Weak decays of triply heavy tetraquarks $b\bar{c}b\bar{q}$

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Abstract We study masses, lifetimes and weak decays of the triply heavy tetraquarks $b\bar{c}b\bar{q}$. The masses of tetraquark are explored under two different models. Further more, our calculations show a possibility of existence for the stable triply heavy tetraquark $b\bar{c}b\bar{q}$ with $J^P = 1^+$ which should be verified in experiment. Following the heavy quark expanding(HQE), the lifetimes of tetraquarks $b\bar{c}b\bar{q}$ can be expressed as the summation of different dimension operators. Particularly, we obtain the lifetimes of $b\bar{c}b\bar{q}$ at the next-to-leading order(NLO) given as $\tau(T_{\{\bar{c}q\}}^{bb}) = 0.70 \times 10^{-12}s$. Besides, we construct the weak decays Hamiltonian of $b\bar{c}b\bar{q}$ in hadronic level under the SU(3) flavor symmetry. The discussion of the Hamiltonian can deduce the decay amplitudes and width relations of the tetraquarks. Following the choosing rules, we collect some golden channels for the mesonic decays of tetraquarks, which will be helpful to search for triply heavy tetraquarks $b\bar{c}b\bar{q}$ in future experiments.

1 Introduction

The quark model could well explain the hadronic states for a long time, in which mesons are handled as the bound states of a constituent quark and anti-quark, and baryons are bound by three constituent quarks. Nevertheless, the Belle collaboration changes the situation in 2003 for the discovery of the X(3872) [1], in $B^\pm \rightarrow K^\pm X (X \rightarrow \pi^+\pi^- J/\psi)$ decays. The new state which contains the $c\bar{c}$ pair is not match with the ordinary quarkonium state, consequently expected to be the candidate of four quark exotic state [2–30]. Subsequently, many more exotic states are observed in processes, for instance, BES-III and Belle collaborations observe charged heavy quarkoniumlike states $Z_c(3900)^\pm$ [31–34], $Z_c(4020)^\pm$ [35,36], $Z_b(10610)^\pm$ [37] and $Z_b(10650)^\pm$ [37]. Until now, the X(3872) is one of the best studied four quark exotic state candidates, whose mass

right at the $D^0\bar{D}^{*0}$ threshold, and spin-parity quantum numbers $J^{PC} = 1^{++}$ [38]. However, there is still a open issue whether the state is a loosely bound molecule or a compact tetraquark.

The four-quark state $b\bar{c}b\bar{q}$ with three heavy quark is apparently different from the discovered quarkoniumlike state, in this respect, it might offer a new platform to study the internal structure of the exotic states. In addition, it can also be an ideal source to understand the hadronic dynamic and QCD factorization approach. Some theoretical and experimental studies have been carried out to explore the properties of the exotic state at present, especially the mass spectrum by the simple quark model [39], QCD sum rule [40]. In this paper, we will concern on the masses, lifetimes and decays channels of the state. The non-relativistic quark model [41] which has successfully used in the mass spectrum of S-wave hadron can be applied in multi-quark system. While the quark state might be one stable tetraquark, its dominant decay modes would be induced by the weak interaction. In the diquark-antidiquark model [42], the lifetimes of lowest lying tetraquarks with the spin-parity 0^+ and 1^+ can be achieved by the operator product expansion(OPE) technique [43,44]. Further more, the flavor SU(3) symmetry analysis which have been successfully applied into heavy mesons or baryons [45–61] can be used to study the weak decays of $b\bar{c}b\bar{q}$.

Generally, the lifetimes of the tetraquarks can be expressed into several matrix elements of effective operators in the OPE technique. In this case, we investigate the lifetimes at the leading and next-to-leading order in the heavy quark expansion. In the other case, the weak decays of the tetraquarks can be considered at the hadronic level Hamiltonian in the SU(3) symmetry. The triply heavy tetraquarks $b\bar{c}b\bar{q}$ and the heavy quark transition can be classified in the SU(3) light quark symmetry. Thus one can construct the Hamiltonian in the hadron level, and the non-perturbative effect can be absorbed into some parameters $a_i, b_i, c_i, \dots$. The discussion of the Hamiltonian can tell us the decay amplitudes and relations between different channels. Such analyses are benefit

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to the searching for the triply heavy tetraquarks in future experiment.

The paper is organized as follows. In Sect. 2, we will study the mass spectrum of tetraquark $b\bar{c}b\bar{q}$ with two different models. Section 3 concerns on the lifetimes of the tetraquark $b\bar{c}b\bar{q}$ with the approach of OPE. The SU(3) flavor symmetry analysis including particle multiplets and possible weak decays will be discussed in Sects. 4 and 5, in which the study of weak decays includes mesonic two-body non-leptonic decays and the three- or four-body semi-leptonic decays. In Sect. 6, we present a collection of golden decay channels. The summary is given in the end.

2 Masses

We will study the mass of the triply heavy tetraquark $bb\bar{c}\bar{q}$ signed as $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ in the section. Since the large ambiguity still be indicated in the quark masses and effective couplings which are actually determined by experiments, the study will no tell the mass of the tetraquark with higher precision. Even though, the calculation may show the possibility of the existence of the stable tetraquark against strong interaction, which is worthwhile to search in future experiment.

The study will be carried on with two models for the tetraquark state $J^P = 0^+, 1^+ (L = 0)$. Model-I is the diquark-antidiquark picture described by the effective Hamiltonian [3], given as

$$\mathcal{H} = m_{bb} + m_{c\bar{q}} + 2(\kappa_{bb})_{\bar{3}}(S_b \cdot S_b) + 2(\kappa_{\bar{c}\bar{q}})_3(S_{\bar{c}} \cdot S_{\bar{q}}) + 4\kappa_{b\bar{c}}(S_b \cdot S_{\bar{c}}) + 4\kappa_{b\bar{q}}(S_b \cdot S_{\bar{q}}), \quad (1)$$

where m_{ij} is the mass of constituent diquark, S_i and κ_{ij} indicate the spin and spin-spin coupling of quarks. Following the idea of “good” and “bad” diquark from Jaffe [45], we introduce the allowed tetraquark states $bb\bar{c}\bar{q}$ with the orbital angular momentum $L = 0$ in the flavor(f) $\otimes$ spin(s) $\otimes$ color(c) spcaes.

$$|bb, 1_f, 1_s, \bar{3}_c\rangle, |\bar{c}\bar{q}, 3_f, 1_s, 3_c\rangle, |3_f, 0_s, 1_c\rangle, \quad J^P = 0^+, \quad (2)$$

$$|bb, 1_f, 1_s, \bar{3}_c\rangle, |\bar{c}\bar{q}, 3_f, 0_s, 3_c\rangle, |3_f, 1_s, 1_c\rangle \quad \text{or} \\ |bb, 1_f, 1_s, \bar{3}_c\rangle, |\bar{c}\bar{q}, 3_f, 1_s, 3_c\rangle, |3_f, 1_s, 1_c\rangle, \quad J^P = 1^+. \quad (3)$$

Here, the quantum numbers of diquark (e.g. $|bb, 1_f, 1_s, \bar{3}_c\rangle, \dots$) and tetraquark ($|\dots, 3_f, 0_s, 1_c\rangle$) are displayed respectively. With the help of Hamiltonian in Eq.(1), we write the corresponding splitting mass matrices for the $J^P = 0^+, 1^+$ tetraquark

$$M(J^P) = m_{bb} + m_{c\bar{q}} + \Delta M(J^P), \quad (4)$$

and

$$\Delta M(0^+) = \frac{1}{2}\kappa_{bb} + \frac{1}{2}\kappa_{\bar{c}\bar{q}} - 2\kappa_{b\bar{c}} - 2\kappa_{b\bar{q}}, \quad (5)$$

$$\Delta M(1^+) = \left(\frac{\frac{1}{2}\kappa_{bb} - \frac{3}{2}\kappa_{\bar{c}\bar{q}}}{\sqrt{2}(\kappa_{b\bar{c}} - \kappa_{b\bar{q}})} \quad \frac{\sqrt{2}(\kappa_{b\bar{c}} - \kappa_{b\bar{q}})}{\frac{1}{2}(\kappa_{bb} + \kappa_{\bar{c}\bar{q}} - 2\kappa_{b\bar{c}} + 2\kappa_{b\bar{q}})} \right). \quad (6)$$

The spin-spin couplings of quark-quark $\kappa_{ij} = (\kappa_{ij})_{\bar{3}}$ and quark-antiquark $\kappa_{ij} = \frac{1}{4}(\kappa_{ij})_0$ are chosen respectively as [62]: $(\kappa_{c\bar{q}})_{\bar{3}} = 22\text{MeV}$, $(\kappa_{cs})_{\bar{3}} = 24\text{MeV}$, $(\kappa_{b\bar{c}})_0 = 20\text{MeV}$, $(\kappa_{b\bar{q}})_0 = 23\text{MeV}$, $(\kappa_{b\bar{s}})_0 = 23\text{MeV}$ and $(\kappa_{b\bar{b}})_0 = 36\text{MeV}$. The diquark masses are adopted as [62]: $m_{bb} = 10.016\text{GeV}$, $m_{c\bar{q}} = 1.975\text{GeV}$. One diagonalizes the mass matrix and collect the results given in Table 1.

Model-II is the simple nonrelativistic quark model which includes the hyperfine (e.g. color-spin) interaction [41,63]. The Hamiltonian of color-magnetic interaction is given as

$$\mathcal{H}_{CM} = \sum_{i < j} \left(-\frac{3}{8} \right) \frac{C_{ij}}{m_i m_j} \vec{\lambda}_i \cdot \vec{\lambda}_j \vec{s}_i \cdot \vec{s}_j, \quad (7)$$

where the coupling constant $C_{ij}/(m_i m_j)$ can be extracted from the hadron mass differences. One gets the corresponding parameters [64]: $C_{c\bar{q}}/m_{c\bar{q}} = 23\text{MeV}$, $C_{b\bar{q}}/(m_b m_q) = 23\text{MeV}$, $C_{cs}/(m_c m_s) = 47\text{MeV}$, $C_{b\bar{s}}/(m_b m_s) = 24\text{MeV}$ and $C_{b\bar{b}}/m_b^2 = 31\text{MeV}$. The $\vec{\lambda}_i$ is the Gell-Mann matrix for color SU(3) group, and $\vec{s}_i = \vec{\sigma}_i/2$ is the spin matrix for spin SU(2) group. We use three different schemes [64] for the effective quark mass, which respectively are SET-1 fitted from the mesons, SET-2 fitted from the baryons and SET-3 widely used in quark potential model. Consistently, we compare the results from different models and schemes in Table 1.

There is small gap prediction between Model-I and Model-II according to the Table 1. Actually, most of the states are above their $B_c B(B_c B_s)$ thresholds, except for the $bb\bar{c}\bar{u}/bb\bar{c}\bar{d}$ with $J^P = 1^+$ in the Model-II in the scheme of SET-3, which below the $B_c B^*$ threshold about 13MeV. It indicates that the state may be one stable particle under the strong interaction. In fact, our results are widely affected by the theoretical uncertainties, such as the effective bottom quark mass and the coupling constant of $C_{bb}/(m_b^2)$. However, it offers a possibility of existence for the stable triply heavy tetraquark which should be verified in experiment. To hunt for the states, we will study their weak decays properties in the next section, which may offer useful help for the future experimental search for these states.

3 Lifetimes

In the section, we will study the lifetimes of the tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ under the OPE. We proceed as follows, the decays

Table 1 The mass of tetraquark $bb\bar{c}\bar{q}$ with $J^P = 0^+, 1^+$ from Model-I and Model-II in units in GeV. Particularly we use three schemes [64] for the calculation from Model-II, which are respectively SET-1 fitted from the mesons, SET-2 fitted from the baryons and SET-3 widely used

Tetraquark(J^P)	Model-I	Model-II			[39]	$B_c B(B_c B_s)$ threshold
		SET-1	SET-2	SET-3		
$bb\bar{c}\bar{u}/bb\bar{c}\bar{d}(0^+)$	11.985	11.931	12.123	11.580	12.1448	11.554
$bb\bar{c}\bar{s}(0^+)$	12.171	12.121	12.312	11.755	12.3227	11.642
$bb\bar{c}\bar{u}/bb\bar{c}\bar{d}(1^+)$	11.963	11.938	12.130	11.587	12.2191	11.600
	12.007	12.011	12.203	11.660	12.1476	
$bb\bar{c}\bar{s}(1^+)$	12.145	12.122	12.308	11.751	12.3965	11.690
	12.193	12.189	12.380	11.823	12.3221	

width of $T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rightarrow X$ with the spin-parity $J^P = (0^+, 1^+)$ can be expressed respectively as

$$\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}}(0^+) \rightarrow X) = \frac{1}{2m_T} \sum_X \int \prod_i \left[\frac{d^3 \vec{p}_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta^4 \left(p_T - \sum_i p_i \right) \sum_\lambda |\langle X | \mathcal{H} | T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rangle|^2, \quad (8)$$

$$\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}}(1^+) \rightarrow X) = \frac{1}{2m_T} \sum_X \int \prod_i \left[\frac{d^3 \vec{p}_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta^4 \left(p_T - \sum_i p_i \right) \frac{1}{3} \sum_\lambda |\langle X | \mathcal{H} | T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rangle|^2, \quad (9)$$

here, m_T , λ and p_T^μ are the mass, spin and four-momentum of tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ respectively. The matrix in full theory Hamiltonian $\mathcal{H}$ can match with that of electro-weak effective Hamiltonian $\mathcal{H}_{eff}^{ew}$, whose Hamiltonian given as

$$\mathcal{H}_{eff}^{ew} = \frac{G_F}{\sqrt{2}} \left[\sum_{q=u,c} V_c^q (C_1 Q_1^q + C_2 Q_2^q) - V_p \sum_{j=3} C_j O_j \right], \quad (10)$$

where C_i and O_i are Wilson coefficient and effective operator. Besides, V_s are the combinations of Cabibbo-Kobayashi-Maskawa(CKM) elements. The total decay width of $\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rightarrow X)$ can be deduced in the optical theorem, rewritten as

$$\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rightarrow X) = \frac{1}{2m_T} \sum_\lambda \langle T_{\{\bar{c}\bar{q}\}}^{\{bb\}} | \mathcal{T} | T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rangle, \quad (11)$$

$$\mathcal{T} = Im \, i \int d^4 x T \{ \mathcal{H}_{eff}(x) \mathcal{H}_{eff}(0) \}. \quad (12)$$

in quark potential model. We also list the theoretical results from Chen [39], in which they give the upper limits of the mass. The last column is the masses estimated from the $B_c B(B_c B_s)$ threshold

Further more, the transition operator can be expanded in the heavy quark expanding(HQE), which given as

$$\mathcal{T} = \sum_{Q=b,c} \frac{G_F^2 m_Q^5}{192\pi^3} |V_{CKM}|^2 \left[c_{3,Q} (\bar{Q} Q) + \frac{c_{5,Q}}{m_Q^2} (\bar{Q} g_s \sigma_{\mu\nu} G^{\mu\nu} Q) + 2 \frac{c_{6,Q}}{m_Q^3} (\bar{Q} q) \Gamma(\bar{q} Q) \Gamma \right], \quad (13)$$

here, G_F is the Fermi constant and V_{CKM} is the CKM element. The coefficients $c_{i,Q}$ which coming from the heavy quark decays are the perturbative coefficients. Therefore, the total decay width of the tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ can be given in the leading dimension contribution as

$$\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}}(0^+) \rightarrow X) = \sum_{Q=b,c} \frac{G_F^2 m_Q^5}{192\pi^3} |V_{CKM}|^2 \sum_\lambda \frac{\langle T_{\{\bar{c}\bar{q}\}}^{\{bb\}} | \bar{Q} Q | T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rangle}{2m_T}, \quad (14)$$

where the heavy quark matrix elements are corresponding with bottom and charm number in the tetraquark state.

$$\sum_\lambda \frac{\langle T_{\{\bar{c}\bar{q}\}}^{\{bb\}} | \bar{b} b | T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rangle}{2m_T} = 2 + \mathcal{O}(1/m_b),$$

$$\sum_\lambda \frac{\langle T_{\{\bar{c}\bar{q}\}}^{\{bb\}} | \bar{c} c | T_{\{\bar{c}\bar{q}\}}^{\{bb\}} \rangle}{2m_T} = 1 + \mathcal{O}(1/m_c). \quad (15)$$

The perturbative short-distance coefficients $c_{3,Q}$ s have been determined as $c_{3,b} = 5.29 \pm 0.35$, $c_{3,c} = 6.29 \pm 0.72$ at the leading order(LO) and $c_{3,b} = 6.88 \pm 0.74$, $c_{3,c} = 11.61 \pm 1.55$ at the next-to-leading order(NLO) [43]. Accordingly, we expect the lifetimes of the tetraquark given as

$$\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}}(0^+)) = \begin{cases} (1.85 \pm 0.16) \times 10^{-12} \text{ GeV}, & \text{LO} \\ (2.83 \pm 0.34) \times 10^{-12} \text{ GeV}, & \text{NLO} \end{cases},$$

$$\Gamma(T_{\{\bar{c}\bar{q}\}}^{\{bb\}}(1^+)) = \begin{cases} (6.18 \pm 0.53) \times 10^{-13} \text{ GeV}, & \text{LO} \\ (9.43 \pm 1.13) \times 10^{-13} \text{ GeV}, & \text{NLO} \end{cases}, \quad (16)$$

Fig. 1 The weight diagrams for the tetraquark $T_{\{c\bar{q}\}}^{\{bb\}}$, bottom meson and charmed meson anti-triplet, anti-charmed meson triplet and light meson octet

$$\begin{aligned}\tau(T_{\{c\bar{q}\}}^{\{bb\}}(0^+)) &= \begin{cases} (3.21 \pm 0.27) \times 10^{-12} \text{ s, LO} \\ (2.10 \pm 0.27) \times 10^{-12} \text{ s, NLO} \end{cases}, \\ \tau(T_{\{c\bar{q}\}}^{\{bb\}}(1^+)) &= \begin{cases} (1.07 \pm 0.09) \times 10^{-12} \text{ s, LO} \\ (0.70 \pm 0.09) \times 10^{-12} \text{ s, NLO} \end{cases}, \end{aligned} \quad (17)$$

where the heavy quark masses $m_c = 1.4 \text{ GeV}$ and $m_b = 4.8 \text{ GeV}$.

4 Particle multiplets

The tetraquark with the quark constituents $bb\bar{c}\bar{q}$ is a flavor SU(3) anti-triplet. In the following, we will concern on the mesonic decays of $T_{\{c\bar{q}\}}^{\{bb\}}$. Therefore, the particle multiplets are given firstly. Following the SU(3) analysis, we give the SU(3) representation of the tetraquark $T_{\{c\bar{q}\}}^{\{bb\}}$,

$$T_{\{c\bar{q}\}}^{\{bb\}} = (T_{\{c\bar{u}\}}^{\{bb\}}, T_{\{c\bar{d}\}}^{\{bb\}}, T_{\{c\bar{s}\}}^{\{bb\}}). \quad (18)$$

The light pseudoscalar mesons can form an octet, which represented as

$$M_8 = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta}{\sqrt{6}} \end{pmatrix}. \quad (19)$$

Besides, we give the SU(3) representations of bottom and charmed mesons,

$$\begin{aligned}B_i &= (B^-, \bar{B}^0, \bar{B}_s^0), \quad D_i = (D^0, D^+, D_s^+), \\ \bar{D}^i &= (\bar{D}^0, D^-, D_s^-). \end{aligned} \quad (20)$$

For completeness, we show the weight diagrams of the states above given in Fig. 1.

5 Weak decays

In this section, we will study the possible weak decay of the tetraquark $T_{\{c\bar{q}\}}^{\{bb\}}$. Generally, we can classify the decays modes by the quantities of CKM matrix elements.

- For the b/c quark semi-leptonic decays, we adopt the following groups.

$$b \rightarrow c/u\ell^-\bar{\nu}_\ell, \quad \bar{c} \rightarrow \bar{d}/\bar{s}\ell^-\bar{\nu}_\ell. \quad (21)$$

The general electro-weak Hamiltonian can be expressed as

$$\begin{aligned}\mathcal{H}_{eff} &= \frac{G_F}{\sqrt{2}} [V_{q'b}\bar{q}'\gamma^\mu(1-\gamma_5)b\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell \\ &\quad + V_{cq}\bar{c}\gamma^\mu(1-\gamma_5)q\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell] + h.c., \end{aligned} \quad (22)$$

where $q' = (u, c)$, $q = (d, s)$. The operators of $b \rightarrow u/c\ell^-\bar{\nu}_\ell$ transition form an SU(3) flavor triplet H'_3 or singlet. In addition, the transition $\bar{c} \rightarrow \bar{q}\ell^-\bar{\nu}$ can form a triplet H_3 .

- For the $\bar{c}$ quark non-leptonic decays, the groups are given as

$$\bar{c} \rightarrow \bar{s}d\bar{u}, \quad \bar{c} \rightarrow \bar{u}d\bar{d}/\bar{s}\bar{s}, \quad \bar{c} \rightarrow \bar{d}s\bar{u}, \quad (23)$$

which are Cabibbo allowed, singly Cabibbo suppressed and doubly Cabibbo suppressed respectively. The operator $\bar{c}q_1\bar{q}_2q_3$ of the transition transforms as $\bar{\mathbf{3}} \otimes \mathbf{3} \otimes \bar{\mathbf{3}} = \bar{\mathbf{3}} \oplus \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \bar{\mathbf{15}}$, marked as H_3 , H_6 and $H_{\bar{15}}$.

- For the b quark non-leptonic decays, the groups are classified as

$$b \rightarrow c\bar{c}d/s, \quad b \rightarrow c\bar{u}d/s, \quad b \rightarrow u\bar{c}d/s, \quad b \rightarrow q_1\bar{q}_2q_3, \quad (24)$$

$q_{1,2,3}$ represent the light quark(d/s). The operator of the transition $b \rightarrow c\bar{c}d/s$ form an triplet H_3 , and the operator of the transition $b \rightarrow c\bar{u}d/s$ can form an octet H_8 . In addition, the operator of transition $b \rightarrow u\bar{c}s/d$ can form an anti-symmetric $\bar{\mathbf{3}}$ plus a symmetric $\mathbf{6}$ tensors. While the charmless tree level operator $(\bar{q}_1b)(\bar{q}_2q_3)$ ($q_i = d, s$) can be decomposed as $\mathbf{3} \otimes \bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{3} \oplus \mathbf{3} \oplus \bar{\mathbf{6}} \oplus \mathbf{15}$, they are signed as H_3 , $H_{\bar{6}}$ and H_{15} .

Fig. 2 The Feynman diagrams for the tetraquark $T_{c\bar{c}q}^{bb}$ semi-leptonic decays into mesons. **a** Represents the one meson in final states. **b–e** Show the processes of two mesons in final states

The parameters of the tensors above can be found in Ref. [59, 64–66], such as $(H_6)_{31}^2 = -(H_6)_{13}^2 = 1$ for the transition of $\bar{c} \rightarrow \bar{s}d\bar{u}$. In the following, we will study the possible weak decay modes of $T_{c\bar{c}q}^{bb}$ in order.

5.1 Semi-leptonic decays

Following the SU(3) analysis, we respectively construct the Hamiltonian of hadronic level for the semi-leptonic decays $b \rightarrow c/u\ell^- \bar{\nu}_\ell$,

$$\begin{aligned} \mathcal{H}_{eff} = & a_1(T_{c\bar{c}q}^{bb})_i(H_3')^i \bar{B}_c \bar{\ell} \nu + a_2(T_{c\bar{c}q}^{bb})_i(H_3')^i D_j \bar{B}^j \bar{\ell} \nu \\ & + a_3(T_{c\bar{c}q}^{bb})_i(H_3')^j D_j \bar{B}^i \bar{\ell} \nu \\ & + a_4(T_{c\bar{c}q}^{bb})_i(H_3')^j M_j^i \bar{B}_c \bar{\ell} \nu, \end{aligned} \quad (25)$$

$$\begin{aligned} \mathcal{H}_{eff} = & a_5(T_{c\bar{c}q}^{bb})_i \bar{B}^i \bar{\ell} \nu + a_6(T_{c\bar{c}q}^{bb})_i \bar{B}^j M_j^i \bar{\ell} \nu \\ & + a_7(T_{c\bar{c}q}^{bb})_i \bar{B}^i J/\psi \bar{\ell} \nu + a_8(T_{c\bar{c}q}^{bb})_i \bar{D}^i \bar{B}_c \bar{\ell} \nu. \end{aligned} \quad (26)$$

The corresponding Feynman diagrams are shown in Fig. 2a–d. Further more, the diagram Fig. 2a is relative to the a_1 and a_5 terms. The diagrams Fig. 2b–d of four-body semi-leptonic decays are respectively corresponding with the remanding terms in the Hamiltonian Eqs. (25) and (26). Expanding the Hamiltonian, we can obtain the decay amplitudes of different decay channels, which are collected in Table 2. The relations of decay widths can be further deduced, when the phase space effect is neglected.

$$\begin{aligned} \Gamma(T_{c\bar{c}u}^{bb} \rightarrow \bar{D}^0 B^- l^- \bar{\nu}) &= \Gamma(T_{c\bar{c}u}^{bb} \rightarrow D^- \bar{B}^0 l^- \bar{\nu}) \\ &= \Gamma(T_{c\bar{c}u}^{bb} \rightarrow D_s^- \bar{B}_s^0 l^- \bar{\nu}), \\ \Gamma(T_{c\bar{c}u}^{bb} \rightarrow \pi^0 B_c^- l^- \bar{\nu}) &= 3\Gamma(T_{c\bar{c}u}^{bb} \rightarrow \eta B_c^- l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}d}^{bb} \rightarrow \pi^+ B_c^- l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}s}^{bb} \rightarrow K^+ B_c^- l^- \bar{\nu}). \\ \Gamma(T_{c\bar{c}u}^{bb} \rightarrow B^- l^- \bar{\nu}) &= \Gamma(T_{c\bar{c}d}^{bb} \rightarrow \bar{B}^0 l^- \bar{\nu}) \end{aligned}$$

$$\begin{aligned} &= \Gamma(T_{c\bar{c}s}^{bb} \rightarrow \bar{B}_s^0 l^- \bar{\nu}), \\ \Gamma(T_{c\bar{c}u}^{bb} \rightarrow B^- J/\psi l^- \bar{\nu}) &= \Gamma(T_{c\bar{c}d}^{bb} \rightarrow \bar{B}^0 J/\psi l^- \bar{\nu}) \\ &= \Gamma(T_{c\bar{c}s}^{bb} \rightarrow \bar{B}_s^0 J/\psi l^- \bar{\nu}), \\ \Gamma(T_{c\bar{c}u}^{bb} \rightarrow D^0 B_c^- l^- \bar{\nu}) &= \Gamma(T_{c\bar{c}d}^{bb} \rightarrow D^+ B_c^- l^- \bar{\nu}) \\ &= \Gamma(T_{c\bar{c}s}^{bb} \rightarrow D_s^+ B_c^- l^- \bar{\nu}), \\ \Gamma(T_{c\bar{c}u}^{bb} \rightarrow B^- \pi^0 l^- \bar{\nu}) &= 3\Gamma(T_{c\bar{c}u}^{bb} \rightarrow B^- \eta l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}u}^{bb} \rightarrow \bar{B}^0 \pi^- l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}u}^{bb} \rightarrow \bar{B}_s^0 K^- l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}d}^{bb} \rightarrow B^- \pi^+ l^- \bar{\nu}) \\ &= \Gamma(T_{c\bar{c}d}^{bb} \rightarrow \bar{B}^0 \pi^0 l^- \bar{\nu}) \\ &= 3\Gamma(T_{c\bar{c}d}^{bb} \rightarrow \bar{B}^0 \eta l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}d}^{bb} \rightarrow \bar{B}_s^0 K^0 l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}s}^{bb} \rightarrow B^- K^+ l^- \bar{\nu}) \\ &= \frac{1}{2}\Gamma(T_{c\bar{c}s}^{bb} \rightarrow \bar{B}^0 K^0 l^- \bar{\nu}) \\ &= \frac{3}{4}\Gamma(T_{c\bar{c}s}^{bb} \rightarrow \bar{B}_s^0 \eta l^- \bar{\nu}). \end{aligned}$$

For the semi-leptonic transition of $\bar{c} \rightarrow \bar{d}/\bar{s}\ell^- \bar{\nu}_\ell$, the Hamiltonian of mesonic decays is easy to construct, given as follows,

$$\mathcal{H}_{eff} = c_1(T_{c\bar{c}q}^{bb})_i(H_3)_j \bar{B}^i \bar{B}^j \bar{\ell} \nu. \quad (27)$$

The corresponding Feynman diagram is drawn in Fig. 2e. More technically, we can expand the Hamiltonian above and collect the decay amplitudes,

$$\begin{aligned} \mathcal{A}(T_{c\bar{c}u}^{bb} \rightarrow \bar{B}^0 B^- l^- \bar{\nu}) &= c_1 V_{cd}, \\ \mathcal{A}(T_{c\bar{c}u}^{bb} \rightarrow \bar{B}_s^0 B^- l^- \bar{\nu}) &= c_1 V_{cs}, \end{aligned}$$

Table 2 Amplitudes for tetraquark $T_{\{\bar{c}q\}}^{\{bb\}}$ semi-leptonic decays into mesons with the transition of $b \rightarrow c/u\ell^- \bar{\nu}_\ell$

Channel	Amplitude	Channel	Amplitude
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B_c^- l^- \bar{\nu}$	$a_1 V_{ub}$	$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow \eta B_c^- l^- \bar{\nu}$	$\frac{a_4 V_{ub}}{\sqrt{6}}$
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow \pi^0 B_c^- l^- \bar{\nu}$	$\frac{a_4 V_{ub}}{\sqrt{2}}$	$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow K^+ B_c^- l^- \bar{\nu}$	$a_4 V_{ub}$
$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \pi^+ B_c^- l^- \bar{\nu}$	$a_4 V_{ub}$	$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow D^- \bar{B}^0 l^- \bar{\nu}$	$(a_2 + a_3) V_{ub}$
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow \bar{D}^0 B^- l^- \bar{\nu}$	$(a_2 + a_3) V_{ub}$	$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 l^- \bar{\nu}$	a_5
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow D_s^- \bar{B}_s^0 l^- \bar{\nu}$	$(a_2 + a_3) V_{ub}$	$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 l^- \bar{\nu}$	a_5
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- l^- \bar{\nu}$	a_5	$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- \pi^0 l^- \bar{\nu}$	$\frac{a_6}{\sqrt{2}}$
$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 l^- \bar{\nu}$	a_5	$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow \bar{B}_s^0 K^- l^- \bar{\nu}$	a_6
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- \pi^0 l^- \bar{\nu}$	$\frac{a_6}{\sqrt{2}}$	$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \pi^0 l^- \bar{\nu}$	$-\frac{a_6}{\sqrt{2}}$
$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \pi^- l^- \bar{\nu}$	a_6	$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}_s^0 K^0 l^- \bar{\nu}$	a_6
$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \eta l^- \bar{\nu}$	$\frac{a_6}{\sqrt{6}}$	$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}^0 K^0 l^- \bar{\nu}$	a_6
$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow B^- K^+ l^- \bar{\nu}$	a_6	$T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 J/\psi l^- \bar{\nu}$	a_7
$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 \eta l^- \bar{\nu}$	$-\sqrt{\frac{2}{3}} a_6$	$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 J/\psi l^- \bar{\nu}$	a_7
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- J/\psi l^- \bar{\nu}$	a_7	$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow D^0 B_c^- l^- \bar{\nu}$	a_8
$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 J/\psi l^- \bar{\nu}$	a_7	$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow D_s^+ B_c^- l^- \bar{\nu}$	a_8
$T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow D^0 B_c^- l^- \bar{\nu}$	a_8		
$T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow D_s^+ B_c^- l^- \bar{\nu}$	a_8		

$$\mathcal{A}(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \bar{B}^0 l^- \bar{\nu}) = 2c_1 V_{cd},$$

$$\mathcal{A}(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \bar{B}_s^0 l^- \bar{\nu}) = c_1 V_{cs},$$

$$\mathcal{A}(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 \bar{B}^0 l^- \bar{\nu}) = c_1 V_{cd},$$

$$\mathcal{A}(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 \bar{B}_s^0 l^- \bar{\nu}) = 2c_1 V_{cs}.$$

The relations of decay widths given as follows.

$$\begin{aligned} \Gamma(T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- \bar{B}^0 l^- \bar{\nu}) &= \frac{1}{2} \Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \bar{B}^0 l^- \bar{\nu}) \\ &= \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}^0 \bar{B}_s^0 l^- \bar{\nu}), \\ \Gamma(T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow \bar{B}_s^0 B^- l^- \bar{\nu}) &= \Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 \bar{B}_s^0 l^- \bar{\nu}) \\ &= \frac{1}{2} \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 \bar{B}_s^0 l^- \bar{\nu}). \end{aligned}$$

5.2 Non-leptonic two-body mesonic decays

In the SU(3) flavor symmetry, the two-body mesonic decays Hamiltonian of the transition $b \rightarrow c\bar{c}d/s$ is constructed as

$$\begin{aligned} \mathcal{H}_{eff} &= a_1 (T_{\{\bar{c}q\}}^{\{bb\}})_i (H_3)^i \bar{B}^j D_j + a_2 (T_{\{\bar{c}q\}}^{\{bb\}})_i (H_3)^j \bar{B}^i D_j \\ &+ a_3 (T_{\{\bar{c}q\}}^{\{bb\}})_i (H_3)^j M_j^i \bar{B}_c \\ &+ a_4 (T_{\{\bar{c}q\}}^{\{bb\}})_i (H_3)^i \bar{B}_c J/\psi. \end{aligned} \quad (28)$$

We draw the corresponding Feynman diagrams shown in Fig. 3a–e. Particularly, the terms a_1 and a_2 with the final states B plus D are related to the diagrams Fig. 3c, e. The terms a_4 is corresponding with the diagrams Fig. 3a, b. In addition, the a_3 term with the light meson and B_c meson in the final states is relevant with the Fig. 3d. Based on the Hamiltonian above, we get the decay amplitudes collected in Table 3. Consistently, the relations between different decay widths given as follows.

$$\begin{aligned} \Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow B^- \bar{D}^0) &= \Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}_s^0 D_s^-), \\ \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow B^- \bar{D}^0) &= \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}^0 D^-), \\ \Gamma(T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- D^-) &= \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow \bar{B}_s^0 D^-), \\ \Gamma(T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B^- D_s^-) &= \Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow \bar{B}^0 D_s^-), \\ \Gamma(T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B_c^- \pi^-) &= 6\Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow B_c^- \eta) \\ &= \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow B_c^- K^0) = 2\Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow B_c^- \pi^0), \\ \Gamma(T_{\{\bar{c}u\}}^{\{bb\}} \rightarrow B_c^- K^-) &= \frac{3}{2} \Gamma(T_{\{\bar{c}s\}}^{\{bb\}} \rightarrow B_c^- \eta) \\ &= \Gamma(T_{\{\bar{c}d\}}^{\{bb\}} \rightarrow B_c^- \bar{K}^0). \end{aligned}$$

The hadronic level Hamiltonian of the transition of $b \rightarrow c\bar{u}d/s$ can be constructed as

Fig. 3 The Feynman diagrams for the tetraquark $T_{\{c\bar{q}\}}^{(bb)}$ non-leptonic decays into mesons

Table 3 Tetraquark $T_{\{c\bar{q}\}}^{(bb)}$ decays into two mesons with the transition of $b \rightarrow c\bar{c}d/s$

Channel	Amplitude	Channel	Amplitude
$T_{\{c\bar{u}\}}^{(bb)} \rightarrow B^- D^-$	$a_2 V_{cd}^*$	$T_{\{c\bar{u}\}}^{(bb)} \rightarrow B^- D_s^-$	$a_2 V_{cs}^*$
$T_{\{c\bar{d}\}}^{(bb)} \rightarrow B^- \bar{D}^0$	$a_1 V_{cd}^*$	$T_{\{c\bar{d}\}}^{(bb)} \rightarrow \bar{B}^0 D^-$	$(a_1 + a_2) V_{cd}^*$
$T_{\{c\bar{d}\}}^{(bb)} \rightarrow \bar{B}^0 D_s^-$	$a_2 V_{cs}^*$	$T_{\{c\bar{d}\}}^{(bb)} \rightarrow \bar{B}_s^0 D^-$	$a_1 V_{cd}^*$
$T_{\{c\bar{s}\}}^{(bb)} \rightarrow B^- \bar{D}^0$	$a_1 V_{cs}^*$	$T_{\{c\bar{s}\}}^{(bb)} \rightarrow \bar{B}^0 D^-$	$a_1 V_{cs}^*$
$T_{\{c\bar{s}\}}^{(bb)} \rightarrow \bar{B}_s^0 D^-$	$a_2 V_{cd}^*$	$T_{\{c\bar{s}\}}^{(bb)} \rightarrow \bar{B}_s^0 D_s^-$	$(a_1 + a_2) V_{cs}^*$
$T_{\{c\bar{u}\}}^{(bb)} \rightarrow B_c^- \pi^-$	$a_3 V_{cd}^*$	$T_{\{c\bar{u}\}}^{(bb)} \rightarrow B_c^- K^-$	$a_3 V_{cs}^*$
$T_{\{c\bar{d}\}}^{(bb)} \rightarrow B_c^- \pi^0$	$-\frac{a_3 V_{cd}^*}{\sqrt{2}}$	$T_{\{c\bar{d}\}}^{(bb)} \rightarrow B_c^- \bar{K}^0$	$a_3 V_{cs}^*$
$T_{\{c\bar{d}\}}^{(bb)} \rightarrow B_c^- \eta$	$\frac{a_3 V_{cd}^*}{\sqrt{6}}$	$T_{\{c\bar{s}\}}^{(bb)} \rightarrow B_c^- K^0$	$a_3 V_{cd}^*$
$T_{\{c\bar{s}\}}^{(bb)} \rightarrow B_c^- \eta$	$-\sqrt{\frac{2}{3}} a_3 V_{cs}^*$		
$T_{\{c\bar{d}\}}^{(bb)} \rightarrow B_c^- J/\psi$	$a_4 V_{cd}^*$	$T_{\{c\bar{s}\}}^{(bb)} \rightarrow B_c^- J/\psi$	$a_4 V_{cs}^*$

$$\begin{aligned} \mathcal{H}_{eff} = & b'_1 (T_{\{c\bar{q}\}}^{(bb)})_i (H_8)_j^i \bar{B}^j J/\psi + b'_2 (T_{\{c\bar{q}\}}^{(bb)})_i (H_8)_j^i \bar{D}^j \bar{B}_c \\ & + b'_3 (T_{\{c\bar{q}\}}^{(bb)})_i (H_8)_j^i M_k^j \bar{B}^k + b'_4 (T_{\{c\bar{q}\}}^{(bb)})_i (H_8)_j^i M_k^j \bar{B}^j \\ & + b'_5 (T_{\{c\bar{q}\}}^{(bb)})_i (H_8)_j^i M_k^j \bar{B}^i. \end{aligned} \quad (29)$$

The corresponding Feynman diagrams are shown in Fig. 3a–e. Further more, we expand the Hamiltonian and obtain the decay amplitudes collected in Table 4. Besides, the relation between different decay channels can be deduced directly, given as follows.

$$\Gamma(T_{\{c\bar{s}\}}^{(bb)} \rightarrow \pi^0 B^-) = \frac{1}{2} \Gamma(T_{\{c\bar{s}\}}^{(bb)} \rightarrow \pi^- \bar{B}^0).$$

We can easily construct the hadronic level Hamiltonian for the transition of $b \rightarrow u\bar{c}d/s$ in the SU(3) symmetry,

$$\mathcal{H}_{eff} = c'_1 (T_{\{c\bar{q}\}}^{(bb)})_i (H_3)^{[ij]} D_j \bar{B}_c. \quad (30)$$

The corresponding Feynman diagrams are shown in Fig. 3a–d. In addition, the amplitudes obtained from the Hamiltonian are collected in Table 4, and the relations between different decay channels can be deduced as

$$\Gamma(T_{\{c\bar{d}\}}^{(bb)} \rightarrow \bar{D}^0 B_c^-) = \Gamma(T_{\{c\bar{u}\}}^{(bb)} \rightarrow D^- B_c^-),$$

$$\Gamma(T_{\{c\bar{s}\}}^{(bb)} \rightarrow \bar{D}^0 B_c^-) = \Gamma(T_{\{c\bar{u}\}}^{(bb)} \rightarrow D_s^- B_c^-).$$

Following the SU(3) analysis, the hadronic level Hamiltonian for the charmless transition of $b \rightarrow q_1 \bar{q}_2 q_3$ can be constructed as

$$\begin{aligned} \mathcal{H}_{eff} = & d'_1 (T_{\{c\bar{q}\}}^{(bb)})_i (H_3)^j M_j^i \bar{B}_c + d'_2 (T_{\{c\bar{q}\}}^{(bb)})_i (H_6)^{[ik]} M_k^j \bar{B}_c \\ & + d'_3 (T_{\{c\bar{q}\}}^{(bb)})_i (H_{15})_j^{[ik]} M_k^j \bar{B}_c \\ & + d'_4 (T_{\{c\bar{q}\}}^{(bb)})_i (H_3)^i D_j \bar{B}^j + d'_5 (T_{\{c\bar{q}\}}^{(bb)})_i (H_3)^j D_j \bar{B}^i \\ & + d'_6 (T_{\{c\bar{q}\}}^{(bb)})_i (H_6)^{[ik]} D_k \bar{B}^j \\ & + d'_7 (T_{\{c\bar{q}\}}^{(bb)})_i (H_{15})_j^{[ik]} D_k \bar{B}^j. \end{aligned} \quad (31)$$

We expand the Hamiltonian and collect the decay amplitudes given in Table 5. In particular, there is no relation between different decay channels.

For the charm quark decay $\bar{c} \rightarrow \bar{q}_1 q_2 \bar{q}_3$, the Hamiltonian of hadronic level can be easily constructed as

$$\mathcal{H}_{eff} = f_1 (T_{\{c\bar{q}\}}^{(bb)})_i (H_{15})_j^{[ik]} \bar{B}^j \bar{B}^k. \quad (32)$$

Table 4 Tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ decays into two mesons with the transition of $b \rightarrow c\bar{u}d/s$ or $b \rightarrow u\bar{c}d/s$

Channel	Amplitude	Channel	Amplitude
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- J/\psi$	$b_1 V_{ud}^*$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- J/\psi$	$b_1 V_{us}^*$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow D^0 B_c^-$	$b_2 V_{ud}^*$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow D^0 B_c^-$	$b_2 V_{us}^*$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow \pi^- B^-$	$(b_4 + b_5) V_{ud}^*$	$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow K^- B^-$	$(b_4 + b_5) V_{us}^*$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \pi^0 B^-$	$\frac{(b_3 - b_4) V_{ud}^*}{\sqrt{2}}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \pi^- \bar{B}^0$	$(b_3 + b_5) V_{ud}^*$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{K}^0 B^-$	$b_4 V_{us}^*$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow K^- \bar{B}^0$	$b_5 V_{us}^*$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow K^- \bar{B}_s^0$	$b_3 V_{ud}^*$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \eta B^-$	$\frac{(b_3 + b_4) V_{ud}^*}{\sqrt{6}}$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \pi^0 B^-$	$\frac{b_3 V_{us}^*}{\sqrt{2}}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \pi^- \bar{B}^0$	$b_3 V_{us}^*$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \pi^- \bar{B}_s^0$	$b_5 V_{ud}^*$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow K^0 B^-$	$b_4 V_{ud}^*$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow K^- \bar{B}_s^0$	$(b_3 + b_5) V_{us}^*$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \eta B^-$	$\frac{(b_3 - 2b_4) V_{us}^*}{\sqrt{6}}$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow D^- B_c^-$	$c_1 V_{cd}^*$	$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow D_s^- B_c^-$	$c_1 V_{cs}^*$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{D}^0 B_c^-$	$-c_1 V_{cd}^*$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{D}^0 B_c^-$	$-c_1 V_{cs}^*$

Table 5 Tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ decays into B_c plus light meson or anti-charm plus B meson induced by the charmless transition $b \rightarrow d$ or $b \rightarrow s$

Channel	Amplitude	Channel	Amplitude
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B_c^- \pi^-$	$d_1 + d_2 + 3d_3$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B_c^- \pi^0$	$-\frac{d_1 + d_2 - 5d_3}{\sqrt{2}}$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B_c^- \eta$	$\frac{d_1 - 3d_2 + 3d_3}{\sqrt{6}}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B_c^- K^0$	$d_1 - d_2 - d_3$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B^- D^-$	$d_5 + d_6 + 3d_7$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- \bar{D}^0$	$d_4 - d_6 + 3d_7$
$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}^0 D^-$	$d_4 + d_5 - 2d_7$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 D_s^-$	$d_4 + d_6 - d_7$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 D^-$	$d_5 - d_6 - d_7$		
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B_c^- K^-$	$d_1 + d_2 + 3d_3$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B_c^- \bar{K}^0$	$d_1 - d_2 - d_3$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B_c^- \pi^0$	$-\sqrt{2} (d_2 - 2d_3)$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B_c^- \eta$	$-\sqrt{\frac{2}{3}} (d_1 - 3d_3)$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B^- D_s^-$	$d_5 + d_6 + 3d_7$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}^0 D_s^-$	$d_5 - d_6 - d_7$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- \bar{D}^0$	$d_4 - d_6 + 3d_7$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}^0 D^-$	$d_4 + d_6 - d_7$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 D_s^-$	$d_4 + d_5 - 2d_7$		

Expanding the Hamiltonian above, one can obtain the decay amplitudes given as follows,

$$\begin{aligned}
 \mathcal{A}(T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- \bar{B}_s^0) &= 2f_1, \mathcal{A}(T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- \bar{B}^0) \\
 &= -2f_1 \sin \theta_C, \mathcal{A}(T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- \bar{B}_s^0) \\
 &= 2f_1 \sin \theta_C, \\
 \mathcal{A}(T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- \bar{B}^0) &= 2f_1 \sin \theta_C^2.
 \end{aligned}$$

Consistently, the relation between different decay channels is deduced as

$$\Gamma(T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- \bar{B}^0) = \Gamma(T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- \bar{B}_s^0).$$

6 Golden decay channels

In order to reconstruct the tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ experimentally, we will discuss the golden channels in the section. In the discussion given in the previous sections, the final meson can be replaced by its corresponding counterpart, which has the same quark constituent but different quantum numbers. For instance, we can replace π^0 by ρ^0 , K^+ by K^{*+} .

Generally, the weak decays are closely related with the CKM element. In addition, the charged particles are easy to detect than neutral one at hadron colliders like LHC. Therefore, one choose the following criteria for the Golden decay channels.

- Branching fractions: One should choose the cabibbo allowed decay modes $c \rightarrow s\ell^+\nu_\ell$ for the charm quark

Table 6 Cabibbo allowed $b\bar{c}b\bar{q}$ decays

<i>b</i> quark decays				
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B^- l^- \bar{\nu}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}^0 l^- \bar{\nu}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 l^- \bar{\nu}$	$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B^- J/\psi l^- \bar{\nu}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}^0 J/\psi l^- \bar{\nu}$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 J/\psi l^- \bar{\nu}$	$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow \bar{B}^0 \pi^- l^- \bar{\nu}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 K^- l^- \bar{\nu}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- \pi^+ l^- \bar{\nu}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 K^0 l^- \bar{\nu}$
$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- K^+ l^- \bar{\nu}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}^0 K^0 l^- \bar{\nu}$	$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow D^0 B_c^- l^- \bar{\nu}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow D^+ B_c^- l^- \bar{\nu}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow D_s^+ B_c^- l^- \bar{\nu}$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B^- D_s^-$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}^0 D_s^-$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B^- \bar{D}^0$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}^0 D^-$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 D_s^-$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow B_c^- K^-$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B_c^- \bar{K}^0$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow B_c^- J/\psi$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- J/\psi$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow D^0 B_c^-$
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow \pi^- B^-$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \pi^- \bar{B}^0$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow K^- \bar{B}_s^0$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \pi^- \bar{B}_s^0$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow K^0 B^-$
$\bar{c}$ quark decays				
$T_{\{\bar{c}\bar{u}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 B^- l^- \bar{\nu}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 \bar{B}^0 l^- \bar{\nu}$	$T_{\{\bar{c}\bar{s}\}}^{\{bb\}} \rightarrow \bar{B}_s^0 \bar{B}_s^0 l^- \bar{\nu}$	$T_{\{\bar{c}\bar{d}\}}^{\{bb\}} \rightarrow B^- \bar{B}_s^0$	

semi-leptonic decays and $c \rightarrow s\bar{d}u$ for the non-leptonic decays. For bottom quark decays, the semi-leptonic decays $b \rightarrow c\ell^- \bar{\nu}_\ell$ and non-leptonic decays $b \rightarrow c\bar{u}d$ or $b \rightarrow c\bar{c}s$ give the largest branching fractions.

- Detection efficiency: Since it is easy to detect the charged particles, one remove the channels with π^0 , η , ϕ , $\rho^\pm (\rightarrow \pi^\pm \pi^0)$, $K^{*\pm} (\rightarrow K^\pm \pi^0)$ and ω , but keep the modes with $\pi^\pm$, $K^0 (\rightarrow \pi^+ \pi^-)$, $\rho^0 (\rightarrow \pi^+ \pi^-)$.

We collect the golden channels with the two-body non-leptonic decays and three- or four-body semi-leptonic decays for the triply heavy tetraquark $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$, given in Table 6.

Some comments are appropriate in the following. For the charm quark decays, the typical branching fraction is at a few percent level. In addition, the reconstruction of final states, such as bottom mesons, is at the order 10^{-3} or even smaller on the experiment side. Therefore the branching fraction for the charm quark decays chains to reconstruct the $T_{\{\bar{c}\bar{q}\}}^{\{bb\}}$ might be at the order 10^{-5} , or smaller. For the bottom quark decays, the typical branching fraction is at the order 10^{-3} . The reconstruction of final states J/ψ or D meson would introduce another factor 10^{-3} . Therefore the branching fraction of these channels could be at the order 10^{-5} .

7 Conclusions

In the paper, we have studied the masses, lifetimes and weak decays of the heavy tetraquarks $b\bar{c}b\bar{q}$. The mass spectrums of tetraquark $b\bar{c}b\bar{q}$ were calculated in non-relativistic model, we found the triply heavy tetraquark with $J^P = 1^+$ could be the stable state which should be verified by further experiments. With the help of heavy quark expanding (HQE), we calculated the lifetimes of $b\bar{c}b\bar{q}$ at the leading order and next-to-leading order. Particularly, the lifetime at NLO is $\tau(T_{\{\bar{c}\bar{q}\}}^{\{bb\}}) = (0.70 \pm 0.09) \times 10^{-12}$ s. In addition, we discussed the weak decays of $b\bar{c}b\bar{q}$ under the SU(3) analysis

which had been successfully applied into heavy meson and baryon processes. Following the building blocks in SU(3) analysis, we constructed the hadronic level Hamiltonian. More technically, the non-perturbative effects are collected into parameters, such as a_i , b_j , ... According to the Hamiltonian, one can derive the relations between different decay channels. Finally, we analyzed the semi-leptonic and non-leptonic mesonic decays of tetraquarks $b\bar{c}b\bar{q}$, and made a collection of the golden channels for the searching of triply heavy tetraquarks $b\bar{c}b\bar{q}$ in further experiments.

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