



Five dimensional Chern–Simons gravity for the expanded (anti)-de Sitter gauge group C_5

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Abstract We study the Hamiltonian dynamics of a five-dimensional Chern–Simons theory for the gauge algebra C_5 of Izaurieta, Rodriguez and Salgado, the so-called S_H -expansão of the 5D (anti)-de Sitter algebra $(a)ds$, based on the cyclic group \mathbb{Z}_4 . The theory consists of a 1-form field containing the $(a)ds$ gravitation variables and 1-form field transforming in the adjoint representation of $(a)ds$. The gravitational part of the action necessarily contains a term quadratic in the curvature, beyond the Einstein–Hilbert and cosmological terms, for any choice of the two independent coupling constants. The total action is also invariant under a new local symmetry, called “crossed diffeomorphisms”, beyond the usual space-time diffeomorphisms. The number of physical degrees of freedom is computed. The theory is shown to be “generic” in the sense of Bañados, Garay and Henneaux, *i.e.*, the constraint associated to the time diffeomorphisms is not independent from the other constraints.

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1 Introduction

In order to understand the universe at the Planck scale, one needs a quantum theory which reduces to General Relativity (GR) in the classical limit. A promising formalism is that of Loop Quantum Gravity (LQG) [1–3]. This approach uses the first-order formalism of GR, which is based on two pillars: Invariance under the local Lorentz transformations and invariance under the space-time diffeomorphisms. In this sense the Chern–Simons theories for gravitation present an encouraging scenario. First, they are also background independent theories, like GR, being invariant under the diffeomorphisms. Second, the actions are defined from invariant polynomials, which are gauge invariant by construction. Finally, they allow us to extend naturally the local Lorentz invariance to a larger symmetry group including the Poincaré or the de Sitter or anti-de Sitter groups – the latter being denoted by (A)dS in this paper. The (A)dS group is a “deformation” of the Poincaré group, the deformation parameter being the cosmological constant Λ , the special value $\Lambda = 0$ corresponding to the Poincaré group. Pioneering works on the subject are those of Witten [4] for 3-dimensional space-time and of Chamseddine [5, 6] for space-times of dimension 5 or higher. More recent references together with a good review may be found in Hassaine and Zanelli’s book [7].

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An analysis of the phenomenological aspects of 5D Chern–Simons gravity models with dimensional reduction to 4D have been investigated, with results indicating their relevance as physical theories [8–13].

In the Hamiltonian formalism of Dirac [14, 15], each local invariance is associated with a constraint which has to be solved at the quantum level. Unfortunately, in gravitation theories, there are many difficulties with respect to the resolution of the Hamiltonian constraint, the one corresponding to the invariance under the time diffeomorphisms [1–3]. In this sense, works by Bañados, Garay and Henneaux [16, 17] have made significant advances, showing the existence of so-called “generic” theories, where the constraint associated with the time diffeomorphisms is no longer independent, but can be seen as a combination of the constraints associated with gauge invariance and spatial diffeomorphisms. They have in particular shown that, among others, the Chern–Simons theory in 5D space-time for the group (A)dS₅ (*i.e.*, SO(1,5) or SO(2,4)) is in fact a generic theory, a result which makes it an interesting candidate for a quantum gravity theory.

Another motivating factor for the present work is found in a series of papers on the “*S*-expansion” of algebras [13, 18–20]. It is shown there that from any Lie algebra \mathcal{G} one can construct a new larger Lie algebra as the direct product $\mathcal{G}_{\text{exp}} = \mathcal{G} \times S$ of the starting Lie algebra with a finite semi-group S . Thus, it is possible, *e.g.*, to obtain a group of symmetry wider than (A)dS₅. This enables the introduction of new fields in the theory beyond the gravitation field. In particular, it was claimed in [18] that, using an alternative expansion process called “*H*-reduction” leading to a symmetry algebra called C_5 , it was possible to obtain a theory that reproduces exactly GR coupled with some matter fields, so that it would be a good candidate for the purpose of obtaining a quantum theory for gravitation consistent with Einstein’s theory.¹

The purpose of the present paper is, therefore, to construct the Chern–Simons action based on the S_H -expanded algebra C_5 of [18]. Our main results are, first, that the theory depends on two independent coupling constants,² second, that it is generic in the sense defined above, and third, that it is invariant under a new class of diffeomorphisms specific to these expanded algebra, which we call “crossed diffeomorphisms”.

The paper is organized as follows. We make a brief review of (a)ds₅ Chern–Simons gravity in five dimensions in Sect. 2, following essentially [5]. The expansion process of Lie alge-

bras together with the calculation of the C_5 invariant tensors are given in Sect. 3. In Sect. 4 we check that the expansion considered by us actually leads to gravitational fields and “fields of matter”. We will see that the pure gravitational part of the action consists of an Einstein–Hilbert action term, a cosmological term and a term quadratic in the curvature, of the Gauss–Bonnet type. In Sect. 5 we make a study of the dynamical structure of the theory, showing that it is also a generic theory. The discussion of the generalized diffeomorphism invariance is presented in the final part of this section. The paper ends with our conclusions. Notations, conventions and some technicalities can be found in Appendices A and B.

The main results presented here constitute the content of a Master thesis defended by one of us [22] at the Federal University of Viçosa.

2 Chern–Simons gravity in 5D

In this section we will present some results known in the literature on the Chern–Simons gravitation theories, important for the understanding of this work. We follow Ref. [5]. The notations and conventions used are given in Appendix A.

2.1 Chern–Simons gravity in 5D for (A)dS₅

Chern–Simons theories occur only in odd dimensions. In this way we will start by treating the Chern–Simons theories in 5D, whose gauge group is that of the transformations which leave invariant the metric $\eta_{MN} = \text{diag}(-1, 1, 1, 1, s)$ of the internal space, where $M, N = 0, 1, \dots, 5$ and $s = \pm 1$. For $s = +1$ we have the de Sitter group $SO(1, 5)$ and for $s = -1$ we have the anti-de Sitter group $SO(2, 4)$. For the moment we will not distinguish between them, simply calling them (A)dS₅.

The associated Lie algebra, denoted by (a)ds₅, consists of the 6×6 matrices X whose elements have the form $X_P^Q = X_{PR}\eta^{RQ}$, with $X_{PR} = -X_{RP}$. A convenient basis is given by the 15 matrices $T^{MN} = -T^{NM}$ defined by

$$(T_{MN})_P^Q = -\eta_{MP}\delta_N^Q + \eta_{NP}\delta_M^Q. \quad (2.1)$$

The commutation relations are

$$[T_{MN}, T_{PQ}] = T_{MP}\eta_{NQ} - T_{MQ}\eta_{NP} - T_{NP}\eta_{MQ} + T_{NQ}\eta_{MP}, \quad (2.2)$$

which lead to the structure constants

$$f_{MN, PQ}^{RS} = \frac{1}{2} \left\{ \eta_{MP} (\delta_N^R \delta_Q^S - \delta_N^S \delta_Q^R) + \eta_{NQ} (\delta_M^R \delta_P^S - \delta_M^S \delta_P^R) + \eta_{PN} (\delta_Q^R \delta_M^S - \delta_Q^S \delta_M^R) \right\}$$

¹ See also [21] for an alternative approach.

² The authors of [21] arrive at the same conclusion, in a somewhat different interpretation frame. But our result enters in contradiction with [18], where the theory is claimed to depend on four independent coupling constants. The consequence of our result is to invalidate the C_5 symmetry of the Einstein–Hilbert model presented in Section 8 of [18].

$$+\eta_{QM} \left(\delta_P^R \delta_N^S - \delta_P^S \delta_N^R \right) \}. \tag{2.3}$$

To construct the Chern–Simons action corresponding to (A)dS₅, we use the formalism of the differential forms. So, the fields are given by the 1-form connection $A = A_\mu dx^\mu$, with $A_\mu = \frac{1}{2} A_\mu^{MN} T_{MN}$, where $\mu = 0, 1, \dots, 4$, transforming under an infinitesimal gauge transformation as

$$\delta A = d\epsilon + [A, \epsilon], \tag{2.4}$$

with $\epsilon = \frac{1}{2} \epsilon^{MN} T_{MN}$ an infinitesimal 0-form Lie algebra valued parameter. Said that, the Chern–Simons action, invariant up to boundary terms, is

$$S = k \epsilon_{MNPQRS} \int \left(A^{MN} \wedge dA^{PQ} \wedge dA^{RS} \wedge \frac{3}{2} A^{MN} \wedge (A^2)^{PQ} \wedge dA^{RS} + \frac{3}{5} A^{MN} \wedge (A^2)^{PQ} \wedge (A^2)^{RS} \right), \tag{2.5}$$

where ϵ_{MNPQRS} , the 6D completely antisymmetric Levi-Civita tensor (with $\epsilon_{012345} = 1$), is an invariant rank 3 tensor³ of (A)dS₅. The equations of motion derived from this action read

$$\epsilon_{MNPQRS} F^{MN} \wedge F^{PQ} = 0, \tag{2.6}$$

where F^{MN} is the curvature 2-form defined by

$$F = dA + A \wedge A = \frac{1}{2} F^{MN} T_{MN}, \\ F^{MN} = dA^{MN} + A^M_U \wedge A^{UN}. \tag{2.7}$$

The solutions of (2.6) are not restricted to the flat ones, $F^{MN} = 0$, as occurs in three-dimensional case [4]. Non-trivial solutions as well as cosmological models were studied in [10,11].

In order to interpret this theory as a gravitation theory, one identifies the 15 generators T_{MN} of the group (A)dS₅ as the 10 generators M_{AB} of the Lorentz group in 5D and the 5 generators P_A of the generalized translations,⁴ where $A, B = 0, \dots, 4$:

$$M_{AB} = T_{AB}, \quad P_A = \frac{1}{l} T_{A5}. \tag{2.8}$$

One writes accordingly the connection as

$$A = \frac{1}{2} \omega^{AB} M_{AB} + e^A P_A \tag{2.9}$$

³ Considering an antisymmetric pair such as M, N as a single (A)dS₅ index taking values from 1 to 15.

⁴ The P_A would be the translation generators in the Poincaré algebra case $s = 0$.

In (2.8), l is a parameter with units of length (in the natural system of units), necessary in order to take into account the difference between the dimensions of the vielbein e^A and of the spin connection ω^{AB} . The commutation relations (2.2) can be rewritten as

$$[M_{AB}, M_{CD}] = M_{AC} \eta_{BD} - M_{AD} \eta_{BC} - M_{BC} \eta_{AD} + M_{BD} \eta_{AC}, \\ [M_{AB}, P_C] = -P_A \eta_{BC} + P_B \eta_{AC}, \\ [P_A, P_B] = \frac{s}{l^2} M_{AB}, \tag{2.10}$$

and the Chern–Simons action (2.5) as⁵

$$S = k \int \epsilon_{ABCDE} \times \left(e^A R^{BC} R^{DE} - \frac{2s}{3l^2} e^A e^B e^C R^{DE} + \frac{1}{5l^4} e^A e^B e^C e^D e^E \right), \tag{2.11}$$

being

$$R^{AB} = d\omega^{AB} + \omega^{AC} \omega_C^B, \tag{2.12}$$

the Riemann curvature 2-form associated with the spin connection. The parameters k and l are related to the Newton’s constant G ($\propto s l^2/k$) and to the cosmological constant Λ ($\propto s/l^2$) [11].

As we can see we have the presence of the Einstein–Hilbert term and the cosmological constant one, in addition to the first term, which is of the Gauss–Bonnet type.

2.2 Dynamics

The dynamics of Chern–Simons theories is best analysed via the Hamiltonian formalism of Dirac, identifying all the constraints of the theory and separating them into first and second class ones. An important concept to understand this dynamics in the context of Loop Quantum Gravity is that of “generic theory”, first presented in [16]. Let us use here the definition of genericity given in [12], which although simpler than the first one, generalizes it. Therefore, we will call a theory as generic if the constraints associated to the time diffeomorphisms is not an independent one, but can be expressed in terms of the other constraints that form a basis of all first-class constraints of the theory.

It has been shown in [17] that the action (2.5) leads to a theory with a total of 75 constraints, 19 of which are first class, of which 15 are associated with the (A)dS gauge transformations and 4 with the spatial diffeomorphisms. The 56 remaining constraints are second-class. This leads to a theory with 13 local degrees of freedom. Thus the theory is generic according to the definition given above. In particular,

⁵ From now on we skip the wedge symbol \wedge for the external product of forms.

the Hamiltonian constraint, associated to the time diffeomorphisms, is not independent, it can be expressed as a combination of the constraints associated with gauge transformations and spatial diffeomorphisms, modulo field equations.

3 The C_5 algebra

3.1 S and S_H expansions

Following the construction of [19], let us suppose that we know a Lie algebra \mathcal{G} with basis $\{T_A\}$ and structure constants f_{AB}^C , the basic commutator being written as

$$[T_A, T_B] = f_{AB}^C T_C. \tag{3.1}$$

Now, given a finite Abelian semi-group $S = \{\lambda_\alpha; \alpha = 1, \dots, N\}$, it has been shown [19] that the direct product $S \times \mathcal{G}$, called the S -expansion of \mathcal{G} , with basis elements

$$T_{A\alpha} = \lambda_\alpha T_A \tag{3.2}$$

and basic commutation rules defined by

$$[T_{A\alpha}, T_{B\beta}] = [\lambda_\alpha T_A, \lambda_\beta T_B] = \lambda_\alpha \lambda_\beta [T_A, T_B], \tag{3.3}$$

is another Lie algebra, of dimension equal to $N \dim(\mathcal{G})$, called S -expanded algebra of \mathcal{G} . So, we obtain a larger algebra having \mathcal{G} as a sub-algebra. Using the multiplication table of S , expressed by the 2-selector $S_{\alpha\beta}^\gamma$:

$$S_{\alpha\beta}^\gamma = \begin{cases} 1, & \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0, & \text{otherwise,} \end{cases} \tag{3.4}$$

we may rewrite the commutation rules (3.3) as

$$[T_{A\alpha}, T_{B\beta}] = f_{A\alpha, B\beta}^{C\gamma} T_{C\gamma}, \tag{3.5}$$

with the structure constants given by

$$f_{A\alpha, B\beta}^{C\gamma} = S_{\alpha\beta}^\gamma f_{AB}^C. \tag{3.6}$$

The authors of [18] introduce another expansion, called S_H -expansion, consisting, in the case of S being the cyclic group of even order \mathbb{Z}_{2n} , in applying the conditions

$$T_{A,i} = \rho T_{A,i+n}, \quad i = 0, \dots, n-1, \tag{3.7}$$

with $\rho = -1$, on the generators $T_{A\alpha}$ ($\alpha = 0, \dots, 2n-1$) of the S -expansion of \mathcal{G} . They show that the resulting algebraic structure is a Lie algebra, denoted by $(\mathbb{Z}_{2n} \times \mathcal{G})_H$. We will use an alternative but equivalent approach. We note that the conditions (3.7) are formally equivalent to the conditions

$$\lambda_i = \rho \lambda_{i+n}, \quad i = 0, \dots, n-1, \tag{3.8}$$

on the elements of the group $S = \mathbb{Z}_{2n}$. Formally, the latter conditions amount to replace the multiplication table of $S = \mathbb{Z}_{2n}$ by the H -reduced one, shown in Table 1 for the special case $n = 2$ in which we will be interested in the following. Let us call \mathcal{S} this reduced group, which is Abelian.

Table 1 Multiplication table of the reduced group \mathcal{S}

	λ_0	λ_1	$\rho\lambda_0$	$\rho\lambda_1$
λ_0	λ_0	λ_1	$\rho\lambda_0$	$\rho\lambda_1$
λ_1	λ_1	$\rho\lambda_0$	$\rho\lambda_1$	λ_0
$\rho\lambda_0$	$\rho\lambda_0$	$\rho\lambda_1$	λ_0	λ_1
$\rho\lambda_1$	$\rho\lambda_1$	λ_0	λ_1	$\rho\lambda_0$

Strictly speaking, the symbols “ $\rho\lambda_i$ ” must be considered as group elements independent of the elements λ_i . It is only after substituting in the definition (3.2) of the expanded generators, that ρ will be considered as a number.

Let us consider the S -expanded algebra $\mathcal{S} \times (\text{ads}_5)$, with (ads_5) the (anti-)de Sitter algebra defined in Sect. 2.1 and \mathcal{S} the Abelian group defined by the multiplication table shown in Table 1. From the 4 generators defined by (3.2), two are independent, which may be taken as

$$T_{MNi} = \lambda_i T_{MN}, \quad i = 0, 1, \tag{3.9}$$

with T_{MN} given by (2.1). Their commutations rules are obtained from (3.3), (3.5):

$$[T_{MNi}, T_{PQj}] = f_{MNi, PQj}^{RSk} T_{RSk}, \tag{3.10}$$

the structure constants being given by

$$f_{MNi, PQj}^{RSk} = S_{ij}^k f_{MN, PQ}^{RS}. \tag{3.11}$$

with $f_{MN, PQ}^{RS}$ given by (2.3). S_{ij}^k ($i, j, k = 0, 1$) is the 2-selector (see definition (3.4)) corresponding to the multiplication rules of λ_0 and λ_1 given in the left upper quadrant of Table 1. Its non-zero components are⁶

$$S_{00}^0 = 1, \quad S_{01}^1 = S_{10}^1 = 1, \quad S_{11}^0 = \rho. \tag{3.12}$$

We now show that the Lie algebra we have constructed is identical, in the case where $\rho = -1$, to the Lie algebra $C_5 = (\mathbb{Z}_4 \times (\text{ads}_5))_H$ of [18]. The structure constants of the latter are given by $(\bar{S}_{ij}^k - \bar{S}_{ij}^{k+2}) f_{MN, PQ}^{RS}$, with the indices i, j, k taking the values 0, 1, and $\bar{S}_{\alpha\beta}^\gamma$ ($\alpha, \beta, \gamma = 0, \dots, 3$) is the 2-selector, as defined by Eq. (3.4), for \mathbb{Z}_4 . It is straightforward to check that they are equal to our structure constants (3.11), which completes the proof.

We observe that giving the value 1 to the factor ρ leads to the S -expanded algebra $\mathbb{Z}_2 \times (\text{ads}_5)$, the selector (3.12) being that of the cyclic group \mathbb{Z}_2 . We shall keep $\rho = \pm 1$ unfixed, treating both cases at the same time, but continuing to call this Lie algebra as C_5 . In both cases we have an algebra with twice as many independent generators (30) as for the (ads_5) algebra, hence the double of gauge fields, which may be divided in 15 gravitation and 15 matter fields, as will be made more precise later on in Sect. 4.

⁶ We thank the authors of Ref. [18] for pointing out to us that this 2-selector, for $\rho = -1$, does not correspond to that of a semi-group, contrarily to what we claimed in a previous version of this work.

3.2 Invariant tensors

As we mentioned, an important ingredient for constructing a Chern–Simons action invariant under a gauge group whose Lie algebra will be denoted by \mathcal{G} , are the \mathcal{G} -invariant tensors. Basically, a tensor in the adjoint representation like g_{XYZ} is invariant if it obeys the relation⁷

$$f_{TX}^U g_{UYZ} + f_{TY}^U g_{XUZ} + f_{TZ}^U g_{XYU} = 0. \tag{3.13}$$

where f_{XY}^Z are the structure constants of \mathcal{G} . We will restrict ourselves to rank 3 symmetric tensors. Thus, the invariance condition of the (a)ds₅ case reads

$$\begin{aligned} & f_{MN, M_1 N_1}{}^{PQ} g_{PQ, M_2 N_2, M_3 N_3} \\ & + f_{MN, M_2 N_2}{}^{PQ} g_{M_1 N_1, PQ, M_3 N_3} \\ & + f_{MN, M_3 N_3}{}^{PQ} g_{M_1 N_1, M_2 N_2, PQ} = 0 \end{aligned} \tag{3.14}$$

From the structure constants (2.3) and the previous relation one can show that $g_{MN, PQ, RS} = \varepsilon_{MNPQRS}$, i.e., the invariant tensor for the (a)ds₅ algebra is the Levi-Civita tensor of six indices. Proceeding in an analogous way, the invariance condition (3.13) for the algebra C_5 leads us to the 16 equations

$$\begin{aligned} & S_{ii_1}{}^j f_{MN, M_1 N_1}{}^{PQ} g_{PQj, M_2 N_2 i_2, M_3 N_3 i_3} \\ & + S_{ii_2}{}^j f_{MN, M_2 N_2}{}^{PQ} g_{M_1 N_1 i_1, PQj, M_3 N_3 i_3} \\ & + S_{ii_3}{}^j f_{MN, M_3 N_3}{}^{PQ} g_{M_1 N_1 i_1 M_2 N_2 i_2, PQj} = 0. \end{aligned} \tag{3.15}$$

with the 2-selector $S_{ij}{}^k$ given in (3.12). The indices i, j, \dots take the values 0, 1. The general solution of this system for the C_5 invariant tensor, calculated in App. B, reads

$$g_{MNi, PQj, RSk} = c_l S_{ijk}{}^l \varepsilon_{MNPQRS}, \tag{3.16}$$

where the 3-selector [18] $S_{ijk}{}^l$ is defined by

$$S_{ijk}{}^l = \begin{cases} 1, & \lambda_i \lambda_j \lambda_k = \lambda_l \\ \rho, & \lambda_i \lambda_j \lambda_k = \rho \lambda_l \\ 0, & \text{otherwise} \end{cases} \tag{3.17}$$

and c_0 and c_1 are two arbitrary constants. So, the most general action is an arbitrary linear combination of two invariant actions⁸

We note that the expression (3.16) is similar to that given by the Theorem 7.1 of [13], with the difference that in our case, the selector $S_{ijk}{}^l$, like the 2-selector defined by (3.12), is not of a semi-group, but refers only to the first two elements of the group defined by the multiplication Table 1.

⁷ The indices X, Y , etc. can be multi-indices such as in Eqs. (3.3), (3.14) or (3.16).

⁸ The authors of reference [18], who consider the case where $\rho = -1$, give an invariant tensor with four independent parameters instead of two. We have checked that their solution satisfies the invariance condition (3.15) only if their four parameters obey two linear conditions and thus reduce to our solution (3.16).

4 Constructing the Chern–Simons action for the C_5 algebra

The Chern–Simons action for the algebra C_5 is constructed in the same way as for the (A)dS₅ case, changing only the symmetry algebra. The gauge connection may be written as follows:

$$\mathcal{A} = \frac{1}{2} \mathcal{A}^{MNi} T_{MNi} = \frac{1}{2} A^{MN} T_{MN0} + \frac{1}{2} B^{MN} T_{MN1}, \tag{4.1}$$

where we have separated the $\alpha = 0$ and $\alpha = 1$ components:

$$A^{MN} := \mathcal{A}^{MN0} \quad B^{MN} := \mathcal{A}^{MN1}. \tag{4.2}$$

It is also useful to write the C_5 connection \mathcal{A} in the form

$$\begin{aligned} \mathcal{A} &= \lambda_0 A + \lambda_1 B, \quad \text{with} \\ A &= \frac{1}{2} A^{MN} T_{MN}, \quad B = \frac{1}{2} B^{MN} T_{MN}, \end{aligned} \tag{4.3}$$

which shows it explicitly as an \mathcal{S} -valued object.

The infinitesimal gauge transformations $\delta \mathcal{A} = d\mathcal{O} + [\mathcal{A}, \mathcal{O}]$, of infinitesimal parameter $\mathcal{O} = \lambda_0 \omega + \lambda_1 \eta$ read, for the fields A and B :

$$\delta A = \delta_\omega A + \delta_\eta A, \quad \delta B = \delta_\eta B + \delta_\omega B, \tag{4.4}$$

where

$$\begin{aligned} \delta_\omega A &= d\omega + [A, \omega], \quad \delta_\omega B = [B, \omega], \\ \delta_\eta A &= \rho[B, \eta], \quad \delta_\eta B = d\eta + [A, \eta]. \end{aligned} \tag{4.5}$$

We observe that $\delta_\omega A$ is the transformation law of a connection for the (A)dS₅ group, whereas the transformation $\delta_\omega B$ is that of a field in the adjoint representation of the same group. This allows us to identify the components of A as the gravitation fields, whereas those of B may be considered as the ‘‘matter’’ fields of the theory. The transformations of parameter η appear due to the process of expansion. Identifying the 5D Lorentz generators $M_{AB} = T_{AB0}$ and the generalized translation generators $P_A = T_{A50}/l$ (in the same way as in (2.8)), we can explicitly identify the Lorentz connection forms ω^{AB} and the 5-bein forms e^A as the components of A defined by Eq. (2.9).

We can now decompose the C_5 curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$ as

$$\mathcal{F} = \lambda_0 F + \lambda_1 G, \tag{4.6}$$

where⁹

$$F := dA + A^2 + \rho B^2, \quad G := dB + [A, B]. \tag{4.7}$$

The next step is to calculate the Chern–Simons action for the expanded algebra C_5 . In terms of the C_5 connection \mathcal{A} , the action is

⁹ We use the multiplication table given by the upper left quadrant of Table 1 for $l\alpha_0, \lambda_1$.

$$\begin{aligned}
 S &= c_l S_{ijk}{}^l \varepsilon_{MNPQRS} \\
 &\int \left(\mathcal{A}^{MNi} d\mathcal{A}^{PQj} d\mathcal{A}^{RSk} \right. \\
 &\quad + \frac{3}{2} \mathcal{A}^{MNi} (\mathcal{A}^2)^{PQj} d\mathcal{A}^{RSk} \\
 &\quad \left. + \frac{3}{5} \mathcal{A}^{MNi} (\mathcal{A}^2)^{PQj} (\mathcal{A}^2)^{RSk} \right). \tag{4.8}
 \end{aligned}$$

The action contains two independent invariants, whose coefficients are the arbitrary coupling constants c_0 and c_1 . In terms of the A and B fields, the action takes the rather complicated form

$$\begin{aligned}
 S &= c_0 \varepsilon_{MNPQRS} \\
 &\int \left\{ A^{MN} dA^{PQ} dA^{RS} + \rho A^{MN} dB^{PQ} dB^{RS} \right. \\
 &\quad + 2\rho B^{MN} dA^{PQ} dB^{RS} \\
 &\quad + \frac{3}{2} \left(A^{MN} (A^2)^{PQ} dA^{RS} + \rho A^{MN} (B^2)^{PQ} dA^{RS} \right. \\
 &\quad + \rho A^{MN} [A, B]^{PQ} dB^{RS} \\
 &\quad + \rho B^{MN} (A^2)^{PQ} dB^{RS} + B^{MN} (B^2)^{PQ} dB^{RS} \\
 &\quad \left. + \rho B^{MN} [A, B]^{PQ} dA^{RS} \right) \\
 &\quad + \frac{3}{5} \left(A^{MN} (A^2)^{PQ} (A^2)^{RS} + A^{MN} (B^2)^{PQ} (B^2)^{RS} \right. \\
 &\quad + 2\rho A^{MN} (A^2)^{PQ} (B^2)^{RS} \\
 &\quad + \rho A^{MN} [A, B]^{PQ} [A, B]^{RS} \\
 &\quad + 2\rho B^{MN} [A, B]^{PQ} (A^2)^{RS} \\
 &\quad \left. + 2B^{MN} [A, B]^{PQ} (B^2)^{RS} \right) \Big\} \\
 &\quad + c_1 \varepsilon_{MNPQRS} \\
 &\int \left\{ B^{MN} dA^{PQ} dA^{RS} + \rho B^{MN} dB^{PQ} dB^{RS} \right. \\
 &\quad + 2A^{MN} dA^{PQ} dB^{RS} \\
 &\quad + \frac{3}{2} \left(A^{MN} (A^2)^{PQ} dB^{RS} + \rho A^{MN} (B^2)^{PQ} dB^{RS} \right. \\
 &\quad + A^{MN} [A, B]^{PQ} dA^{RS} \\
 &\quad + B^{MN} (A^2)^{PQ} dA^{RS} + \rho B^{MN} (B^2)^{PQ} dA^{RS} \\
 &\quad \left. + \rho B^{MN} [A, B]^{PQ} dB^{RS} \right) \\
 &\quad + \frac{3}{5} \left(B^{MN} (A^2)^{PQ} (A^2)^{RS} + B^{MN} (B^2)^{PQ} (B^2)^{RS} \right. \\
 &\quad + 2\rho B^{MN} (A^2)^{PQ} (B^2)^{RS} \\
 &\quad + \rho B^{MN} [A, B]^{PQ} [A, B]^{RS} \\
 &\quad + 2A^{MN} [A, B]^{PQ} (A^2)^{RS} \\
 &\quad \left. + 2\rho A^{MN} [A, B]^{PQ} (B^2)^{RS} \right) \Big\}. \tag{4.9}
 \end{aligned}$$

If we put to zero the ‘‘matter’’ field B , we recover the action for pure Chern–Simons gravitation in 5D given by Eq. (2.5), or Eq. (2.11) in terms of the 5-bein e^A and the Riemann curvature R^{AB} , with c_0 as the coupling constant. We see in par-

ticular that the Gauss–Bonnet term cannot be avoided, in contradiction with the claim of [18]. Let us mention however the possibility of achieving the separation of the Gauss–Bonnet term by assuming an alternative identification between the C_5 connection components \mathcal{A}^{MNi} and the gravity components ω^{AB} , e^A and the ‘‘matter’’ ones [21]. But in this case the (a)ds invariance of the pure gravity part of the action does not hold any more.

5 Dynamics

The dynamics of the C_5 theory is formally analogous to that of the $(A)dS_5$ theory exposed in [16]. So, adapting the discussion of [16] to our present case, we will assume that the 5D manifold admits the topology $\mathbb{R} \times \Sigma$, where the real line \mathbb{R} corresponds to time $x^0 = t$, while Σ is the dimension four space sheet with coordinates x^a , $a = 1, \dots, 4$. Decomposing the C_5 connection according to this foliation,

$$\mathcal{A}^{MNi} = \mathcal{A}_0^{MNi} dx^0 + \mathcal{A}_a^{MNi} dx^a, \tag{5.1}$$

allows us to rewrite the Chern–Simons action in the form

$$S = \int \left(l_{MNi}^a \partial_t \mathcal{A}_a^{MNi} - \mathcal{A}_0^{MNi} K_{MNi} \right), \tag{5.2}$$

where K_{MNi} depends on the space components of the C_5 curvature \mathcal{F} and l_{MNi}^a on the space components of the connection \mathcal{A} and of the curvature \mathcal{F} :

$$K_{MNi} = -\frac{1}{32} c_l S_{ijk}{}^l \varepsilon_{MNPQRS} \varepsilon^{abcd} \mathcal{F}_{ab}^{PQj} \mathcal{F}_{cd}^{RSk}, \tag{5.3}$$

$$l_{MNi}^a = -\frac{1}{4} c_l S_{ijk}{}^l \varepsilon_{MNPQRS} \varepsilon^{abcd} \mathcal{A}_b^{PQj} \mathcal{F}_{cd}^{RSk}. \tag{5.4}$$

The functional variations of the action with respect to \mathcal{A}_0^{MNi} and \mathcal{A}_a^{MNi} yield the field equations

$$K_{MNi} = 0, \tag{5.5}$$

$$\Omega_{MNi, PQj}^{ab} (\partial_t \mathcal{A}_b^{PQj} - D_b \mathcal{A}_0^{PQj}) = 0, \tag{5.6}$$

where

$$\Omega_{MNi, PQj}^{ab} = -\frac{1}{2} c_l S_{ijk}{}^l \varepsilon_{MNPQRS} \varepsilon^{abcd} \mathcal{F}_{cd}^{RSk}. \tag{5.7}$$

5.1 Hamiltonian formalism and constraints

In order to pass from the Lagrangian to the Hamiltonian approach [14, 15], we need the momenta conjugated to the generalized coordinates, *i.e.*, to the fields \mathcal{A}_a^{MNi} and \mathcal{A}_0^{MNi} :

$$P_{MNi}^a = l_{MNi}^a, \tag{5.8}$$

$$P_{MNi}^0 = 0, \tag{5.9}$$

with l_{MNi}^a given by (5.4). These 150 relations between momenta and generalized coordinates are primary constraints, which read

$$\phi_{MNi}^a := p_{MNi}^a - l_{MNi}^a \approx 0, \tag{5.10}$$

$$\phi_{MNi}^0 := p_{MNi}^0 \approx 0. \tag{5.11}$$

The Hamiltonian of the system is obtained by the usual Legendre transform and the addition of the constraints multiplied by Legendre multiplier fields λ :

$$H = \int \left(A_0^{MNi} K_{MNi} + \lambda_a^{MNi} \phi_{MNi}^a + \lambda_0^{MNi} \phi_{MNi}^0 \right), \tag{5.12}$$

with K_{MNi} given by (5.3). The stability of the constraints (5.11) requires $\dot{\phi}_{MNi}^0 = \{\phi_{MNi}^0, H\} \approx 0$, which implies 30 secondary constraints given by

$$K_{MNi} \approx 0. \tag{5.13}$$

The fields A_0^{MNi} play now the role of Lagrange multipliers, the constraints (5.11) becoming thus irrelevant. So we are left with a total of 150 constraints in our theory. Those given by (5.13) are conditions on the curvatures \mathcal{F} . Thus not all the \mathcal{F} 's are independents. On the other end, the constraints (5.10) determine the p_{MNi}^a in terms of the other variables. Equation (5.13) for $i = 0$ and $i = 1$ read, in terms of the fields A and B and with the definitions (4.7):

$$K_{MN0} = -\frac{1}{32} \varepsilon_{MNPQRS} \varepsilon^{abcd} \times \left[\left(F_{ab}^{PQ} F_{cd}^{RS} + \rho G_{ab}^{PQ} G_{cd}^{RS} \right) c_0 + 2F_{ab}^{PQ} G_{cd}^{RS} c_1 \right] \approx 0, \tag{5.14}$$

$$K_{MN1} = -\frac{1}{32} \varepsilon_{MNPQRS} \varepsilon^{abcd} \times \left[2\rho F_{ab}^{PQ} G_{cd}^{RS} c_0 + \left(F_{ab}^{PQ} F_{cd}^{RS} + \rho G_{ab}^{PQ} G_{cd}^{RS} \right) c_1 \right] \approx 0. \tag{5.15}$$

However it will be opportune to substitute the constraint (5.13) by the equivalent constraint

$$\mathcal{G}_{MNi} = -K_{MNi} + D_a \phi_{MNi}^a. \tag{5.16}$$

Indeed, the Poisson parentheses of \mathcal{A} with \mathcal{G} ,

$$\left\{ \mathcal{A}_a^{MNi}(\vec{x}), \int \mathcal{O}^{PQj}(\vec{y}) \mathcal{G}_{PQj}(\vec{y}) \right\} = -D_a \mathcal{O}^{MNi}(\vec{x}), \tag{5.17}$$

with \mathcal{O}^{MNi} an infinitesimal field in the C_5 algebra, show that the new constraints (5.16) generate the gauge transformations (4.4). The \mathcal{G}_{MNi} and ϕ_{MNi}^a form a basis for the constraints and obey the (open) algebra relations

$$\begin{aligned} \{ \mathcal{G}_{MNi}(\vec{x}), \mathcal{G}_{PQj}(\vec{y}) \} &= S_{ij}^k f_{MN,PQ}{}^{RS} \mathcal{G}_{RSk}(\vec{x}) \delta(\vec{x} - \vec{y}), \\ \{ \phi_{MNi}^a(\vec{x}), \mathcal{G}_{PQj}(\vec{y}) \} &= S_{ij}^k f_{MN,PQ}{}^{RS} \phi_{RSk}^a(\vec{x}) \delta(\vec{x} - \vec{y}), \\ \{ \phi_{MNi}^a(\vec{x}), \phi_{PQj}^b(\vec{y}) \} &= \Omega_{MNi,PQj}^{ab}(\vec{x}) \delta(\vec{x} - \vec{y}), \end{aligned} \tag{5.18}$$

where the $S_{ij}^k f_{MN,PQ}{}^{RS}$ are the C_5 structure constants and $\Omega_{MNi,PQj}^{ab}$ is given by (5.7). The constraints \mathcal{G}_{MNi} appear here explicitly as first class ones,¹⁰ whereas a quantity N_2 of the 120 ϕ_{MNi}^a will be second class, where N_2 is the rank of Ω considered as a 120×120 matrix indexed by the multi-indices $\{^a_{MNi}\}$ and $\{^b_{PQj}\}$. The theory does not allow for any further independent constraint. Indeed, the stability of the constraints \mathcal{G}_{MNi} given by $\dot{\mathcal{G}}_{MNi} = 0$ is automatically satisfied by the fact that they are first-class, while the stability of ϕ_{MNi}^a only carries us to restrictions on the Lagrange multipliers.

We turn now to the calculation of the rank N_2 of Ω . We already know the existence of four first class constraints, given by the generators of the four spatial diffeomorphisms [16], which are linear combinations of the ϕ 's:

$$H_a = \mathcal{F}_{ab}^{MNi} \phi_{MNi}^b = F_{ab}^{MN} \phi_{MN0}^b + G_{ab}^{MN} \phi_{MN1}^b, \tag{5.19}$$

which implies that $N_2 \leq 120 - 4 = 116$. Inspired by the example of [16], we take a special case of curvature with the following configuration:

$$\begin{aligned} F^{12} &= G^{12} = dx^1 dx^2 + dx^3 dx^4, \\ F^{34} &= G^{34} = dx^1 dx^2 - dx^3 dx^4, \\ F^{56} &= G^{56} = dx^1 dx^3 + dx^2 dx^4, \\ &(\text{other components} = 0), \end{aligned} \tag{5.20}$$

which can be derived from the potentials

$$\begin{aligned} A^{12} &= B^{12} = x^1 dx^2 + x^3 dx^4, \\ A^{34} &= B^{34} = x^1 dx^2 - x^3 dx^4, \\ A^{56} &= B^{56} = x^1 dx^3 + x^2 dx^4, \\ &(\text{other components} = 0). \end{aligned}$$

It is easy to verify that this configuration obeys the constraints (5.14) and (5.15). Indeed, one sees that $F^{12} F^{34} = G^{12} G^{34} = F^{12} G^{34} = G^{12} F^{34} = 0$, as well $F^{12} F^{56} = G^{12} G^{56} = F^{12} G^{56} = G^{12} F^{56} = 0$ and $F^{34} F^{56} = G^{34} G^{56} = F^{34} G^{56} = G^{34} F^{56} = 0$, hence $K_{MN0} = 0$ and $K_{MN1} = 0$. We finally check that the corresponding matrix Ω has rank $N_2 = 112$. besides this particular field configuration, we have analysed many numerical examples, all yielding a value ≤ 112 [22]. Thus we conclude, for the time being, that

$$112 \leq N_2 \leq 116. \tag{5.21}$$

A more careful analysis of diffeomorphism invariance performed in the next subsection will show that this rank is in fact equal to 112.

5.2 Crossed diffeomorphisms

We know that infinitesimal diffeomorphisms $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ are given as Lie derivatives $\mathcal{L}_{\xi} = i_{\xi} d + di_{\xi}$, where i_{ξ}

¹⁰ Since the Hamiltonian H is a combination of constraints, we have that the Poisson brackets of the \mathcal{G} 's with it are also weakly zero.

is the contraction along the (infinitesimal) vector ξ . Remembering that the C_5 connection is S -valued (see (4.3)), we consider an S -valued vector

$$\xi = \lambda_i \xi^i = \lambda_0 u + \lambda_1 v, \quad \text{with } \xi^0 = u, \xi^1 = v, \quad (5.22)$$

It is easy to see that the action as given by (4.8) in terms of the C_5 connection \mathcal{A} is invariant under the generalized diffeomorphisms defined as the Lie derivative along the S -valued vector (5.22):

$$\delta_\xi \mathcal{A} = \mathcal{L}_\xi \mathcal{A}. \quad (5.23)$$

Indeed, since $S = \int Q_5$ and Q_5 is a 5-form in five-dimensional space-time, its exterior derivative vanishes. Therefore

$$\delta_\xi S = \int (i_\xi dQ_5 + di_\xi Q_5) = \int di_\xi Q_5 \quad (5.24)$$

is a boundary term.

From now on we specialize on the space generalized diffeomorphisms, with $u = (u^a)$ and $v = (v^a)$, $a = 1, \dots, 4$. Using $i_\xi = \lambda_0 i_u + \lambda_1 i_v$, we can write (5.23) for the components A and B :

$$\delta A = \mathcal{L}_u A + \rho \mathcal{L}_v B, \quad \delta B = \mathcal{L}_u B + \mathcal{L}_v A, \quad (5.25)$$

where $\rho = \pm 1$ is the parameter entering in the multiplication laws given in Table 1.

This way we have that

$$\delta_u A = \mathcal{L}_u A, \quad \delta_u B = \mathcal{L}_u B \quad (5.26)$$

corresponds to the usual spatial diffeomorphisms, while the new symmetry transformations

$$\delta_v^\times A = \rho \mathcal{L}_v B, \quad \delta_v^\times B = \mathcal{L}_v A, \quad (5.27)$$

appearing as a result of the expansion process, will be called ‘‘crossed diffeomorphisms’’. Since u and v are 4-vectors, we have a total of 8 generalized diffeomorphisms.

These generalized diffeomorphisms obey commutation rules deduced from the transformation law (5.23) and the Lie derivative commutator $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$ where $[X, Y]$ is the Lie bracket of the vectors X and Y :

$$\begin{aligned} [\delta_u, \delta_{u'}] &= -\delta_{[u, u']}, & [\delta_u, \delta_v^\times] &= -\delta_{[u, v]}^\times, \\ [\delta_v^\times, \delta_{v'}^\times] &= -\rho \delta_{[v, v']}. \end{aligned} \quad (5.28)$$

5.3 Constraints associated with the crossed diffeomorphisms

Four first class constraints generating the usual spatial diffeomorphism are given by (5.19). We want now to find four first class constraints, linear combinations of the primary constraints of ϕ_{MNi}^a , which generate the crossed spatial diffeomorphisms. For this we will use the fact that the diffeomorphisms (5.23) given by the Lie derivative, can be equiva-

lently represented by so-called improved diffeomorphisms [16], given by

$$\delta_\xi^{\text{impr}} \mathcal{A}_\mu^a = \xi^\nu \mathcal{F}_{\nu\mu}^a, \quad (5.29)$$

since these differ from the Lie derivative only by a gauge transformation:

$$\xi^\nu \mathcal{F}_{\nu\mu} = \mathcal{L}_\xi \mathcal{A}_\mu - \delta_{(i, \mathcal{A})}^{\text{gauge}} \mathcal{A}_\mu.$$

Recalling that \mathcal{A} as well as ξ depend on the group elements λ_0 and λ_1 (see (4.3) and (5.22)), we can rewrite (5.29) for the usual (parameter u) and crossed (parameter v) diffeomorphisms of the fields A and B , with F and G their associated curvatures (4.7)

$$\begin{aligned} \delta^{\text{impr}} A &= i_u F, & \delta^{\text{impr}} B &= i_u G, \\ \delta^{\times, \text{impr}} A &= \rho i_v G, & \delta^{\times, \text{impr}} B &= i_v F \end{aligned} \quad (5.30)$$

The expression (5.19) for the usual diffeomorphism constraint and a comparison between the two lines of (5.30) suggest the expression

$$H_a^\times = \rho G_{ab}^{MN} \phi_{MN0}^b + F_{ab}^{MN} \phi_{MN1}^b, \quad (5.31)$$

for the generators of the crossed spatial diffeomorphism constraints. This is readily checked to be true:

$$\begin{aligned} &\left\{ \mathcal{A}_a^{MNi}(x), \int d^4x v^c(y) \right. \\ &\quad \times \left. \left(\rho G_{cd}^{PQ}(y) \phi_{PQ0}^d(y) + F_{cd}^{PQ}(y) \phi_{PQ1}^d(y) \right) \right\} \\ &= v^c \left(\rho G_{ca}^{MN} + F_{ca}^{MN} \right) = i_v \left(\rho G^{MN} + F^{MN} \right)_a. \end{aligned}$$

An important consequence of these considerations is that the C_5 theory is indeed generic according to the definition of genericity given at the beginning of Sect. 2.2.

5.4 Counting the constraints and the degrees of freedom

Having thus found 8 first class constraints associated with the generalized diffeomorphisms, all of them being linear combinations of the 120 primary constraints ϕ_{MNi}^a , we conclude that the rank N_2 of the matrix Ω , hence the number of second class constraints, cannot exceed 112. In view of the inequality (5.21), we conclude that we have exactly $N_2 = 112$ second class constraints. Hence the number of first class constraints is $N_1 = 38$: the 30 constraints (5.16), the 4 constraints (5.19) and the 4 constraints (5.31) generating, respectively, the C_5 gauge transformations, the spatial diffeomorphisms and the crossed spatial diffeomorphisms.

The number of physical degrees of freedom $N_{\text{d.o.f}}$ is given by the formula [16, 17]

$$N_{\text{d.o.f}} = \frac{1}{2} (D_{\text{phase}} - 2N_1 - N_2), \quad (5.32)$$

where $D_{\text{phase}} = 240$ is the dimension of the original phase space of generalized coordinates and momenta \mathcal{A}_a^{MNi} and p_{MNi}^a . The theory thus has 26 physical degrees of freedom.

6 Conclusions

Our first result concerns a fact about the algebra C_5 defined as the expansion of the 5D (anti-)de Sitter by the reduced group \mathcal{S} whose multiplication table is displayed in Table 1: we have shown that there are two independent symmetric invariant tensors of rank 3 in its adjoint representation, given by the Eq. (3.16), instead of four as claimed by the authors of [18], who consider the same gauge group. This result provides indeed a counter-example to the Theorem 2 in Section 5 of [18].

The theory thus depends on two coupling constants, appearing in the Chern–Simons action we have constructed for the C_5 gauge invariance. The C_5 connection 1-form \mathcal{A} decomposes into the (anti-)de Sitter connection 1-form A and a 1-form B transforming in the adjoint representation of the latter algebra. The geometric part of the action, obtained by taking $B = 0$, is identical to the (a)ds₅ one, as originally given in [5] and summarized in Sect. 2. When written in terms of the 5-bein e and the 5D Lorentz connection ω (see (2.11)), the latter action shows, beyond the Einstein–Hilbert and cosmological terms a Gauss–Bonnet type term present which cannot be eliminated by any choice of the coupling constants – in contradiction with the result of [18].

In third place, we have performed a complete canonical analysis of the dynamics of the theory, separating the total of 150 constraints into 112 second class ones and 38 first class ones, 30 of the latter being the generators of the C_5 gauge transformations and 8 being the generators of the generalized four-dimensional space diffeomorphisms. This result has allowed us to count the number of physical degrees of freedom of the theory, 26.

It is the canonical analysis which has led us to the extra local symmetry, called crossed diffeomorphism invariance, which, together with the usual diffeomorphism invariance, constitutes the generalized diffeomorphism invariance. Another important by-product is that the theory is generic, *i.e.*, there is no independent constraint corresponding to the – usual and crossed – temporal diffeomorphisms.

A possible unfolding of this work may be a realistic phenomenological analysis of the theory, making a dimensional reduction of 5D to 4D, in a way similar to the one performed in [10], which presents solutions of the Schwarzschild type and others compatible with the Λ CDM cosmological model, despite the presence of the Gauss–Bonnet term. Another prospect can be the quantization of the model. It would avoid the difficulty, found in the quantization of General Relativity in 4D, of solving the constraint associated with the invariance

under the temporal diffeomorphisms, thanks to the genericity of the C_5 theory.

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Appendices

A Notations and Conventions

- Units adopted are such that $c = 1$.
- The indices $M, N, P, Q, \dots = 0, 1, \dots, 5$ are indices referring to (A)dS₅.
- The indices $A, B, C, D, \dots = 0, 1, \dots, 4$ are Lorentz indices in 5D.
- The indices $\alpha, \beta, \dots = 0, \dots, 3$ are indices referring to the group \mathbb{Z}_4 .
- The indices $i, j, \dots = 0, 1$ are indices referring to the first two elements of the group \mathbb{Z}_4 .
- The indices $\mu, \nu, \dots = 0, 1, \dots, 4$ are space-time indices.
- The indices $a, b, \dots = 1, 2, 3, 4$ are spatial indices.
- The metric for (A)dS₅ is $\eta_{MN} = \text{diag}(-1, 1, 1, 1, s)$, with $s = \pm 1$ (+1 refers to de Sitter and –1 to anti-de Sitter).

B Invariant symmetric rank 3 tensors of C_5

There is a unique invariant symmetric rank 3 tensor of the Lie algebra (a)ds₅, solution of the conditions (3.14), given by

$$g_{MN,PQ,RS} = \varepsilon_{MNPQRS}, \tag{B.1}$$

where ε_{MNPQRS} is the 6D Levi-Civita antisymmetric tensor, and each antisymmetric pair MN , etc. has to be considered as a multi-index, equivalent to an (a)ds₅ index taking values from 1 to 15.

In the case of the C_5 algebra, the invariance condition for the symmetric rank 3 invariant tensor g_{MNi_1,PQi_2,RSi_3} is given by (3.15), where we have now multi-indices MNi taking 30 values. Since the indexes $i, i_1, i_2, i_3 = 0, 1$ only, we have 16 possibilities of combining these indices, leading to 16 equations that the invariant tensors must obey. Let us begin by the equation corresponding to $(i, i_1, i_2, i_3) = (0, 0, 0, 0)$:

$$\begin{aligned} &S_{00}^0 f_{MN,M_1N_1}{}^{PQ} g_{PQO,M_2N_2O,M_3N_3O} \\ &+ S_{00}^0 f_{MN,M_2N_2}{}^{PQ} g_{M_1N_1O,PQO,M_3N_3O} \\ &+ S_{00}^0 f_{MN,M_3N_3}{}^{PQ} g_{M_1N_1O,M_2N_2O,PQO} = 0, \end{aligned} \tag{B.2}$$

Here, the f_{\dots} are the (a)ds₅ structure constants (2.3). The coefficients $S_{ij}{}^k$ being here all equal to 1, we see that $g_{MN0,PQ0,RS0}$ obeys the (a)ds₅ invariance condition (3.14). Hence $g_{MN0,PQ0,RS0} = x_{000} \varepsilon_{MNPQRS}$, where x_{000} is an arbitrary coefficient. Proceeding in the same way for the seven other cases $(0, i_1, i_2, i_3)$, we arrive at

$$g_{MNi_1,PQi_2,RSi_3} = x_{i_1i_2i_3} \varepsilon_{MNPQRS}. \tag{B.3}$$

The coefficients $x_{i_1i_2i_3}$ are completely symmetric in there indices due to the required symmetry of the tensor g . We have thus four independent parameters $x_{000}, x_{100}, x_{110}$ and x_{111} , for the time being. Consider now the equations (3.15) for $(i, i_1, i_2, i_3) = (1, 1, 0, 0)$, using the result (B.3):

$$\begin{aligned} &\rho f_{MN,M_1N_1}{}^{PQ} x_{000} \\ &+ (f_{MN,M_2N_2}{}^{PQ} + f_{MN,M_3N_3}{}^{PQ}) x_{110} = 0, \end{aligned}$$

where $\rho = \pm 1$ is the value of the 2-selector S_{11}^0 , see (3.12). Due to the invariance condition for the Levi-Civita tensor ε (see Eq. (3.14) with $g_{\dots\dots} = \varepsilon_{\dots\dots}$), the latter equation reduces to

$$f_{MN,M_1N_1}{}^{PQ} (\rho x_{000} - x_{110}) = 0,$$

hence $x_{110} = \rho x_{000}$. In the same way, now with $(i, i_1, i_2, i_3) = (1, 1, 1, 0)$, we find $x_{111} = \rho x_{100}$.

Thus, the most general symmetric rank 3 tensor in the adjoint representation of (a)ds₅ depends on two parameters, $c_0 = x_{000}$ and $c_1 = x_{100}$, so that $x_{110} = \rho c_0$ and $x_{111} = \rho c_1$. With the 3-selector $S_{ijk}{}^l$ given by (3.17), we can now write $x_{i_1i_2i_3} = c_l S_{i_1i_2i_3}{}^l$, as can readily be verified. This, with the use of the result (B.3), allows us to express the invariant tensor in the compact form

$$g_{MNi,PQj,RSk} = c_l S_{ijk}{}^l \varepsilon_{MNPQRS}.$$

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