THE EUROPEAN
PHYSICAL JOURNAL C

Robustness of the quantum BKL scenario

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Abstract The quantum Belinski–Khalatnikov–Lifshitz scenario presents an unitary evolution of the system. However, the affine coherent states quantization applied to the quantization of the underlying classical scenario depends on the choice of the group parametrization. Using the two simplest parameterizations of the affine group, we show that qualitative features of our quantum system do not depend on the choice. It means that the quantum bounce replacing a singular classical scenario is expected to be a generic feature of the considered system. This paper complements our recent article (Góźdź et al. in Eur Phys J C 79:45, 2019).

1 Introduction

Recently, we have found that the affine coherent states (ACS) quantization depends on the parametrization of the affine group [2]. Since our paper [1] concerning the quantization of the Belinski–Khalatnikov–Lifshitz (BKL) scenario is based on the ACS quantization, it is reasonable to examine the dependence of the results on the group parametrization. This is the main motivation of the present paper. To have analytical results, as in a previous paper [1], we consider the second, the most popular parametrization of the affine group.

It is worth to recall that the BKL scenario concerns the generic singularity of general relativity (see [3–8]). The resolution of the singularity at the quantum level is of primary importance for the quantum gravity programme. Recently, we have found [1] that the quantum BKL scenario presents an unitary process, so that the singular classical BKL evolution is replaced by a regular quantum bounce. However, the quantization method we have applied is not unique (as any quantization scheme). Thus, the examination of the robust-

ness of the obtained results is an important issue that cannot be omitted.

2 Classical dynamics

For self-consistency of the present paper, we first recall the main results of Ref. [1].

The two form Ω defining the Hamiltonian formulation, devoid of the dynamical constraints, is given by

$$\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + dt \wedge dH, \tag{1}$$

where the variables $\{q_1, q_2, p_1, p_2\}$ parameterize the phase space, H is the Hamiltonian generating the dynamics, and where t is an evolution parameter (time) corresponding to the specific choice of H. The Hamiltonian reads

$$
\begin{aligned}
H(t, q_1, q_2, p_1, p_2) := -q_2 - \ln\Bigg[&-e^{2q_1} - e^{q_2-q_1} \\
&-\frac{1}{4}(p_1^2 + p_2^2 + t^2) \\
&+\frac{1}{2}(p_1 p_2 + p_1 t + p_2 t)\Bigg] \\
=: -q_2 &- \ln F(t, q_1, q_2, p_1, p_2), \quad (2)
\end{aligned}
$$

where $F(t, q_1, q_2, p_1, p_2) > 0$.

The examination of the topology of the phase space and well-definedness of the logarithmic function in (2) requires [1]: (i) $(p_1, p_2) \in \mathbb{R}_+^2$, where $\mathbb{R}_+ := \{p \in \mathbb{R} \mid p > 0\}$, and (ii) $p_1 \to 0$ and $p_2 \to 0$ implies $t \to 0^+$. Thus, the considered gravitational system evolves away from the singularity at $t = 0$. The range of the variables q_1 and q_2 results from the physical interpretation ascribed to them [9] so that $(q_1, q_2) \in \mathbb{R}^2$. Thus, the physical phase space Π consists of the two half planes:

$$
\begin{aligned}
\Pi &= \Pi_1 \times \Pi_2 \\
&:= \{(q_1, p_1) \in \mathbb{R} \times \mathbb{R}_+\} \times \{(q_2, p_2) \in \mathbb{R} \times \mathbb{R}_+\}. \tag{3}
\end{aligned}
$$

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It is important to notice that only the subspace

$$\tilde{\Pi} = \{(q_1, p_1, q_2, p_2) \colon F(t, q_1, p_1, q_2, p_2) > 0\} \subset \Pi \quad (4)$$

is available to the dynamics. It is due to the logarithmic function in the expression defining the Hamiltonian. To make this restriction explicit, we rewrite the Hamiltonian (2) in the form

$$
\begin{aligned}
&H(t, q_1, q_2, p_1, p_2) \\
&= \begin{cases} -q_2 - \ln F(t, q_1, q_2, p_1, p_2), & \text{for } F(t, q_1, q_2, p_1, p_2) > 0 \\ 0, & \text{for } F(t, q_1, q_2, p_1, p_2) < 0 \end{cases} \quad (5)
\end{aligned}
$$

with $\lim_{F \to 0^-} H = 0$ and $\lim_{F \to 0^+} H = +\infty$.

3 Hilbert space and quantum observables

Each Π_k ($k = 1, 2$) can be identified with the manifold of the affine group $G := \text{Aff}(\mathbb{R})$ acting on $\mathbb{R}$, which is sometimes denoted as "$px + q$". In the case considered in [1] the actions of this group on $\mathbb{R}_+$ are defined to be

$$x' = (\tilde{q}, \tilde{p}) \cdot x = \tilde{p}x + \tilde{q}, \quad \text{where} \quad (\tilde{q}, \tilde{p}) \in \mathbb{R} \times \mathbb{R}_+, \quad (6)$$

and the corresponding multiplication law of the group G reads

$$(\tilde{q}', \tilde{p}') \cdot (\tilde{q}, \tilde{p}) = (\tilde{p}'\tilde{q} + \tilde{q}', \tilde{p}'\tilde{p}). \quad (7)$$

In the present paper we apply another simple parametrization, considered in [2,10], with the action of the group G on $\mathbb{R}_+$ defined as

$$x' = (q, p) \cdot x = x/p + q, \quad \text{where} \quad (q, p) \in \mathbb{R} \times \mathbb{R}_+. \quad (8)$$

The corresponding multiplication law of the group is defined to be

$$(q', p') \cdot (q, p) = (q/p' + q', p'p). \quad (9)$$

The affine group $G = \text{Aff}(\mathbb{R})$ has two (nontrivial) inequivalent irreducible unitary representations [11–13], defined in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) := dx/x$. In what follows, we choose the one defined by

$$U(q, p)\Psi(x) := e^{iqx}\Psi(x/p), \quad (10)$$

where $\Psi \in L^2(\mathbb{R}_+, d\nu(x))$.

Integration over the affine group is defined as

$$\int_G d\mu(q, p) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \int_0^{\infty} dp, \quad (11)$$

where the measure in (11) is left invariant.

Any coherent state can be obtained:

$$\langle x|q, p \rangle = U(q, p)\Phi(x), \quad (12)$$

where $L^2(\mathbb{R}_+, d\nu(x)) \ni \Phi(x) = \langle x|\Phi\rangle$, with $\langle \Phi|\Phi\rangle = 1$, is the so-called fiducial vector.

The resolution of the identity in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$ reads

$$\int_G d\mu(q, p)|q, p\rangle\langle q, p| = A_\Phi \mathbb{I}, \quad (13)$$

where

$$A_\Phi = \int_0^{\infty} \frac{dx}{x^2}|\Phi(x)|^2 < \infty. \quad (14)$$

3.1 Affine coherent states for the entire system

Here, we again recall some essentials of the formalism of [1], and insert suitable modifications resulting from the different parametrization (8) of the affine group.

In the Cartesian product $\Pi = \Pi_1 \times \Pi_2$, the partial phase spaces Π_1 or Π_2 are identified with the corresponding affine groups $G_1 = \text{Aff}_1(\mathbb{R})$ or $G_2 = \text{Aff}_2(\mathbb{R})$. The product of both affine groups $G_\Pi = G_1 \times G_2$ can be identified with the whole phase space and its action reads

$$
\begin{aligned}
\Pi \ni (\xi_1, \xi_2) \to |\xi_1, \xi_2\rangle &= U(\xi_1, \xi_2)|\Phi\rangle \\
&:= U_1(\xi_1) \otimes U_2(\xi_2)|\Phi\rangle \in \mathcal{H},
\end{aligned}
$$
$$(15)$$

where $\xi_k = (q_k, p_k)$ (with $k = 1, 2$), and where the entire Hilbert space is the tensor product of two Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\nu(x_1, x_2))$ with the measure $d\nu(x_1, x_2) = d\nu(x_1)d\nu(x_2)$. The scalar product in $\mathcal{H}$ is defined as

$$\langle \psi_2|\psi_1\rangle = \int_0^{\infty} d\nu(x_1) \int_0^{\infty} d\nu(x_2)\psi_1(x_1, x_2)^{\star}\psi_2(x_1, x_2). \quad (16)$$

The fiducial vector $\langle x_1, x_2|\Phi\rangle = \Phi(x_1, x_2)$ is a product of two fiducial vectors $\Phi(x_1, x_2) = \Phi_1(x_1)\Phi_2(x_2)$. See [1] for some subtleties concerning the choice of the vector Φ.

Finally, the explicit form of the action of the group G_Π on the vector $\langle x_1, x_2|\Psi\rangle = \Psi(x_1, x_2) \in \mathcal{H}$, in the parametrization (8), reads [1]

$$U(q_1, p_1, q_2, p_2)\Psi(x_1, x_2) = e^{iq_1 x_1}e^{iq_2 x_2}\Psi(x_1/p_1, x_2/p_2). \quad (17)$$

3.2 Quantum observables

Making use of the resolution of identity in the Hilbert space $\mathcal{H}$, we define the quantization of a classical observable f defined in the phase space Π as follows [1]:

$$\hat{f}(t) = \frac{1}{A_{\Phi_1}A_{\Phi_2}} \int_{G_\Pi} d\mu(\xi_1, \xi_2)|\xi_1, \xi_2\rangle f(\xi_1, \xi_2)\langle \xi_1, \xi_2|, \quad (18)$$

where $d\mu(\xi_1, \xi_2) := d\mu(q_1, p_1)d\mu(q_2, p_2)$.

We recommend [1] for a discussion of the properties of the mapping that leads to (18). If the operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ is unbounded, its possible self-adjoint extensions require further examination [14–16].

4 Quantum dynamics

The mapping (18) applied to the classical Hamiltonian reads

$$\hat{H}(t) = \frac{1}{A_{\Phi_1} A_{\Phi_2}} \int_{G_\Pi} d\mu(\xi_1, \xi_2) |\xi_1, \xi_2\rangle H(t, \xi_1, \xi_2) \langle \xi_1, \xi_2|, \tag{19}$$

where t is an evolution parameter of the classical level and where

$$\int_{G_\Pi} d\mu(\xi_1, \xi_2)$$
$$:= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_2 \int_0^{+\infty} dp_1 \int_0^{+\infty} dp_2. \tag{20}$$

In our article [1] we apply the reduced phase space quantization. It means that we quantize the classical system with a dynamical constraint already resolved. Its Hamilton dynamics, corresponding to (1), includes the generator of the evolution in the physical phase space, i.e. the Hamiltonian H, and the corresponding evolution parameter t. As this Hamiltonian system has no dynamical constraint, no quantum constraint occurs. This is quite different from the Dirac quantization where the classical constraint[1] is kept unsolved and is promoted to the quantum level so that it leads to an operator type equation. The latter serves as the quantum transformation that sometimes can be used to define a kind of quantum evolution, but in most cases it stays timeless (see [17] for more details).

As H and t is a classical canonical pair in (1), it is reasonable to assume that the quantum operator $\hat{H}$ corresponding to H is a generator of the evolution of the system in the Hilbert space $\mathcal{H}$. More precisely, the operator $\hat{H}$ is the generator of translations of the wave function of our quantum system with the corresponding shift parameter τ. It is natural to identify the classical shift parameter t and the quantum shift parameter τ, i.e. we assume $\tau = t$. This is a reasonable assumption as t changes monotonically [1], and it introduces consistency between the classical and quantum levels. Assuming the above identification of the evolution parameters, the translation of the system from t_0 to t is represented by the unitary operator $U(t, t_0)$ generated by $\hat{H}(t)$. The standard properties of the unitary evolution operators in the Hilbert space $\mathcal{H}$,

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[1] For simplicity we assume there is only one constraint.

$$U(t, t) = 1, \quad U(t, t_0)^\dagger = U(t_0, t) = U(t, t_0)^{-1},$$
$$U(t_2, t_0) = U(t_2, t_1) U(t_1, t_0), \tag{21}$$

and continuity imply $\Psi(t) = U(t, t_0) \Psi(t_0)$. It further means that the quantum evolution of our gravitational system can be equivalently defined by a Schrödinger type equation:

$$i \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle. \tag{22}$$

The classical time t occurs in (22) because it enters the integrand of (19). We do not quantize the classical time t. In the case t were a quantum observable, it would be mapped into a quantum operator [18], but we do not consider here such a case.

4.1 Classical dynamics near the singularity

Near the gravitational singularity, the terms $\exp(2q_1)$ and $\exp(q_2 - q_1)$ in the function F can be neglected (see [1] for more details) so that we have

$$F(t, q_1, q_2, p_1, p_2) \longrightarrow F_0(t, p_1, p_2)$$
$$:= p_1 p_2 - \frac{1}{4}(t - p_1 - p_2)^2. \tag{23}$$

This form of F leads to the simplified form of the Hamiltonian (5) which now reads

$$H_0(t, q_2, p_1, p_2)$$
$$:= \begin{cases} -q_2 - \ln F_0(t, p_1, p_2), & \text{for} \quad F_0(t, p_1, p_2) > 0, \\ 0, & \text{for} \quad F_0(t, p_1, p_2) < 0, \end{cases} \tag{24}$$

with $\lim_{F_0 \to 0^-} H_0 = 0$ and $\lim_{F_0 \to 0^+} H_0 = +\infty$. In fact, the condition

$$F_0(t, p_1, p_2) > 0 \tag{25}$$

defines the available part of the physical phase space Π for the classical dynamics, defined by (3), which corresponds to the approximation (23). Equations (23)–(24) define the approximation to our original Hamiltonian system describing the dynamics in the close vicinity of the singularity.

4.2 Quantum dynamics near the singularity

Calculations similar to the ones carried out in our paper [1], applied to the Hamiltonian (24), lead to the Schrödinger equation (22) in the form

$$i \frac{\partial}{\partial t} \Psi(t, x_1, x_2)$$
$$= \left(i \frac{\partial}{\partial x_2} - \frac{i}{2x_2} - \tilde{K}(t, x_1, x_2) \right) \Psi(t, x_1, x_2), \tag{26}$$

where $\Psi(t, x_1, x_2) := \langle x_1, x_2|\Psi(t)\rangle$. The function $\tilde{K}$ reads

$$
\tilde{K}(t, x_1, x_2) := \frac{1}{A_{\Phi_1} A_{\Phi_2}}
$$
$$
\times \int_0^\infty dp_1 \int_0^\infty dp_2 \, \widetilde{\ln}\big(F_0(t, p_1, p_2)\big)
$$
$$
\times |\Phi_1(x_1/p_1)|^2 |\Phi_2(x_2/p_2)|^2, \tag{27}
$$

where

$$
\widetilde{\ln}\big(F_0(t, p_1, p_2)\big)
$$
$$
:= \begin{cases} \ln\big(F_0(t, p_1, p_2)\big), & \text{for} \quad F_0(t, p_1, p_2) > 0, \\ 0, & \text{for} \quad F_0(t, p_1, p_2) < 0. \end{cases} \tag{28}
$$

The fiducial function $\Phi_2(x) \in \mathbb{R}$ should satisfy the conditions

$$
\Phi_2(x) =: x\tilde{\Phi}(x), \quad \lim_{x \to 0^+} \tilde{\Phi}(x) = 0, \quad \lim_{x \to +\infty} \tilde{\Phi}(x) = 0, \tag{29}
$$

and the solution to (26) is expected to have the properties

$$
\Psi(t, x_1, x_2) =: \sqrt{x_2}\,\tilde{\Psi}(t, x_1, x_2),
$$
$$
\lim_{x_2 \to 0^+} \tilde{\Psi}(t, x_1, x_2) = 0, \quad \lim_{x_2 \to +\infty} \tilde{\Psi}(t, x_1, x_2) = 0. \tag{30}
$$

The mathematical structure of Eq. (26) is similar to the corresponding one of Ref. [1]. The difference concerns just one part of these equations, namely the actual function $\tilde{K}(t, x_1, x_2)$ and the function $K(t, x_1, x_2)$ in [1]. Therefore, the general solution to (26) reads

$$
\Psi(t, x_1, x_2) = \eta(x_1, x_2 + t - t_0)
$$
$$
\sqrt{\frac{x_2}{x_2 + t - t_0}} \exp\left(i \int_{t_0}^t \tilde{K}(t', x_1, x_2 + t - t')\,dt'\right), \tag{31}
$$

where $t \geq t_0 > 0$, and where $\eta(x_1, x_2) := \Psi(t_0, x_1, x_2)$ is the initial state satisfying the condition

$$
\eta(x_1, x_2) = 0 \quad \text{for} \quad x_2 < t_H, \tag{32}
$$

with $t_H > 0$ being the parameter of our model. This condition is consistent with (30) and for $t < t_H$ we get (see [1])

$$
\langle\Psi(t)|\Psi(t)\rangle = \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty \frac{dx_2}{x_2} |\eta(x_1, x_2)|^2, \tag{33}
$$

so that the inner product is time independent, which implies that the quantum evolution is unitary. Due to (32), the probability of finding the system in the region with $x_2 < t_H$ vanishes so that this region does not contribute to the expectation values of observables. These results are consistent with the results of [1].

Since the mathematical structure of the dynamics presented here and in [1] are quite similar, the operation of time

reversal turns (26) into the equation

$$
i\frac{\partial}{\partial t}\tilde{\Psi}(t, x_1, x_2)
$$
$$
= \left(-i\frac{\partial}{\partial x_2} + \frac{i}{2x_2} - \tilde{K}(-t, t, x_1, x_2)\right)\tilde{\Psi}(t, x_1, x_2), \tag{34}
$$

where $\tilde{\Psi}(t, x_1, x_2) := \Psi(-t, x_1, x_2)^*$. Consequently, the solution to (34) for $t < 0$ reads

$$
\tilde{\Psi}(t, x_1, x_2) = \eta(x_1, x_2 + |t| - |t_0|)
$$
$$
\sqrt{\frac{x_2}{x_2 + |t| - |t_0|}} \exp\left(i\int_{t_0}^t \tilde{K}(-t', x_1, x_2 - t + t')\,dt'\right), \tag{35}
$$

where $|t| \geq |t_0|$, and where $\eta(x_1, x_2) := \tilde{\Psi}(t_0, x_1, x_2)$ is the initial state.

The unitarity of the evolution (with $t_0 = 0$) can be obtained again if

$$
\eta(x_1, x_2) = 0 \quad \text{for} \quad x_2 < |t_H|, \tag{36}
$$

which corresponds to the condition (32).

Since the solutions (31) and (35) differ only by the corresponding phases, the probability density is continuous at $t = 0$, which means that we are dealing with a quantum bounce at $t = 0$ (which marks the classical singularity).

5 Conclusions

The quantum dynamics we have obtained does not depend essentially on the applied parametrization of the affine group. Two different parametrizations give qualitatively the same results, which differ only slightly quantitatively. The latter is meaningless if we only insist on the main result which is the resolution of the classical singularity.

We have applied the simplest two group parametrizations. The general one can be presented in the form of the action of the group on $\mathbb{R}_+$ as follows [2]:

$$
\mathbb{R}_+ \ni x \to x' = \xi(p, q) \cdot x + \eta(p, q) \cdot p \in \mathbb{R}_+. \tag{37}
$$

We expect that in the case $\xi(p, q) = \tilde{\xi}(q)$ and $\eta(p, q) = \tilde{\eta}(p)$, the result of quantization will be qualitatively the same as the one obtained in the present paper.

The effect of using a quite general parametrization considered in [2], applied to the quantization of our gravitational system, would need separate examination and is beyond the scope of the present paper. We may stay with the simplest parametrizations if we do not test the quantization method as such, but we intend to get the result with satisfactory physics. After experimental or observational data on quantum grav-

ity become available, the way of choosing the most suitable group parametrization will obtain a sound guideline.

References

1. A. Góźdź, W. Piechocki, G. Plewa, Quantum Belinski–Khalatnikov-Lifshitz scenario. Eur. Phys. J. C **79**, 45 (2019)
2. A. Góźdź, W. Piechocki, T. Schmitz, On the dependence of the affine coherent states quantization on the parametrization of the affine group. arXiv:1908.100039 [math-ph]
3. V.A. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology. Adv. Phys. **19**, 525 (1970)
4. V.A. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, A general solution of the Einstein equations with a time singularity. Adv. Phys. **31**, 639 (1982)
5. V. Belinski, M. Henneaux, *The Cosmological Singularity* (Cambridge University Press, Cambridge, 2017)
6. V.A. Belinskii, I.M. Khalatnikov, M.P. Ryan, The oscillatory regime near the singularity in Bianchi-type IX universes, vol. 469 (1971), Landau Institute for Theoretical Physics, Moscow (unpublished) **(preprint)**
7. V.A. Belinskii, I.M. Khalatnikov, M.P. Ryan, Published as Secs. 1 and 2 in M. P. Ryan. Ann. Phys. **70**, 301 (1971)
8. C. Kiefer, N. Kwidzinski, W. Piechocki, On the dynamics of the general Bianchi IX spacetime near the singularity. Eur. Phys. J. C **78**, 691 (2018)
9. E. Czuchry, W. Piechocki, Bianchi IX model: reducing phase space. Phys. Rev. D **87**, 084021 (2013)
10. C.R. Almeida, H. Bergeron, J.-P. Gazeau, A.C. Scardua, Three examples of quantum dynamics on the half-line with smooth bouncing. Ann. Phys. **392**, 206 (2018)
11. I.M. Gel'fand, M.A. Naïmark, Unitary representations of the group of linear transformations of the straight line. Dokl. Akad. Nauk. SSSR **55**, 567 (1947)
12. E.W. Aslaksen, J.R. Klauder, Unitary Representations of the Affine Group. J. Math. Phys. **9**, 206 (1968)
13. E.W. Aslaksen, J.R. Klauder, Continuous representation theory using unitary affine group. J. Math. Phys. **10**, 2267 (1969)
14. M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, vols I and II (Academic Press, San Diego, 1980),
15. K. Schmüdgen, *Unbounded Self-adjoint Operatotrs on Hilbert Space* (Springer, San Francisko, 2012)
16. D.M. Gitman, I.V. Tyutin, B.L. Voronov, *Self-adjoint Extensions in Quantum Mechanics* (Springer, Birkhäuser, 2012)
17. M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992)
18. A. Góźdź, M. Góźdź, A. Pędrak, Projection evolution of quantum states. Part I. arXiv:1910.111987 [quant-ph]