



Remarks on the non-Riemannian sector in Double Field Theory

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Abstract Taking $\mathbf{O}(D, D)$ covariant field variables as its truly fundamental constituents, Double Field Theory can accommodate not only conventional supergravity but also non-Riemannian gravities that may be classified by two non-negative integers, (n, \bar{n}) . Such non-Riemannian backgrounds render a propagating string chiral and anti-chiral over n and \bar{n} dimensions respectively. Examples include, but are not limited to, Newton–Cartan, Carroll, or Gomis–Ooguri. Here we analyze the variational principle with care for a generic (n, \bar{n}) non-Riemannian sector. We recognize a nontrivial subtlety for $n\bar{n} \neq 0$ that infinitesimal variations generically include those which change (n, \bar{n}) . This seems to suggest that the various non-Riemannian gravities should better be identified as different solution sectors of Double Field Theory rather than viewed as independent theories. Separate verification of our results as string worldsheet beta-functions may enlarge the scope of the string landscape far beyond Riemann.

Contents

1	Introduction	1
1.1	Double Field Theory as the $\mathbf{O}(D, D)$ completion of General Relativity	2
1.2	Doubled but at the same time gauged string action	3
2	Review of [1]: classification of the non-Riemannian DFT geometries	4
3	Variational principle around non-Riemannian backgrounds	6
3.1	Variations of the DFT-metric around a generic (n, \bar{n}) background	7
3.2	Einstein Double Field Equations still hold for non-Riemannian sectors	8
4	What if we keep (n, \bar{n}) fixed once and for all?	9
4.1	Variational principle with fixed (n, \bar{n})	9

4.2	Difference between keeping (n, \bar{n}) fixed or not	10
4.3	Non-Riemannian differential geometry as book-keeping device	11
5	Conclusion	14
Appendix		15
A	Proof on the equivalence among (4.7), (4.8), (4.9), and (4.10)	15
B	Derivation of the non-Riemannian differential tool kit from DFT	16
References		17

1 Introduction

This paper is a sequel to [1] which proposed to classify all the possible geometries of Double Field Theory (DFT) [2–7] by two non-negative integers, (n, \bar{n}) . The outcome – which we shall review in Sect. 2 – is that only the case of $(0, 0)$ corresponds to conventional supergravity based on Riemannian geometry. Other generic cases of $(n, \bar{n}) \neq (0, 0)$ do not admit any invertible Riemannian metric and hence are *non-Riemannian* by nature. Strings propagating on these backgrounds become chiral and anti-chiral over n and \bar{n} dimensions respectively.

The non-Riemannian property is a point-wise or local statement [8–12] and differs from the global notion of ‘non-geometry’ [13–17] which is also well described by DFT [18–32]. Possible examples of non-Riemannian geometries include Newton–Cartan geometry [33–35] as $(1, 0)$, stringy Newton–Cartan [36] as $(1, 1)$, (wonderland) Carroll geometry [37, 38] as $(D - 1, 0)$, and non-relativistic Gomis–Ooguri string theory [39] as $(1, 1)$. These are of continuous interest, e.g. [40–58]. Further, the fully $\mathbf{O}(D, D)$ symmetric vacua of Double Field Theory turn out to be ‘maximally’ non-Riemannian, being of either $(D, 0)$ or $(0, D)$ type, compelling string to be completely chiral or anti-chiral. A remarkable insight from [11] is that, the ordinary Riemannian space-time arises after spontaneous symmetry breaking of these

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fully $\mathbf{O}(D, D)$ symmetric vacua while identifying the Riemannian metric, $g_{\mu\nu}$, as a Nambu–Goldstone boson.

In this work we attempt to explore the dynamics of the generic (n, \bar{n}) sector in Double Field Theory. We analyze with care the relevant variational principle and recognize a nontrivial subtlety: when $n\bar{n} \neq 0$, the resulting Euler–Lagrangian equations of motion depend whether the variations of the action keep the values of (n, \bar{n}) fixed or not. This rather unexpected subtle discrepancy contrasts DFT with the traditional approaches to the various non-Riemannian gravities.

The organization of the present paper is as follows.

In the remaining of this Introduction, to put the present work into context and set up notation, we describe DFT as the $\mathbf{O}(D, D)$ completion of General Relativity along with a relevant doubled string action.

In Sect. 2, we review the (n, \bar{n}) classification of the non-Riemannian DFT geometries from [1].

In Sect. 3, we revisit the variational principle in DFT and confirm that the known Euler–Lagrangian equations, or ‘Einstein Double Field Equations’ (1.3) are still valid for non-Riemannian sectors.

In Sect. 4, now keeping (n, \bar{n}) fixed, we reanalyze the variational principle and show that the full Einstein Double Equations are not necessarily implied when $n\bar{n} \neq 0$. We explain the discrepancy, and further propose a non-Riemannian differential tool kit as a ‘bookkeeping device’ to expound the equations.

We conclude in Sect. 5, followed by Appendix A and B.

1.1 Double Field Theory as the $\mathbf{O}(D, D)$ completion of General Relativity

While the initial motivation of Double Field Theory was to reformulate supergravity in an $\mathbf{O}(D, D)$ manifest manner [2–7] ([59–61] for reviews), through subsequent further developments [62–65], DFT has evolved and can be now identified as a pure gravitational theory that string theory seems to predict foremost¹ and may differ from General Relativity as it is capable of describing non-Riemannian geometries [1]. Specifically, DFT is the string theory based, $\mathbf{O}(D, D)$ completion of General Relativity (GR): taking the $\mathbf{O}(D, D)$ symmetry as the first principle, DFT geometrises not merely the Riemannian metric but the whole massless NS-NS sector of closed string as the fundamental gravitational multiplet, hence ‘completing’ GR. Further, the $\mathbf{O}(D, D)$ symmetry principle fixes its coupling to other superstring sectors (R-R [68–71], R-NS [72], and heterotic Yang–Mills [73–75]). Having said that, regardless of supersymmetry, it can also couple to various matter fields which may appear in lower dimensional effective field theories [72, 76, 77], just as

GR does so. In particular, supersymmetric extensions have been completed to the full (i.e. quartic) order in fermions for $D = 10$ cases powered by ‘1.5 formalism’ [78, 79], and the pure Standard Model without any extra physical degrees of freedom can easily couple to DFT in an $\mathbf{O}(D, D)$ symmetric manner [80].

Schematically, governed by the $\mathbf{O}(D, D)$ symmetry principle, DFT may couple to generic matter fields, say collectively Υ , which should be also in $\mathbf{O}(D, D)$ representations:

$$\int \frac{1}{16\pi G} e^{-2d} S_{(0)} + \mathcal{L}_{\text{matter}}(\Upsilon, \nabla_A \Upsilon). \tag{1.1}$$

Here, d is the $\mathbf{O}(D, D)$ singlet DFT-dilaton, $S_{(0)}$ is the DFT scalar curvature, and $\nabla_A \Upsilon$ denotes the covariant derivative of a matter field. To manifest the $\mathbf{O}(D, D)$ symmetry, the action is equipped with an $\mathbf{O}(D, D)$ invariant metric,

$$\mathcal{J}_{AB} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{1.2}$$

which, with its inverse \mathcal{J}^{AB} , is going to be always used to lower and raise the $\mathbf{O}(D, D)$ vector indices (Latin capital letters). It splits the doubled coordinates into two parts, $x^A = (\tilde{x}_\mu, x^\nu)$ and $\partial_A = (\tilde{\partial}^\mu, \partial_\nu)$. Note that the doubling of the coordinates is crucial to manifest the $\mathbf{O}(D, D)$ symmetry in DFT. Like GR, the General Covariance (DFT-diffeomorphisms) of the action (1.1) naturally gives rise to the definitions of the $\mathbf{O}(D, D)$ completions of the Einstein curvature, G_{AB} [81] and also the Energy-Momentum tensor, T_{AB} [65], of which the former and the latter are respectively off-shell and on-shell conserved. Equating the two, they comprise the $\mathbf{O}(D, D)$ completion of the Einstein field equations, or the Einstein Double Field Equations (EDFEs) [65, 82],

$$G_{AB} = 8\pi G T_{AB}. \tag{1.3}$$

We summarize the basic geometrical notation of DFT in Table 1,² while the DFT-diffeomorphisms are generated by the so-called generalized Lie derivative [3, 7]: acting on a tensor density with weight ω_T ,

$$\begin{aligned} \delta_\xi T_{A_1 \dots A_n} &= \hat{\mathcal{L}}_\xi T_{A_1 \dots A_n} = \xi^B \partial_B T_{A_1 \dots A_n} + \omega_T \partial_B \xi^B T_{A_1 \dots A_n} \\ &\quad + \sum_{j=1}^n (\partial_{A_j} \xi^B - \partial_B \xi_{A_j}) T_{A_1 \dots A_{j-1}{}^B{}_{A_{j+1} \dots A_n}. \end{aligned} \tag{1.4}$$

In particular, being a scalar density with weight one ($\omega_T = 1$), the exponentiation e^{-2d} is the integral measure of DFT.

It is noteworthy and relevant to this work that, all the geometrical notation of the covariant derivative, ∇_A , and the

² The expression of S_{AB} in Table 1 is newly derived from [63] using $\Gamma_{ACD}\Gamma^{CBD} = \Gamma^{BCD}\Gamma_{CAD} = \frac{1}{2}\Gamma_{ACD}\Gamma^{BCD}$ and $\Gamma_{CAD}\Gamma^{DBC} = \Gamma_{CAD}\Gamma^{CBD} - \frac{1}{2}\Gamma_{ACD}\Gamma^{BCD}$ which hold due to the symmetric properties, $\Gamma_{[ABC]} = 0$ and $\Gamma_{A(BC)} = 0$.

¹ At least formally let alone its phenomenological validity, c.f. [66, 67].

Table 1 Geometric notation for DFT. For latest exposition see e.g. section 2 of [65]

Integral measure	e^{-2d} (weight one scalar density)
Projectors	$P_{AB} = P_{BA} = \frac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB}), \quad \bar{P}_{AB} = \bar{P}_{BA} = \frac{1}{2}(\mathcal{J}_{AB} - \mathcal{H}_{AB})$ $P_A{}^B P_B{}^C = P_A{}^C, \quad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C, \quad P_A{}^B \bar{P}_B{}^C = 0$
Christoffel symbols	$\Gamma_{CAB} = 2(P\partial_C P\bar{P})_{[AB]} + 2(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E)\partial_D P_{EC}$ $-4\left(\frac{1}{P_M{}^M-1}P_{C[A}P_{B]}{}^D + \frac{1}{\bar{P}_M{}^M-1}\bar{P}_{C[A}\bar{P}_{B]}{}^D\right)(\partial_D d + (P\partial^E P\bar{P})_{[ED]})$
Covariant derivatives	$P_A{}^C \bar{P}_B{}^D \nabla_C V_D, \quad \bar{P}_A{}^C P_B{}^D \nabla_C V_D, \quad P^{AB} \nabla_A V_B, \quad \bar{P}^{AB} \nabla_A V_B$
Semi-covariant derivative	$\nabla_C V_D = \partial_C V_D - \omega_V \Gamma^E{}_{EC} V_D + \Gamma_{CD}{}^E V_E$
Compatibility	$\nabla_C P_{AB} = \nabla_C \bar{P}_{AB} = \nabla_C \mathcal{J}_{AB} = 0, \quad \nabla_C d = -\frac{1}{2}e^{2d} \nabla_C (e^{-2d}) = 0$
Scalar curvature	$S_{(0)} = \mathcal{H}^{AB} S_{AB}$
Ricci curvature	$(PS\bar{P})_{AB} = P_A{}^C \bar{P}_B{}^D S_{CD}$
Einstein curvature	$G_{AB} = 4P_{[A}{}^C \bar{P}_{B]}{}^D S_{CD} - \frac{1}{2}\mathcal{J}_{AB} S_{(0)}$
Semi-covariant curvature	$S_{AB} = 2\partial_A \partial_B d - e^{2d} \partial_C (e^{-2d} \Gamma_{(AB)}{}^C) + \frac{1}{2}\Gamma_{ACD} \Gamma_B{}^{CD} - \frac{1}{2}\Gamma_{CDA} \Gamma^C{}^D{}_B$
Variational property	$\delta S_{AB} = \nabla_{[A} \delta \Gamma_{C]B}{}^C + \nabla_{[B} \delta \Gamma_{C]A}{}^C$
Energy-Momentum tensor	$T^{AB} = e^{2d} \left(8\bar{P}^{[A}{}^C P^{B]}{}_D \frac{\delta \mathcal{L}_{matter}}{\delta \mathcal{H}_{CD}} - \frac{1}{2}\mathcal{J}^{AB} \frac{\delta \mathcal{L}_{matter}}{\delta d}\right)$
Conservation	$\nabla_A G^{AB} = 0$ (off-shell), $\nabla_A T^{AB} = 0$ (on-shell)
EDFEs	$G_{AB} = 8\pi G T_{AB}$

curvatures, $S_{(0)}, G_{AB}$, can be constructed strictly in terms of $\mathbf{O}(D, D)$ covariant field variables, notably the $\mathbf{O}(D, D)$ invariant DFT-dilaton, d , and the $\mathbf{O}(D, D)$ covariant DFT-metric, \mathcal{H}_{AB} (“generalized metric”), or more powerfully $\mathbf{O}(D, D)$ covariant DFT-vielbeins, without necessarily referring to conventional, undoubled $\mathbf{O}(D, D)$ breaking supergravity variables. Similarly, a doubled string action can be constructed in terms of $\mathbf{O}(D, D)$ covariant objects as we review below.

1.2 Doubled but at the same time gauged string action

One of the characteristics of DFT is the imposition of the ‘section condition’: acting on arbitrary functions in DFT, say Φ_r , and their products like $\Phi_s \Phi_t$, the $\mathbf{O}(D, D)$ invariant Laplacian should vanish

$$\partial_A \partial^A = 0 : \partial_A \partial^A \Phi_r = 0, \quad \partial_A \Phi_s \partial^A \Phi_t = 0. \tag{1.5}$$

We remind the reader that the $\mathbf{O}(D, D)$ indices are raised with \mathcal{J}^{AB} . Upon imposing the section condition, the generalized Lie derivative (1.4) is closed by commutators [3, 7],

$$\begin{aligned} [\hat{\mathcal{L}}_\zeta, \hat{\mathcal{L}}_\xi] &= \hat{\mathcal{L}}_{[\zeta, \xi]_C}, \\ [\zeta, \xi]_C^M &= \zeta^N \partial_N \xi^M - \xi^N \partial_N \zeta^M \\ &\quad + \frac{1}{2} \xi^N \partial^M \zeta_N - \frac{1}{2} \zeta^N \partial^M \xi_N. \end{aligned} \tag{1.6}$$

The section condition is mathematically equivalent to the following translational invariance [8, 83],

$$\Phi_r(x) = \Phi_r(x + \Delta), \quad \Delta^A \partial_A = 0, \tag{1.7}$$

where the shift parameter, Δ^A , is *derivative-index-valued*, meaning that its superscript index should be identifiable as a derivative index, for example $\Delta^A = \Phi_s \partial^A \Phi_t$. This insight on the section condition may suggest that the doubled coordinates of DFT are in fact gauged by an equivalence relation,

$$x^A \sim x^A + \Delta^A, \quad \Delta^A \partial_A = 0. \tag{1.8}$$

Each gauge orbit, i.e. equivalence class, represents a single physical point. As a matter of fact in DFT, the usual infinitesimal one-form of coordinates, dx^A , is not DFT-diffeomorphism covariant,

$$\begin{aligned} \delta(dx^A) &= d(\delta x^A) = d\xi^A \\ &= dx^B \partial_B \xi^A \neq dx^B (\partial_B \xi^A - \partial^A \xi_B). \end{aligned} \tag{1.9}$$

However, if we gauge the one-form by introducing a derivative-index-valued connection, we can have a DFT-diffeomorphism covariant one-form, provided that the gauge potential transforms appropriately,

$$\begin{aligned} Dx^A &= dx^A - \mathcal{A}^A, \quad \mathcal{A}^A \partial_A = 0, \\ \delta(Dx^A) &= Dx^B (\partial_B \xi^A - \partial^A \xi_B), \\ \delta \mathcal{A}^A &= Dx^B \partial^A \xi_B. \end{aligned} \tag{1.10}$$

It is also a singlet of the coordinate gauge symmetry (1.8): $\delta x^A = \Delta^A, \delta \mathcal{A}^A = d\Delta^A, \delta(Dx^A) = 0$. The gauged one-form then naturally allows to construct a perfectly symmetric doubled string action [8, 84],

$$\begin{aligned} \frac{1}{4\pi\alpha'} \int d^2\sigma \left[-\frac{1}{2} \sqrt{-hh^{\alpha\beta}} D_\alpha x^A D_\beta x^B \mathcal{H}_{AB} \right. \\ \left. - \epsilon^{\alpha\beta} D_\alpha x^A \mathcal{A}_{\beta A} \right], \end{aligned} \tag{1.11}$$

which enjoys symmetries like global $\mathbf{O}(D, D)$, target spacetime DFT-diffeomorphisms, worldsheet diffeomorphisms, Weyl symmetry, and the coordinate gauge symmetry.³ All the background information is encoded in the DFT-metric, \mathcal{H}_{AB} .

2 Review of [1]: classification of the non-Riemannian DFT geometries

The section condition can be generically solved, up to $\mathbf{O}(D, D)$ rotations, by enforcing the tilde coordinate independency: $\tilde{\partial}^\mu \equiv 0 \Rightarrow \partial_A \partial^A = 2\partial_\mu \tilde{\partial}^\mu \equiv 0$. Choosing $\Delta^A = c_\mu \partial^A x^\mu = (c_\mu, 0)$ for (1.8) and similarly $\mathcal{A}^A = A_\mu \partial^A x^\mu = (A_\mu, 0)$, we note that the tilde coordinates are indeed gauged: $(\tilde{x}_\mu, x^\nu) \sim (\tilde{x}_\mu + c_\mu, x^\nu)$, $Dx^A = (d\tilde{x}_\mu - A_\mu, dx^\nu)$. With respect to this choice of the section, the well-known parametrization of the DFT-metric and the DFT-dilaton in terms of the conventional massless NS-NS field variables [88,89],

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\sigma} B_{\sigma\lambda} \\ B_{\kappa\rho} g^{\rho\nu} & g_{\kappa\lambda} - B_{\kappa\rho} g^{\rho\sigma} B_{\sigma\lambda} \end{pmatrix},$$

$$e^{-2d} = e^{-2\phi} \sqrt{|g|}, \tag{2.1}$$

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma} B_{\sigma\lambda} + Y_i^\mu X_\lambda^i - \bar{Y}_{\bar{i}}^\mu \bar{X}_{\bar{\lambda}}^{\bar{i}} \\ B_{\kappa\rho} H^{\rho\nu} + X_\kappa^i Y_i^\nu - \bar{X}_{\bar{\kappa}}^{\bar{i}} \bar{Y}_{\bar{i}}^\nu & K_{\kappa\lambda} - B_{\kappa\rho} H^{\rho\sigma} B_{\sigma\lambda} + 2X_{(\kappa}^i B_{\lambda)\rho} Y_i^\rho - 2\bar{X}_{(\bar{\kappa}}^{\bar{i}} B_{\bar{\lambda})\rho} \bar{Y}_{\bar{i}}^\rho \end{pmatrix}, \tag{2.5}$$

reduces DFT to supergravity. In this case, the single expression of the EDFEs (1.3) unifies all the equations of motion of the three fields, $\{g_\mu, B_{\mu\nu}, \phi\}$. Further, after Gaussian integration of the auxiliary gauge potential, A_μ , the doubled-yet-gauged string action (1.11) reproduces the standard undoubled string action.

Yet, this is not the full story. The above parametrization (2.1) is merely one particular solution to the defining relations of the DFT-metric:

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}, \quad \mathcal{H}_A^C \mathcal{H}_B^D \mathcal{J}_{CD} = \mathcal{J}_{AB}. \tag{2.2}$$

DFT and the doubled-yet-gauged string action work well, provided these conditions are fulfilled. For example, instead of (2.1), we may let the DFT-metric coincide with the $\mathbf{O}(D, D)$ invariant metric,

$$\mathcal{H}_{AB} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{2.3}$$

such that $\mathcal{H}_A^B = \delta_A^B$. This is a vacuum solution to DFT, or to the ‘matter-free’ EDFEs, $G_{AB} = 0$ (1.3), which is very special in several aspects. Firstly, compared with (2.1), there cannot be any associated Riemannian metric $g_{\mu\nu}$ and hence it

does not allow any conventional or Riemannian interpretation at all. It is maximally non-Riemannian. Secondly, it is fully $\mathbf{O}(D, D)$ symmetric, being one of the two most symmetric vacua of DFT, $\mathcal{H}_{AB} = \pm \mathcal{J}_{AB}$. Thirdly, it is moduli-free since it does not admit any infinitesimal fluctuation, $\delta\mathcal{H}_{AB} = 0$ [75].⁴ And lastly but not leastly, upon this background, the auxiliary gauge potential, A_μ , appears linearly rather than quadratically in the doubled-yet-gauged string action (1.11). Consequently it serves as a Lagrange multiplier to prescribe that all the untilded target spacetime coordinates should be chiral [8] (c.f. [90,91]),

$$\partial_\alpha x^\mu + \frac{1}{\sqrt{-h}} \epsilon_{\alpha\beta} \partial_\beta x^\mu = 0. \tag{2.4}$$

An intriguing insight from [11] is then that, the usual supergravity fields in (2.1) would be the Nambu–Goldstone modes of the perfectly $\mathbf{O}(D, D)$ symmetric vacuum (2.3).

Given the Riemannian and maximally non-Riemannian backgrounds, (2.1) v.s. (2.3), one may wonder about the existence of more generic non-Riemannian geometries (c.f. [8, 10] for other examples and also [22] for ‘timelike’ duality rotations). This question was answered in [1]: the most general solutions to the defining properties of the DFT-metric (2.2) can be classified by two non-negative integers, (n, \bar{n}) ,

where $i, j = 1, 2, \dots, n, \bar{i}, \bar{j} = 1, 2, \dots, \bar{n}$ and $0 \leq n + \bar{n} \leq D$.

- (i) While the B -field is skew-symmetric as usual, $H^{\mu\nu}$ and $K_{\mu\nu}$ are symmetric tensors whose kernels are spanned by linearly independent vectors, $\{X_\mu^i, \bar{X}_{\bar{\nu}}^{\bar{i}}\}$ and $\{Y_j^\mu, \bar{Y}_{\bar{j}}^\nu\}$, respectively,

$$\begin{aligned} H^{\mu\nu} X_\nu^i &= 0, & H^{\mu\nu} \bar{X}_{\bar{\nu}}^{\bar{i}} &= 0, \\ K_{\mu\nu} Y_j^\nu &= 0, & K_{\mu\nu} \bar{Y}_{\bar{j}}^\nu &= 0. \end{aligned} \tag{2.6}$$

- (ii) A completeness relation must be satisfied

$$H^{\mu\rho} K_{\rho\nu} + Y_i^\mu X_\nu^i + \bar{Y}_{\bar{i}}^\mu \bar{X}_{\bar{\nu}}^{\bar{i}} = \delta_\nu^\mu. \tag{2.7}$$

From the linear independency of the zero-eigenvectors, $\{X_\mu^i, \bar{X}_{\bar{\nu}}^{\bar{i}}\}$, orthogonal/algebraic relations follow

$$\begin{aligned} Y_i^\mu X_\mu^j &= \delta_i^j, & \bar{Y}_{\bar{i}}^\mu \bar{X}_{\bar{\mu}}^{\bar{j}} &= \delta_{\bar{i}}^{\bar{j}}, \\ Y_i^\mu \bar{X}_{\bar{\mu}}^{\bar{j}} &= \bar{Y}_{\bar{i}}^\mu X_\mu^j = 0, \\ H^{\rho\mu} K_{\mu\nu} H^{\nu\sigma} &= H^{\rho\sigma}, & K_{\rho\mu} H^{\mu\nu} K_{\nu\sigma} &= K_{\rho\sigma}. \end{aligned} \tag{2.8}$$

³ See also [85] for Green–Schwarz doubled superstring, [66] for doubled point particle, and [86,87] for ‘exceptional’ extensions.

⁴ Put $\mathcal{H}_A^B = \delta_A^B$ in (3.5).

Intriguingly, the B -field (hence ‘Courant algebra’) is universally present regardless of the values of (n, \bar{n}) , and contributes to the DFT-metric through an $\mathbf{O}(D, D)$ adjoint action:

$$\mathcal{H}_{AB} = \mathcal{B}_A^C \mathcal{B}_B^D \mathcal{H}_{CD}, \tag{2.9}$$

where \mathcal{H} corresponds to the ‘ B -field-free’ DFT-metric,

$$\mathcal{H}_{AB}^{\circ} = \begin{pmatrix} H & Y_i(X^i)^T - \bar{Y}_{\bar{i}}(\bar{X}^{\bar{i}})^T \\ X^i(Y_i)^T - \bar{X}^{\bar{i}}(\bar{Y}_{\bar{i}})^T & K \end{pmatrix}, \tag{2.10}$$

and \mathcal{B} is an $\mathbf{O}(D, D)$ element containing the B -field,

$$\mathcal{B}_A^B = \begin{pmatrix} \delta^{\mu}_{\sigma} & 0 \\ B_{\rho\sigma} & \delta_{\rho\tau} \end{pmatrix}, \quad \mathcal{B}_A^C \mathcal{B}_B^D \mathcal{J}_{CD} = \mathcal{J}_{AB}. \tag{2.11}$$

It is also worth while to note the ‘vielbeins’ or ‘square-roots’ of $K_{\mu\nu}$ and $H^{\mu\nu}$:

$$\begin{aligned} K_{\mu\nu} &= K_{\mu}^a K_{\nu}^b \eta_{ab}, \\ H^{\mu\nu} &= H^{\mu}_a H^{\nu}_b \eta^{ab}, \\ K_{\mu}^a H^{\mu}_b &= \delta^a_b, \\ K_{\mu a} H^{\nu a} &= K_{\mu\rho} H^{\rho\nu}, \end{aligned} \tag{2.12}$$

where a, b are $(D - n - \bar{n})$ -dimensional indices subject to a flat metric, say η_{ab} , whose signature is arbitrary. Essentially, $\{K_{\mu}^a, X_{\mu}^i, \bar{X}_{\mu}^{\bar{i}}\}$ form a $D \times D$ invertible square matrix whose inverse is given by $\{H^{\mu}_a, Y_i^{\mu}, \bar{Y}_{\bar{i}}^{\mu}\}$ as

$$K_{\mu}^a H^{\nu}_a + X_{\mu}^i Y_i^{\nu} + \bar{X}_{\mu}^{\bar{i}} \bar{Y}_{\bar{i}}^{\nu} = \delta_{\mu}^{\nu}. \tag{2.13}$$

In fact, the analysis of the DFT-vielbeins corresponding to the (n, \bar{n}) DFT-metric (2.5) carried out in [1] shows that the local Lorentz symmetry group, i.e. spin group is

$$\mathbf{Spin}(t + n, s + n) \times \mathbf{Spin}(s + \bar{n}, t + \bar{n}). \tag{2.14}$$

Here (t, s) is the arbitrary signature of η_{ab} or the nontrivial signature of $H^{\mu\nu}$ and $K_{\mu\nu}$ satisfying $t + s + n + \bar{n} = D$. Of course, once the spin group of any given theory is specified, it is fixed once and for all. Thus, each sum, $t + n, s + n, s + \bar{n}$, and $t + \bar{n}$, should be constant. For example, the Minkowskian $D = 10$ maximally supersymmetric DFT [85] and the doubled-yet-gauged Green-Schwarz superstring action [79], both having the local Lorentz group of $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$, can accommodate $(0, 0)$ Riemannian and $(1, 1)$ non-Riemannian sectors only (see [12] for examples of supersymmetric non-Riemannian backgrounds). Nevertheless, we may readily relax the Majorana–Weyl condition therein [79,85] and impose the Weyl condition only on spinors, such that the local Lorentz group can take any of $\mathbf{Spin}(\hat{t}, \hat{s}) \times \mathbf{Spin}(\hat{s}, \hat{t})$ with $\hat{t} + \hat{s} = 10$. The allowed non-Riemannian geometries will be then (n, n) types with $n = \bar{n}$ running from zero to $\min(\hat{t}, \hat{s})$ [1]. On the other hand, bosonic DFT does not care

about spin groups and hence should be free from such constraints. It can admit more generic (n, \bar{n}) non-Riemannian geometries.

Crucially, the (n, \bar{n}) parametrization of the DFT-metric (2.5) possesses two local symmetries, namely $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ rotations and Milne-shift transformations. The $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ symmetry rotates the i, j, \dots and \bar{i}, \bar{j}, \dots indices of the component fields: with infinitesimal local parameters, w_i^j and $\bar{w}_{\bar{i}}^{\bar{j}}$,

$$\begin{aligned} \delta_{\mathbf{GL}} X_{\mu}^i &= X_{\mu}^j w_j^i, & \delta_{\mathbf{GL}} Y_i^{\mu} &= -w_i^j Y_j^{\mu}, \\ \delta_{\mathbf{GL}} \bar{X}_{\mu}^{\bar{i}} &= \bar{X}_{\mu}^{\bar{j}} \bar{w}_{\bar{j}}^{\bar{i}}, & \delta_{\mathbf{GL}} \bar{Y}_{\bar{i}}^{\mu} &= -\bar{w}_{\bar{i}}^{\bar{j}} \bar{Y}_{\bar{j}}^{\mu}, \\ \delta_{\mathbf{GL}} d &= 0, & \delta_{\mathbf{GL}} H^{\mu\nu} &= 0, & \delta_{\mathbf{GL}} K_{\mu\nu} &= 0, & \delta_{\mathbf{GL}} B_{\mu\nu} &= 0. \end{aligned} \tag{2.15}$$

The Milne-shift symmetry generalizes the so-called ‘Galilean boost’ in the Newtonian gravity literature [40,41]. It acts with infinitesimal local parameters, $V_{\mu i}$ and $\bar{V}_{\mu \bar{i}}$,⁵

$$\begin{aligned} \delta_{\mathbf{M}} Y_i^{\mu} &= H^{\mu\nu} V_{\nu i}, & \delta_{\mathbf{M}} \bar{Y}_{\bar{i}}^{\mu} &= H^{\mu\nu} \bar{V}_{\nu \bar{i}}, \\ \delta_{\mathbf{M}} K_{\mu\nu} &= -2X_{(\mu}^i (K_{\nu)\rho} H^{\rho\sigma} V_{\sigma i} - 2\bar{X}_{(\mu}^{\bar{i}} (K_{\nu)\rho} H^{\rho\sigma} \bar{V}_{\sigma \bar{i}}, \\ \delta_{\mathbf{M}} B_{\mu\nu} &= -2X_{[\mu}^i V_{\nu]i} + 2\bar{X}_{[\mu}^{\bar{i}} \bar{V}_{\nu]\bar{i}} \\ &\quad + 2X_{[\mu}^i \bar{X}_{\nu]}^{\bar{i}} (Y_i^{\rho} \bar{V}_{\rho \bar{i}} + \bar{Y}_{\bar{i}}^{\rho} V_{\rho i}), \\ \delta_{\mathbf{M}} d &= 0, & \delta_{\mathbf{M}} X_{\mu}^i &= 0, & \delta_{\mathbf{M}} \bar{X}_{\mu}^{\bar{i}} &= 0, & \delta_{\mathbf{M}} H^{\mu\nu} &= 0. \end{aligned} \tag{2.16}$$

Remarkably, both transformations, (2.15) and (2.16), leave the DFT-metric invariant,

$$\delta_{\mathbf{GL}} \mathcal{H}_{AB} = 0, \quad \delta_{\mathbf{M}} \mathcal{H}_{AB} = 0, \tag{2.17}$$

as the two local symmetries are actually parts of the underlying local Lorentz symmetries (2.14).

Upon the (n, \bar{n}) background, the doubled-yet-gauged worldsheet string action (1.11) reduces to

$$\begin{aligned} \frac{1}{2\pi\alpha'} \int d^2\sigma \left[-\frac{1}{2} \sqrt{-h} h^{\alpha\beta} \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu} K_{\mu\nu} \right. \\ \left. + \frac{1}{2} \epsilon^{\alpha\beta} \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu} B_{\mu\nu} + \frac{1}{2} \epsilon^{\alpha\beta} \partial_{\alpha} \tilde{x}_{\mu} \partial_{\beta} x^{\mu} \right], \end{aligned} \tag{2.18}$$

which should be supplemented by the chiral and anti-chiral constraints over the n and \bar{n} directions,

$$\begin{aligned} X_{\mu}^i \left(\partial_{\alpha} x^{\mu} + \frac{1}{\sqrt{-h}} \epsilon_{\alpha}^{\beta} \partial_{\beta} x^{\mu} \right) &= 0, \\ \bar{X}_{\mu}^{\bar{i}} \left(\partial_{\alpha} x^{\mu} - \frac{1}{\sqrt{-h}} \epsilon_{\alpha}^{\beta} \partial_{\beta} x^{\mu} \right) &= 0. \end{aligned} \tag{2.19}$$

These constraints are prescribed by the integrated-out auxiliary gauge potential \mathcal{A}^A (1.10).

⁵ Through exponentiations, finite Milne-shift transformations can be achieved, which turn out to get truncated at finite orders, for example $e^{\delta_{\mathbf{M}}} Y_i^{\mu} = Y_i^{\mu} + H^{\mu\nu} V_{\nu i}$. See Eq.(2.16) of [1] for the full list.

Comment 1. Matching with the content of the non-Riemannian component fields,

$$\{H^{\mu\nu}, K_{\rho\sigma}, X^i_\mu, \bar{X}^i_\nu, Y^\rho_j, \bar{Y}^\sigma_j, B_{\mu\nu}\}, \tag{2.20}$$

and the undoubled string worldsheet action resulting from (1.11), one can identify the original Newton–Cartan [33–35] as (1, 0), Stringy Newton–Cartan [36] as (1, 1), Carroll [37, 38] as (D–1, 0), and Gomis–Ooguri [39] as (1, 1): see [1, 11, 57] for the details of the identifications. Further, the isometry of the (1, 1) flat DFT-metric matches with the non-relativistic symmetry algebra such as Bargmann algebra [10], while the notion of T-duality persists to make sense in the non-relativistic string theory [47]. These all seem to suggest that DFT may be the *home*, i.e. the unifying framework, to describe various known as well as yet-unknown non-Riemannian gravities.⁶ Having said that there are also a few novel ingredients from DFT, such as the local $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ symmetry (2.15), the notion of ‘Milne-shift covariance’ as we shall discuss below (2.24), (2.26), and the very existence of the DFT-dilaton of which the exponentiation, e^{-2d} , gives the integral measure in DFT being a scalar density with weight one,

$$\begin{aligned} \delta_\xi d &= -\frac{1}{2} e^{-2d} \mathcal{L}_\xi (e^{-2d}) = -\frac{1}{2} e^{-2d} \partial_\mu (\xi^\mu e^{-2d}) \\ &= \xi^\mu \partial_\mu d - \frac{1}{2} \partial_\mu \xi^\mu. \end{aligned} \tag{2.21}$$

Comment 2. It is worth while to generalize the decomposition (2.9) to an arbitrary DFT tensor,

$$\begin{aligned} \mathring{T}_{A_1 A_2 \dots A_n} &:= (\mathcal{B}^{-1})_{A_1}{}^{B_1} (\mathcal{B}^{-1})_{A_2}{}^{B_2} \\ &\quad \dots (\mathcal{B}^{-1})_{A_n}{}^{B_n} T_{B_1 B_2 \dots B_n}, \quad T_{A_1 \dots A_n} \\ &= \mathcal{B}_{A_1}{}^{B_1} \dots \mathcal{B}_{A_n}{}^{B_n} \mathring{T}_{B_1 \dots B_n}. \end{aligned} \tag{2.22}$$

Under diffeomorphisms, while the DFT tensor $T_{A_1 \dots A_n}$ is surely subject to the generalized Lie derivative (1.4), the circled quantity, $\mathring{T}_{A_1 \dots A_n}$, is now governed by the undoubled ordinary Lie derivative which can be conveniently obtained as the truncation of the generalized Lie derivative by choosing the section, $\tilde{\partial}^\mu \equiv 0$, and setting the parameter, $\xi^A = (0, \xi^\mu)$ as $\tilde{\xi}_\nu \equiv 0$:

$$\begin{aligned} \delta_\xi \mathring{T}_{A_1 \dots A_n} &= \mathcal{L}_\xi \mathring{T}_{A_1 \dots A_n} \\ &= \xi^\mu \partial_\mu \mathring{T}_{A_1 \dots A_n} + \omega_T \partial_\mu \xi^\mu \mathring{T}_{A_1 \dots A_n} \\ &\quad + \sum_{j=1}^n (\partial_{A_j} \xi_B - \partial_B \xi_{A_j}) \mathring{T}_{A_1 \dots A_{j-1} B A_{j+1} \dots A_n}. \end{aligned} \tag{2.23}$$

Further, by construction, a DFT tensor is Milne-shift *invariant*. Yet, the circled one is Milne-shift *covariant* in the following manner,

$$\begin{aligned} \delta_M T_{A_1 \dots A_n} &= 0, \quad \delta_M \mathring{T}_{A_1 \dots A_n} \\ &= \sum_{j=1}^n -\delta_M \mathcal{B}_{A_j}{}^B \mathring{T}_{A_1 \dots A_{j-1} B A_{j+1} \dots A_n}. \end{aligned} \tag{2.24}$$

Explicitly, for a DFT vector, $V_A = \mathcal{B}_A{}^B \mathring{V}_B$, we have (c.f. [76, 93])

$$\begin{aligned} \delta_\xi \mathring{V}_A &= \begin{pmatrix} \delta_\xi \mathring{V}^\mu \\ \delta_\xi \mathring{V}_\nu \end{pmatrix} = \begin{pmatrix} \mathcal{L}_\xi \mathring{V}^\mu \\ \mathcal{L}_\xi \mathring{V}_\nu \end{pmatrix} \\ &= \xi^\rho \partial_\rho \mathring{V}_A + \omega_\nu \partial_\rho \xi^\rho \mathring{V}_A \\ &\quad + (\partial_A \xi_B - \partial_B \xi_A) \mathring{V}^B = \mathcal{L}_\xi \mathring{V}_A, \\ \delta_M \mathring{V}_A &= \begin{pmatrix} \delta_M \mathring{V}^\mu \\ \delta_M \mathring{V}_\nu \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta_M B_{\nu\rho} \mathring{V}^\rho \end{pmatrix} \\ &= -\delta_M \mathcal{B}_A{}^B \mathring{V}_B. \end{aligned} \tag{2.25}$$

That is to say, the circled quantities, $\mathring{T}_{A_1 \dots A_n}, \mathring{V}_A$, are ‘B-field free’, subject to the ordinary Lie derivative, and Milne-shift covariant rather than invariant. More specifically, the undoubled lower Greek indices are Milne-shift covariant, while the upper ones are invariant: from (2.10), (2.16), (2.25),

$$\begin{aligned} \delta_M \mathring{V}_\nu &= -\delta_M B_{\nu\rho} \mathring{V}^\rho, \\ \delta_M \mathring{V}^\mu &= 0, \\ \delta_M K_{\mu\nu} &= \delta_M \mathring{\mathcal{H}}_{\mu\nu} = -\delta_M B_{\mu\rho} \mathring{\mathcal{H}}^\rho{}_\nu - \delta_M B_{\nu\rho} \mathring{\mathcal{H}}_{\mu}{}^\rho \\ &= -\delta_M B_{\mu\rho} (Y_i{}^\rho X^i_\nu - \bar{Y}_i{}^\rho \bar{X}^i_\nu) - \delta_M B_{\nu\rho} (Y_i{}^\rho X^i_\mu - \bar{Y}_i{}^\rho \bar{X}^i_\mu), \\ \delta_M (Y_i{}^\mu X^i_\nu - \bar{Y}_i{}^\mu \bar{X}^i_\nu) &= \delta_M \mathring{\mathcal{H}}^\mu{}_\nu \\ &= -\delta_M B_{\nu\rho} \mathring{\mathcal{H}}^{\mu\rho} = \delta_M \mathring{\mathcal{H}}^\mu{}_\nu = -\delta_M B_{\nu\rho} \mathring{\mathcal{H}}^{\rho\mu} = -\delta_M B_{\nu\rho} H^{\mu\rho}, \\ \delta_M H^{\mu\nu} &= \delta_M \mathring{\mathcal{H}}^{\mu\nu} = 0. \end{aligned} \tag{2.26}$$

For consistency, we also note for the $\mathbf{O}(D, D)$ invariant metric,

$$\begin{aligned} \mathcal{J}_{AB} &= \mathring{\mathcal{J}}_{AB}, \\ \delta_M \mathring{\mathcal{J}}_{AB} &= -\delta_M \mathcal{B}_A{}^C \mathring{\mathcal{J}}_{CB} - \delta_M \mathcal{B}_B{}^C \mathring{\mathcal{J}}_{AC} = 0. \end{aligned} \tag{2.27}$$

3 Variational principle around non-Riemannian backgrounds

Here we revisit with care the variational principle for a general DFT action coupled to matter (1.1) especially around non-Riemannian backgrounds. While the variations of the matter fields lead to their own Euler–Lagrange equations of motion, the variations of the DFT-metric and the DFT-dilaton give [65]

⁶ Similarly, inequivalent parametrizations of the DFT-vielbeins, or U-duality-covariant generalized metric, correspond to the conventional distinctions between IIA and IIB [79, 85], or IIB and M-“theories” [92].

$$\begin{aligned} &\delta \int \frac{1}{16\pi G} e^{-2d} S_{(0)} + \mathcal{L}_{\text{matter}} \\ &= \frac{1}{16\pi G} \int e^{-2d} \left[\delta \mathcal{H}_{AB} \{ (PG\bar{P})^{AB} - 8\pi G(PT\bar{P})^{AB} \} \right. \\ &\quad \left. + \frac{2}{D} \delta d (G_A^A - 8\pi G T_A^A) \right]. \end{aligned} \tag{3.1}$$

Here G_{AB} and T_{AB} are respectively the stringy or $\mathbf{O}(D, D)$ completions of the Einstein curvature [81] and the Energy-Momentum tensor [65], as summarized in Table 1. The above result is easy to obtain once we neglect a boundary contribution arising from a total derivative [63]:

$$\frac{1}{16\pi G} e^{-2d} \mathcal{H}^{AB} \delta S_{AB} = \partial_A \left(e^{-2d} \frac{1}{8\pi G} \mathcal{H}^{B[A} \delta \Gamma_{CB}^{C]} \right), \tag{3.2}$$

and take into account a well-known identity which the infinitesimal variation of the DFT-metric should satisfy [7, 62, 94],

$$\delta \mathcal{H}_{AB} = 2P_{(A}^C \bar{P}_{B)}^D \delta \mathcal{H}_{CD}. \tag{3.3}$$

Equation (3.2) holds due to the nice variational property of the semi-covariant curvature, $\delta S_{AB} = \nabla_{[A} \delta \Gamma_{C]B}^C + \nabla_{[B} \delta \Gamma_{C]A}^C$, and the compatibility of the derivative, $\nabla_A \mathcal{J}_{BC} = 0$, $\nabla_A \mathcal{H}_{BC} = 0$, $\nabla_A d = 0$, see Table 1. Equation (3.3) holds because the DFT-metric is constrained to be a symmetric $\mathbf{O}(D, D)$ element (2.2), see also (3.5) below. This is the reason why in the variation of the action (3.1) $\delta \mathcal{H}_{AB}$ is contracted with a projected quantity, i.e. $(PG\bar{P})^{AB} - 8\pi G(PT\bar{P})^{AB}$. Equation (3.1) is then supposed to give the EDFEs, $G_{AB} = 8\pi G T_{AB}$ (1.3) [65], as the two variations, $\delta \mathcal{H}_{AB}$ and δd , give the projected part and the trace part separately,

$$(PG\bar{P})_{AB} = 8\pi G(PT\bar{P})_{AB}, \quad G_A^A = 8\pi G T_A^A, \tag{3.4}$$

which comprise the full EDFEs. While there is no issue on the equation of motion of the DFT-dilaton, i.e. the trace part in (3.4), there might be some ambiguity on the DFT-metric variation especially around a non-Riemannian background. For example, let us take one of the two maximally non-Riemannian, fully $\mathbf{O}(D, D)$ symmetric vacua, as $\mathcal{H}_{AB} = \mathcal{J}_{AB}$. Because it does not allow any infinitesimal variation or moduli, $\delta \mathcal{H}_{AB} = 0$ [75], the induced variation of the action is null and therefore it should not generate any nontrivial Euler–Lagrange equation of motion. Nevertheless, in this case the ‘barred’ projector vanishes automatically, $\bar{P}_{AB} = 0$, and the projected part of the EDFEs in (3.4) is satisfied rather trivially. It appears that we have a slightly puzzling situation for the non-Riemannian background, $\mathcal{H}_{AB} = \mathcal{J}_{AB}$: it allows no infinitesimal variation $\delta \mathcal{H}_{AB} = 0$ and hence one may expect that the variation of the action should be trivial and there should be no nontrivial Euler–Lagrange equation of motion of the DFT-metric. This is all true, but nevertheless

the full EDFEs are still valid! (though in a trivial manner as $\bar{P} = 0$).

Below, through Sects. 3.1 and 3.2, we shall rigorously revisit the variational principle of DFT around a generic non-Riemannian background. Basically, we are questioning whether it is really safe from (3.3) to put $\delta \mathcal{H}_{AB} = 2P_{(A}^C \bar{P}_{B)}^D \mathcal{M}_{CD}$ and read off the Euler–Lagrange equation of motion of the DFT-metric as if \mathcal{M}_{CD} is a generic symmetric matrix. To answer this, we shall directly identify the truly independent degrees of freedom in the infinitesimal fluctuations of an arbitrary (n, \bar{n}) non-Riemannian DFT-metric, as (3.12). We shall confirm that the full Einstein Double Field Equations are still valid for non-Riemannian sectors, either trivially due to projection properties or nontrivially from the genuine variational principle.

3.1 Variations of the DFT-metric around a generic (n, \bar{n}) background

Here we shall identify the most general form of the infinitesimal fluctuations around a generic (n, \bar{n}) DFT-metric (2.5). The fluctuations must respect the defining properties of the DFT-metric (2.2) and hence satisfy

$$\delta \mathcal{H}_{AB} = \delta \mathcal{H}_{BA}, \quad \delta \mathcal{H}_A^B \mathcal{H}_B^C + \mathcal{H}_A^B \delta \mathcal{H}_B^C = 0. \tag{3.5}$$

It follows that $\delta \mathcal{H}_A^B = -\mathcal{H}_A^C \delta \mathcal{H}_C^D \mathcal{H}_D^B$, and hence equivalent (3.3) holds. In particular, $\delta \mathcal{H}_A^A$ is traceless,

$$\delta \mathcal{H}_A^A = 0. \tag{3.6}$$

That is to say, the trace of the DFT-metric, $\mathcal{H}_A^A = 2(n - \bar{n})$, is invariant under continuous deformations.

Without loss of generality, like (2.9), we put

$$\begin{aligned} \delta \mathcal{H}_{AB} &= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix} \begin{pmatrix} 1 - B \\ 0 & 1 \end{pmatrix}, \\ \alpha &= \alpha^T, \quad \beta = \beta^T. \end{aligned} \tag{3.7}$$

With this ansatz, the former condition in (3.5) is met and the latter gives

$$\begin{aligned} &\gamma Y_i (X^i)^T - \gamma \bar{Y}_i (\bar{X}^i)^T + \alpha K \\ &\quad + Y_i (X^i)^T \gamma - \bar{Y}_i (\bar{X}^i)^T \gamma + H \beta = 0, \\ &\beta Y_i (X^i)^T - \beta \bar{Y}_i (\bar{X}^i)^T \\ &\quad + \gamma^T K + K \gamma + X^i (Y_i)^T \beta - \bar{X}^i (\bar{Y}_i)^T \beta = 0, \\ &\gamma H + \alpha X^i (Y_i)^T - \alpha \bar{X}^i (\bar{Y}_i)^T + Y_i (X^i)^T \alpha \\ &\quad - \bar{Y}_i (\bar{X}^i)^T \alpha + H \gamma^T = 0. \end{aligned} \tag{3.8}$$

We need to solve these three constraints. For this, we utilize the completeness relation (2.13), and decompose each of $\{\alpha, \beta, \gamma\}$ into mutually orthogonal pieces,

$$\begin{aligned}
 \alpha^{\mu\nu} &= H^\mu{}_a H^\nu{}_b \alpha^{ab} + Y_i^\mu Y_j^\nu \alpha^{ij} + \bar{Y}_i^\mu \bar{Y}_j^\nu \alpha^{\bar{i}\bar{j}} \\
 &\quad + 2H^{(\mu} Y_i^{\nu)} \alpha^{ai} + 2H^{(\mu} \bar{Y}_i^{\nu)} \alpha^{a\bar{i}} + 2Y^{(\mu} \bar{Y}_i^{\nu)} \alpha^{\bar{i}i}, \\
 \beta_{\mu\nu} &= K_\mu{}^a K_\nu{}^b \beta_{ab} + X_\mu{}^i X_\nu{}^j \beta_{ij} + \bar{X}_\mu{}^i \bar{X}_\nu{}^j \beta_{\bar{i}\bar{j}} \\
 &\quad + 2K_{(\mu} X_\nu{}^i \beta_{ai} + 2K_{(\mu} \bar{X}_\nu{}^i \beta_{a\bar{i}} + 2X_{(\mu}{}^i \bar{X}_\nu{}^j \beta_{\bar{i}j}, \\
 \gamma^\mu{}_\nu &= H^\mu{}_a K_\nu{}^b \gamma^a{}_b + H^\mu{}_a X_\nu{}^i \gamma^a{}_i \\
 &\quad + H^\mu{}_a \bar{X}_\nu{}^i \gamma^a{}_{\bar{i}} + Y_i^\mu K_\nu{}^a \gamma^i{}_a + Y_i^\mu X_\nu{}^j \gamma^i{}_j + Y_i^\mu \bar{X}_\nu{}^j \gamma^i{}_{\bar{j}} \\
 &\quad + \bar{Y}_i^\mu K_\nu{}^a \gamma^i{}_a + \bar{Y}_i^\mu X_\nu{}^j \gamma^i{}_j + \bar{Y}_i^\mu \bar{X}_\nu{}^j \gamma^i{}_{\bar{j}}, \tag{3.9}
 \end{aligned}$$

where, since α, β are symmetric,

$$\begin{aligned}
 \alpha^{ab} &= \alpha^{ba}, \quad \alpha^{ij} = \alpha^{ji}, \\
 \alpha^{\bar{i}\bar{j}} &= \alpha^{\bar{j}\bar{i}}, \quad \beta_{ab} = \beta_{ba}, \\
 \beta_{ij} &= \beta_{ji}, \quad \beta_{\bar{i}\bar{j}} = \beta_{\bar{j}\bar{i}}. \tag{3.10}
 \end{aligned}$$

We remind the readers that, using the $(D-n-\bar{n})$ -dimensional flat metric, η_{ab} , we freely raise or lower the indices, a, b . Now, with the decomposition (3.9), it is straightforward to see that (3.8) implies

$$\begin{aligned}
 \alpha_a{}^i + \gamma^i{}_a &= 0, \quad \alpha_a{}^{\bar{i}} - \gamma^{\bar{i}}{}_a = 0, \\
 \beta_{ai} + \gamma_{ai} &= 0, \quad \beta_{a\bar{i}} - \gamma_{a\bar{i}} = 0, \\
 \alpha_{ab} + \beta_{ab} &= 0, \quad \gamma_{ab} + \gamma_{ba} = 0, \quad \alpha^{ij} = 0, \\
 \alpha^{\bar{i}\bar{j}} = 0, \beta_{ij} = 0, \quad \beta_{\bar{i}\bar{j}} = 0, \gamma^i{}_j = 0, \quad \gamma^{\bar{i}}{}_{\bar{j}} = 0. \tag{3.11}
 \end{aligned}$$

Thus, the independent degrees of freedom for the fluctuations consist of

$$\begin{aligned}
 \alpha_{(ab)} &= -\beta_{(ab)}, \quad \gamma_{[ab]}, \quad \gamma^a{}_i = -\beta^a{}_i, \\
 \gamma^a{}_{\bar{i}} &= \beta^a{}_{\bar{i}}, \quad \gamma^i{}_a = -\alpha_a{}^i, \\
 \gamma^{\bar{i}}{}_a &= \alpha_a{}^{\bar{i}}, \quad \alpha^{\bar{i}i}, \quad \beta_{j\bar{j}}, \gamma^i{}_{\bar{i}}, \quad \gamma^{\bar{j}}{}_{j}. \tag{3.12}
 \end{aligned}$$

In total, as counted sequently as

$$\begin{aligned}
 &\frac{1}{2}(D-n-\bar{n})(D-n-\bar{n}+1) \\
 &\quad + \frac{1}{2}(D-n-\bar{n})(D-n-\bar{n}-1) \\
 &\quad + 2(D-n-\bar{n})(n+\bar{n}) + 4n\bar{n} = D^2 - (n-\bar{n})^2, \tag{3.13}
 \end{aligned}$$

there are $D^2 - (n - \bar{n})^2$ number of degrees of freedom which matches precisely the dimension of the underlying coset [11],

$$\frac{\mathbf{O}(D, D)}{\mathbf{O}(t+n, s+n) \times \mathbf{O}(s+\bar{n}, t+\bar{n})}. \tag{3.14}$$

Furthermore, if we employ the DFT-vielbeins,⁷ $V_{Ap}, \bar{V}_{A\bar{p}}$, the projected part of the EDFEs (3.4) is equivalent to

$$[(PG\bar{P})_{AB} - 8\pi G(PT\bar{P})_{AB}] V^A{}_p \bar{V}^B{}_{\bar{p}} = 0. \tag{3.15}$$

⁷ The only required property of the DFT-vielbeins is $V_{Ap} V_B{}^p + \bar{V}_{A\bar{p}} \bar{V}_B{}^{\bar{p}} = \mathcal{J}_{AB}$. See [75] for a related discussion.

As the local Lorentz vector indices p and \bar{p} run from one to $D+n-\bar{n}$ and $D-n+\bar{n}$ respectively, there are in total $(D+n-\bar{n}) \times (D-n+\bar{n}) = D^2 - (n-\bar{n})^2$ number of components in (3.15) which coincides with the total number of independent fluctuations of the (n, \bar{n}) DFT-metric (3.13). As the number of the equations and the fluctuations are the same, we may well expect that the former should be implied by the variational principle generated by the latter. Below, we confirm this directly through explicit computation, without using the DFT-vielbeins.

3.2 Einstein Double Field Equations still hold for non-Riemannian sectors

Now, we proceed to organize the variation of the action induced by that of the (n, \bar{n}) DFT-metric (3.1) in terms of the independent degrees of freedom for the fluctuations (3.12).

We apply the prescription (2.22) and write a pair of circled ‘ B -field-free’ symmetric projectors,

$$\begin{aligned}
 \mathring{P}_{AB} &= \mathring{P}_{BA} = (\mathcal{B}^{-1})_A{}^C (\mathcal{B}^{-1})_B{}^D P_{CD} \\
 &= \frac{1}{2} \begin{pmatrix} H & HK + 2Y_i(X^i)^T \\ KH + 2X^i(Y_i)^T & K \end{pmatrix}, \\
 \mathring{\bar{P}}_{AB} &= \mathring{\bar{P}}_{BA} = (\mathcal{B}^{-1})_A{}^C (\mathcal{B}^{-1})_B{}^D \bar{P}_{CD} \\
 &= \frac{1}{2} \begin{pmatrix} -H & HK + 2\bar{Y}_i(\bar{X}^i)^T \\ KH + 2\bar{X}^i(\bar{Y}_i)^T & -K \end{pmatrix}, \tag{3.16}
 \end{aligned}$$

which satisfy $\mathring{P}_A{}^B + \mathring{\bar{P}}_A{}^B = \delta_A{}^B$, $\mathring{P}_A{}^B \mathring{\bar{P}}_B{}^C = 0$, and useful identities,

$$\begin{aligned}
 K_{\mu a} \mathring{P}^\mu{}_A &= H^\mu{}_a \mathring{P}_{\mu A}, \quad \bar{X}^{\bar{i}}_\mu \mathring{P}^\mu{}_A = 0, \\
 \bar{Y}^{\bar{i}}_\mu \mathring{P}^\mu{}_A &= 0, \\
 K_{\mu a} \mathring{\bar{P}}^\mu{}_A &= -H^\mu{}_a \mathring{\bar{P}}_{\mu A}, \quad X^i_\mu \mathring{\bar{P}}^\mu{}_A = 0, \\
 Y^i_\mu \mathring{\bar{P}}^\mu{}_A &= 0. \tag{3.17}
 \end{aligned}$$

We also introduce a shorthand notation for the Einstein Double Field Equations,

$$\begin{aligned}
 E_{AB} &:= G_{AB} - 8\pi G T_{AB}, \\
 \widehat{E}_{AB} &:= (\mathcal{B}^{-1})_A{}^C (\mathcal{B}^{-1})_B{}^D E_{CD}. \tag{3.18}
 \end{aligned}$$

Hereafter, hatted quantities contain generically the \mathbb{H} -flux,

$$\mathbb{H}_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}, \tag{3.19}$$

but, like the circled ones, there is no apparent bare B -field in them.

It is now straightforward to compute the variation in (3.1),

$$\delta\mathcal{H}_{AB}(PE\bar{P})^{AB} = 2\gamma^a{}_i X^i_\mu (\hat{P}\hat{E}\hat{P})^\mu{}_\nu H^{\nu a} + 2\gamma^a{}_{\bar{i}} H^\mu{}_a (\hat{P}\hat{E}\hat{P})^\mu{}_\nu \bar{X}^{\bar{i}}_\nu - 2\gamma^i{}_a Y_i^\mu (\hat{P}\hat{E}\hat{P})_{\mu\nu} H^{\nu a} + 2\gamma^{\bar{i}}{}_a H^\mu{}_a (\hat{P}\hat{E}\hat{P})_{\mu\nu} \bar{Y}^{\bar{i}}_\nu + \alpha^{\bar{i}i} Y_i^\mu (\hat{P}\hat{E}\hat{P})_{\mu\nu} \bar{Y}^{\bar{i}}_\nu + \gamma^i{}_{\bar{i}} Y_i^\mu (\hat{P}\hat{E}\hat{P})^\nu{}_\mu \bar{X}^{\bar{i}}_\nu + \gamma^{\bar{i}}{}_i X^i_\mu (\hat{P}\hat{E}\hat{P})^\mu{}_\nu \bar{Y}^{\bar{i}}_\nu + \beta_{\bar{i}i} X^i_\mu (\hat{P}\hat{E}\hat{P})^{\mu\nu} \bar{X}^{\bar{i}}_\nu + 2(\alpha^{(ab)} - \gamma^{[ab]}) H^\mu{}_a (\hat{P}\hat{E}\hat{P})_{\mu\nu} H^{\nu b}. \tag{3.20}$$

Each term is independent and thus, from the variational principle, should vanish individually on-shell,

$$\begin{aligned} X^i_\mu (\hat{P}\hat{E}\hat{P})^\mu{}_\nu H^{\nu a} &= 0, & H^\mu{}_a (\hat{P}\hat{E}\hat{P})^\mu{}_\nu \bar{X}^{\bar{i}}_\nu &= 0, \\ Y_i^\mu (\hat{P}\hat{E}\hat{P})_{\mu\nu} H^{\nu a} &= 0, \\ H^\mu{}_a (\hat{P}\hat{E}\hat{P})_{\mu\nu} \bar{Y}^{\bar{i}}_\nu &= 0, & Y_i^\mu (\hat{P}\hat{E}\hat{P})_{\mu\nu} \bar{Y}^{\bar{i}}_\nu &= 0, \\ Y_i^\mu (\hat{P}\hat{E}\hat{P})^\nu{}_\mu \bar{X}^{\bar{i}}_\nu &= 0, \\ X^i_\mu (\hat{P}\hat{E}\hat{P})^\mu{}_\nu \bar{Y}^{\bar{i}}_\nu &= 0, & X^i_\mu (\hat{P}\hat{E}\hat{P})^{\mu\nu} \bar{X}^{\bar{i}}_\nu &= 0, \\ H^\mu{}_a (\hat{P}\hat{E}\hat{P})_{\mu\nu} H^{\nu b} &= 0. \end{aligned} \tag{3.21}$$

In total, as counted sequentially as,

$$2(D - n - \bar{n})(n + \bar{n}) + 4n\bar{n} + (D - n - \bar{n})^2 = D^2 - (n - \bar{n})^2, \tag{3.22}$$

there is $D^2 - (n - \bar{n})^2$ number of independent on-shell relations, or EDFEs, in consistent with (3.13).

Up to the completeness relations (2.7), (2.13), and the identities (3.17), the first and the seventh in (3.21), the first and the eighth, the third and the fifth, the third and the sixth, the second and the last, the fourth and the last imply respectively,

$$\begin{aligned} X^i_\mu (\hat{P}\hat{E}\hat{P})^\mu{}_\nu &= 0, & X^i_\mu (\hat{P}\hat{E}\hat{P})^{\mu\nu} &= 0, \\ Y_i^\mu (\hat{P}\hat{E}\hat{P})_{\mu\nu} &= 0, & Y_i^\mu (\hat{P}\hat{E}\hat{P})^\nu{}_\mu &= 0, \\ H^\mu{}_a (\hat{P}\hat{E}\hat{P})^\nu{}_\mu &= K_{\mu a} (\hat{P}\hat{E}\hat{P})^{\mu\nu} = 0, \\ H^\mu{}_a (\hat{P}\hat{E}\hat{P})_{\mu\nu} &= K_{\mu a} (\hat{P}\hat{E}\hat{P})^\mu{}_\nu = 0. \end{aligned} \tag{3.23}$$

Finally, the first and the last, the second and the fifth, the third and the last, the fourth and the fifth give

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^\mu{}_\nu &= 0, & (\hat{P}\hat{E}\hat{P})^{\mu\nu} &= 0, \\ (\hat{P}\hat{E}\hat{P})_{\mu\nu} &= 0, & (\hat{P}\hat{E}\hat{P})^\nu{}_\mu &= 0. \end{aligned} \tag{3.24}$$

In this way, all the components of $(\hat{P}\hat{E}\hat{P})_{AB}$ vanish and the full EDFEs persist to be valid universally for arbitrary (n, \bar{n}) backgrounds.

Comment. From (3.17), off-shell relations hold among the components of the EDFEs,

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^\nu{}_\mu &= K_{\mu\rho} (\hat{P}\hat{E}\hat{P})^{\rho\nu} + X^i_\mu Y_i^\rho (\hat{P}\hat{E}\hat{P})^\rho{}_\nu, \\ (\hat{P}\hat{E}\hat{P})^\mu{}_\nu &= -(\hat{P}\hat{E}\hat{P})^{\mu\rho} K_{\rho\nu} + (\hat{P}\hat{E}\hat{P})^\mu{}_\rho \bar{Y}^{\bar{i}}_\rho \bar{X}^{\bar{i}}_\nu, \\ (\hat{P}\hat{E}\hat{P})_{\mu\nu} &= -K_{\mu\rho} (\hat{P}\hat{E}\hat{P})^{\rho\sigma} K_{\sigma\nu} + K_{\mu\rho} (\hat{P}\hat{E}\hat{P})^\rho{}_\sigma \bar{Y}^{\bar{i}}_\sigma \bar{X}^{\bar{i}}_\nu - X^i_\mu Y_i^\rho (\hat{P}\hat{E}\hat{P})^\sigma{}_\rho K_{\sigma\nu} + X^i_\mu Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}^{\bar{i}}_\sigma \bar{X}^{\bar{i}}_\nu, \end{aligned} \tag{3.25}$$

such that the full EDFEs are satisfied if

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^{\mu\nu} &= 0, & Y_i^\mu (\hat{P}\hat{E}\hat{P})^\nu{}_\mu &= 0, \\ (\hat{P}\hat{E}\hat{P})^\mu{}_\nu \bar{Y}^{\bar{i}}_\nu &= 0, & Y_i^\mu (\hat{P}\hat{E}\hat{P})_{\mu\nu} \bar{Y}^{\bar{i}}_\nu &= 0, & \hat{E}_A{}^A &= 0. \end{aligned} \tag{3.26}$$

4 What if we keep (n, \bar{n}) fixed once and for all?

As it is a outstandingly hard problem to construct an action principle for non-Riemannian gravity (c.f. [45,46,48] for recent proposals), we may ask if the DFT action restricted to a fixed (n, \bar{n}) sector might serve as the desired target spacetime gravitational action, c.f. (4.21). In this section, seeking for the answer to this question, we reanalyze the variational principle of DFT, crucially keeping (n, \bar{n}) fixed. To our surprise, we obtain a subtle discrepancy with the previous section where the most general variations of the DFT-metric were analyzed. We shall see that, when the values of (n, \bar{n}) are kept fixed and $n\bar{n} \neq 0$, not all the components of the EDFEs (3.26) are implied by the variational principle.

4.1 Variational principle with fixed (n, \bar{n})

We start with (3.1) which gives the variation of the general DFT action induced by the DFT-metric. With fixed (n, \bar{n}) , the variation of the DFT-metric therein should comprise the variations of the (n, \bar{n}) component fields:

$$\delta\mathcal{H} = \mathcal{B} \left(\delta_B H + \delta [X^i(Y_i)^T - \bar{X}^{\bar{i}}(\bar{Y}_{\bar{i}})^T] \delta K + \delta B [Y_i(X^i)^T - \bar{Y}_{\bar{i}}(\bar{X}^{\bar{i}})^T] - [(X^i(Y_i)^T - \bar{X}^{\bar{i}}(\bar{Y}_{\bar{i}})^T) \delta B] \right) \mathcal{B}^T. \tag{4.1}$$

Further, from their defining relations, (2.6), (2.7), the variations of the (n, \bar{n}) component fields are not entirely independent. They must meet

$$\begin{aligned} \delta Y_i^\mu &= -H^{\mu\rho} \delta K_{\rho\sigma} Y_i^\sigma - Y_j^\mu \delta X_\rho^j Y_i^\rho - \bar{Y}_j^\mu \delta \bar{X}_\rho^j Y_i^\rho, \\ \delta \bar{Y}_i^\mu &= -H^{\mu\rho} \delta K_{\rho\sigma} \bar{Y}_i^\sigma - Y_j^\mu \delta X_\rho^j \bar{Y}_i^\rho - \bar{Y}_j^\mu \delta \bar{X}_\rho^j \bar{Y}_i^\rho, \\ \delta X_\mu^i &= -K_{\mu\rho} \delta H^{\rho\sigma} X_\sigma^i - X_\mu^j \delta Y_j^\rho X_\rho^i - \bar{X}_\mu^j \delta \bar{Y}_j^\rho X_\rho^i, \\ \delta \bar{X}_\mu^i &= -K_{\mu\rho} \delta H^{\rho\sigma} \bar{X}_\sigma^i - X_\mu^j \delta Y_j^\rho \bar{X}_\rho^i - \bar{X}_\mu^j \delta \bar{Y}_j^\rho \bar{X}_\rho^i, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \delta H^{\mu\nu} &= -H^{\mu\rho} \delta K_{\rho\sigma} H^{\sigma\nu} - 2Y_i^{(\mu} H^{\nu)\rho} \delta X_\rho^i \\ &\quad - 2\bar{Y}_i^{(\mu} H^{\nu)\rho} \delta \bar{X}_\rho^i, \\ \delta K_{\mu\nu} &= -K_{\mu\rho} \delta H^{\rho\sigma} K_{\sigma\nu} - 2\delta Y_i^\rho K_{\rho(\mu} X_{\nu)}^i \\ &\quad - 2\delta \bar{Y}_i^\rho K_{\rho(\mu} \bar{X}_{\nu)}^i. \end{aligned} \tag{4.3}$$

From (2.12), we also note

$$\delta K_{\mu\nu} = 2K_{(\mu}{}^a \delta K_{\nu)a}, \quad \delta H^{\mu\nu} = 2H^{(\mu}{}^a \delta H^{\nu)a}, \tag{4.4}$$

which imply in particular,

$$\delta K_{\mu\nu} Y_i^\mu \bar{Y}_i^\nu = 0, \quad \delta H^{\mu\nu} X_\mu^i \bar{X}_i^\nu = 0. \tag{4.5}$$

It is then evident from (4.2), (4.3), and (4.4) that we have freedom to choose either $\{\delta K_\mu{}^a, \delta X_\rho^i, \delta \bar{X}_\sigma^i\}$ or $\{\delta H^\mu{}_a, \delta Y_i^\rho, \delta \bar{Y}_i^\sigma\}$ as independent variations. Each of them has (formally) D^2 number of degrees of freedom.

Now, we substitute (4.1) into (3.1), and utilize (4.2), (4.3), (4.4), (4.5) to obtain

$$\begin{aligned} &\delta \int \frac{1}{16\pi G} e^{-2d} S_{(0)} + \mathcal{L}_{\text{matter}} \\ &= \int \frac{1}{4\pi G} e^{-2d} \left[2\delta K_{\nu a} K_\mu{}^a (\hat{P}\hat{E}\hat{P})^{(\mu\nu)} \right. \\ &\quad \left. + Y_i^\rho (\hat{P}\hat{E}\hat{P})_\rho{}^\mu \delta X_\mu^i - \delta \bar{X}_\mu^i (\hat{P}\hat{E}\hat{P})^\rho{}_\mu \delta \bar{Y}_i^\rho \right. \\ &\quad \left. - \delta B_{\mu\nu} (\hat{P}\hat{E}\hat{P})^{\mu\nu} \right] \\ &= \int \frac{1}{4\pi G} e^{-2d} \left[2\delta H^{\nu a} H^\mu{}_a (\hat{P}\hat{E}\hat{P})_{(\mu\nu)} \right. \\ &\quad \left. + X_\rho^i (\hat{P}\hat{E}\hat{P})^\rho{}_\mu \delta Y_i^\mu \right. \\ &\quad \left. - \delta \bar{Y}_i^\mu (\hat{P}\hat{E}\hat{P})_\mu{}^\rho \delta \bar{X}_\rho^i - \delta B_{\mu\nu} (\hat{P}\hat{E}\hat{P})^{\mu\nu} \right]. \end{aligned} \tag{4.6}$$

$$\begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix} = \begin{pmatrix} \delta H & -H\delta B + \delta [Y_i(X^i)^T - \bar{Y}_i(\bar{X}^i)^T] \\ \delta BH + \delta [X^i(Y_i)^T - \bar{X}^i(\bar{Y}_i)^T] \delta K + \delta B [Y_i(X^i)^T - \bar{Y}_i(\bar{X}^i)^T] - [(X^i(Y_i)^T - \bar{X}^i(\bar{Y}_i)^T) \delta B] \end{pmatrix}. \tag{4.12}$$

The variational principle implies either from the second line of (4.6),

$$\begin{aligned} K_{\mu\rho} (\hat{P}\hat{E}\hat{P})^{\rho\nu} + (\hat{P}\hat{E}\hat{P})^{\nu\rho} K_{\rho\mu} &= 0, \quad Y_i^\rho (\hat{P}\hat{E}\hat{P})_\rho{}^\mu = 0, \\ (\hat{P}\hat{E}\hat{P})^\mu{}_\rho \bar{Y}_i^\rho &= 0, \quad (\hat{P}\hat{E}\hat{P})^{[\mu\nu]} = 0, \end{aligned} \tag{4.7}$$

or alternatively from the third line of (4.6),

$$\begin{aligned} H^{\mu\rho} (\hat{P}\hat{E}\hat{P})_{\rho\nu} + (\hat{P}\hat{E}\hat{P})_{\nu\rho} H^{\rho\mu} &= 0, \quad X_\rho^i (\hat{P}\hat{E}\hat{P})^\rho{}_\mu = 0, \\ (\hat{P}\hat{E}\hat{P})_\mu{}^\rho \bar{X}_\rho^i &= 0, \quad (\hat{P}\hat{E}\hat{P})^{[\mu\nu]} = 0. \end{aligned} \tag{4.8}$$

Although (4.7) and (4.8) appear seemingly different, they are – as should be – equivalent. In fact, they are both equivalent to

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^{\mu\nu} &= 0, \quad (\hat{P}\hat{E}\hat{P})^\mu{}_\nu = 0, \\ (\hat{P}\hat{E}\hat{P})_\mu{}^\nu &= 0, \quad (\hat{P}\hat{E}\hat{P})_{\mu\nu} = X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}_i^\sigma \bar{X}_i^\nu, \end{aligned} \tag{4.9}$$

which are, from (3.25), further equivalent to more concise ones,

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^{\mu\nu} &= 0, \quad (\hat{P}\hat{E}\hat{P})^\mu{}_\nu \bar{Y}_i^\nu = 0, \\ Y_i^\mu (\hat{P}\hat{E}\hat{P})_\mu{}^\nu &= 0. \end{aligned} \tag{4.10}$$

Appendix A carries our proof.

The surprise which is manifest in (4.9) is that, when $n\bar{n} \neq 0$ the variational principle with fixed (n, \bar{n}) does not imply the full EDFEs (3.26): it does not constrain $Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}_i^\sigma$. However, as we have shown in the previous section, within the DFT frame they should vanish on-shell, $Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}_i^\sigma = 0$, and the full EDFEs should hold. We shall continue to discuss and conclude in the final Sect. 5.

4.2 Difference between keeping (n, \bar{n}) fixed or not

In order to understand the discrepancy in the resulting Euler–Lagrangian equations, (3.24) vs. (4.9), here we investigate how the infinitesimal variations of the component fields of the (n, \bar{n}) DFT-metric (4.1),

$$\{\delta H^{\mu\nu}, \delta K_{\rho\sigma}, \delta X_\mu^i, \delta Y_j^\nu, \delta \bar{X}_\rho^i, \delta \bar{Y}_j^\sigma, \delta B_{\mu\nu}\}, \tag{4.11}$$

contribute actually to the α, β, γ variables defined in the generic variation of the DFT-metric (3.7),

With (3.9), one can identify the contributions thoroughly:

$$\begin{aligned} \alpha_{ab} &= -H^\mu{}_a H^\nu{}_b \delta K_{\mu\nu} = -2\delta K_{\rho(a} H^{\rho}{}_{b)}, \\ \beta_{ab} &= -\alpha_{ab} = -2K_{\rho(a} \delta H^{\rho}{}_{b)} = -K_{\mu a} K_{\nu b} \delta H^{\mu\nu}, \\ \alpha^{ai} &= -H^{\rho a} \delta X_\rho^i, \\ \beta_{ai} &= -K_{\rho a} \delta Y_i^\rho + H^{\rho}{}_a \delta B_{\rho\sigma} Y_i^\sigma, \end{aligned}$$

$$\begin{aligned}
 \alpha^{a\bar{i}} &= -H^{\rho a} \delta \bar{X}_{\rho}^{\bar{i}}, \\
 \beta_{a\bar{i}} &= -K_{\rho a} \delta \bar{Y}_{\bar{i}}^{\rho} - H^{\rho a} \delta B_{\rho\sigma} \bar{Y}_{\bar{i}}^{\sigma}, \\
 \alpha^{ij} &= 0, \quad \beta_{ij} = 0, \\
 \alpha^{\bar{i}\bar{j}} &= 0, \quad \beta_{\bar{i}\bar{j}} = 0, \\
 \boxed{\alpha^{i\bar{i}} = 0}, \quad \beta_{i\bar{i}} &= -2Y_i^{\rho} \delta B_{\rho\sigma} \bar{Y}_{\bar{i}}^{\sigma},
 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
 \gamma^a_i &= K_{\rho}^a \delta Y_i^{\rho} - H^{\rho a} \delta B_{\rho\sigma} Y_i^{\sigma} = -\beta^a_i, \\
 \gamma^a_{\bar{i}} &= -K_{\rho}^a \delta \bar{Y}_{\bar{i}}^{\rho} - H^{\rho a} \delta B_{\rho\sigma} \bar{Y}_{\bar{i}}^{\sigma} = \beta^a_{\bar{i}}, \\
 \gamma^i_a &= -X^i_{\rho} \delta H^{\rho\sigma} K_{\sigma a} = -\alpha^i_a, \\
 \gamma^{\bar{i}}_a &= \bar{X}^{\bar{i}}_{\rho} \delta H^{\rho\sigma} K_{\sigma a} = \alpha^{\bar{i}}_a, \quad \gamma^i_j = 0, \\
 \gamma^{\bar{i}}_{\bar{j}} &= 0, \quad \gamma^i_{\bar{i}} = -X^i_{\rho} \delta \bar{Y}_{\bar{i}}^{\rho}, \\
 \gamma^{\bar{i}}_i &= \bar{X}^{\bar{i}}_{\rho} \delta Y_i^{\rho}, \quad \gamma_{ab} = -\gamma_{ba} = -H^{\rho a} H^{\sigma b} \delta B_{\rho\sigma}.
 \end{aligned} \tag{4.14}$$

This is consistent with the general result of (3.11). However, one surprise is that $\alpha^{i\bar{i}}$ must be trivial when the (n, \bar{n}) component fields (4.11) are varied while keeping (n, \bar{n}) fixed.

To identify the significance of the $\alpha^{i\bar{i}}$ parameter, we focus on the induced transformation of $H^{\mu\nu}$,

$$H^{\mu\nu} \longrightarrow H'^{\mu\nu} \simeq H^{\mu\nu} + 2Y_i^{(\mu} \bar{Y}_{\bar{i}}^{\nu)} \alpha^{i\bar{i}}. \tag{4.15}$$

Geometrically the deformation of $2Y_i^{(\mu} \bar{Y}_{\bar{i}}^{\nu)} \alpha^{i\bar{i}}$ is ‘orthogonal’ to $H^{\mu\nu}$, and thus we expect it should reduce the kernel of $H^{\mu\nu}$. To verify this explicitly, we solve for the eigenvectors of $H'^{\mu\nu}$ with zero eigenvalue,

$$H'^{\mu\nu} \mathcal{X}_\nu = 0. \tag{4.16}$$

Without loss of generality, utilizing the completeness relation, $K_{\mu a} H^{\nu a} + X^i_{\mu} Y_i^{\nu} + \bar{X}^{\bar{i}}_{\mu} \bar{Y}_{\bar{i}}^{\nu} = \delta_{\mu}^{\nu}$, we decompose the zero-eigenvector,

$$\mathcal{X}_\nu = K_{\nu a} c^a + X^i_{\nu} c_i + \bar{X}^{\bar{i}}_{\nu} \bar{c}_{\bar{i}}, \tag{4.17}$$

substitute this ansatz into (4.16), and acquire the conditions the coefficients should satisfy,

$$c^a = 0, \quad \alpha^{i\bar{i}} c_i = 0, \quad \alpha^{i\bar{i}} \bar{c}_{\bar{i}} = 0. \tag{4.18}$$

This shows that there are in total $(n - \text{rank}[\alpha^{i\bar{i}}]) + (\bar{n} - \text{rank}[\alpha^{i\bar{i}}]) = n + \bar{n} - 2 \times \text{rank}[\alpha^{i\bar{i}}]$ number of zero-eigenvectors. Moreover, from the invariance, $\delta \mathcal{H}_A^A = 0$ (3.6), we note that the deformation by the $\alpha^{i\bar{i}}$ parameter actually changes the type of the ‘non-Riemannianity’ as

$$(n, \bar{n}) \longrightarrow (n - \text{rank}[\alpha^{i\bar{i}}], \bar{n} - \text{rank}[\alpha^{i\bar{i}}]). \tag{4.19}$$

This essentially explains why $\alpha^{i\bar{i}}$ vanishes in (4.13) where the (n, \bar{n}) component field variables are varied with fixed values of (n, \bar{n}) , or fixed ‘non-Riemannianity’. It is intriguing to

note that the deformation makes the DFT-metric always *less* non-Riemannian.⁸

4.3 Non-Riemannian differential geometry as bookkeeping device

This subsection is the last one before Conclusion, and is somewhat out of context. At first reading, readers may glimpse (4.21) in comparison with (4.20), and skip to the final Sect. 5.

While the various (n, \bar{n}) non-Riemannian geometries are universally well described by DFT through $\mathbf{O}(D, D)$ covariant tensors – as summarized in Table 1 – it may be desirable in practical computations to break the manifest $\mathbf{O}(D, D)$ symmetry spontaneously by fixing the section, $\tilde{\partial}^\mu \equiv 0$, and dismantle the $\mathbf{O}(D, D)$ covariant tensors or curvatures into smaller *modules* which should be still covariant under undoubled ordinary diffeomorphisms, B -field gauge symmetry, and $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local rotations. We remind the readers that in the case of the $(0, 0)$ Riemannian sector, the $\mathbf{O}(D, D)$ singlet DFT scalar curvature reduces to four modules (c.f. [27, 95, 96]):

$$\begin{aligned}
 S_{(0)} \Big|_{(0,0) \text{ Riemannian}} &= R_g - \frac{1}{12} \mathbb{H}^{\lambda\mu\nu} \mathbb{H}_{\lambda\mu\nu} \\
 &+ 4\Box\phi - 4\partial^\mu\phi\partial_\mu\phi.
 \end{aligned} \tag{4.20}$$

Here in this last section, we propose an undoubled non-Riemannian differential tool kit, such as covariant derivative and curvature, for an arbitrary (n, \bar{n}) sector. It descends from the DFT geometry, or the so-called ‘semi-covariant formalism’ [63], and generalizes the standard Riemannian geometry underlying (4.20) in a consistent manner. It breaks the manifest $\mathbf{O}(D, D)$ symmetry spontaneously, but preserves the ordinary diffeomorphisms, B -field gauge symmetry, and the $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local symmetries as desired. In particular, it enables us to extend the Riemannian expression of (4.20) in a way ‘continuously’ to the generic (n, \bar{n}) non-Riemannian case,

$$\begin{aligned}
 S_{(0)} \Big|_{(n,\bar{n}) \text{ fixed}} &= R - \frac{1}{12} H^{\lambda\rho} H^{\mu\sigma} H^{\nu\tau} \mathbb{H}_{\lambda\mu\nu} \mathbb{H}_{\rho\sigma\tau} \\
 &- \mathbb{H}_{\lambda\mu\nu} H^{\lambda\rho} \left(Y_i^\mu \mathbb{D}^\nu X_\rho^i - \bar{Y}_{\bar{i}}^\mu \mathbb{D}^\nu \bar{X}_{\bar{i}}^\rho \right) \\
 &+ 4K_{\mu\nu} (\mathbb{D}^\mu \mathbb{D}^\nu d - \mathbb{D}^\mu d \mathbb{D}^\nu d).
 \end{aligned} \tag{4.21}$$

We commence our explanation. First of all, \mathbb{D}^μ is our proposed ‘upper-indexed’ covariant derivative:

$$\mathbb{D}^\mu = H^{\mu\rho} \partial_\rho + \Omega^\mu + \Upsilon^\mu + \tilde{\Upsilon}^\mu, \tag{4.22}$$

⁸ In a way, on the space of full DFT geometries, the $(0, 0)$ Riemannian geometry corresponds to an open set as $\det(H^{\mu\nu}) \neq 0$, while the genuine non-Riemannian geometries form a closed set, $\det(H^{\mu\nu}) = 0$. Infinitesimally, it is impossible to leave an open set but possible to leave a closed set.

which preserves both the undoubled diffeomorphisms (2.23) and the $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local symmetries (2.15) as is equipped with proper connections: for undoubled ordinary diffeomorphisms,

$$\begin{aligned} \Omega^{\mu\nu}{}_{\lambda} &= -\frac{1}{2}\partial_{\lambda}H^{\mu\nu} - H^{\rho[\mu}\partial_{\rho}H^{\nu]\sigma}K_{\sigma\lambda} \\ &\quad - H^{\rho[\mu}\partial_{\rho}Y_i^{\nu]}X_{\lambda}^i - H^{\rho[\mu}\partial_{\rho}\bar{Y}_i^{\nu]}\bar{X}_{\lambda}^{\bar{i}} \\ &\quad + \left(2H^{\rho[\mu}Y_i^{\nu]}\partial_{[\tau}X_{\rho]}^i - 2H^{\rho[\mu}\bar{Y}_i^{\nu]}\partial_{[\tau}\bar{X}_{\rho]}^{\bar{i}}\right) \\ &\quad \times \left(Y_j^{\tau}X_{\lambda}^j - \bar{Y}_j^{\tau}\bar{X}_{\lambda}^{\bar{j}}\right), \end{aligned} \tag{4.23}$$

and for $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ rotations,

$$\begin{aligned} \Upsilon^{\mu i}{}_j &= -2H^{\mu\rho}Y_j^{\sigma}\partial_{[\rho}X_{\sigma]}^i, \quad \bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}} \\ &= -2H^{\mu\rho}\bar{Y}_j^{\sigma}\partial_{[\rho}\bar{X}_{\sigma]}^{\bar{i}}. \end{aligned} \tag{4.24}$$

We also denote a diffeomorphism-only preserving covariant derivative by

$$\mathfrak{D}^{\mu} = H^{\mu\rho}\partial_{\rho} + \Omega^{\mu}, \tag{4.25}$$

and write for (4.22) and (4.24),

$$\begin{aligned} \mathbb{D}^{\mu} &= \mathfrak{D}^{\mu} + \Upsilon^{\mu} + \bar{\Upsilon}^{\mu}, \\ \Upsilon^{\mu i}{}_j &= X_{\rho}^i\mathfrak{D}^{\mu}Y_j^{\rho} = -Y_j^{\rho}\mathfrak{D}^{\mu}X_{\rho}^i, \\ \bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}} &= \bar{X}_{\rho}^{\bar{i}}\mathfrak{D}^{\mu}\bar{Y}_j^{\rho} = -\bar{Y}_j^{\rho}\mathfrak{D}^{\mu}\bar{X}_{\rho}^{\bar{i}}. \end{aligned} \tag{4.26}$$

Taking care of both spacetime and $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ indices, \mathbb{D}^{μ} acts on general tensor densities in a standard manner:

$$\begin{aligned} \mathbb{D}^{\lambda}T^{\mu\bar{i}}{}_{\nu j\bar{j}} &= H^{\lambda\rho}\partial_{\rho}T^{\mu\bar{i}}{}_{\nu j\bar{j}} - \omega_{\tau}\Omega^{\lambda\rho}{}_{\rho}T^{\mu\bar{i}}{}_{\nu j\bar{j}} \\ &\quad + \Omega^{\lambda\mu}{}_{\rho}T^{\rho\bar{i}}{}_{\nu j\bar{j}} - \Omega^{\lambda\rho}{}_{\nu}T^{\mu\bar{i}}{}_{\rho j\bar{j}} \\ &\quad + \Upsilon^{\lambda i}{}_kT^{\mu\bar{i}}{}_{\nu j\bar{j}} + \bar{\Upsilon}^{\lambda\bar{i}}{}_{\bar{k}}T^{\mu\bar{i}}{}_{\nu j\bar{j}} \\ &\quad - \Upsilon^{\lambda k}{}_jT^{\mu\bar{i}}{}_{\nu k\bar{j}} - \bar{\Upsilon}^{\lambda\bar{k}}{}_{\bar{j}}T^{\mu\bar{i}}{}_{\nu j\bar{k}}. \end{aligned} \tag{4.27}$$

On the other hand, \mathfrak{D}^{μ} cares only the spacetime indices and ignores any $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ indices,

$$\begin{aligned} \mathfrak{D}^{\lambda}T^{\mu\bar{i}}{}_{\nu j\bar{j}} &= H^{\lambda\rho}\partial_{\rho}T^{\mu\bar{i}}{}_{\nu j\bar{j}} - \omega_{\tau}\Omega^{\lambda\rho}{}_{\rho}T^{\mu\bar{i}}{}_{\nu j\bar{j}} \\ &\quad + \Omega^{\lambda\mu}{}_{\rho}T^{\rho\bar{i}}{}_{\nu j\bar{j}} - \Omega^{\lambda\rho}{}_{\nu}T^{\mu\bar{i}}{}_{\rho j\bar{j}}. \end{aligned} \tag{4.28}$$

For example, we have explicitly

$$\begin{aligned} \mathbb{D}^{\mu}X_v^i &= H^{\mu\rho}\partial_{\rho}X_v^i - X_{\rho}^i\Omega^{\mu\rho}{}_{\nu} + \Upsilon^{\mu i}{}_jX_v^j \\ &= H^{\mu\rho}(KH)_{\nu}{}^{\sigma}\partial_{[\rho}X_{\sigma]}^i, \\ \mathbb{D}^{\mu}\bar{X}_v^{\bar{i}} &= H^{\mu\rho}\partial_{\rho}\bar{X}_v^{\bar{i}} - \bar{X}_{\rho}^{\bar{i}}\Omega^{\mu\rho}{}_{\nu} + \bar{\Upsilon}^{\mu\bar{i}}{}_{\bar{j}}\bar{X}_v^{\bar{j}} \\ &= H^{\mu\rho}(KH)_{\nu}{}^{\sigma}\partial_{[\rho}\bar{X}_{\sigma]}^{\bar{i}}. \end{aligned} \tag{4.29}$$

It is instructive to see that the far right resulting expressions in (4.29) are clearly covariant under both diffeomorphisms and $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local rotations, as the ρ, σ indices therein are skew-symmetrized and also contracted with $H^{\mu\rho}, (KH)_{\nu}{}^{\sigma}$. However, without the $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ connections, we note

$$\begin{aligned} \mathfrak{D}^{\mu}X_v^i &= H^{\mu\rho}\partial_{\rho}X_v^i - \Omega^{\mu\rho}{}_{\nu}X_v^i \\ &= H^{\mu\rho}\left[(KH)_{\nu}{}^{\sigma} + 2X_v^jY_j^{\sigma}\right]\partial_{[\rho}X_{\sigma]}^i, \end{aligned} \tag{4.30}$$

and this breaks the $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local symmetry.

Further, for the DFT-dilaton we should have

$$\begin{aligned} \mathbb{D}^{\mu}d &= \mathfrak{D}^{\mu}d = -\frac{1}{2}e^{2d}\mathfrak{D}^{\mu}\left(e^{-2d}\right) \\ &= H^{\mu\rho}\partial_{\rho}d + \frac{1}{2}\Omega^{\mu\rho}{}_{\rho}, \end{aligned} \tag{4.31}$$

where we have explicitly

$$\begin{aligned} \Omega^{\mu\rho}{}_{\rho} &= H^{\mu\nu}\left(\frac{1}{2}H^{\rho\sigma}\partial_{\nu}K_{\rho\sigma} + Y_i^{\rho}\partial_{\rho}X_v^i + \bar{Y}_i^{\rho}\partial_{\rho}\bar{X}_v^{\bar{i}}\right) \\ &= -\frac{1}{2}K_{\rho\sigma}\partial^{\mu}H^{\rho\sigma} + K_{\rho\sigma}\partial^{\sigma}H^{\mu\rho} - \partial_{\rho}H^{\mu\rho}. \end{aligned} \tag{4.32}$$

Because $H^{\mu\nu}$ and $K_{\rho\sigma}$ are generically degenerate, the conventional relation (2.1) between the DFT-dilaton, d , and the string dilaton, ϕ , cannot hold. We stick to use the DFT-dilaton all the way.⁹

The connections do the job as they transform properly under the diffeomorphisms (2.23), (2.25) and the $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local rotations (2.15),

$$\begin{aligned} \delta_{\xi}\Omega^{\mu\nu}{}_{\lambda} &= \mathcal{L}_{\xi}\Omega^{\mu\nu}{}_{\lambda} + H^{\mu\rho}\partial_{\rho}\delta_{\lambda}\xi^{\nu}, \\ \delta_{\mathbf{GL}}\Omega^{\mu\nu}{}_{\lambda} &= 0, \\ \delta_{\xi}\Upsilon^{\mu i}{}_j &= \mathcal{L}_{\xi}\Upsilon^{\mu i}{}_j, \\ \delta_{\mathbf{GL}}\Upsilon^{\mu i}{}_j &= \Upsilon^{\mu k}w_k^i - w_j^k\Upsilon^{\mu i}{}_k - H^{\mu\rho}\partial_{\rho}w_j^i, \\ \delta_{\xi}\bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}} &= \mathcal{L}_{\xi}\bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}}, \\ \delta_{\mathbf{GL}}\bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}} &= \bar{\Upsilon}^{\mu \bar{k}}\bar{w}_{\bar{k}}^{\bar{i}} - \bar{w}_{\bar{j}}^{\bar{k}}\bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{k}} - H^{\mu\rho}\partial_{\rho}\bar{w}_{\bar{j}}^{\bar{i}}. \end{aligned} \tag{4.33}$$

In particular, $X_{\mu}^i\Omega^{\mu\nu}{}_{\lambda}, \bar{X}_{\mu}^{\bar{i}}\Omega^{\mu\nu}{}_{\lambda}$, and $H^{\rho[\lambda}\Omega^{\mu]\nu}{}_{\rho}$ are covariant tensors which might be viewed as ‘‘torsions’’.

Finally, we define an upper-indexed Ricci curvature,

$$\begin{aligned} R^{\mu\nu} &:= H^{\mu\rho}\partial_{\rho}\Omega^{\sigma\nu}{}_{\sigma} - H^{\sigma\rho}\partial_{\rho}\Omega^{\mu\nu}{}_{\sigma} + \Omega^{\mu\nu}{}_{\rho}\Omega^{\sigma\rho}{}_{\sigma} \\ &\quad - \Omega^{\sigma\mu}{}_{\rho}\Omega^{\rho\nu}{}_{\sigma} \\ &\quad + 2\left(Y_i^{\sigma}\mathbb{D}^{\mu}X_{\rho}^i + \bar{Y}_i^{\sigma}\mathbb{D}^{\mu}\bar{X}_{\rho}^{\bar{i}}\right)\Omega^{\rho\nu}{}_{\sigma}, \end{aligned} \tag{4.34}$$

which is diffeomorphism and $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ covariant, as it comes from the following commutator relation that is clearly also covariant,

$$\begin{aligned} [\mathbb{D}^{\mu}, \mathbb{D}^{\nu}]T_{\nu} &+ 4\left(Y_i^{\sigma}\mathbb{D}^{\mu}X_{\rho}^i + \bar{Y}_i^{\sigma}\mathbb{D}^{\mu}\bar{X}_{\rho}^{\bar{i}}\right)H^{\rho\nu}\partial_{[\sigma}T_{\nu]} \\ &+ 2\left(Y_i^{\nu}\mathbb{D}^{\mu}X_{\rho}^i + \bar{Y}_i^{\nu}\mathbb{D}^{\mu}\bar{X}_{\rho}^{\bar{i}}\right)\mathbb{D}^{\rho}T_{\nu} = -R^{\mu\nu}T_{\nu}. \end{aligned} \tag{4.35}$$

A scalar curvature follows naturally,

$$R := K_{\mu\nu}R^{\mu\nu}, \tag{4.36}$$

which debuted in (4.21).

⁹ We tend to believe that the conventional string dilaton, ϕ , is an artifact of the (0, 0) Riemannian geometry and the DFT-dilaton, d , is more fundamental as being an $\mathbf{O}(D, D)$ singlet.

Our covariant derivative is “compatible” with the (n, \bar{n}) component fields in a generalized fashion:

$$\begin{aligned} \mathbb{D}^\lambda H^{\mu\nu} + 2Y_i^{(\mu} H^{\nu)\rho} \mathbb{D}^\lambda X_\rho^i + 2\bar{Y}_i^{(\mu} H^{\nu)\rho} \mathbb{D}^\lambda \bar{X}_\rho^{\bar{i}} &= 0, \\ Y_i^\rho \mathbb{D}^\mu X_\rho^i &= 0, \quad \bar{Y}_i^\rho \mathbb{D}^\mu \bar{X}_\rho^{\bar{i}} = 0, \\ \mathbb{D}^\lambda K_{\mu\nu} + 2X_{(\mu}^i K_{\nu)\rho} \mathbb{D}^\lambda Y_i^\rho + 2\bar{X}_{(\mu}^{\bar{i}} K_{\nu)\rho} \mathbb{D}^\lambda \bar{Y}_i^\rho &= 0, \\ \mathbb{D}^\lambda \delta_\mu^\nu &= 0, \quad \mathbb{D}^\lambda \delta_i^j = 0, \quad \mathbb{D}^\lambda \delta_{\bar{i}}^{\bar{j}} = 0. \end{aligned} \tag{4.37}$$

Another characteristic is that, if we add one more torsion linear in the \mathbb{H} -flux to the Ω -connection,

$$\begin{aligned} \widehat{\Omega}^{\mu\nu\lambda} &:= \Omega^{\mu\nu\lambda} + \frac{1}{2} H^{\mu\rho} H^{\nu\sigma} \mathbb{H}_{\rho\sigma\tau} \left(Y_j^\tau X_\lambda^j - \bar{Y}_j^\tau \bar{X}_\lambda^{\bar{j}} \right), \\ \widehat{\mathbb{D}}^\mu &:= H^{\mu\rho} \partial_\rho + \widehat{\Omega}^\mu, \end{aligned} \tag{4.38}$$

the hatted new connection becomes Milne-shift covariant as well, in the sense of (2.16), (2.25), (2.26),

$$\delta_M \widehat{\Omega}^{\mu\nu\lambda} = -\frac{1}{2} \delta_M B_{\lambda\rho} \widehat{\mathbb{H}}^{\mu\nu\rho}, \quad \delta_M \widehat{\mathbb{H}}^{\lambda\mu\nu} = 0, \tag{4.39}$$

where $\widehat{\mathbb{H}}^{\lambda\mu\nu}$ is a diffeomorphism covariant, $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ invariant, and Milne-shift invariant \mathbb{H} -flux,

$$\begin{aligned} \widehat{\mathbb{H}}^{\lambda\mu\nu} &= \widehat{\mathbb{H}}^{[\lambda\mu\nu]} := H^{\lambda\rho} H^{\mu\sigma} H^{\nu\tau} H_{\rho\sigma\tau} + 6H^{\rho[\lambda} Y_i^\mu \mathbb{D}^{\nu]} X_\rho^i \\ &\quad - 6H^{\rho[\lambda} \bar{Y}_i^\mu \mathbb{D}^{\nu]} \bar{X}_\rho^{\bar{i}}. \end{aligned} \tag{4.40}$$

The $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ connections (4.26) are inert to the addition of the \mathbb{H} -flux-valued-torsion (4.38) as

$$\begin{aligned} \Upsilon^{\mu i}{}_j &= X_\rho^i \mathbb{D}^\mu Y_j^\rho = X_\rho^i \widehat{\mathbb{D}}^\mu Y_j^\rho = \widehat{\Upsilon}^{\mu i}{}_j, \\ \bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}} &= \bar{X}_\rho^{\bar{i}} \mathbb{D}^\mu \bar{Y}_j^\rho = \bar{X}_\rho^{\bar{i}} \widehat{\mathbb{D}}^\mu \bar{Y}_j^\rho = \widehat{\bar{\Upsilon}}^{\mu \bar{i}}{}_{\bar{j}}, \end{aligned} \tag{4.41}$$

while they transform under the Mine-shift as $\delta_M \Upsilon^{\mu i}{}_j = -2H^{\rho\sigma} V_{\rho j} \mathbb{D}^\mu X_\sigma^i$, $\delta_M \bar{\Upsilon}^{\mu \bar{i}}{}_{\bar{j}} = -2H^{\rho\sigma} \bar{V}_{\rho \bar{j}} \mathbb{D}^\mu \bar{X}_\sigma^{\bar{i}}$.

After all, in terms of a hatted covariant derivative,

$$\widehat{\mathbb{D}}^\mu := H^{\mu\rho} \partial_\rho + \widehat{\Omega}^\mu + \widehat{\Upsilon}^\mu + \widehat{\bar{\Upsilon}}^\mu, \tag{4.42}$$

we can dismantle the DFT curvatures into a \mathbb{H} -flux-free (circled) term and evidently \mathbb{H} -flux-valued ones:

$$\begin{aligned} S_{(0)} &= \hat{S}_{(0)} - \frac{1}{12} H^{\lambda\rho} H^{\mu\sigma} H^{\nu\tau} \mathbb{H}_{\lambda\mu\nu} \mathbb{H}_{\rho\sigma\tau} \\ &\quad - \mathbb{H}_{\lambda\mu\nu} H^{\lambda\rho} \left(Y_i^\mu \widehat{\mathbb{D}}^\nu X_\rho^i - \bar{Y}_i^\mu \widehat{\mathbb{D}}^\nu \bar{X}_\rho^{\bar{i}} \right), \\ Y_i^\mu (\hat{P} \hat{S} \hat{P})_\mu{}^\nu &= Y_i^\mu (\hat{P} \hat{S} \hat{P})_\mu{}^\nu + Y_i^\mu \left[\mathbb{H}_{\mu\rho\sigma} \left(\bar{Y}_i^{\bar{i}} \widehat{\mathbb{D}}^{\nu]} \bar{X}_\lambda^{\bar{i}} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} Y_j^\rho \widehat{\mathbb{D}}^\nu X_\lambda^j \right) H^{\lambda\sigma} + \frac{1}{4} H^{\nu\sigma} e^{2d} \widehat{\mathbb{D}}^\rho \left(e^{-2d} \mathbb{H}_{\rho\sigma\mu} \right) \right], \\ (\hat{P} \hat{S} \hat{P})^\mu{}_\nu \bar{Y}_i^\nu &= (\hat{P} \hat{S} \hat{P})^\mu{}_\nu \bar{Y}_i^\nu + \left[\mathbb{H}_{\rho\sigma\nu} \left(Y_i^{\bar{i}} \widehat{\mathbb{D}}^{\mu]} X_\lambda^{\bar{i}} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \bar{Y}_j^{\bar{j}} \widehat{\mathbb{D}}^\mu \bar{X}_\lambda^{\bar{j}} \right) H^{\lambda\sigma} + \frac{1}{4} H^{\mu\sigma} e^{2d} \widehat{\mathbb{D}}^\rho \left(e^{-2d} \mathbb{H}_{\rho\sigma\nu} \right) \right] \bar{Y}_i^\nu, \\ Y_i^\mu (\hat{P} \hat{S} \hat{P})_{\mu\nu} \bar{Y}_i^\nu &= Y_i^\mu (\hat{P} \hat{S} \hat{P})_{\mu\nu} \bar{Y}_i^\nu \\ &\quad + \frac{1}{2} Y_i^\mu \bar{Y}_i^\nu \left[e^{2d} \widehat{\mathbb{D}}^\rho \left(e^{-2d} \mathbb{H}_{\rho\mu\nu} \right) \right. \\ &\quad \left. + \frac{1}{2} H^{\alpha\beta} H^{\gamma\delta} \mathbb{H}_{\mu\alpha\gamma} \mathbb{H}_{\nu\beta\delta} \right], \\ (\hat{P} \hat{S} \hat{P})^{\mu\nu} &= (\hat{P} \hat{S} \hat{P})^{\mu\nu} - \frac{1}{8} e^{2d} \partial_\lambda \left(e^{-2d} \widehat{\mathbb{H}}^{\lambda\mu\nu} \right) \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{16} H^{\mu\rho} H^{\nu\sigma} H^{\alpha\beta} H^{\gamma\delta} \mathbb{H}_{\rho\alpha\gamma} \mathbb{H}_{\sigma\beta\delta} \\ &+ \frac{3}{8} \left[H^{\mu\rho} \left(H^{\lambda[\nu} Y_i^\sigma \widehat{\mathbb{D}}^\tau X_\lambda^i - H^{\lambda[\nu} \bar{Y}_i^\sigma \widehat{\mathbb{D}}^\tau \bar{X}_\lambda^{\bar{i}} \right) \right. \\ &\quad \left. + (\mu \leftrightarrow \nu) \right] \mathbb{H}_{\rho\sigma\tau}, \end{aligned} \tag{4.43}$$

where, as it should be obvious from our notation, we set $\hat{S}_{AB} := (\mathcal{B}^{-1})_A{}^C (\mathcal{B}^{-1})_B{}^D S_{CD}$, and the circled quantities are all \mathbb{H} -flux free: from Table 1 or [63,65],

$$\begin{aligned} \hat{S}_{AB} &= 2\partial_A \partial_B d - e^{2d} \partial_C \left(e^{-2d} \hat{\Gamma}_{(AB)}{}^C \right) \\ &\quad + \frac{1}{2} \hat{\Gamma}_{ACD} \hat{\Gamma}_B{}^{CD} - \frac{1}{2} \hat{\Gamma}_{CDA} \hat{\Gamma}^{CD}{}_B, \end{aligned} \tag{4.44}$$

and, with (3.16),

$$\begin{aligned} \hat{\Gamma}_{CAB} &:= 2(\hat{P} \hat{\partial}_C \hat{P} \hat{P})_{[AB]} + 2(\hat{P}_{[A}{}^D \hat{P}_{B]}{}^E \\ &\quad - \hat{P}_{[A}{}^D \hat{P}_{B]}{}^E) \partial_D \hat{P}_{EC} - 4 \left(\frac{1}{\hat{P}_G{}^{G-1}} \hat{P}_{C[A} \hat{P}_{B]}{}^D \right. \\ &\quad \left. + \frac{1}{\hat{P}_G{}^{G-1}} \hat{P}_{C[A} \hat{P}_{B]}{}^D \right) \left(\partial_D d + (\hat{P} \hat{\partial}^E \hat{P} \hat{P})_{[ED]} \right). \end{aligned} \tag{4.45}$$

While we organize the \mathbb{H} -flux-valued parts in terms of the hatted covariant derivative, like (4.41), we have

$$\begin{aligned} \widehat{\mathbb{D}}^\mu X_\nu^i &= \mathbb{D}^\mu X_\nu^i, \quad \widehat{\mathbb{D}}^\mu \bar{X}_\nu^{\bar{i}} = \mathbb{D}^\mu \bar{X}_\nu^{\bar{i}}, \\ \widehat{\mathbb{D}}^\mu d &= \mathbb{D}^\mu d, \quad \widehat{\mathbb{D}}^\mu \widehat{\mathbb{D}}^\nu d = \mathbb{D}^\mu \mathbb{D}^\nu d. \end{aligned} \tag{4.46}$$

The only nontrivial distinction lies in

$$\begin{aligned} \widehat{\mathbb{D}}^\rho \left(e^{-2d} \mathbb{H}_{\rho\mu\nu} \right) &= \mathbb{D}^\rho \left(e^{-2d} \mathbb{H}_{\rho\mu\nu} \right) + H^{\rho\alpha} H^{\sigma\beta} H_{\rho\sigma\tau} \\ &\quad \times \left(H_{\alpha\beta[\mu} X_{\nu]}^i Y_i^\tau - H_{\alpha\beta[\mu} \bar{X}_{\nu]}^{\bar{i}} \bar{Y}_i^\tau \right). \end{aligned} \tag{4.47}$$

Since $e^{-2d} \widehat{\mathbb{H}}^{\lambda\mu\nu}$ carries a unit weight, its contraction with the ordinary derivative, $\partial_\lambda (e^{-2d} \widehat{\mathbb{H}}^{\lambda\mu\nu})$, is also by itself diffeomorphism covariant. In this way, every single term in (4.43) is symmetric under both undoubled diffeomorphisms and $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ local rotations. On the other hand, as we have singled out the \mathbb{H} -flux-valued terms from the \mathbb{H} -flux-free parts, each individual term is not necessarily Milne-shift covariant.

As advertised in (4.21), we may further dismantle $\hat{S}_{(0)}$ as well as $(\hat{P} \hat{S} \hat{P})^{\mu\nu}$ into more elementary modules:

$$\begin{aligned} \hat{S}_{(0)} &= R + 4K_{\mu\nu} (\mathbb{D}^\mu \mathbb{D}^\nu d - \mathbb{D}^\mu d \mathbb{D}^\nu d), \\ (\hat{P} \hat{S} \hat{P})^{\mu\nu} &= -\frac{1}{4} R^{(\mu\nu)} - \frac{1}{4} (Y_i^\mu \mathbb{D}^\rho X_\sigma^i \\ &\quad - \bar{Y}_i^\mu \mathbb{D}^\rho \bar{X}_\sigma^{\bar{i}}) (Y_j^\nu \mathbb{D}^\sigma X_\rho^j \\ &\quad - \bar{Y}_j^\nu \mathbb{D}^\sigma \bar{X}_\rho^{\bar{j}}) - \frac{1}{2} \mathbb{D}^{(\mu} \mathbb{D}^{\nu)} d. \end{aligned} \tag{4.48}$$

From (3.26), vanishing of all the five quantities in (4.43) characterizes the (n, \bar{n}) vacuum geometry of DFT.

Comment 1. It is worth while to note

$$e^{-2d} K_{\mu\nu} (\mathbb{D}^\mu \mathbb{D}^\nu d - 2\mathbb{D}^\mu d \mathbb{D}^\nu d) = \partial_\mu (e^{-2d} \mathbb{D}^\mu d), \quad (4.49)$$

and rewrite the ‘kinetic term’ of the DFT-dilaton in (4.21),

$$4e^{-2d} K_{\mu\nu} (\mathbb{D}^\mu \mathbb{D}^\nu d - \mathbb{D}^\mu d \mathbb{D}^\nu d) = 4e^{-2d} K_{\mu\nu} \mathbb{D}^\mu d \mathbb{D}^\nu d + 4\partial_\mu (e^{-2d} \mathbb{D}^\mu d). \quad (4.50)$$

Comment 2. Especially when $n + \bar{n} = D$, i.e. in the maximally non-Riemannian cases, all the quantities like $H^{\mu\nu}, K_{\rho\sigma}, \Omega^{\lambda\mu}_\nu, \widehat{\mathbb{H}}^{\lambda\mu\nu}, \mathbb{D}^\mu d, R^{\mu\nu}, S_{(0)}, (\mathring{P}\hat{S}\mathring{P})^{\mu\nu}$ are trivial except the term of interest, $Y_i^\mu (\mathring{P}\hat{S}\mathring{P})_{\mu\nu} \bar{Y}_i^\mu$.

Comment 3. Restricted to the (0, 0) Riemannian case, we have $K_{\mu\nu} = g_{\mu\nu}, H^{\mu\nu} = g^{\mu\nu}, K_{\mu\rho} H^{\rho\mu} = \delta_\mu^\nu$,

and the vectors, $\{X^i_\mu, \bar{X}^i_\nu, Y_j^\rho, \bar{Y}_j^\sigma\}$, are trivially absent.

Both the Ω and $\widehat{\Omega}$ connections (4.23, 4.38) coincide with nothing but the standard Christoffel symbols with one index raised by the Riemannian metric,

$$\widehat{\Omega}^{\mu\nu}_\lambda \equiv \Omega^{\mu\nu}_\lambda \equiv g^{\mu\rho} \left\{ \begin{matrix} \nu \\ \rho \lambda \end{matrix} \right\}. \quad (4.51)$$

Consequently, the proposed covariant derivative (4.25) and Ricci curvature (4.34) reduce to the standard covariant derivative and Ricci curvature in Riemannian geometry,

$$\mathcal{D}^\mu \equiv g^{\mu\nu} \nabla_\nu = g^{\mu\nu} (\partial_\nu + \{ \cdot \}_\nu), \\ R^{\mu\nu} \equiv g^{\mu\rho} g^{\nu\sigma} R^\nabla_{\rho\sigma}. \quad (4.52)$$

Comment 4. Besides $(\mathring{P}\hat{S}\mathring{P})^{\mu\nu}$, we have not been able to dismantle other circled \mathbb{H} -flux-free DFT Ricci curvatures which carry at least one lower index. In addition to $\mathcal{D}^\mu = H^{\mu\rho} \partial_\rho + \Omega^\mu$, separate type of covariant derivatives containing $Y_i^\mu \partial_\mu$ or $\bar{Y}_i^\mu \partial_\mu$ might help, c.f. (B.13).

Comment 5. Appendix B sketches how we have arrived at the above proposal of the non-Riemannian differential tool kit starting from the semi-covariant formalism of DFT. In any case, our proposal is meant to provide a bookkeeping device to expound the EDFEs into smaller modules and to single out the \mathbb{H} -fluxes. The actual computation of the variations of the action, even with (n, \bar{n}) fixed, are still powered by the semi-covariant formalism, specifically (3.2).

5 Conclusion

The very gravitational theory string theory predicts may be the Double Field Theory with non-Riemannian surprises, rather than General Relativity based on Riemannian geometry. The underlying mathematical structure of DFT unifies supergravity with various non-Riemannian gravities including (stringy) Newton–Cartan geometry, ultra-relativistic Carroll geometry, and non-relativistic Gomis–Ooguri string theory. The non-Riemannian geometries of DFT can be classified by two non-negative integers, (n, \bar{n}) [1].

We have analyzed with care the variational principle. We have shown that the most general infinitesimal variations of an arbitrary (n, \bar{n}) DFT-metric have $D^2 - (n - \bar{n})^2$ number of degrees of freedom, which matches with the dimension of the underlying coset [11], $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n) \times \mathbf{O}(s+\bar{n},t+\bar{n})}$ (3.14). Through action principle, these variations imply the full Einstein Double Field Equations (3.22), (3.24). However, $n\bar{n}$ number of them change the value of (n, \bar{n}) , i.e. the type of non-Riemannianity (4.19). Consequently, if we keep (n, \bar{n}) fixed once and for all, the variational principle gets restricted and fails to reproduce the full EDFEs: the specific part, $Y_i^\mu (PE\bar{P})_{\mu\nu} \bar{Y}_i^\nu$, does not have to vanish on-shell (4.9).¹⁰

The EDFEs are supposed to arise as the string worldsheet beta-functions [97,98]. For the doubled-yet-gauged string action (1.11) upon an arbitrarily chosen (n, \bar{n}) background, the (n, \bar{n}) -changing variations of the DFT-metric would correspond to marginal deformations. We must stress that these deformations could not be realized by merely varying the background component fields with fixed (n, \bar{n}) (4.13), c.f. [52,54,56]. Nevertheless, it is natural to expect that $n\bar{n}$ number of $Y_i^\mu (\mathring{P}\hat{E}\mathring{P})_{\mu\nu} \bar{Y}_i^\nu$ arise as the corresponding beta-functions too. That is to say, at least for $n\bar{n} \neq 0$, the quantum consistency with the worldsheet string theory seems to forbid us to fix (n, \bar{n}) rigidly. We conclude that the various non-Riemannian gravities should be identified as different solution sectors of Double Field Theory rather than viewed as independent theories. Quantum consistency of the non-Riemannian geometries calls for thorough investigation, which may enlarge the scope of the string theory landscape far beyond Riemann.

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¹⁰ As can be seen from (4.44), $Y_i^\rho (\mathring{P}\hat{E}\mathring{P})_{\rho\sigma} \bar{Y}_i^\sigma$ contains a second order derivative of the DFT-dilaton along the Y_i^μ and \bar{Y}_i^ν directions, i.e. $Y_i^\mu \bar{Y}_i^\nu \partial_\mu \partial_\nu d$.

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Appendix

A Proof on the equivalence among (4.7), (4.8), (4.9), and (4.10)

Taking $\{\delta K_\mu^a, \delta X_\rho^i, \delta \bar{X}_\sigma^i\}$ as independent variations, from the second line of (4.6), the variational principle implies (4.7) which we enumerate here:

$$K_{\mu\rho}(\hat{P}\hat{E}\hat{P})^{\rho\nu} + (\hat{P}\hat{E}\hat{P})^{\nu\rho}K_{\rho\mu} = 0, \tag{A.1}$$

$$Y_i^\rho(\hat{P}\hat{E}\hat{P})_\rho^\mu = 0, \tag{A.2}$$

$$(\hat{P}\hat{E}\hat{P})^\mu_\rho \bar{Y}_i^\rho = 0, \tag{A.3}$$

$$(\hat{P}\hat{E}\hat{P})^{[\mu\nu]} = 0. \tag{A.4}$$

Alternatively taking $\{\delta H^\mu_a, \delta Y_i^\rho, \delta \bar{Y}_i^\sigma\}$ as independent variations, we acquire from the third line of (4.6),

$$H^{\mu\rho}(\hat{P}\hat{E}\hat{P})_{\rho\nu} + (\hat{P}\hat{E}\hat{P})_{\nu\rho}H^{\rho\mu} = 0, \tag{A.5}$$

$$X_\rho^i(\hat{P}\hat{E}\hat{P})^\rho_\mu = 0, \tag{A.6}$$

$$(\hat{P}\hat{E}\hat{P})_\mu^\rho \bar{X}_\rho^i = 0, \tag{A.7}$$

$$(\hat{P}\hat{E}\hat{P})^{[\mu\nu]} = 0. \tag{A.8}$$

Henceforth we show that Eqs. (A.1, A.2, A.3, A.4) and Eqs. (A.5, A.6, A.7, A.8) are all equivalent to (4.9) as well as to (4.10). The equivalence between (4.9) and (4.10) should be obvious from the off-shell relation (3.25), and therefore we focus on (4.9) which we recall for quick reference:

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^{\mu\nu} = 0, \quad (\hat{P}\hat{E}\hat{P})^\mu_\nu = 0, \\ (\hat{P}\hat{E}\hat{P})_\mu^\nu = 0, \quad (\hat{P}\hat{E}\hat{P})_{\mu\nu} = X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}_i^\sigma \bar{X}_\nu^i. \end{aligned} \tag{A.9}$$

Proof It is manifest that (A.9) implies both Eqs. (A.1, A.2, A.3, A.4) and Eqs. (A.5, A.6, A.7, A.8). Thus, we only need to show the converse. Equations (A.4) and (A.8) are common and give, combined with (3.17),

$$X_\mu^i (\hat{P}\hat{E}\hat{P})^{\mu\nu} = 0, \quad (\hat{P}\hat{E}\hat{P})^{\mu\nu} \bar{X}_\nu^i = 0. \tag{A.10}$$

With these in mind we first focus on the former set of Eqs. (A.1, A.2, A.3, A.4), of which the first and the last imply

$$K_{\mu\rho}(\hat{P}\hat{E}\hat{P})^{\rho\nu} = 0, \quad (\hat{P}\hat{E}\hat{P})^{\nu\rho}K_{\rho\mu} = 0. \tag{A.11}$$

Consequently, with the completeness relation (2.7), the identities from (3.17), and (A.2), we note

$$(\hat{P}\hat{E}\hat{P})_\mu^\nu = (KH)_\mu^\rho (\hat{P}\hat{E}\hat{P})_\rho^\nu + X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_\rho^\nu$$

$$= K_{\mu\rho}(\hat{P}\hat{E}\hat{P})^{\rho\nu} + X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_\rho^\nu = 0. \tag{A.12}$$

Similarly we get with (A.3),

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^\mu_\nu = (\hat{P}\hat{E}\hat{P})^\mu_\rho (HK)_\nu^\rho + (\hat{P}\hat{E}\hat{P})^\mu_\rho \bar{Y}_i^\rho \bar{X}_\nu^i \\ = -(\hat{P}\hat{E}\hat{P})^{\mu\rho}K_{\rho\nu} + (\hat{P}\hat{E}\hat{P})^\mu_\rho \bar{Y}_i^\rho \bar{X}_\nu^i = 0, \end{aligned} \tag{A.13}$$

and with (3.17), (A.4),

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})^{\mu\nu} = (HK)^\mu_\rho (\hat{P}\hat{E}\hat{P})^{\rho\nu} \\ = H^{\mu\rho}(\hat{P}\hat{E}\hat{P})_\rho^\nu = 0. \end{aligned} \tag{A.14}$$

It follows that

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})_{\mu\nu} = \left[(KH)_\mu^\rho + X_\mu^i Y_i^\rho \right] (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \\ \left[(HK)^\sigma_\nu + \bar{Y}_i^\sigma \bar{X}_\nu^i \right] \\ = -K_{\mu\rho}(\hat{P}\hat{E}\hat{P})^{\rho\sigma}K_{\sigma\nu} + K_{\mu\rho}(\hat{P}\hat{E}\hat{P})^\rho_\sigma \bar{Y}_i^\sigma \bar{X}_\nu^i \\ - X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_\rho^\sigma K_{\sigma\nu} + X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}_i^\sigma \bar{X}_\nu^i \\ = X_\mu^i Y_i^\rho (\hat{P}\hat{E}\hat{P})_{\rho\sigma} \bar{Y}_i^\sigma \bar{X}_\nu^i. \end{aligned} \tag{A.15}$$

Thus, Eqs. (A.1, A.2, A.3, A.4) are equivalent to (A.9).

We now turn to the latter set of equations (A.5, A.6, A.7, A.8). In a parallel manner to (A.12), (A.13), we note from (A.6), (A.7),

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})_\nu^\mu = (\hat{P}\hat{E}\hat{P})_{\nu\rho} (KH)^\rho_\mu + (\hat{P}\hat{E}\hat{P})_{\nu\rho} \bar{X}_\rho^i \bar{Y}_i^\mu \\ = -(\hat{P}\hat{E}\hat{P})_{\nu\rho}H^{\rho\mu}, \\ (\hat{P}\hat{E}\hat{P})^\mu_\nu = (HK)^\mu_\rho (\hat{P}\hat{E}\hat{P})_\nu^\rho + Y_i^\mu X_\rho^i (\hat{P}\hat{E}\hat{P})_\nu^\rho \\ = H^{\mu\rho}(\hat{P}\hat{E}\hat{P})_{\rho\nu}, \end{aligned} \tag{A.16}$$

which imply with (A.5),

$$(\hat{P}\hat{E}\hat{P})_\mu^\nu = (\hat{P}\hat{E}\hat{P})^\nu_\mu, \tag{A.17}$$

and hence in particular,

$$\begin{aligned} Y_i^\mu (\hat{P}\hat{E}\hat{P})_\mu^\nu = 0, \quad (\hat{P}\hat{E}\hat{P})_\mu^\nu \bar{X}_\nu^i = 0, \\ X_\mu^i (\hat{P}\hat{E}\hat{P})^\mu_\nu = 0, \quad (\hat{P}\hat{E}\hat{P})^\mu_\nu \bar{Y}_i^\nu = 0. \end{aligned} \tag{A.18}$$

Then with (A.8) and from

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})_\mu^\nu = (KH)_\mu^\rho (\hat{P}\hat{E}\hat{P})_\rho^\nu = K_{\mu\rho}(\hat{P}\hat{E}\hat{P})^{\rho\nu} \\ = (\hat{P}\hat{E}\hat{P})^{\nu\rho}K_{\rho\mu} = -(\hat{P}\hat{E}\hat{P})^\nu_\rho (HK)^\rho_\mu \\ = -(\hat{P}\hat{E}\hat{P})^\nu_\mu, \end{aligned} \tag{A.19}$$

we note

$$\begin{aligned} (\hat{P}\hat{E}\hat{P})_\mu^\nu = 0, \\ (\hat{P}\hat{E}\hat{P})^\mu_\nu = 0. \end{aligned} \tag{A.20}$$

It follows then, with (A.8), (A.10),

$$\begin{aligned}
 (\mathring{P}\hat{E}\mathring{P})^{\mu\nu} &= (HK)^\mu{}_\rho (\mathring{P}\hat{E}\mathring{P})^{\rho\nu} \\
 &= H^{\mu\rho} (\mathring{P}\hat{E}\mathring{P})_\rho{}^\nu = 0.
 \end{aligned}
 \tag{A.21}$$

Finally, as in (A.15), we have

$$\begin{aligned}
 (\mathring{P}\hat{E}\mathring{P})_{\mu\nu} &= K_{\mu\rho} (\mathring{P}\hat{E}\mathring{P})^\rho{}_\nu + X^i{}_\mu Y_i{}^\rho (\mathring{P}\hat{E}\mathring{P})_{\rho\sigma} [(HK)^\sigma{}_\nu \\
 &\quad + \bar{Y}_i{}^\sigma \bar{X}_\nu{}^i] = X^i{}_\mu Y_i{}^\rho (\mathring{P}\hat{E}\mathring{P})_{\rho\sigma} \bar{Y}_i{}^\sigma \bar{X}_\nu{}^i.
 \end{aligned}
 \tag{A.22}$$

Thus, Eqs. (A.5, A.6, A.7, A.8) are also equivalent to (A.9), and this completes our *Proof*. \square

B Derivation of the non-Riemannian differential tool kit from DFT

The non-Riemannian differential geometry we have proposed in section 4.3, in particular the hatted $\widehat{\Omega}$ connection (4.38), descends from the known covariant derivatives in the DFT semi-covariant formalism [63]: specifically,¹¹

$$P_A{}^C \bar{P}_B{}^D \nabla_C V_D, \quad \bar{P}_A{}^C P_B{}^D \nabla_C V_D.
 \tag{B.1}$$

In order to convert these into undoubled ordinary covariant quantities—or to get rid of the bare B -field in them—we multiply \mathcal{B}^{-1} as in (2.22) and write

$$\begin{aligned}
 (\mathcal{B}^{-1}P)^{AC} (\mathcal{B}^{-1}\bar{P})^{BD} \nabla_C V_D &= \mathring{P}^{AC} \mathring{P}^{BD} \hat{\nabla}_C \hat{V}_D, \\
 (\mathcal{B}^{-1}\bar{P})^{AC} (\mathcal{B}^{-1}P)^{BD} \nabla_C V_D &= \mathring{P}^{AC} \mathring{P}^{BD} \hat{\nabla}_C \hat{V}_D.
 \end{aligned}
 \tag{B.2}$$

Here we set

$$\begin{aligned}
 \hat{\nabla}_A \hat{V}_B &:= (\mathcal{B}^{-1})_A{}^C (\mathcal{B}^{-1})_B{}^D \nabla_C V_D \\
 &= \partial_A \hat{V}_B + \hat{\Gamma}_{ABC} \hat{V}^C,
 \end{aligned}
 \tag{B.3}$$

and $\hat{\Gamma}_{ABC}$ is a naturally induced – or ‘twisted’ [100], c.f. [101] – new connection,¹²

$$\begin{aligned}
 \hat{\Gamma}_{CAB} &:= (\mathcal{B}^{-1})_C{}^D (\mathcal{B}^{-1})_A{}^E (\mathcal{B}^{-1})_B{}^F \Gamma_{DEF} + \partial_C \mathcal{B}_{AB} \\
 &= \mathring{\Gamma}_{CAB} + (\mathring{P}_C{}^\rho \mathring{P}_A{}^\sigma \mathring{P}_B{}^\tau + \mathring{P}_C{}^\rho \mathring{P}_A{}^\sigma \mathring{P}_B{}^\tau) \mathbb{H}_{\rho\sigma\tau} \\
 &\quad + (\mathring{P} + \bar{\mathring{P}})_{CAB}{}^{DEF} \partial_D \mathcal{B}_{EF}.
 \end{aligned}
 \tag{B.4}$$

The very last term on the second line involves certain six-indexed projectors formed by $\mathring{P}_A{}^B, \bar{\mathring{P}}_A{}^B$ (c.f. Eq. (17) of [63] and Eq. (2.26) of [65]), and is actually irrelevant as it is always projected out in the final results. Using the new connection (B.4) we can conveniently separate the B -field contributions and eventually acquire the results (4.43).

Now, remembering $\mathring{P}^{\mu\nu} = -\bar{\mathring{P}}^{\mu\nu} = \frac{1}{2} H^{\mu\nu}$ (3.16) and $\mathring{P}_A{}^B + \bar{\mathring{P}}_A{}^B = \delta_A{}^B$, we subtract the two quantities in

¹¹ While $\nabla_C V_D$ itself is not covariant, the projected ones in (B.1) are covariant, and hence the name, ‘semi-covariant formalism’.

¹² Note $\mathcal{B}_A{}^B \partial_B = \partial_A$ as $\bar{\partial}^\mu \equiv 0$.

(B.2), and acquire a desired covariant derivative, or $\widehat{\mathcal{D}}^\mu = H^{\mu\rho} \partial_\rho + \widehat{\Omega}^\mu$ (4.38):

$$\begin{aligned}
 &2 \left[(\mathcal{B}^{-1}P)^{\lambda C} (\mathcal{B}^{-1}\bar{P})^{BD} - (\mathcal{B}^{-1}\bar{P})^{\lambda C} (\mathcal{B}^{-1}P)^{BD} \right] \nabla_C V_D \\
 &= \left(\widehat{\mathcal{D}}^\lambda \hat{V}^\mu - \widehat{\Phi}^{\lambda\rho\mu} \hat{V}^\rho \right),
 \end{aligned}
 \tag{B.5}$$

where, with shorthand notation,

$$\begin{aligned}
 (\mathring{P}\hat{\Gamma}\mathring{P})_{ABC} &:= \mathring{P}_A{}^D \hat{\Gamma}_{DBE} \mathring{P}^E{}_C, \\
 (\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})_{ABC} &:= \bar{\mathring{P}}_A{}^D \hat{\Gamma}_{DBE} \bar{\mathring{P}}^E{}_C,
 \end{aligned}
 \tag{B.6}$$

we set, extending (4.39),

$$\begin{aligned}
 \widehat{\Phi}^{\lambda}{}_{\mu\nu} &= 2(\mathring{P}\hat{\Gamma}\mathring{P})^\lambda{}_{\mu\nu} - 2(\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})^\lambda{}_{\mu\nu}, \\
 \delta_M \widehat{\Phi}^{\lambda}{}_{\mu\nu} &= -\delta_M B_{\mu\rho} \Omega^{\lambda\rho}{}_\nu + \delta_M B_{\nu\rho} \widehat{\Omega}^{\lambda\rho}{}_\mu, \\
 \widehat{\Omega}^{\mu\nu}{}_\lambda &= 2(\mathring{P}\hat{\Gamma}\mathring{P})^{\mu\nu}{}_\lambda - 2(\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})^{\mu\nu}{}_\lambda, \\
 \delta_M \widehat{\Omega}^{\mu\nu}{}_\lambda &= -\frac{1}{2} \delta_M B_{\lambda\rho} \mathbb{H}^{\mu\nu\rho}, \\
 \mathbb{H}^{\lambda\mu\nu} &= 4(\mathring{P}\hat{\Gamma}\mathring{P})^{\lambda\mu\nu} - 4(\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})^{\lambda\mu\nu}, \quad \delta_M \mathbb{H}^{\lambda\mu\nu} = 0.
 \end{aligned}
 \tag{B.7}$$

With $\partial_A = (0, \partial_\mu)$ and $\xi^A = (0, \xi^\mu)$ (2.23), using Eq. (2.43) of [65], we get under diffeomorphisms,

$$\begin{aligned}
 \delta_\xi (\mathring{P}\hat{\Gamma}\mathring{P})_{ABC} &= \mathcal{L}_\xi (\mathring{P}\hat{\Gamma}\mathring{P})_{ABC} + \mathring{P}_A{}^\rho \mathring{P}_C{}^\sigma \partial_\rho \partial_\sigma \xi_B \\
 &\quad - \mathring{P}_A{}^\rho \mathring{P}_{C\sigma} \partial_\rho \partial_B \xi^\sigma, \\
 \delta_\xi (\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})_{ABC} &= \mathcal{L}_\xi (\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})_{ABC} + \bar{\mathring{P}}_A{}^\rho \bar{\mathring{P}}_C{}^\sigma \partial_\rho \partial_\sigma \xi_B \\
 &\quad - \bar{\mathring{P}}_A{}^\rho \bar{\mathring{P}}_{C\sigma} \partial_\rho \partial_B \xi^\sigma.
 \end{aligned}
 \tag{B.8}$$

Hence both $\widehat{\Phi}^{\lambda}{}_{\mu\nu}$ and $\mathbb{H}^{\lambda\mu\nu}$ are diffeomorphism covariant (and surely $\mathbf{GL}(n) \times \mathbf{GL}(\bar{n})$ invariant) tensors.

Further, due to identities,

$$\begin{aligned}
 (\mathring{P}\hat{\Gamma}\mathring{P})_A{}^\mu{}_B &= (\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})_B{}^\mu{}_A, \\
 (\mathring{P}\hat{\Gamma}\mathring{P})^\mu{}_{(AB)} &= (\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})^\mu{}_{(AB)},
 \end{aligned}
 \tag{B.9}$$

$\mathbb{H}^{\lambda\mu\nu}$ and $\widehat{\Phi}^{\lambda}{}_{\mu\nu}$ are skew-symmetric,

$$\mathbb{H}^{\lambda\mu\nu} = \mathbb{H}^{[\lambda\mu\nu]}, \quad \widehat{\Phi}^{\lambda}{}_{\mu\nu} = \widehat{\Phi}^{\lambda}{}_{[\mu\nu]},
 \tag{B.10}$$

and we may express $\widehat{\Omega}^{\mu\nu}{}_\lambda$ in different ways,

$$\begin{aligned}
 \widehat{\Omega}^{\mu\nu}{}_\lambda &= -2(\mathring{P}\hat{\Gamma}\mathring{P})^\mu{}_\lambda{}^\nu + 2(\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})^\mu{}_\lambda{}^\nu \\
 &= -2(\mathring{P}\hat{\Gamma}\mathring{P})_\lambda{}^{\nu\mu} + 2(\bar{\mathring{P}}\hat{\Gamma}\bar{\mathring{P}})_\lambda{}^{\nu\mu}.
 \end{aligned}
 \tag{B.11}$$

In particular, when the circled vector, $\mathring{V}_A = (0, \mathring{V}_\mu)$, is derivative-index-valued as $\mathring{V}^\mu \equiv 0$, from (2.26) \mathring{V}_μ becomes Milne-shift invariant and so does $\widehat{\mathcal{D}}^\lambda \mathring{V}_\mu$,

$$\delta_M \mathring{V}_\mu = 0, \quad \delta_M (\widehat{\mathcal{D}}^\lambda \mathring{V}_\mu) = 0.
 \tag{B.12}$$

Alternative combination of (B.1), rather than (B.5), can give different type of covariant derivatives,

$$\begin{aligned}
 (\mathcal{D}_i V)^\mu &:= Y_i{}^\rho (H^{\mu\sigma} \partial_\rho V_\sigma - \Omega^{\mu\sigma}{}_\rho V_\sigma), \\
 (\bar{\mathcal{D}}_i V)^\mu &:= \bar{Y}_i{}^\rho (H^{\mu\sigma} \partial_\rho V_\sigma - \Omega^{\mu\sigma}{}_\rho V_\sigma).
 \end{aligned}
 \tag{B.13}$$

However, these can act only on one-form fields, and appear not so useful.

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