introduced through the æther field present in the gravitational action integral [30]. Mathematically, the æther field is described by a unit time-like vector field, with its kinematic quantities coupled to gravity in the Einstein-Hilbert action. By definition Einstein-æther gravity is a second-order theory however, because of the existence of nonlinear terms in the æther field, there are very few exact solutions. For applications of Einstein-æther gravity in cosmological studies we refer the reader to [31–35].

Recently, in [36] exact solutions were found for the case of Friedmann–Lemaître–Robertson–Walker (FLRW) and LRS Bianchi III spacetimes in Einstein-æther theory. It was there proved that exact solutions exist only for specific ranges of values of the free coupling parameters for the æther field. These exact solutions have been applied in [37] for the determination of inhomogeneous cosmological solutions in Einstein-æther theory. Other studies which can be found in the literature on anisotropic Einstein-æther gravitational models are based on the method of the critical point analysis, where special solutions are determined at the critical points which describe asymptotic behavior for the evolution of the field equations [38–42]. The plan of the paper is as follows:

In Sect. 2 we briefly present the field equations in Einstein-æther gravity. Section 3 includes the main material of our study where we present the solution of the field equations for the case where the æther field is either parallel to the comoving observer (Class A), or tilted (Class B). For these two classes we find the relations between the coupling constants c_i of the theory in order that the field equations do admit solutions. For both Classes there are anisotropic solu-

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tions, whereas in the first class, where the aether field is parallel to the comoving observer, there is an isotropic solution which corresponds to a FLRW spacetime with nonzero spatial curvature. Furthermore, the physical description of the corresponding energy momentum tensor is calculated in all cases and is then used to understand the physical properties of the resulting spacetimes. Finally, at Sect. 4 we draw our conclusions.

2 Einstein-æther gravity

The action integral in Einstein-æther gravity is

$$S_{AE} = \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} (K^{\alpha\beta\mu\nu} u_{\mu;\alpha} u_{\nu;\beta} + \lambda (u^\alpha u_\alpha + 1)), \quad (2.1)$$

where u^α is the aether field i.e. a unit time-like vector field, which defines the preferred frame. The Lagrange multiplier λ ensures the unitarity of the aether field, $u^\alpha u_\alpha = -1$. Finally, the Lorentz violating terms are introduced by the tensor $K^{\alpha\beta\mu\nu}$ defined as

$$K^{\alpha\beta\mu\nu} \equiv c_1 g^{\alpha\beta} g^{\mu\nu} + c_2 g^{\alpha\mu} g^{\beta\nu} + c_3 g^{\alpha\nu} g^{\beta\mu} + c_4 g^{\mu\nu} u^\alpha u^\beta. \quad (2.2)$$

where c_1, c_2, c_3 and c_4 are the dimensionless coupling parameters of the aether field with the gravity.

In [43] the first two terms of action integral (2.1) have been written with the use of the kinematic quantities of the aether field in 1 + 3 analysis as follows

$$S_{EA} = \int \sqrt{-g} d^4x \left(R + c_\theta \theta^2 + c_\sigma \sigma^2 + c_\omega \omega^2 + c_a \alpha^2 + \lambda (u^\alpha u_\alpha + 1) \right) \quad (2.3)$$

where

$$\alpha_\mu = u_{\mu;\nu} u^\nu, \quad \theta = \frac{1}{3} u_{\mu;\nu} h^{\mu\nu}, \quad \sigma_{\mu\nu} = u_{(\alpha;\beta)} h_\mu^\alpha h_\nu^\beta - \theta h_{\mu\nu}, \quad (2.4)$$

$$\omega_{\mu\nu} = u_{[\alpha;\beta]} h_\mu^\alpha h_\nu^\beta, \quad \sigma^2 = \sigma^{\mu\nu} \sigma_{\mu\nu}, \quad \omega^2 = \omega^{\mu\nu} \omega_{\mu\nu}, \quad (2.5)$$

with α^μ the acceleration, θ the expansion rate, $\sigma_{\mu\nu}$ the shear tensor, $\omega_{\mu\nu}$ the vorticity tensor and $h_{\mu\nu}$ the projective tensor defined as $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$. The coefficient constants $c_\theta, c_\sigma, c_\omega$ and c_a are related with c_1, c_2, c_3 and c_4 as follows

$$c_\theta = \frac{1}{3} (3c_2 + c_1 + c_3), \quad c_\sigma = c_1 + c_3, \quad c_\omega = c_1 - c_3, \quad c_a = c_4 - c_1. \quad (2.6)$$

The gravitational field equations of the Einstein-æther theory are obtained by variation of (2.1) with respect to the metric $g^{\mu\nu}$, fielding

$$G_{\mu\nu} = T_{\mu\nu}^{\text{æ}}, \quad (2.7)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}^{\text{æ}}$ is the effective æther energy-momentum tensor which is defined as

$$\begin{aligned} T_{\mu\nu}^{\text{æ}} = & \frac{1}{2} g_{\mu\nu} K^{\alpha\beta\rho\sigma} u_{\rho;\alpha} u_{\sigma;\beta} + \frac{1}{2} g_{\mu\nu} \lambda (g^{\alpha\beta} u_\alpha u_\beta + 1) \\ & - c_1 g^{\alpha\beta} (u_{\alpha;\mu} u_{\beta;\nu} + u_{\mu;\alpha} u_{\nu;\beta}) \\ & - c_2 g^{\alpha\beta} (u_{\mu;\nu} u_{\alpha;\beta} + u_{\beta;\alpha} u_{\nu;\mu}) \\ & - c_3 g^{\alpha\beta} (u_{\alpha;\mu} u_{\nu;\beta} + u_{\mu;\alpha} u_{\nu;\beta}) \\ & - c_4 u^\alpha u^\beta u_{\mu;\alpha} u_{\nu;\beta} \\ & - c_4 (g^{\lambda\alpha} u_\mu u^\beta u_{\lambda;\nu} u_{\alpha;\beta} + g^{\beta\lambda} u^\alpha u_\nu u_{\beta;\alpha} u_{\lambda;\mu}) - \lambda u_\mu u_\nu \\ & + \left(\frac{1}{2} K^{\alpha\beta\lambda\kappa} u^\rho u_{\kappa;\beta} g_{\rho\mu} g_{\lambda\nu} + \frac{1}{2} K^{\alpha\beta\lambda\kappa} u^\rho u_{\kappa;\beta} g_{\rho\nu} g_{\lambda\mu} \right)_{;\alpha} \\ & + \left(\frac{1}{2} K^{\alpha\beta\lambda\kappa} u^\rho u_{\kappa;\beta} g_{\rho\mu} g_{\alpha\nu} + \frac{1}{2} K^{\alpha\beta\lambda\kappa} u^\rho u_{\kappa;\beta} g_{\rho\nu} g_{\alpha\mu} \right)_{;\lambda} \\ & - \left(\frac{1}{2} K^{\alpha\beta\lambda\kappa} u^\rho u_{\kappa;\beta} g_{\lambda\mu} g_{\alpha\nu} + \frac{1}{2} K^{\alpha\beta\lambda\kappa} u^\rho u_{\kappa;\beta} g_{\lambda\nu} g_{\alpha\mu} \right)_{;\rho}. \end{aligned} \quad (2.8)$$

The equation of motion for the aether field follows by varying u_a

$$c_4 g^{\mu\nu} u^\alpha u_{\nu;\beta} u_{\mu;\alpha} g^{\kappa\beta} - c_4 g^{\mu\kappa} g^{\alpha\lambda} u_{\lambda;\beta} u^\beta u_{\mu;\alpha} - c_4 g^{\mu\kappa} u^\alpha u_{\beta;\mu} u_{\alpha;\kappa} - K^{\alpha\beta\mu\kappa} u_{\mu;\alpha;\beta} + \lambda g^{\alpha\kappa} u_\alpha = 0 \quad (2.9)$$

while its constraining unit condition is obtained by varying λ

$$u^a u_a + 1 = 0. \quad (2.10)$$

The above equations make it clear that Einstein-æther gravity is a second-order theory; in a four dimensional manifold this system comprises fifteen equations. It is important to mention that in our consideration we have not assumed any matter content and the effective æther energy-momentum tensor is produced from the second term of the action (2.1).

2.1 Physical interpretation

In order to assign a possible physical meaning to the solutions obtained we calculate, on mass-shell, the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ and interpret it as an effective energy-momentum tensor of a perfect fluid by writing $G_{\mu\nu} = T_{\mu\nu}^{(eff)}$. The energy momentum tensor in the 1+3 decomposition can be written as follows

$$T_{\mu\nu}^{(eff)} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \pi_{\mu\nu}, \quad (2.11)$$

where ρ is the energy density of the fluid, u_μ the 4-velocity, q_μ the heat flux vector, p the pressure and $\pi_{\mu\nu}$ the anisotropic

stress tensor. The relations that make the identification possible are

$$\Pi_{\mu\nu} = G_{\alpha\beta} h^\alpha_\mu h^\beta_\nu = p h_{\mu\nu} + \pi_{\mu\nu},$$

$$\pi_{\mu\nu} = \Pi_{\mu\nu} - \frac{1}{3} \Pi^\kappa_\kappa h_{\mu\nu} = \Pi_{\mu\nu} - p h_{\mu\nu} \quad (2.12)$$

$$\rho = G_{\mu\nu} u^\mu u^\nu, \quad p = \frac{1}{3} \Pi^\kappa_\kappa \quad (2.13)$$

$$q_\nu = -G_{\alpha\beta} u^\alpha h^\beta_\nu, \quad (2.14)$$

in which $h_{\mu\nu}$ is the projection tensor orthogonal to velocity u^μ defined by

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad \text{with } u_\mu u^\mu = -1 \quad (2.15)$$

In the next section we begin our analysis by selecting the underlying geometry to be that of a Bianchi Type V space-time; in this case the field equations are reduced to ordinary, coupled differential equations with time as the independent variable.

3 Solution space

The general diagonal Bianchi Type V line element is

$$ds^2 = -M(t)^2 dt^2 + a(t)^2 b(t)^2 dx^2 + e^{-2x} \left(a(t)^4 dy^2 + b(t)^4 dz^2 \right), \quad (3.1)$$

with the corresponding isometry generating vector fields:

$$\xi_1 = \frac{\partial}{\partial y}, \quad \xi_2 = \frac{\partial}{\partial z}, \quad \xi_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (3.2)$$

In order to find a compatible initial form for the æther vector field we first note that it should commute with the above generators, which dictates that $u_\mu = \{u_0(t), u_1(t), e^{-x} u_2(t), e^{-x} u_3(t)\}$. A further restriction is brought about by the assumption that it should be curl-free, i.e. $u_{\mu;\nu} - u_{\nu;\mu} = 0$. Of course, this extra restriction kills the possibly interesting longitudinal degree of freedom of the æther field, but it also has several advantages: It ensures that in the solutions found u_μ (thus also u^μ) will be orthogonal to the hyper-surfaces defined by the corresponding potential, thus allowing the recovery of metrics with higher symmetry (e.g. FLRW in an appropriate limit) in which this condition is mandatory; It also makes feasible the possible contact with Horava gravity in which the existence of a physical time T (here identifiable to the aforesaid potential) is of main importance; Last, but not least, it simplifies the (otherwise intractable) equations to be solved. These requirements, along with the use of (2.10), result in the following initial form (λ_1 is an integration constant):

$$u^a = \frac{u_0(t)}{-M^2} \delta^a_t + \frac{\lambda_1}{a(t)^2 b(t)^2} \delta^a_x,$$

$$u_0(t) = \sqrt{a(t)^2 b(t)^2 + \lambda_1^2}. \quad (3.3)$$

We next make use of the inherent time re-parameterization covariance and select the time gauge so that $M(t) = a(t)b(t)$; thus, the second component of the Eq. (2.9) assumes the form

$$\lambda_1 a^{-6} b^{-6} \left((c_1 + c_3) \left(-b^2 a'^2 - ab(ba'' + 4a'b') \right) + a^2 \left(-bb'' - b'^2 + 2b^2 \right) \right) + a(t)^4 b^4 \lambda, \quad (3.4)$$

implying that we have to separately investigate the two classes of solutions where: Class A when $\lambda_1 = 0$ and Class B when $\lambda_1 \neq 0$.

The general approach we adopt in order to reveal the solution space is to algebraically solve two of the equations in terms of the accelerations $a''(t)$, $b''(t)$ and substitute the result into the rest. In doing so some particular branches appear when the denominators of the corresponding expressions vanish. This may happen either for specific value ranges of the constants or for particular relations among $a(t)$, $b(t)$. In what follows we present all the cases that emerge.

3.1 Class A solutions

The strategy for uncovering the solution space is, in a way, similar to that adopted in [36]: one algebraically solves three of the ensuing equations in terms of $\lambda(t)$, $a''(t)$, $b''(t)$ and substitutes into the rest, thus obtaining a number of equations involving only up to first time derivatives. The consistency of these equations along with the vanishing of the various denominators appearing in the entire process imply that the existence of the solution to the field equations is directly related to the ranges of values of the coefficients c_1 , c_2 , c_3 and c_4 of the theory. In this Class there are three possible cases of study, corresponding to

$$c_1 + 3c_2 + c_3 - 2 = 0 \quad (\text{Case } A_1)$$

$$c_1 + 3c_2 + c_3 - 2 \neq 0 \quad q = 0 \quad (\text{Case } A_2)$$

$$c_1 + 3c_2 + c_3 - 2 \neq 0 \quad q \neq 0 \quad (\text{Case } A_3)$$

where $q = \frac{1}{5}(5c_1 + 9c_2 + 5c_3 - 4)$. The only non-zero component of (2.9) fixes the Lagrange multiplier to $\lambda = \frac{1}{5a(t)^4 b(t)^4} \left(a(t)b(t) \left(2(6c_2 + 5q + 4)a'(t)b'(t) - 15c_2 a(t)b''(t) \right) - 5b(t)^2 \left(3c_2 a(t)a''(t) + (3c_2 - 5q - 4)a'(t)^2 \right) + 5a(t)^2 (-3c_2 + 5q + 4)b'(t)^2 \right)$ which is then substituted into (2.7), giving the final set of equations to be solved in this class.

3.1.1 Case A_1

The assumption leads to $q = \frac{6(1-c_2)}{5}$ and substituting in (2.7) we calculate the difference $(0, 0) - (1, 1) = -4$. So there is no solution in this case.

3.1.2 Case A_2

The assumptions of this case make the $(0, 0)$ component of (2.7)

$$\frac{12}{5}(1-c_2)\frac{a'b'}{ab} - 1 = 0, \quad (3.5)$$

indicating that $c_2 \neq 1$ must hold for solutions to exist. As we can easily see the above equation admits the scaling symmetry $a \rightarrow \omega_1 a$, $b \rightarrow \omega_2 b$; thus, if we make the replacement

$$\begin{aligned} a(t) &= \exp \left[\int (a_1(t) + b_1(t)) dt \right], \\ b(t) &= \exp \left[\int (a_1(t) - b_1(t)) dt \right], \end{aligned} \quad (3.6)$$

the aforementioned equation transforms to

$$\frac{12}{5}(1-c_2)(a_1(t)^2 - b_1(t)^2) - 1 = 0, \quad (3.7)$$

which, being a quadratic form in $a_1(t), b_1(t)$, can be parametrized by

$$\begin{aligned} a_1(t) &= \frac{m}{2} \sinh(f(t)), \quad b_1(t) = \frac{m}{2} \cosh(f(t)), \\ m &= \sqrt{\frac{5}{3(c_2 - 1)}}. \end{aligned} \quad (3.8)$$

If we substitute the above forms of $a_1(t), b_1(t)$ into the rest of (2.7) we find that a solution exists if $f(t)$ satisfies the differential equation

$$2 \cosh f(t) (2m \cosh f(t) + f'(t)) = 0, \quad (3.9)$$

with solutions

$$\begin{aligned} f(t) &= \frac{i\pi}{2} (2\kappa + 1), \quad \kappa \in \mathbb{Z}, \quad \text{or} \\ f(t) &= -2 \tanh^{-1} (\tan(m(t - t_0))), \end{aligned} \quad (3.10)$$

- If $f(t) = \frac{i\pi}{2} (2\kappa + 1)$, the solution is real if $c_2 < 1$ and then $a(t), b(t)$ becomes

$$a(t) = b(t) = \exp \frac{\epsilon m}{2} t, \quad \epsilon = \pm 1, \quad (3.11)$$

yielding the line element

$$ds^2 = e^{2\epsilon m t} (-dt^2 + dx^2 + e^{-2x} dy^2 + e^{-2x} dz^2). \quad (3.12)$$

The Riemann tensor is zero when $c_2 = -\frac{2}{3}$, so we obtain the Minkowski space-time in this case. The corresponding Ricci scalar is

$$R = -\frac{2(3c_2 + 2)}{c_2 - 1} \exp(-2\epsilon m t) \quad (3.13)$$

which has a curvature singularity at $t \rightarrow \pm\infty$ depending on the value of ϵ . For this solution the physical components of the energy momentum tensor describe a perfect fluid with energy density and pressure component as follows

$$\rho = \frac{3c_2 + 2}{c_2 - 1} \exp(-2\epsilon m t), \quad p = -\frac{1}{3}\rho. \quad (3.14)$$

- If $f(t) = -2 \tanh^{-1} (\tan(m(t - t_0)))$, the solution of the field equations is given by the functions

$$\begin{aligned} a(t) &= \sqrt{\sin(m(t - t_0)) + \cos(m(t - t_0))}, \\ b(t) &= \sqrt{\sin(m(t - t_0)) - \cos(m(t - t_0))}, \end{aligned} \quad (3.15)$$

Hence, the line element reads

$$\begin{aligned} ds^2 &= \cos(2m(t - t_0)) (dt^2 - dx^2) \\ &\quad + e^{-2x} (1 + \sin(2m(t - t_0))) dy^2 \\ &\quad - e^{-2x} (1 - \sin(2m(t - t_0))) dz^2 \end{aligned} \quad (3.16)$$

with $c_2 > 1$. The corresponding Ricci scalar is $(3(m^2 + 1) \cos(4m(t - t_0)) + 7m^2 + 3) \sec^3(2m(t - t_0))$ which develops curvature singularities at finite intervals.

It is interesting to note that as the argument $\tau \equiv 2m(t - t_0)$ spans the prime interval $[0, 2\pi]$ the role of the coordinates t, x as time-like and space-like respectively is interchanged (i.e. when $\frac{\pi}{2} < \tau < \frac{3\pi}{2}$ t is time-like while when $\frac{3\pi}{2} < \tau < \frac{\pi}{2}$ x is time-like). The g_{yy} component is always positive (or zero for $\tau = 0, 2\pi$) with the g_{zz} component exhibiting the opposite behaviour. Thus, in the entire prime interval of τ , the signature of the line element is neutral.

The norms $|\xi_i|^2 \equiv g_{\mu\nu} \xi_i^\mu \xi_i^\nu$ of the Killing vectors are $|\xi_1|^2 = e^{-2x} (\sin(\tau) + 1)$, $|\xi_2|^2 = e^{-2x} (\sin(\tau) - 1)$, $|\xi_3|^2 = e^{-2x} (-e^{2x} \cos(\tau) + \sin(\tau) (y^2 + z^2) + y^2 - z^2)$. The first two vanish only at the endpoints of the prime interval $\tau = \frac{3\pi}{2}, \frac{\pi}{2}$ respectively, while the third becomes zero inside the interval of interest. All these can be suggestive that horizons or domain walls exist; However, the neutral character of the signature somewhat obscures the issue.

The above line element corresponds to a perfect fluid

solution with

$$\rho = \frac{3 - m^2 + 3(1 + m^2) \cos(4m(t - t_0))}{2 \cos^3(m(t - t_0))} \quad (3.17)$$

$$p = -\frac{1 + 5m^2 + (1 + m^2) \cos(4m(t - t_0))}{2 \cos^3(m(t - t_0))} \quad (3.18)$$

with equation of state parameter

$$w = -\frac{1 + 5m^2 + (1 + m^2) \cos(4m(t - t_0))}{3 - m^2 + 3(1 + m^2) \cos(4m(t - t_0))}. \quad (3.19)$$

3.1.3 Case A_3

In this case we first solve the (2,2) and (3,3) components of (2.7) in terms of the accelerations $a''(t)$, $b''(t)$. The denominator has the term $6(c_2 - 1) + 5q$, so we have to check what happens if $q = \frac{6}{5}(1 - c_2)$; in this case from the components (0,0) and (1,1) of (2.7) we have $(0, 0) - (1, 1) = -4$, thus we run into incompatibility.

With $q \neq \frac{6}{5}(1 - c_2)$ we substitute the above mentioned accelerations into the equation (2.7). The (0,0) component of the latter equation admits a scaling symmetry $a \rightarrow \omega_1 a$, $b \rightarrow \omega_2 b$; thus, if we make a replacement similar to (3.6), namely $a(t) = \exp\left[\int (-a_1(t) + b_1(t)) dt\right]$, $b(t) = \exp\left[\int (a_1(t) + b_1(t)) dt\right]$, the (0,0) component of the field equations is simplified as

$$\begin{aligned} & \frac{1}{30} a_1(t)^2 (8(9c_2 - 5q - 9)) \\ & - \frac{1}{30} b_1(t)^2 (12(6c_2 + 5q - 6)) = 1. \end{aligned} \quad (3.20)$$

In order to parametrize the above equation, we have to check the case $q = \frac{1}{5}(9c_2 - 9)$. Note that the alternative $(6c_2 + 5q - 6) = 0$ implies $q = \frac{6}{5}(1 - c_2)$ which leads to incompatibility as mentioned in the beginning of this section.

- If $q = \frac{9}{5}(c_2 - 1)$ then the Eq. (3.20) gives

$$b_1(t) = \pm \left(\sqrt{6} \sqrt{1 - c_2} \right)^{-1}. \quad (3.21)$$

Then we substitute the above form of $a(t)$, $b(t)$ into the equations of the accelerations and we arrive at a differential equation for $a_1(t)$

$$a_1'(t) \pm 4 \left(\sqrt{6} \sqrt{1 - c_2} \right)^{-1} a_1(t) = 0, \quad (3.22)$$

which has the solution $a_1(t) = m_1 \exp(\mp 4nt)$ with m_1 an integration constant and $n^{-1} = \sqrt{6(1 - c_2)}$.

Finally the scale factors $a(t)$, $b(t)$ are given by

$$\begin{aligned} a(t) &= \exp \left(2nt + c e^{-4nt} \right), \\ b(t) &= \exp \left(2nt - c e^{-4nt} \right), \end{aligned} \quad (3.23)$$

where $\epsilon^2 = 1$, c is a redefinition of m_1 and $c_2 < 1$. The corresponding Ricci scalar reads

$$R = 2e^{-4nt(\epsilon+2)} \left(64c^2 n^2 - 3e^{8nt}(1 - 4n^2) \right), \quad (3.24)$$

which has curvature singularities at $t \rightarrow \pm\infty$ for $\epsilon = -1$ and at $t \rightarrow -\infty$ for $\epsilon = 1$.

The above solution describes a perfect fluid when $\epsilon = 1$, and a non perfect one for $\epsilon = -1$.

The energy density and the pressure component are given by

$$\begin{aligned} \rho &= e^{-4n(\epsilon+2)t} \left(3e^{8nt}(4n^2 - 1) - 64c^2 n^2 \right), \\ p &= -e^{-4n(\epsilon+2)t} \left(e^{8nt}(4n^2 - 1) + 64c^2 n^2 \right), \end{aligned} \quad (3.25)$$

while the nonzero components of the stress tensor $\pi_{\mu\nu}$ are

$$\begin{aligned} \pi_{yy} &= -\pi_{zz} = -32cn^2(\epsilon - 1) \\ &\exp \left(-4nt - 2x + 4c\epsilon e^{-4nt} \sqrt{2n^2 + 1} \right) \end{aligned} \quad (3.26)$$

- If $(9c_2 - 5q - 9) \neq 0$ Eq. (3.20) can be parametrized as

$$\begin{aligned} a_1(t) &= \frac{\sqrt{15} \cosh(f(t))}{2\sqrt{(9c_2 - 5q - 9)}}, \\ b_1(t) &= \frac{\sqrt{5} \sinh(f(t))}{\sqrt{2(6c_2 + 5q - 6)}}. \end{aligned} \quad (3.27)$$

Now, if we substitute the above values of $a_1(t)$, $b_1(t)$ into the rest of (2.7) we find that a solution exists if $f(t)$ satisfies the differential equation

$$\begin{aligned} & \cosh(f(t)) \left(\sqrt{10}(6c_2 + 5q - 6) f'(t) \right. \\ & \left. + 20\sqrt{6c_2 + 5q - 6} \cosh(f(t)) \right) = 0. \end{aligned} \quad (3.28)$$

- When $\cosh(f(t)) = 0$ we have

$$f(t) = \frac{i\pi}{2} (2\kappa + 1), \quad \kappa \in \mathbb{Z} \quad (3.29)$$

For these functions $f(t)$, the solution of the field equations is determined to be

$$a(t) = b(t) = e^{\epsilon kt}, \quad k = \sqrt{\frac{5}{2(6 - 5q - 6c_2)}}, \quad (3.30)$$

and the line element becomes

$$ds^2 = e^{4\epsilon kt} \left(-dt^2 + dx^2 + e^{-2x} dy^2 + e^{-2x} dz^2 \right). \quad (3.31)$$

In the special case when $q = -\frac{1}{5}(6c_2 + 4)$ the Riemann tensor is 0, and we thus obtain the Minkowski space-time.

For any other value of q the corresponding Ricci scalar is

$$R = -6e^{-4\epsilon kt} (4k^2 - 1), \quad (3.32)$$

which has a curvature singularity at $t \rightarrow \pm\infty$ depending on ϵ , being plus or minus 1. The energy momentum tensor describes a perfect fluid, similarly to the previous case.

- When $\cosh f(t) \neq 0$ the solution of the differential Eq. (3.28) is

$$f(t) = -2 \tanh^{-1} (\tan(A(t - t_0))), \\ A = \sqrt{10(6c_2 + 5q - 6)^{-1}}, \quad (3.33)$$

which implies that,

$$a(t) = \cos^{1/4} 2A(t - t_0) \left(\frac{1 + \tan A(t - t_0)}{1 - \tan A(t - t_0)} \right)^{-B}, \\ b(t) = \cos^{1/4} 2A(t - t_0) \left(\frac{1 + \tan A(t - t_0)}{1 - \tan A(t - t_0)} \right)^B \quad (3.34)$$

with $B = \sqrt{3(8(6 - 5A^2q))^{-1}}$.

The corresponding line element is

$$ds^2 = \cos(2At) \left(-dt^2 + dx^2 + e^{-2x} \left(\left(\frac{1 + \tan At}{1 - \tan At} \right)^{-4B} dy^2 + \left(\frac{1 + \tan At}{1 - \tan At} \right)^{4B} dz^2 \right) \right), \quad (3.35)$$

where we removed t_0 by a redefinition of the coordinate $t \mapsto t + t_0$. It is easy to see that this line element describes a perfect fluid solution.

The Ricci scalar is given by

$$R = - \left(3 + A^2(9 - 32B^2) + 3(A^2 + 1) \cos(4At) \right) \sec^3(2At), \quad (3.36)$$

which diverges for $t = \frac{\pi}{4A}(2\kappa + 1)$, $\kappa \in \mathbb{Z}$.

We proceed our analysis with the derivation of the second-class of solutions in which u^a is not the comoving observer.

3.2 Class B solutions

For the second class of solutions where $\lambda_1 \neq 0$, the strategy is now to first solve the second component of (2.9) for the Lagrange multiplier $\lambda(t)$ obtaining:

$$\lambda(t) = -(c_1 + c_3) \left(-ab^2a'' - 4aba'b'^2a'^2 - a^2bb''^2b'^2 + 2a^2b^2 \right) (ab)^{-4}. \quad (3.37)$$

Next we substitute this $\lambda(t)$ into the first component of (2.9) and also into the components of (2.7).

If, from the resulting equations, we solve the first component of (2.9) and the (2,2) component of (2.7) in terms of the accelerations $a''(t)$, $b''(t)$, the possible vanishing of the following expression

$$(a^2b^2(c_1 + c_3 + 1) + \lambda_1^2(c_1 + c_3)) \\ (a(t)^2b(t)^2(c_1 + 3c_2 + c_3) + 2c_2\lambda_1^2)$$

appearing in the denominators of $a''(t)$, $b''(t)$ must be considered, giving rise to the following cases

$$a^2b^2(c_1 + c_3 + 1) + \lambda_1^2(c_1 + c_3) = 0 \quad c_1 + c_3 + 1 \neq 0 \quad (\text{Case } B_1)$$

$$c_2 = 0 \quad c_1 + c_3 = 0 \quad (\text{Case } B_2)$$

$$a^2b^2(c_1 + 3c_2 + c_3) + 2c_2\lambda_1^2 = 0 \quad c_1 + 3c_2 + c_3 \neq 0, q \neq 0 \quad (\text{Case } B_3)$$

$$a^2b^2(c_1 + 3c_2 + c_3) + 2c_2\lambda_1^2 = 0 \quad c_1 + c_3 \neq 0 \quad (\text{Case } B_4)$$

3.2.1 Case B_1

This case finally results into a, b being both constant functions. When we substitute these constant values in (2.7), we end up with $(0, 0) + (1, 1) = -2(c_1 + c_3 + 1) = 0$ which is impossible; thus there is no solution in this case.

3.2.2 Case B_2

With the above assumption the constrain equation, i.e. the (0,0) component of (2.7), becomes

$$\frac{8a'b'}{ab} + \frac{2a'^2}{a^2} + \frac{2b'^2}{b^2} - 3 = 0. \quad (3.38)$$

As before, the existing scaling symmetry indicates that the replacement

$$\begin{aligned} a(t) &= \exp \left[\frac{1}{2} \int \left(\sqrt{3} \sinh(f(t)) + \cosh(f(t)) \right) dt \right], \\ b(t) &= \exp \left[\frac{1}{2} \int \left(\cosh(f(t)) - \sqrt{3} \sinh(f(t)) \right) dt \right], \end{aligned} \quad (3.39)$$

satisfies the above expression, while the rest of the equations (2.7) are solved if $f(t)$ obeys the first order differential equation

$$-2 \sinh(f(t)) (f'(t) + 2 \sinh(f(t))) = 0. \quad (3.40)$$

Its solutions read

$$f(t) = i\kappa\pi, \kappa \in \mathbb{Z}, \quad \text{or} \quad f(t) = 2 \coth^{-1} e^{2t-m_1}, \quad (3.41)$$

where m_1 is an integration constant.

- If $f(t) = i\kappa\pi$ the solution is

$$a(t) = e^{\frac{t\epsilon}{2}}, \quad b(t) = e^{\frac{t\epsilon}{2}}, \quad (3.42)$$

and the line element becomes

$$ds^2 = e^{2\epsilon t} \left(-dt^2 + dx^2 + e^{-2x} dy^2 + e^{-2x} dz^2 \right), \quad (3.43)$$

with the Riemann tensor equal to zero, indicating the Minkowski space time.

- If $f(t) = 2 \coth^{-1} e^{2t-m_1}$ the solution is:

$$\begin{aligned} a(t) &= e^{-\frac{t}{2}} \left(\frac{e^{m_1} - e^{2t}}{e^{m_1} + e^{2t}} \right)^{\frac{\sqrt{3}}{4}} \sqrt[4]{1 - e^{4t-2m_1}}, \\ b(t) &= e^{-\frac{t}{2}} \left(\frac{e^{m_1} - e^{2t}}{e^{m_1} + e^{2t}} \right)^{-\frac{\sqrt{3}}{4}} \sqrt[4]{1 - e^{4t-2m_1}}, \end{aligned} \quad (3.44)$$

with the corresponding line-element

$$\begin{aligned} ds^2 &= \kappa^2 \sinh(2\tau) \left(d\tau^2 - dx^2 + e^{-2x} \tanh^{\sqrt{3}}(\tau) dy^2 \right. \\ &\quad \left. + e^{-2x} \tanh^{-\sqrt{3}}(\tau) dz^2 \right), \end{aligned} \quad (3.45)$$

where $\kappa^2 = 2e^{-m_1}$ and $\tau = t + \ln(\frac{\kappa}{\sqrt{2}})$.

The Ricci tensor is derived to be $R_{\mu\nu} = 0$; thus the above line element, while being a vacuum of the present theory, also describes a GR vacuum solution. The Kretschmann scalar equals

$$Kr \equiv R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} = \frac{96}{\kappa^4} \text{csch}^6(2\tau), \quad (3.46)$$

signaling the existence of a singularity at $\tau = 0$. Note that κ is a gravitational essential constant and that the time-like character between τ, x is interchanged as τ spans the interval $(-\infty, \infty)$. However the components g_{yy}, g_{zz} of the metric described by the above line element become imaginary for $\tau \in (-\infty, 0)$; consequently, the allowed range is $\tau \in (0, \infty)$ with the singularity residing at $\tau = 0$. In this range the signature is $(+, -, +, +)$ indicating that x is the time-like variable. The norms $|\xi_i|^2 \equiv g_{\mu\nu} \xi_i^\mu \xi_i^\nu$ of the Killing vectors are $|\xi_1|^2 = k^2 e^{-2x} \sinh(2\tau) \tanh^{\sqrt{3}}(\tau)$, $|\xi_2|^2 = k^2 e^{-2x} \sinh(2\tau) \tanh^{-\sqrt{3}}(\tau)$, $|\xi_3|^2 = k^2 \sinh(2\tau) \left(e^{-2x} y^2 \tanh^{\sqrt{3}}(\tau) + e^{-2x} z^2 \tanh^{-\sqrt{3}}(\tau) - 1 \right)$. The first two are positively defined in the allowed range for τ and $x \in (-\infty, +\infty)$, but the third vanishes on the hypersurface $e^{2x} = y^2 \tanh^{\sqrt{3}}(\tau) + z^2 \tanh^{-\sqrt{3}}(\tau)$ indicating a complicated partition of the spacetime in causally disconnected regions. Of course, a more detailed description requires the investigation of the geodesic structure which lies well outside the scope of the present work.

3.2.3 Case B₃

In this case we end up with $a(t)b(t) = \lambda_1 \sqrt{-2c_2} (c_1 + 3c_2 + c_3)^{-1/2}$ and if we substitute this expression into the components of (2.7) we can see that $a(t)$ must be constant, which entails the result $(0, 0) + (1, 1) = -2(c_1 + c_3 + 1) = 0$. However, this relationship makes the rest of Eq. (2.7) incompatible with the assumptions of the given branch; we thus conclude that there is no solution.

The above analysis corresponds to the last case where the denominators vanish. Thus we can now use the forms of the accelerations $a''(t), b''(t)$ and replace them into Eq. (2.7). This action results in the (0,1) component being

$$\begin{aligned} &2\lambda_1 (c_1 + c_3) (\lambda_1 b a' \\ &+ a \left(b \sqrt{a^2 b^2 + \lambda_1^2} + \lambda_1 b' \right)) (ab)^{-3} = 0. \end{aligned} \quad (3.47)$$

Various cases will emerge from the above equation. When $c_1 + c_3 = 0$, the only independent components of (2.7) are the (0,0) and (1,1) which are quadratic in the velocities. If we differentiate these with respect to t , use again the accelerations and subtract appropriate linear combinations of the expressions themselves, we end up with the following two cases:

- If $c_2 \neq 0, c_1 + c_3 = 0, ab = c_b$ then the combination (0,0)-(1,1) of the components of (2.7) reads $-\frac{4c_2\lambda_1^2}{cb^2} - 4 = 0$ implying $cb = i\sqrt{c_2}\lambda_1$ and thus a neutral signature. With this value for cb the (0,0) component of (2.7) becomes

$$-\frac{4a'(t)^2}{a(t)^2} - 1 = 0, \quad (3.48)$$

which has the solution

$$a(t) = e^{m_1 + \frac{it\epsilon}{2}} \quad (3.49)$$

with $\epsilon = \pm 1$, m_1 an integration constant, and all equations have now been satisfied.

At this stage the redefinitions $\left\{ y \rightarrow \mu e^{-2m_1} Y, z \rightarrow \frac{e^{2m_1} Z}{\mu}, t \rightarrow iT, x \rightarrow X, c_2 \rightarrow \frac{\mu^2}{\lambda_1^2} \right\}$ cast the line element into the simple final form

$$ds^2 = \mu^2 \left(-dT^2 - dX^2 + e^{-2T\epsilon - 2X} dY^2 + e^{2T\epsilon - 2X} dZ^2 \right). \quad (3.50)$$

while the aether one-form transform into $u_\mu = \left\{ -\sqrt{\mu^2 - \lambda_1^2}, \lambda_1, 0, 0 \right\}$ which, in order to be real, implies the restriction $\lambda_1^2 < |\mu|^2$.

The above line-element represents a singularity free geometry, on account of having vanishing covariant derivative of its Riemann tensor and all fourteen curvature scalars monomials of the Ricci scalar $R = \frac{8}{\mu^2}$. Note that, since the space admits only one extra Killing field $\xi_4 = (1, 0, \epsilon Y, -\epsilon Z)$, we have the interesting case of a CSI space time [44]

Despite the non physical character of a neutral singnature line element, it is not unreasonable to investigate the resulting effecting $T_{\mu\nu}$ in terms of perfect fluid:

$$\rho = \frac{2}{\mu^2}, \quad p = -\frac{2}{\mu^2}, \quad q_\kappa = 0, \quad \pi_{\mu\nu} = 0 \quad (3.51)$$

inferring that it corresponds to an isotropic perfect fluid with an equation of state $p + \rho = 0$ and zero heat conduction, i.e. it mimics that of a cosmological constant.

- If $c_2 \neq 0$, $c_1 + c_3 = 0$, $ab \neq c_b$ then with the application of the same steps as before we end up with $ab = \text{const.}$, which contradicts the initial assumption of this case. Thus there is no solution here although, of course, the previous solution holds true.

3.2.4 Case B_4

Now we can solve Eq. (3.47) as a differential equation for $b(t)$ obtaining

$$b(t) = \frac{2\lambda_1^2 e^{m_1 + t}}{e^{2t} a(t) - \lambda_1^2 e^{2m_1} a(t)}, \quad (3.52)$$

with m_1 an integration constant.

When this is substituted into (2.7) we get a differential equation for $a(t)$ which leads to the final scale factors

$$\begin{aligned} a(t) &= \frac{m_2 e^{t/2}}{\sqrt{e^{2t} - \lambda_1^2 e^{2m_1}}}, \\ b(t) &= \frac{2\lambda_1^2 e^{m_1 + \frac{t}{2}}}{m_2 \sqrt{e^{2t} - \lambda_1^2 e^{2m_1}}} \end{aligned} \quad (3.53)$$

under the assumption $c_1 + 3c_2 + c_3 = 2$.

The resulting line element is

$$ds^2 = \lambda_1^2 \text{csch}^2 t \left(-dt^2 + dx^2 + e^{-2x} dy^2 + e^{-2x} dz^2 \right), \quad (3.54)$$

where we have used the transformation $t \mapsto t + m_1 + \ln \lambda_1$, $x \mapsto x + m_1^2 + \ln 2$, $y \mapsto 4m_2^{-2} \lambda_1^2 y$, $z \mapsto e^{-2m_1} \lambda_1^{-2} m_2^2 z$.

Furthermore, it holds that $R_{\lambda\mu\nu\kappa} = \frac{1}{\lambda_1^2} (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu})$, revealing the space-time as maximally symmetric with Ricci scalar $R = \frac{12}{\lambda_1^2}$. If we further investigate the physical properties of the energy-momentum tensor for the above solution, we find that

$$\rho = \frac{3}{\lambda_1^2}, \quad p = -\frac{3}{\lambda_1^2}, \quad q_\kappa = 0, \quad \pi_{\mu\nu} = 0 \quad (3.55)$$

inferring that it corresponds to an isotropic perfect fluid with an equation of state $p + \rho = 0$ and zero heat conduction, i.e. it mimics that of a cosmological constant.

4 Conclusion

In this work we studied the existence of exact solutions for the field equations in the Lorentz violating theory known as Einstein-aether gravity. For the background geometry we assumed that of a Bianchi Type V spacetime, with a diagonal scale factor matrix dictated by the vanishing of the G_{tx} constraint equation. Note that this form of the line-element, despite having two independent functions of time, is not Locally Rotationally Symmetric, i.e. it does not admit a fourth Killing field. We have also restricted the form of the aether field by imposing on it the symmetries of the geometry, thus arriving at two possible cases, that of tilted or non-tilted u_μ . For both models of our study we determined the field equations and exhibited their solution space. Like in the previous similar work of ours concerning Bianchi Type III geometry [36], we find here also that exact solutions exist only for specific values of the coupling parameters of the Einstein-aether theory.

When the aether field is parallel to the comoving observer, we found that there exists an exact isotropic solution in case

Table 1 Physically acceptable exact Bianchi V spacetimes in Einstein-aether theory

Case	Condition	Isotropic spacetime	Ricciscalar R
A_{3i}	$(9c_2 - 5q - 9) \neq 0, f(t) = \frac{i\pi}{2}(2\kappa + 1)$	Yes	$-6e^{-4\kappa t}(4\kappa^2 - 1)$
A_{3ii}	$(9c_2 - 5q - 9) \neq 0, f(t) = -2 \tanh^{-1}(\tan(A(t - t_0)))$	No	$-(\rho_0 + \rho_1 \cos(4\rho_2 t)) \sec^3(2\rho_2 t)$
B_{2i}	$c_2 = 0, f(t) = i\kappa\pi$	Yes (Minkowski)	0
B_{2ii}	$c_2 = 0, f(t) = 2 \coth^{-1} e^{2t-m_1}$	No	$\frac{96}{\kappa^4} \csc h^6(2\tau)$
B_3	$c_1 + c_3 = 0, a(t)b(t) = c_b$	No	$\frac{8}{\mu^2}$ (singularity free)
B_4	$c_1 + 3c_2 + c_3 = 2$	Yes	$\frac{12}{\lambda_1^2}$ (singularity free)

A_2 , when $c_1 + 3c_2 + c_3 - 2 \neq 0, q = 0$. In this case, the spacetime reduces to the Friedmann–Lemaître–Robertson–Walker spacetime with non-zero spatial curvature. On the other hand, in case A_3 , when $c_1 + 3c_2 + c_3 - 2 \neq 0, q \neq 0$ both anisotropic and isotropic solutions are found; the latter case being when $(9c_2 - 5q - 9) \neq 0$, an isotropic solution is found.

In a similar way when the aether field is not parallel to the comoving observer anisotropic and isotropic exact solutions are found for specific relations between the coupling constants of the Einstein-aether theory. There are two possible constraining relations for the coupling constants, when exact solutions exist. These occur in case B_2 where $c_2 = 0, c_1 + c_3 = 0$, and in case B_6 where $c_1 + c_3 \neq 0$.

Lastly, we would like to briefly comment on the LRS case: The scale factor matrix is now $\gamma_{\mu\nu} = \text{diag}\{a^2, a^2, b^2\}$ which results in a non-zero G_{tx} component of the Einstein tensor. Now the geometry admits the fourth Killing field $\xi_4 = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$. The aether field remains the same, since it commutes with ξ_4 , and thus the classification according to λ_1 being zero or non-zero is also valid. There is however a substantial difference in the solution space. For example in the class A case ($\lambda_1 = 0$) there is only one family of solutions described by the scale factors $a(t) = m_2 \exp \frac{(\sqrt{2}t)\epsilon}{m_1\sqrt{(2-Q)}}$ and $b(t) = m_1 a(t)$ where $\epsilon = \pm 1, Q = c_1 + 3c_2 + c_3 < 2$ for the solution to be real, with the Minkowski space-time corresponding to the value $Q = 0$. Note also that, if a free electromagnetic field was considered instead of the aether field, the line element used in the present work would lead only to the vacuum solution whereas the LRS element would have non-trivial solutions. All these issues are under consideration and will be presented in a future publication.

Before we end our discussion, a brief word is pertinent, concerning the viability of the exact solutions here found when they are examined in the light of the values and/or value ranges of the aether coupling constants given by observations (see, e.g. [45–47]). In [45] the gravitational wave data of the event GW170817 have been applied, while in [46] the constraints have been produced by using ultra-high energy cosmic rays.

With the use of the results of these works, we conclude that the solution of case A_2 is not viable since c_2 is not in the range of values as given by [45]. For the case A_3 , only the solution with $(9c_2 - 5q - 9) \neq 0$ is physically acceptable, while the exact solution in which $q = \frac{9}{5}(c_2 - 1)$ is rejected by the observations.

As for the family B of solutions, B_2 and B_3 are acceptable solutions if they are seen in their lower limits $c_2 \rightarrow 0$ and $c_1 + c_3 \rightarrow 0$ respectively. Finally, for case B_4 , only the de Sitter solution does not satisfy the constraints. The results are summarized in the Table 1.

Data Availability Statement This manuscript has associated data in a data repository. [Authors' comment: This manuscript has associated data in [arXiv:1911.03660](https://arxiv.org/abs/1911.03660) [gr-qc].]

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