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Analytical investigations on formations of hairy neutral reflecting shells in the scalar-Gauss–Bonnet gravity

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Abstract We study scalarization of spherically symmetric neutral reflecting shells in the scalar-tensor gravity. We consider neutral static massless scalar fields non-minimally coupled to the Gauss–Bonnet invariant. We obtain a relation representing the existence regime of hairy neutral reflecting shells. For parameters unsatisfying this relation, the massless scalar field cannot exist outside the neutral reflecting shell. In the parameter region where this relation holds, we get analytical solutions of scalar field hairs outside neutral reflecting shells.

1 Introduction

One well known property of classical black holes is the famous no hair theorem, which states that spherically symmetric black holes cannot support static scalar field hairs in the asymptotically flat background, see references [1–9] and reviews [10,11]. The belief in this no hair behavior is partly based on the existence of black hole absorbing horizons. According to some candidate quantum-gravity models, quantum effects may prevent the formation of stable black-hole horizons [12–16]. And horizonless compact objects with reflective boundary conditions have been proposed as alternatives to the familiar (classical) black-hole spacetimes [17–23]. So it is interesting to study properties of horizonless reflecting objects.

Interestingly, no hair theorem also holds in such horizonless reflecting object backgrounds. Hod firstly proved that massive static scalar field hairs cannot form in the background of neutral horizonless reflecting objects [24]. This no hair theorem for neutral horizonless reflecting objects was also extended to the case of massless scalar field hairs [25,26]. Considering a positive cosmological constant, it was found that the no hair theorem still holds in the background of neutral horizonless reflecting objects [27]. This no hair theorem for composed system of scalar fields and neutral horizonless reflecting objects was further generalized by including couplings between scalar fields and Ricci curvature [25,28]. However, when horizonless reflecting objects are charged, analytical and numerical results showed that scalar field hairs can exist [29–38]. From front progress, we conclude that no static scalar field hair behavior is a very general property in the background of neutral horizonless reflecting objects.

In other modified gravities, whether static scalar field hairs could exist outside neutral horizonless reflecting objects is a question to be answered. On the other side of black holes, usual ways to introduce scalar hairs are considering stationary scalar fields or adding a confinement to the system [39–48]. Recently, a novel approach to trigger black hole scalar hairs was provided by considering non-minimal couplings between scalar fields and the Gauss–Bonnet invariant [49–55]. Moreover, it was found that this scalar-Gauss–Bonnet coupling can lead to scalar condensations in various black hole models [56–64]. Inspired by these black hole properties, in the background of neutral reflecting compact stars, we have constructed scalar hairy configurations by including scalar-Gauss–Bonnet couplings with numerical methods [65]. In particular, reflecting shell backgrounds usually allow fully analytical studies, which showed that neutral reflecting shells cannot support static scalar hairs [29,30]. As a further step, it is interesting to examine whether scalar fields can condense outside neutral reflecting shells in the model generalized by including scalar-Gauss–Bonnet couplings.

This work is organized as follows. We firstly construct a system with static massless scalar fields outside neutral horizonless reflecting shells in the scalar-Gauss–Bonnet gravity. Then we obtain a relation representing existence regime of hairy shells. For parameters satisfying this relation, we get analytical solutions of scalar field hairs outside neutral reflecting shells. The analytical solutions presented in this paper are valid only in the linearized regime of the scalar fields. At last, we give the main conclusion.

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2 Scalar condensation behaviors around neutral Dirichlet reflecting shells

2.1 A characteristic relation for scalar hairy neutral reflecting shells

We now write down the model with static massless scalar fields non-minimally coupled to the Gauss–Bonnet invariant. The Lagrangian density of this scalar-tensor gravity is described by [49–55]

$$\mathcal{L} = R - |\nabla_\mu \psi|^2 + f(\psi)\mathcal{R}^2_{GB}, \tag{1}$$

where R is the Ricci curvature, $\psi(r)$ is a real scalar field, $f(\psi)$ is the coupling function and $\mathcal{R}^2_{GB}$ is the source term. In the linear regime, without generality, we can take the coupling function in the form

$$f(\psi) = \eta\psi^2 \tag{2}$$

with η describing the coupling strength [51,52]. The source term is the Gauss–Bonnet invariant given by

$$\mathcal{R}^2_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{3}$$

When neglecting matter fields' backreaction on the metric, the Gauss–Bonnet invariant term is

$$\mathcal{R}^2_{GB} = \frac{48M^2}{r^6}. \tag{4}$$

We consider spherically symmetric static neutral spacetimes. The metric ansatz in Schwarzschild coordinates is of the form [52]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{5}$$

where the metric function is $g(r) = 1 - \frac{2M}{r}$ with M corresponding to the ADM mass. The shell radius is imposed at the radial coordinate $r = r_s$. Since we concentrate on the horizonless spacetime, the shell radii satisfy the relation $r_s > 2M$. The spherically symmetric angular coordinates are labeled as θ and ϕ.

With variation methods, we get the exact linearized scalar equation [49–55]

$$\nabla^\nu\nabla_\nu\psi + \eta\mathcal{R}^2_{GB}\psi = 0. \tag{6}$$

By employing the line element (5), the scalar equation takes the form [65]

$$\psi'' + \left(\frac{2}{r} + \frac{g'}{g}\right)\psi' + \frac{\eta\mathcal{R}^2_{GB}}{g}\psi = 0 \tag{7}$$

with $g = 1 - \frac{2M}{r}$ and $\mathcal{R}^2_{GB} = \frac{48M^2}{r^6}$.

In the limit case of $M \ll r_s$, the functions are $g(r) = 1 - \frac{2M}{r} \to 1$ and $g'(r) = \frac{2M}{r^2} \to 0$ as assumed in Refs. [29,31]. In the large-r regime, $g'(r)$ is neglected and $g(r)$ is set to be 1. The presence of a coupling parameter η is crucial for the existence of a non-trivial analytical solution. With the nonzero term $\eta\mathcal{R}^2_{GB} = \frac{48\eta M^2}{r^6}$, η appears in the scalar field equation. It means that we study the scalar condensation in the large η regime. In this shell background, the Eq. (7) can be expressed as

$$\psi'' + \frac{2}{r}\psi' + \frac{48\eta M^2}{r^6}\psi = 0. \tag{8}$$

In order to solve the equation, we need boundary conditions of the scalar field. The asymptotic behavior of the massless scalar field near the infinity boundary is

$$\psi \propto \frac{1}{r} \quad for \quad r \to \infty. \tag{9}$$

So the infinity boundary condition is

$$\psi(\infty) = 0. \tag{10}$$

At the shell radius, we impose Dirichlet reflecting boundary conditions that the scalar field vanishes. So the scalar field condition at the surface is

$$\psi(r_s) = 0. \tag{11}$$

We introduce a new radial function $\tilde{\psi} = \sqrt{r}\psi$. According to (8), $\tilde{\psi}$ satisfies the differential equation

$$r^2\tilde{\psi}'' + r\tilde{\psi}' + \left(-\frac{1}{4} + \frac{48\eta M^2}{r^4}\right)\tilde{\psi} = 0. \tag{12}$$

With relations (9) and (11), we get boundary conditions

$$\tilde{\psi}(r_s) = 0, \qquad \tilde{\psi}(\infty) = 0. \tag{13}$$

From boundary conditions (13), one deduces that the function $\tilde{\psi}$ must possess one extremum point $r = r_{peak}$ in the range (r_s, ∞). At this extremum point, the scalar field satisfies relations [24]

$$\{\tilde{\psi}' = 0 \quad and \quad \tilde{\psi}\tilde{\psi}'' \leqslant 0\} \quad for \quad r = r_{peak}. \tag{14}$$

Relations (12) and (14) yield the following inequality

$$-\frac{1}{4} + \frac{48\eta M^2}{r^4} \geqslant 0 \quad for \quad r = r_{peak}. \tag{15}$$

This inequality can be transformed into

$$\frac{\sqrt{\eta}M}{r^2} \geqslant \frac{1}{8\sqrt{3}} \quad for \quad r = r_{peak}. \tag{16}$$

Considering $r_s \leqslant r_{peak}$, we conclude that scalar hairy shells should satisfy the relation

$$\frac{\sqrt{\eta}M}{r_s^2} \geqslant \frac{1}{8\sqrt{3}}. \tag{17}$$

Table 1 Radii of Dirichlet reflecting scalar hairy shells

i	1	2	3	4	5
r_{si}	$1.1161\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.7658\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.6189\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.5332\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.4755\eta^{\frac{1}{4}}M^{\frac{1}{2}}$

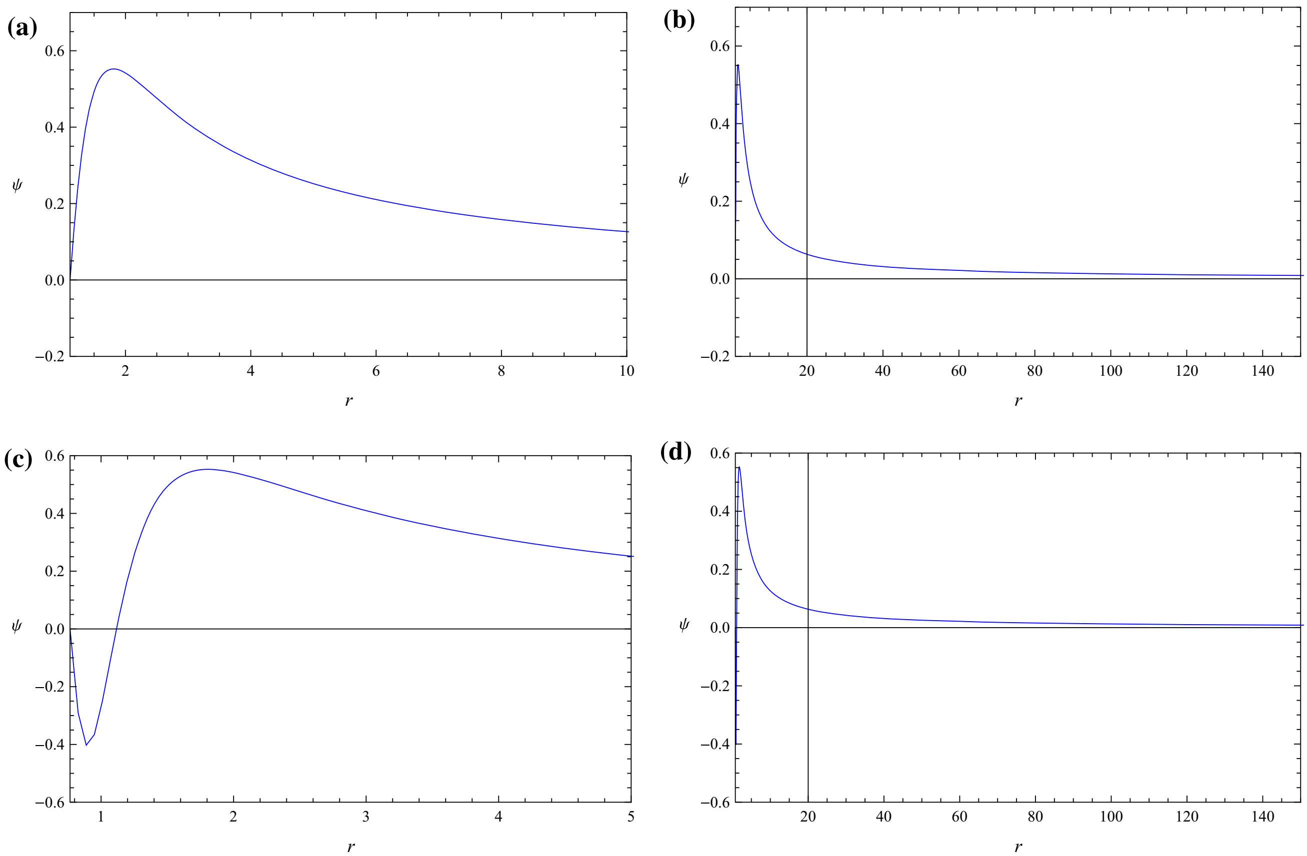

Fig. 1 We plot the scalar field solution around Dirichlet reflecting shells. We take the value $\eta M^2 = 1$ with various r_s as: **a** the case $r_{s1} = 1.1161$ in small region, **b** the case $r_{s1} = 1.1161$ in large region, **c** the case $r_{s2} = 0.7658$ in small region, **d** the case $r_{s2} = 0.7658$ in large region

2.2 Construction of massless scalar field hairy neutral Dirichlet reflecting shells

In this section, we apply analytical methods to get solutions of scalar field hairs outside Dirichlet reflecting shells in the scalar-Gauss–Bonnet gravity. The general solutions of Eq. (12) can be expressed with Bessel functions in the form [66]

$$\tilde{\psi}(r) = A \cdot J_{-\frac{1}{4}}\left(\frac{2\sqrt{3}\eta M}{r^2}\right) + B \cdot J_{\frac{1}{4}}\left(\frac{2\sqrt{3}\eta M}{r^2}\right) \quad (18)$$

with A and B as integral constants.

At the infinity, the solution (18) asymptotically behaves as

$$\tilde{\psi}(r) \propto A \cdot \sqrt{r} + B \cdot \frac{1}{\sqrt{r}}. \quad (19)$$

According to the condition (13), the first coefficient A is zero: $A = 0$. So the bound-state neutral massless scalar fields are

$$\psi = \sqrt{\frac{1}{r}}\tilde{\psi}(r) = B \cdot \sqrt{\frac{1}{r}} J_{\frac{1}{4}}\left(\frac{2\sqrt{3}\eta M}{r^2}\right). \quad (20)$$

With the scalar reflecting condition (11), we get the characteristic scalar field equation

$$J_{\frac{1}{4}}\left(\frac{2\sqrt{3}\eta M}{r_s^2}\right) = 0. \quad (21)$$

If we find parameters satisfying (21), scalar field hairs exist. Defining a new parameter $x = \frac{\sqrt{\eta}M}{r_s^2}$, there is $x \geqslant \frac{1}{8\sqrt{3}}$ according to (17). The remaining question is to solve the equation

$$J_{\frac{1}{4}}(2\sqrt{3}x) = 0 \quad (22)$$

in the region $x \geqslant \frac{1}{8\sqrt{3}}$. With numerical methods, the condition (22) determines discrete values of x_i

$$\cdots > x_3 > x_2 > x_1 = x_{min} \geqslant \frac{1}{8\sqrt{3}}. \quad (23)$$

Table 2 Radii of Dirichlet reflecting scalar hairy shells

i	1	2	3	4	5
r_{si}	$1.8090\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.8992\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.6823\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.5720\eta^{\frac{1}{4}}M^{\frac{1}{2}}$	$0.5022\eta^{\frac{1}{4}}M^{\frac{1}{2}}$

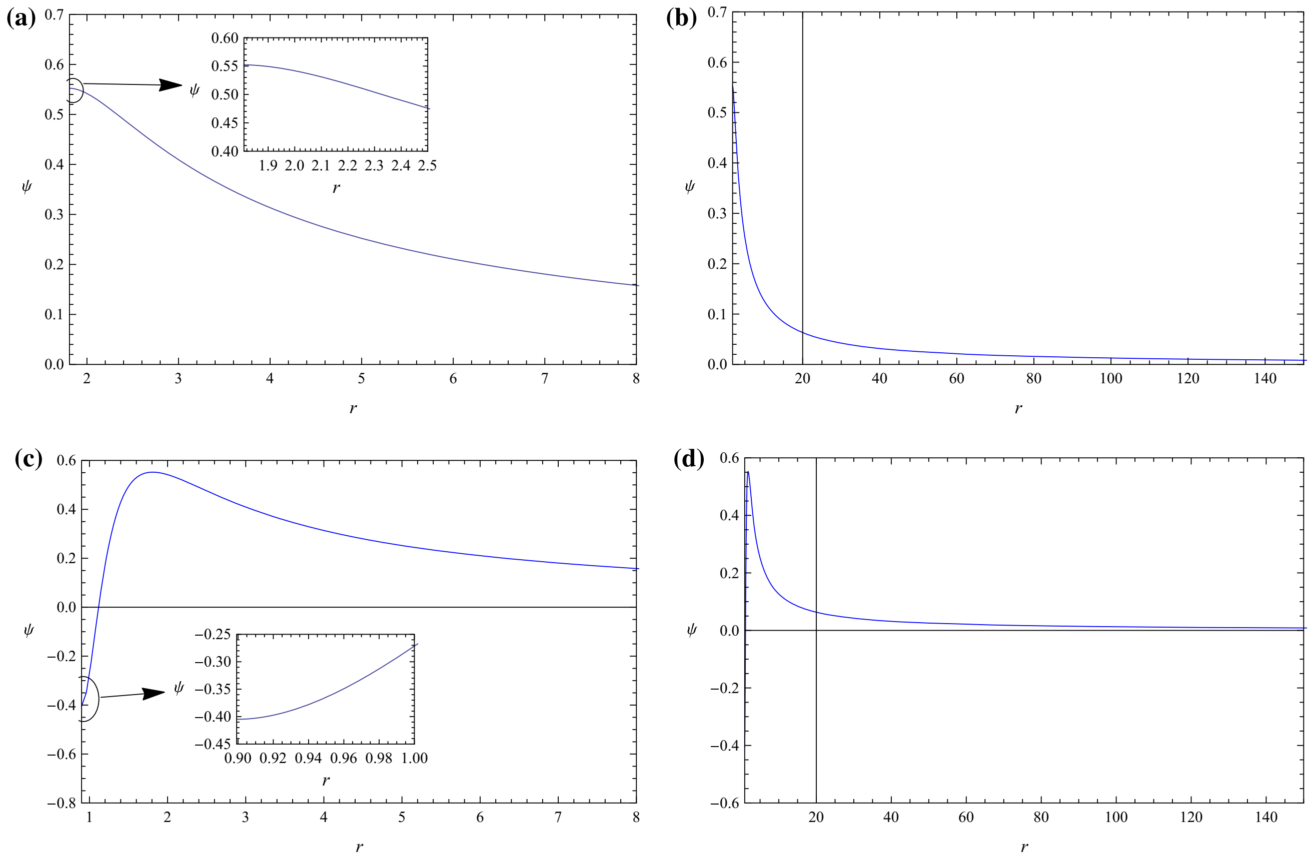

Fig. 2 We show behaviors of the scalar field around Neumann reflecting shells. We take the value $\eta M^2 = 1$ with various r_s as: **a** the case $r_{s1} = 1.8090$ in small region, **b** the case $r_{s1} = 1.8090$ in large region, **c** the case $r_{s2} = 0.8992$ in small region, **d** the case $r_{s2} = 0.8992$ in large region

Fixing shell radii at $r_{si} = \frac{\eta^{\frac{1}{4}}M^{\frac{1}{2}}}{x_i^{\frac{1}{2}}}$, the corresponding scalar field is $\psi \propto \sqrt{\frac{1}{r}}J_{\frac{1}{4}}(\frac{2\sqrt{3}\eta M}{r^2})$ in the form of (20). In Table 1, we show various values of r_{si} with respect to i. We plot the first two solutions of scalar fields in the background of Dirichlet reflecting shells in Fig. 1. The scalar fields start from zero at the shell radii and asymptotically approach zero at the infinity.

3 Scalar condensation behaviors around neutral Neumann reflecting shells

Now we turn to study scalar hair formations in the background of neutral reflecting shells with Neumann surface boundary conditions. At the surface, we impose the Neumann reflecting condition $\psi'(r_s) = 0$. The derivative of the function $\tilde{\psi}$ satisfies boundary conditions

$$\tilde{\psi}'(r_s) = (\sqrt{r}\psi)'|_{r=r_s} = \frac{1}{2\sqrt{r_s}}\psi(r_s) + \sqrt{r_s}\psi'(r_s)$$
$$= \frac{1}{2\sqrt{r_s}}\psi(r_s) = \frac{1}{2r_s}\sqrt{r_s}\psi(r_s) = \frac{1}{2r_s}\tilde{\psi}(r_s). \quad (24)$$

The case of $\tilde{\psi}(r_s) = 0$ is just the model studied in Sect. 2. In this part, we focus on the case of $\tilde{\psi}(r_s) \neq 0$. In the case of $\tilde{\psi}(r_s) > 0$, the function $\tilde{\psi}$ increases to be more positive around the surface and then decreases asymptotically to be zero. In another case of $\tilde{\psi}(r_s) < 0$, the function decreases to be more negative around the surface and then increases to be zero at the infinity. For both cases, one extremum point $r = r_{peak}$ satisfying (14) exists. Following analysis in part A of Sect. 2, for scalar hairy Neumann reflecting shells, we can easily get the same relation as (17) in the form

$$\frac{\sqrt{\eta}M}{r_s^2} \geqslant \frac{1}{8\sqrt{3}}. \quad (25)$$

With the scalar field solution (20), we can express the Neumann reflecting condition as

$$\frac{d\psi}{dr}\,|_{r=r_s} = \frac{d}{dr}\left[\sqrt{r}\widetilde{\psi}\right]|_{r=r_s} = \frac{d}{dr}\left[\sqrt{r}J_{\frac{1}{4}}\left(\frac{2\sqrt{3\eta}M}{r^2}\right)\right]|_{r=r_s} = 0. \tag{26}$$

The Eq. (26) can be solved through numerical methods. In the parameter regime obeying (25), we obtain discrete values of shell radii which can support the existence of static neutral massless scalar fields. We give the discrete shell radii in Table 2. We also plot the first two solutions of scalar fields outside Neumann reflecting shells in Fig. 2. The scalar fields start with $\psi'(r_s) = 0$ at the radii and asymptotically approaches zero in the large r region.

4 Conclusions

We studied condensations of static massless scalar fields non-minimally coupled to the Gauss–Bonnet invariant outside neutral reflecting shells. At the shell radii, we imposed scalar reflecting boundary conditions. We took two types of reflecting conditions, which are Dirichlet and Neumann reflecting boundary conditions. For both types of conditions, we analytically obtained a characteristic relation for hairy shells in the form $\frac{\sqrt{\eta}M}{r_s^2} \geqslant \frac{1}{8\sqrt{3}}$, where r_s is the shell radius, M is the shell mass and η is the coupling parameter. For parameters unsatisfying this relation, there is no scalar hair theorem. For parameters obeying this relation, we obtained analytical solutions of massless neutral scalar field hairs.

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