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$T\bar{T}$ deformation of chiral bosons and Chern–Simons AdS_3 gravity

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Abstract We study the $T\bar{T}$ deformation of the chiral bosons and show the equivalence between the chiral bosons of opposite chiralities and the scalar fields at the Hamiltonian level under the deformation. We also derive the deformed Lagrangian of more generic theories which contain an arbitrary number of chiral bosons to all orders. By using these results, we derive the $T\bar{T}$ deformed boundary action of the AdS_3 gravity theory in the Chern–Simons formulation. We compute the deformed one-loop torus partition function, which satisfies the $T\bar{T}$ flow equation up to the one-loop order. Finally, we calculate the deformed stress–energy tensor of a solution describing a BTZ black hole in the boundary theory, which coincides with the boundary stress–energy tensor derived from the BTZ black hole with a finite cutoff.

1 Introduction

The deformation by the $T\bar{T}$ operator [1] has drawn much attention, because of its solvability and the relation with gravity theory. Although the $T\bar{T}$ deformation is an irrelevant deformation, it is possible to derive the deformed Lagrangian, finite size spectrum and the S-matrix from the ones of the original theory [2–4], which does not require the integrability in many cases. Based on the finite size spectrum, one could compute the torus partition function of the $T\bar{T}$ deformation [5–7], which is still modular invariant but not conformal invariant. The $T\bar{T}$ deformation is related to the gravity theory in several aspects. On the one hand, the deformed theory can be interpreted as the original theory coupled to a topological gravity [5,8,9]. More concretely, one finds a one-to-one map between the equations of motion (EOM) in the deformed theories and those of the original theories [10,11], which enables one to derive the all-order deformed Lagrangians [12]. On the other hand, the two-dimensional $T\bar{T}$ deformed holographic CFT is proposed to correspond to the gravity theory with a finite cutoff under the Dirichlet boundary condition, where the cutoff is explicitly related to the deformation parameter [13]. More discussions on the holography under the $T\bar{T}$ deformation can be found in [14–20]. See also [21] for an interesting review and related topics.

The $T\bar{T}$ deformation of the two-dimensional scalar theory has been well studied. In particular, the all-order deformed action of the N massless free bosons has the form of the Nambu–Goto action in the static gauge of $N+2$-dimension [4]. The deformation of a scalar with an arbitrary potential was shown in [4,22] and more examples of Lagrangians of $T\bar{T}$ deformed theories was presented in [23]. In this paper, we study the chiral bosons, which are interesting in many aspects such as string theory and condensed matter. Even though the chiral bosons are not manifestly Lorentz invariant, the sum of a left and a right chiral bosons reproduces the scalar theory [24,25]. The $T\bar{T}$ deformed action of a general system of chiral bosons, scalars and fermions was studied in [26], where the first-order action for chiral bosons and canonical stress–energy tensor were used. We are interested here in the Floreanini–Jackiw action [27] and the covariant stress–energy tensor.

A remarkable connection between chiral Wess–Zumino–Witten (WZW) models and Chern–Simons theories was established in [28–30]. In particular, the AdS_3 Einstein gravity theory can be reformulated as a $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ Chern–Simons theory [31]. Much attention has been paid to the exact boundary action [32–35], due to its connection with two-dimensional conformal field theory [36]. The AdS_3 Chern–Simons action can be reduced to two chiral $SL(2,\mathbb{R})$ Wess–Zumino–Witten (WZW) models on the boundary, and the AdS_3 boundary condition implements certain constraints on the chiral WZW model [32]. In [35], the exact boundary action was shown to be a quantum field theory of reparametrizations, which analogous to the Schwarzian action of the nearly AdS_2 gravity. Moreover, the torus parti-

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tion is one-loop exact and shows a shift in the central charge of 13. The present work aims to study the $T\bar{T}$ deformation of the boundary action. We derive the $T\bar{T}$ deformed Lagrangian of generic chiral boson theories by explicitly solving the flow equation. Then we focus on the $T\bar{T}$ deformed action of the constrained chiral WZW model associated with the AdS_3 Chern–Simons theory. We will see how the exact boundary action and the one-loop torus partition function change under the $T\bar{T}$ deformation. We also calculate the deformed stress–energy tensor of the boundary theory for the BTZ black hole, and compare it with the boundary stress–energy tensor derived from the BTZ black hole with a finite cutoff. Our results provide a concrete realization of the $T\bar{T}$ deformation on the boundary of the Chern–Simons AdS_3 gravity and may shed light on the holography dual of the deformation.

This paper is organized as follows. In Sect. 2, we present the $T\bar{T}$ deformed Lagrangian of chiral boson theories. We show the equivalence between the sum of two chiral bosons of opposite chiralities and a massless free non-chiral scalar under the $T\bar{T}$ deformation at the Hamiltonian level. In Sect. 3, we review the relation between the AdS_3 Chern–Simons theory and the sum of two constrained $SL(2,\mathbb{R})$ chiral WZW model of opposite chiralities, and derive the corresponding deformed Lagrangian. We then compute the one-loop torus $T\bar{T}$ deformed partition function, which is found to satisfy the flow equations of $T\bar{T}$ deformation in all-order of deformation parameter up to one-loop level. We also compute the deformed stress–energy tensor for a solution describing a BTZ black hole in the deformed field theory and compare it with the boundary stress–energy tensor of the BTZ black hole at a finite cutoff. Section 4 is devoted to conclusions and discussions. In Appendix A, we consider the $J\bar{J}$ and $T\bar{J}$ deformation of the chiral bosons. In Appendix B, we study the solutions to the EOMs of $T\bar{T}$ deformed WZW models.

2 $T\bar{T}$ deformed Lagrangian of chiral bosons

In this section, we will study the $T\bar{T}$ deformation of chiral bosons. The Floreanini–Jackiw action [27] of a left-moving chiral boson is

$$S_{\text{left}} = \int d^2x \frac{1}{2}(\partial_t\phi\partial_\theta\phi - \partial_\theta\phi\partial_\theta\phi). \quad (1)$$

As a warm-up, we will first consider the simple case of two chiral bosons of opposite chiralities and solve the flow equation induced by the $T\bar{T}$ deformation. More complicated theories of chiral bosons will also be considered, which will be useful in the study of the AdS_3 Chern–Simons theory.

2.1 Two chiral bosons of opposite chiralities

Let us begin with the undeformed Lagrangian of a left and a right chiral boson

$$S_0 = \int d^2x \mathcal{L}_0, \quad (2)$$

$$\mathcal{L}_0 = -\frac{1}{2}\left(-\partial_t\phi\partial_\theta\phi + \frac{E_t^+}{E_\theta^+}\partial_\theta\phi\partial_\theta\phi + \partial_t\bar{\phi}\partial_\theta\bar{\phi} - \frac{E_t^-}{E_\theta^-}\partial_\theta\bar{\phi}\partial_\theta\bar{\phi}\right), \quad (3)$$

where E^a is the zweibein and the metric is $g_{\mu\nu} = E_\mu^+ E_\nu^- + E_\mu^- E_\nu^+$. We couple the zweibein to the fields such that the undeformed action is invariant under the transformation

$$\begin{aligned} \delta\phi &= \epsilon_+^\theta \partial_\theta\phi, \\ \delta\bar{\phi} &= \epsilon_-^\theta \partial_\theta\bar{\phi}, \\ \delta E_\mu^+ &= \epsilon_+^\theta \partial_\theta E_\mu^+ + E_\theta^+ \partial_\mu \epsilon_+^\theta. \\ \delta E_\mu^- &= \epsilon_-^\theta \partial_\theta E_\mu^- + E_\theta^- \partial_\mu \epsilon_-^\theta, \end{aligned} \quad (4)$$

where $\epsilon_\pm^\theta$ are coordinate dependent transformation parameters. The translation symmetry generated by constant $\epsilon_\pm^\theta$ enables us to define the stress–energy tensor as

$$T^\mu_\nu = -\frac{1}{\det E}\frac{\delta S}{\delta E_\mu^A}E_\nu^A. \quad (5)$$

In this paper we focus on $T\bar{T}$ deformation in flat spacetime, so E_μ^A can be set to be constants after deriving the stress–energy tensor. When E_μ^A are constants the conserved law can be written as $\partial_\mu T^\mu_\nu = 0$.

The $T\bar{T}$ deformation of a two-dimensional field theory is induced by the $T\bar{T}$ operator which is defined as minus the determinant of the stress–energy tensor. Concretely, the $T\bar{T}$ deformed Lagrangian $\mathcal{L}_\lambda$ is the solution to the flow equation

$$\frac{\partial\mathcal{L}_\lambda}{\partial\lambda} = \det E \det T_\lambda, \quad (6)$$

with the initial condition (3). Here λ is the deformation parameter and T_λ is the deformed stress–energy tensor of the deformed theory. We can solve the equation by making a perturbative expansion in small λ and then guessing the exact solution. Skipping the boring details, the solution is given by

$$\begin{aligned} \mathcal{L}_\lambda =& \frac{1}{2}(\partial_t\phi\partial_\theta\phi - \partial_t\bar{\phi}\partial_\theta\bar{\phi}) \\ &- \frac{(E_\theta^- E_t^+ + E_t^- E_\theta^+)(\partial_\theta\phi\partial_\theta\phi - \partial_\theta\bar{\phi}\partial_\theta\bar{\phi})}{4E_\theta^+ E_\theta^-} \\ &+ \frac{\det E}{2\lambda}(\mathcal{S} - 1), \end{aligned} \quad (7)$$

with

$$\mathcal{S}=\sqrt{1-\frac{(\partial_\theta\phi\partial_\theta\phi+\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})}{E_\theta^- E_\theta^+}\lambda+\frac{(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})^2}{4E_\theta^{-2}E_\theta^{+2}}\lambda^2}. \tag{8}$$

The deformed theory still has the conservation laws $\partial_\mu(T^\mu_\nu)_\lambda = 0$, which corresponds to the symmetries

$$\delta\phi=\epsilon_1\partial_\theta\phi+\epsilon_0\partial_\theta\phi\frac{2E_\theta^+E_\theta^- -\lambda(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})}{2E_\theta^+E_\theta^-\mathcal{S}}, \tag{9}$$

$$\delta\bar{\phi}=\epsilon_1\partial_\theta\bar{\phi}-\epsilon_0\partial_\theta\bar{\phi}\frac{2E_\theta^+E_\theta^- +\lambda(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})}{2E_\theta^+E_\theta^-\mathcal{S}}, \tag{10}$$

where ϵ_i are constant parameters. We also consider the $J\bar{J}$ and $T\bar{J}$ deformation and the results are shown in Appendix A.

2.2 Equivalence to the $T\bar{T}$ deformation of a non-chiral free scalar

In the undeformed theory, the sum of a left moving chiral boson and a right moving chiral boson is equivalent to a free massless scalar [24,37]. We now show that the equivalence still holds under $T\bar{T}$ deformation.

We now restrict our attention to flat spacetime so we can set $E_t^+ = E_\theta^+ = E_\theta^- = -E_t^- = 1/\sqrt{2}$ after solving the flow equation. Therefore the Lagrangian (7) becomes

$$\mathcal{L}_\lambda=\frac{1}{2}(\partial_t\phi\partial_\theta\phi-\partial_t\bar{\phi}\partial_\theta\bar{\phi})+\frac{1}{2\lambda}(\mathcal{S}-1), \tag{11}$$

with

$$\mathcal{S}=\sqrt{1-2(\partial_\theta\phi\partial_\theta\phi+\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})\lambda+(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})^2\lambda^2}, \tag{12}$$

The stress–energy tensor of the deformed theory becomes

$$(T_\lambda)^\mu_{\ \nu}=\begin{pmatrix}\frac{1-\mathcal{S}}{2\lambda} & \frac{(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})}{2}\\ -\frac{(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})}{2} & \frac{\mathcal{S}-1}{2\lambda\mathcal{S}}+\frac{(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})^2\lambda}{2\mathcal{S}}\end{pmatrix}, \tag{13}$$

from which we obtain the corresponding energy and momentum

$$H_\lambda=\int d\theta\frac{1}{2\lambda}\Big(1-\sqrt{1-2(\partial_\theta\phi\partial_\theta\phi+\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})\lambda+(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi})^2\lambda^2}\Big), \tag{14}$$

$$P_\lambda=\int d\theta\frac{1}{2}(\partial_\theta\phi\partial_\theta\phi-\partial_\theta\bar{\phi}\partial_\theta\bar{\phi}). \tag{15}$$

The Hamiltonian density of the system can be written as

$$\mathcal{H}=\frac{1}{2\lambda}\Big(1-\sqrt{1-8(\pi^2+\bar{\pi}^2)\lambda+16(\pi^2-\bar{\pi}^2)^2\lambda^2}\Big), \tag{16}$$

where $\pi = \frac{1}{2}\partial_\theta\phi$ and $\bar{\pi} = -\frac{1}{2}\partial_\theta\bar{\phi}$ are the canonical momenta of the fields.

Let us turn to the $T\bar{T}$ deformed free massless non-chiral scalar. The Lagrangian is given by [4]

$$\mathcal{L}_\lambda^{\text{scalar}}=\frac{1}{2\lambda}\Big(\sqrt{1+2\lambda(\partial_t\varphi\partial_t\varphi-\partial_\theta\varphi\partial_\theta\varphi)}-1\Big). \tag{17}$$

The associated Hamiltonian density is

$$\mathcal{H}_\lambda^{\text{scalar}}=\frac{1}{2\lambda}\Big(1-\sqrt{(1-2\lambda\partial_\theta\varphi^2)(1-2\lambda\pi_\varphi^2)}\Big), \tag{18}$$

where the canonical moment of φ is defined as

$$\pi_\varphi=\frac{\partial_t\varphi}{\sqrt{1+2\lambda(\partial_t\varphi\partial_t\varphi-\partial_\theta\varphi\partial_\theta\varphi)}}. \tag{19}$$

One can check that the Hamiltonian densities (16) and (18) are equivalent via the relation

$$\varphi=\frac{1}{\sqrt{2}}(\phi+\bar{\phi}),\quad \pi_\varphi=\sqrt{2}(\pi+\bar{\pi}). \tag{20}$$

The $T\bar{T}$ deformed Lorentz invariant free massless scalar is related to the undeformed model via a field dependent coordinate transformation [10,11]. To obtain a solution to the deformed theory, one can start with a solution

$$\varphi(\tilde{t},\tilde{\theta})=f(\tilde{x}^+)+g(\tilde{x}^-), \tag{21}$$

where $\tilde{x}^\pm = \tilde{\theta}\pm\tilde{t}$, to the equation of motion of the undeformed model

$$(\partial_{\tilde{t}}^2-\partial_{\tilde{\theta}}^2)\varphi=0. \tag{22}$$

Then one need to solve the equations

$$\begin{aligned} t &=\tilde{t}+\frac{\lambda}{2}\big(G(\tilde{x}^-)-F(\tilde{x}^+)\big),\\ \theta &=\tilde{\theta}+\frac{\lambda}{2}\big(G(\tilde{x}^-)+F(\tilde{x}^+)\big), \end{aligned} \tag{23}$$

to express $\tilde{x}^\pm$ in terms of t and θ, where the derivatives of F and G are the components of stress–energy tensor in the specific classical solution

$$F'(x)=2f'(x)f'(x),\quad G'(x)=2g'(x)g'(x). \tag{24}$$

A solution to the equation of motion of the deformed model is then given by

$$\varphi(t,\theta)=f\big(\tilde{x}^+(t,\theta)\big)+g\big(\tilde{x}^-(t,\theta)\big). \tag{25}$$

Let us return to the chiral boson model. Though we have not found a coordinate transformation which maps the equations of motion of the model (11) directly to those of the undeformed model, one can check that

$$\begin{aligned}\phi(t,\theta) &= h(t)+\sqrt{2}f(\tilde{x}^{+}(t,\theta)),\\ \bar{\phi}(t,\theta) &= \bar{h}(t)+\sqrt{2}g(\tilde{x}^{-}(t,\theta)),\end{aligned}\tag{26}$$

is a solution to the equations of motion. Here t,θ are still related to $\tilde{x}^{\pm}$ by (23). $h(t)$ and $\bar{h}(t)$ are arbitrary functions of t. We show this in a more general case in Appendix B. The energy and momentum corresponding to the solution (26) are

$$H_\lambda=\int d\theta\,\frac{f'(\tilde{x}^+)^2-g'(\tilde{x}^-)^2-4\lambda f'(\tilde{x}^+)^2g'(\tilde{x}^-)^2}{1-4\lambda^2f'(\tilde{x}^+)^2g'(\tilde{x}^-)^2},\tag{27}$$

$$P_\lambda=\int d\theta\,\frac{f'(\tilde{x}^+)^2-g'(\tilde{x}^-)^2}{1-4\lambda^2f'(\tilde{x}^+)^2g'(\tilde{x}^-)^2}.\tag{28}$$

We now put the deformed model on a circle of length L. Then the fields should be periodic in coordinate θ. We take periodicities of f and g to be L and consider the solutions with the following form

$$\varphi=f(n(\lambda)\tilde{x}^+(t,\theta))+g(m(\lambda)\tilde{x}^-(t,\theta)),\tag{29}$$

where we introduce $n(\lambda)$ and $m(\lambda)$ such that the periodicity of φ is L in coordinate θ. It is not difficult to show that

$$\tilde{t}(t,L)-\tilde{t}(t,0)=\lambda P_\lambda,\quad \tilde{x}(t,L)-\tilde{x}(t,0)=-\lambda H_\lambda.\tag{30}$$

Then we have

$$n(\lambda)=\frac{L}{L-\lambda(H_\lambda-P_\lambda)},\quad m(\lambda)=\frac{L}{L-\lambda(H_\lambda+P_\lambda)}.\tag{31}$$

Using Eqs. (23) and $F(L)-F(0)=H_0+P_0$, $G(L)-G(0)=H_0-P_0$, we get

$$-H_0L+H_\lambda L\pm P_0L\mp P_\lambda L-\lambda H_\lambda^2+\lambda P_\lambda^2=0.\tag{32}$$

Finally we get

$$H_\lambda=\frac{L-\sqrt{L^2-4H_0L\lambda+4P_0^2\lambda^2}}{2\lambda},\quad P_\lambda=P_0,\tag{33}$$

which is a classical version of the general quantum spectrum in [3,4]. The significance of the sign of the deformation parameter λ is well-known in the literature. When $\lambda>0$, the deformed energy can become complex if H_0 is large. This regime of λ is related to holography. For $\lambda<0$ the deformed energy spectrum is real and there are Hagedorn growth of density of states [38].

2.3 General theory of chiral bosons

To solve the flow equation, the field contents and details of the potentials are not important. We can study the $T\bar{T}$ deformed Lagrangian of more general model of chiral bosons with the initial translational invariant Lagrangian

$$\begin{aligned}\mathcal{L}_0 &= C-\frac{E_t^+}{2E_\theta^+}K_++\frac{E_t^-}{2E_\theta^-}K_-+E_\theta^+V_+\\ &\quad +E_\theta^-V_-+E_t^+W_++E_t^-W_-,\end{aligned}\tag{34}$$

where $K_\pm$, $W_\pm$ and $V_\pm$ are the functions of the fields. We require that the equations of motion are consistent with the conservation of the stress–energy tensor defined by (5). We can again solve the flow equation (6) using a perturbative approach. The all-order solution can be written as

$$\begin{aligned}\mathcal{L}_\lambda &= C+\frac{\tilde{E}_\theta^-\tilde{E}_t^++\tilde{E}_t^-\tilde{E}_\theta^+}{4\tilde{E}_\theta^+\tilde{E}_\theta^-}(K_--K_+)\\ &\quad +\frac{1}{2\lambda}\left(\frac{E_\theta^+E_\theta^-}{\tilde{E}_\theta^+\tilde{E}_\theta^-}\det\tilde{E}\mathcal{S}+\det\tilde{E}-2\det E\right),\end{aligned}\tag{35}$$

with

$$\tilde{E}_t^\pm=E_t^\pm\mp\lambda V_\mp,\ \tilde{E}_\theta^\pm=E_\theta^\pm\mp\lambda W_\mp,\tag{36}$$

$$\mathcal{S}=\sqrt{\frac{4\tilde{E}_\theta^{+2}\tilde{E}_\theta^{-2}-4(K_-+K_+)\tilde{E}_\theta^+\tilde{E}_\theta^-\lambda+(K_--K_+)^2\lambda^2}{4E_\theta^+E_\theta^-}}.\tag{37}$$

As a particular example, we consider a generalized chiral bosons theory

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}\left(F_{t\theta}-\frac{E_t^+}{E_\theta^+}F_{\theta\theta}-\bar{F}_{t\theta}+\frac{E_t^-}{E_\theta^-}\bar{F}_{\theta\theta}\right)\\ &\quad -E_\theta^+V(\phi)-E_\theta^-\bar{V}(\bar{\phi}),\end{aligned}\tag{38}$$

$$F_{\mu\nu}=G^{IJ}(\phi)\partial_\mu\phi_I\partial_\nu\phi_J,\quad I,J=1,2,\ldots,N,\tag{39}$$

$$\bar{F}_{\mu\nu}=\bar{G}^{\bar{I}\bar{J}}(\bar{\phi})\partial_\mu\bar{\phi}_{\bar{I}}\partial_\nu\bar{\phi}_{\bar{J}},\quad \bar{I},\bar{J}=1,2,\ldots,\bar{N},\tag{40}$$

where G and $\bar{G}$ are non-degenerate matrices. The $T\bar{T}$ deformation of the chiral boson theory (38) is therefore

$$\begin{aligned}\mathcal{L}_\lambda &= \frac{1}{2}(F_{t\theta}-\bar{F}_{t\theta})-E_\theta^-\bar{V}-E_\theta^+V\\ &\quad -\frac{(E_\theta^-E_t^++E_t^-E_\theta^++\lambda(E_\theta^+V-E_\theta^-\bar{V}))(F_{\theta\theta}-\bar{F}_{\theta\theta})}{4E_\theta^+E_\theta^-}\\ &\quad +\frac{\det E-\lambda(E_\theta^-\bar{V}+E_\theta^+V)}{2\lambda}(\mathcal{S}-1),\end{aligned}\tag{41}$$

with

$$\mathcal{S}=\sqrt{1-\frac{(F_{\theta\theta}+\bar{F}_{\theta\theta})}{E_\theta^-E_\theta^+}\lambda+\frac{(F_{\theta\theta}-\bar{F}_{\theta\theta})^2}{4E_\theta^{-2}E_\theta^{+2}}\lambda^2}.\tag{42}$$

One can also add an arbitrary number of Weyl–Majorana fermions to the theory (38). The undeformed Lagrangian is

$$\begin{aligned}\mathcal{L}_0 = &\frac{1}{2}\left(F_{t\theta} - \frac{E_t^+}{E_\theta^+}F_{\theta\theta} - \bar{F}_{t\theta} + \frac{E_t^-}{E_\theta^-}\bar{F}_{\theta\theta}\right)\\ &- E_\theta^+ V(\phi) - E_\theta^- \bar{V}(\bar{\phi})\\ &+ B_{MN}\psi^M(E_t^+\partial_\theta - E_\theta^+\partial_t)\psi^N\\ &- \bar{B}_{MN}\bar{\psi}^M(E_t^-\partial_\theta - E_\theta^-\partial_t)\bar{\psi}^N,\end{aligned} \tag{43}$$

where ψ and $\bar{\psi}$ are Weyl–Majorana left and right fermions respectively. The deformed theory is then given by (41) with

$$\begin{aligned}&C = \frac{1}{2}\left(F_{t\theta} - \bar{F}_{t\theta}\right), \quad K_+ = F_{\theta\theta}, \quad K_- = \bar{F}_{\theta\theta},\\ &V_+ = V(\phi) + B_{MN}\psi^M\partial_t\psi^N,\\ &V_- = \bar{V}(\phi) - \bar{B}_{MN}\bar{\psi}^M\partial_t\bar{\psi}^N,\\ &W_+ = B_{MN}\psi^M\partial_\theta\psi^N, \quad W_- = -\bar{B}_{MN}\bar{\psi}^M\partial_\theta\bar{\psi}^N.\end{aligned} \tag{44}$$

This action differs from the one obtained in [26] using the canonical stress–energy tensor. It was argued in [26] that there should be a field redefinition which would make the $T\bar{T}$ deformed action driven by the canonical stress–energy tensor coincide with the one driven by the covariant stress–energy tensor. It would be interesting to find such a field redefinition explicitly.

3 $T\bar{T}$ deformation and Chern–Simons gravity

The three-dimensional Einstein gravity theory with a negative cosmological constant can be reformulated as a Chern–Simons action with a gauge group $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ [31]. It was shown in [32] that the Chern–Simons action is equivalent to two copies of constrained $SL(2,\mathbb{R})$ chiral WZW models of opposite chiralities on the boundary, which can be combined into a non-chiral Liouville field theory. The chiral description is more convenient to deal with the zero modes and leads to geometric actions associated with coadjoint orbits of the Virasoro group [34,35]. In this section, we will focus on the $T\bar{T}$ deformed action of the constrained chiral WZW model. Since the original Lagrangian is a special case of (38), we can get the all-order $T\bar{T}$ deformation of the boundary action using the results in the previous section.[1]

3.1 AdS$_3$ Chern–Simons theory

Let us recall the connection between AdS$_3$ gravity and the chiral WZW model derived in [32]. The AdS$_3$ Einstein gravity with metric

$$ds^2 = -(r^2+1)dt^2 + r^2 d\theta^2 + \frac{dr^2}{r^2+1}, \tag{45}$$

can be reformulated as the Chern–Simons action [35]

$$\begin{aligned}&S = S[A] - S[\bar{A}] + S_{\text{bdy}},\\ &S[A] = -\frac{k}{2\pi}\int_{\mathcal{M}} dt\wedge \text{Tr}\Big(-\frac{1}{2}\tilde{A}\wedge\dot{\tilde{A}} + A_0\tilde{F}\Big),\\ &S_{\text{bdy}} = -\frac{k}{4\pi}\int_{\partial\mathcal{M}} dx^2\Big(\frac{E_t^+}{E_\theta^+}\text{Tr}(A_\theta^2) - \frac{E_t^+}{E_\theta^+}\text{Tr}(\bar{A}_\theta^2)\Big),\end{aligned} \tag{46}$$

where $k = \frac{1}{4G}$ and we couple the boundary terms to the boundary zweibein E^a. We will take $E_t^+ = E_\theta^+ = E_\theta^- = -E_t^- = 1/\sqrt{2}$. The gauge fields A and $\bar{A}$ are expressed by using the $SL(2)$ generators and related with the bulk dreibein e^a and the bulk spin connection ω

$$A - \bar{A} = 2e, \quad A + \bar{A} = 2\omega. \tag{47}$$

In this action, $A = A_0 dt + \tilde{A}_i dx^i$ and $\bar{A} = \bar{A}_0 dt + \tilde{\bar{A}}_i dx^i$ are separated into the temporal and spatial parts. The boundary conditions of the gauge fields are fixed to be $A_- = A_t - A_\theta = 0$ and $\bar{A}_- = \bar{A}_t + \bar{A}_\theta = 0$, which are chosen to match the asymptotics of the AdS$_3$ geometry

$$\begin{aligned}&A = \begin{pmatrix} \frac{1}{2}\Omega + \frac{dr}{2r} + O(r^{-2}) & O(r^{-1}) \\ \sqrt{2}rE^+dx^+ + O(r^{-1}) & -\frac{1}{2}\Omega - \frac{dr}{2r} + O(r^{-2})\end{pmatrix},\\ &\bar{A} = \begin{pmatrix} \frac{1}{2}\Omega - \frac{dr}{2r} + O(r^{-2}) & -\sqrt{2}rE^-dx^- + O(r^{-1}) \\ O(r^{-1}) & -\frac{1}{2}\Omega + \frac{dr}{2r} + O(r^{-2})\end{pmatrix},\end{aligned} \tag{48}$$

where Ω is the boundary spin connection. For simplicity, we consider $\Omega = 0$ in this paper. The boundary term S_{bdy} is necessary for a consistency variation principle.

Since the spatial field strength $\tilde{F}$ is flat, one can parametrize the $\tilde{A}$ and $\tilde{\bar{A}}$ as

$$\tilde{A} = g^{-1}\tilde{d}g, \quad \tilde{\bar{A}} = \bar{g}^{-1}\tilde{d}\bar{g}, \tag{49}$$

where $\tilde{d}$ is the spatial exterior derivative. g and $\bar{g}$ are elements of $SL(2)$ and can be written in the Gauss parameterization:

$$\begin{aligned}&g = \begin{pmatrix}1 & 0\\ F & 1\end{pmatrix}\begin{pmatrix}e^\varphi & 0\\ 0 & e^{-\varphi}\end{pmatrix}\begin{pmatrix}1 & \Psi\\ 0 & 1\end{pmatrix},\\ &\bar{g} = \begin{pmatrix}1 & -\bar{F}\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{-\bar{\varphi}} & 0\\ 0 & e^{\bar{\varphi}}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\bar{\Psi} & 1\end{pmatrix}.\end{aligned} \tag{50}$$

The gauge fields can be written as

[1] See also [39] for the $T\bar{T}$-deformation of the classical Liouville field theory.

$$
\begin{aligned}
\tilde{A} &= g^{-1}dg = \begin{pmatrix} A^3 & A^- \\ A^+ & -A^3 \end{pmatrix} \\
&= \begin{pmatrix} -e^{2\varphi}\Psi dF + d\varphi & -e^{2\varphi}\Psi^2 dF + 2\Psi d\varphi + d\Psi \\ e^{2\varphi} dF & e^{2\varphi}\Psi dF - d\varphi \end{pmatrix}, \\
\tilde{\bar{A}} &= \bar{g}^{-1}d\bar{g} = \begin{pmatrix} \bar{A}^3 & \bar{A}^- \\ \bar{A}^+ & -\bar{A}^3 \end{pmatrix} \\
&= \begin{pmatrix} e^{2\bar{\varphi}}\bar{\Psi} d\bar{F} - d\bar{\varphi} & -e^{2\bar{\varphi}} d\bar{F} \\ e^{2\bar{\varphi}}\bar{\Psi}^2 d\bar{F} - 2\bar{\Psi} d\bar{\varphi} - d\bar{\Psi} & -e^{2\bar{\varphi}}\bar{\Psi} d\bar{F} + d\bar{\varphi} \end{pmatrix}.
\end{aligned} \tag{51}
$$

The action (46) thus can be evaluated as

$$
S = \frac{k}{\pi}\int_{\partial\mathcal{M}} d^2x \mathcal{L}_0^{\mathrm{WZW}}, \tag{52}
$$

where $\mathcal{L}_0^{\mathrm{WZW}}$ has the form of Eq. (38) with $E_t^+ = E_\theta^+ = E_\theta^- = -E_t^- = 1/\sqrt{2}$ and

$$
\begin{aligned}
F_{t\theta} &= \partial_t\varphi\partial_\theta\varphi + e^{2\varphi}\partial_\theta F \partial_t \Psi, \\
\bar{F}_{t\theta} &= \partial_t\bar{\varphi}\partial_\theta\bar{\varphi} + e^{2\bar{\varphi}}\partial_\theta \bar{F} \partial_t \bar{\Psi}, \\
F_{\theta\theta} &= A_\theta^3 A_\theta^3 + A_\theta^+ A_\theta^-, \\
\bar{F}_{\theta\theta} &= \bar{A}_\theta^3 \bar{A}_\theta^3 + \bar{A}_\theta^+ \bar{A}_\theta^-, \quad V = \bar{V} = 0.
\end{aligned} \tag{53}
$$

The fields in the expression of g and $\bar{g}$ are not independent. The boundary condition (3.1) imposes the constrains

$$
A_\theta^3 = \bar{A}_\theta^3 = 0, \quad A_\theta^+ - \sqrt{2}E_\theta^+ r = 0, \quad \bar{A}_\theta^- + \sqrt{2}E_\theta^- r = 0 \tag{54}
$$

on the AdS boundary. The constrains can be expressed as

$$
\begin{aligned}
e^{\varphi} &= \sqrt{\frac{r}{\partial_\theta F}}, \quad \Psi = -\frac{\partial_\theta^2 F}{2r\partial_\theta F}, \\
e^{\bar{\varphi}} &= \sqrt{\frac{r}{\partial_\theta \bar{F}}}, \quad \bar{\Psi} = -\frac{\partial_\theta^2 \bar{F}}{2r\partial_\theta \bar{F}}.
\end{aligned} \tag{55}
$$

By using the conditions (55), we could express the action S in terms of F and $\bar{F}$, which we parameterize as

$$
F = \tan\frac{\phi}{2}, \quad \bar{F} = \tan\frac{\bar{\phi}}{2}. \tag{56}
$$

where ϕ and $\bar{\phi}$ are elements of $\mathrm{Diff}(S^1)/PSL(2,\mathbb{R})$ and we get two copies of the Alekseev–Shatashvili quantization of coadjoint orbit $\mathrm{Diff}(S^1)/PSL(2,\mathbb{R})$ of the Virasoro group [40]. If we parameterize F and $\bar{F}$ as

$$
F = \tan\frac{\alpha\phi}{2}, \quad \bar{F} = \tan\frac{\alpha\bar{\phi}}{2}, \tag{57}
$$

with $\alpha \neq n, n \in \mathbb{Z}$, we get the orbit $\mathrm{Diff}(S^1)/U(1)$. See [41] for further discussion.

3.2 $T\bar{T}$ deformation of the boundary action

With the solution (41) to the flow equation induced by the $T\bar{T}$ deformation at hand, we are now ready to get the all-order $T\bar{T}$ deformation of the boundary action. Simply plugging (53) into (41), we obtain a $T\bar{T}$ deformed WZW model denoted by $\mathcal{L}_\lambda^{\mathrm{WZW}}$.[2] However, the action (52) is constrained. It is a non-trivial question whether the constrains are deformed by the $T\bar{T}$. To treat the constraints carefully, we introduce Lagrange multipliers in the undeformed action:

$$
\begin{aligned}
\mathcal{L}_0^{\mathrm{cWZW}} &= \mathcal{L}_0^{\mathrm{WZW}} - a_3 A_\theta^3 - \bar{a}_3 \bar{A}_\theta^3 - a_+(A_\theta^+ - \sqrt{2}E_\theta^+ r) \\
&\quad - \bar{a}_-(\bar{A}_\theta^- + \sqrt{2}E_\theta^- r).
\end{aligned} \tag{58}
$$

Here we keep $E_\theta^\pm$ unfixed which is necessary when we apply the solution (41). The terms with coefficient $E_\theta^\pm$ can be view as potentials. By using (41) again with $V = -\sqrt{2}ra_+$ and $\bar{V} = \sqrt{2}r\bar{a}_-$, we find the deformed constrained Lagrangian

$$
\begin{aligned}
\mathcal{L}_\lambda^{\mathrm{cWZW}} &= \frac{1}{2}(F_{t\theta} - \bar{F}_{t\theta}) + \frac{1}{2\lambda}(\mathcal{S} - 1) - a_+ \\
&\quad \times \left(A_\theta^+ - \frac{r + r\mathcal{S}}{2} - \frac{r\lambda(F_{\theta\theta} - \bar{F}_{\theta\theta})}{2}\right) \\
&\quad - \bar{a}_- \left(\bar{A}_\theta^- + \frac{r + r\mathcal{S}}{2} - \frac{r\lambda(F_{\theta\theta} - \bar{F}_{\theta\theta})}{2}\right) \\
&\quad - a_3 A_\theta^3 - \bar{a}_3 \bar{A}_\theta^3,
\end{aligned} \tag{59}
$$

where $F_{\mu\nu}$ are given in (53) and we have set $E_t^+ = E_\theta^+ = E_\theta^- = -E_t^- = 1/\sqrt{2}$. The constraints $A_\theta^3 = \bar{A}_\theta^3 = 0$ are solved by

$$
\Psi = \frac{e^{-2\varphi}\partial_\theta\varphi}{\partial_\theta F}, \quad \bar{\Psi} = \frac{e^{-2\bar{\varphi}}\partial_\theta\bar{\varphi}}{\partial_\theta \bar{F}}. \tag{60}
$$

Plugging $A_\theta^3 = \bar{A}_\theta^3 = 0$ into the rest two constraints, we get

$$
A_\theta^+ = r + r^2\lambda\bar{A}_\theta^+, \quad \bar{A}_\theta^- = -r + r^2\lambda A_\theta^-, \tag{61}
$$

which are similar to (5.16) in [42]. The explicit expressions are

$$
\begin{aligned}
&p - 1 - \frac{\lambda}{4\bar{p}^3}(4\bar{p}^2\bar{s} + 3(\partial_\theta\bar{p})^2 - 2\bar{p}\partial_\theta^2\bar{p}) = 0, \\
&\bar{p} - 1 - \frac{\lambda}{4p^3}(4p^2 s + 3(\partial_\theta p)^2 - 2p\partial_\theta^2 p) = 0,
\end{aligned} \tag{62}
$$

where

$$
e^{2\varphi} = \frac{rp}{\partial_\theta F}, \quad e^{2\bar{\varphi}} = \frac{r\bar{p}}{\partial_\theta \bar{F}}, \quad s = \frac{1}{2}\{F,\theta\}, \quad \bar{s} = \frac{1}{2}\{\bar{F},\theta\}. \tag{63}
$$

[2] In principle λ should be rescaled to keep the flow equation invariant due to the coefficient k/π in front of the action. But for convenience we are not going to rescale λ here.

s and $\bar{s}$ are halves of the Schwarzian derivatives defined by

$$\{f,\theta\}=\frac{\partial_\theta^3 f}{\partial_\theta f}-\frac{3}{2}\left(\frac{\partial_\theta^2 f}{\partial_\theta f}\right)^2. \tag{64}$$

Suppose the solution p and $\bar{p}$ satisfying (62) is known, we then substitute this solution to the $\mathcal{L}_\lambda^{\rm cWZW}$ and obtain the all-order $T\bar{T}$ deformed Lagrangian

$$\begin{aligned}\mathcal{L}_\lambda^{\rm cWZW}=&\frac{s}{2p}+\frac{\bar{s}}{2\bar{p}}-\frac{\dot{F}''}{4F'}+\frac{3\dot{F}'F''}{8F'^2}+\frac{\dot{\bar{F}}''}{4\bar{F}'}\\&-\frac{3\dot{\bar{F}}'\bar{F}''}{8\bar{F}'^2}+\frac{3\left(p'\right)^2}{8p^3}-\frac{p''}{4p^2}\\&-\frac{3\dot{p}p'}{8p^2}+\frac{\dot{p}'}{4p}+\frac{3\left(\bar{p}'\right)^2}{8\bar{p}^3}-\frac{\bar{p}''}{4\bar{p}^2}+\frac{3\dot{\bar{p}}\bar{p}'}{8\bar{p}^2}-\frac{\dot{\bar{p}}'}{4\bar{p}},\end{aligned} \tag{65}$$

where the overdot and prime denote the derivative with respect to t and θ respectively. The deformed stress–energy tensor is given by

$$(T_\lambda)^\mu_{\ \nu}=\frac{k}{\pi}\begin{pmatrix}-\frac{p+\bar{p}-2}{2\lambda} & \frac{p-\bar{p}}{2\lambda}\\ -\frac{p-\bar{p}}{2\lambda} & \frac{p^2-2\bar{p}p+p+\bar{p}^2+\bar{p}-2}{2\lambda(p+\bar{p}-1)}\end{pmatrix}. \tag{66}$$

Using the parameterization

$$F=\tan\frac{\alpha\phi}{2},\quad \bar{F}=\tan\frac{\alpha\bar{\phi}}{2}. \tag{67}$$

We have

$$s=\frac{\phi^{(3)}}{2\phi'}-\frac{3\phi''^2}{4\phi'^2}+\frac{\alpha\phi'^2}{4},\quad \bar{s}=\frac{\bar{\phi}^{(3)}}{2\bar{\phi}'}-\frac{3\bar{\phi}''^2}{4\bar{\phi}'^2}+\frac{\alpha\bar{\phi}'^2}{4}, \tag{68}$$

where $f^{(n)}$ denotes the nth derivative of f with respect to θ. We also define

$$u=\frac{\dot{\phi}''}{2\phi'}-\frac{3\dot{\phi}'\phi''}{4\phi'^2}+\frac{\alpha\dot{\phi}\phi'}{4},\quad \bar{u}=\frac{\dot{\bar{\phi}}''}{2\bar{\phi}'}-\frac{3\dot{\bar{\phi}}'\bar{\phi}''}{4\bar{\phi}'^2}+\frac{\alpha\dot{\bar{\phi}}\bar{\phi}'}{4}. \tag{69}$$

Dropping total derivatives, the Lagrangian can be written as

$$\begin{aligned}\mathcal{L}=&\frac{s}{2p}+\frac{\bar{s}}{2\bar{p}}-\frac{u}{2}+\frac{\bar{u}}{2}+\frac{3\left(p'\right)^2}{8p^3}-\frac{p''}{4p^2}-\frac{3\dot{p}p'}{8p^2}\\&+\frac{\dot{p}'}{4p}+\frac{3\left(\bar{p}'\right)^2}{8\bar{p}^3}-\frac{\bar{p}''}{4\bar{p}^2}+\frac{3\dot{\bar{p}}\bar{p}'}{8\bar{p}^2}-\frac{\dot{\bar{p}}'}{4\bar{p}},\end{aligned} \tag{70}$$

where p and $\bar{p}$ are determined by s and $\bar{s}$ through the constraints (62).

Though the constraints (62) are difficult to solve to all orders in λ, we can solve p and $\bar{p}$ in the first few orders of small λ

$$p=1+\lambda\bar{s}+\lambda^2\left(-s\bar{s}-\frac{s''}{2}\right)+O(\lambda)^3, \tag{71}$$

$$\bar{p}=1+\lambda s+\lambda^2\left(-s\bar{s}-\frac{\bar{s}''}{2}\right)+O(\lambda)^3, \tag{72}$$

which leads to

$$\begin{aligned}\mathcal{L}_\lambda^{\rm cWZW}=&\frac{s}{2}+\frac{\bar{s}}{2}-\frac{\dot{F}''}{2F'}+\frac{3\dot{F}'F''}{4F'^2}+\frac{\dot{\bar{F}}''}{2\bar{F}'}-\frac{3\dot{\bar{F}}'\bar{F}''}{4\bar{F}'^2}-\lambda s\bar{s}\\&+\frac{1}{8}\lambda^2(4\bar{s}s''+8s^2\bar{s}+8s'\bar{s}'+3\bar{s}'^2-3\dot{\bar{s}}\bar{s}'\\&-2\bar{s}\dot{\bar{s}}'+4s\bar{s}''+6\bar{s}\bar{s}''+8s\bar{s}^2\\&+6ss''+3s'^2+3\dot{s}s'+2s\dot{s}')+O(\lambda^3).\end{aligned} \tag{73}$$

When $\lambda=0$, this reproduces the original Lagrangian. The first order term $s\bar{s}$ is nothing but the $T\bar{T}$ operator of the undeformed action.

At the end of this subsection, let us comment on the deformed constrains (61) and their relation with finite cut-off AdS. Since (55) is derived from the boundary condition (3.1) of gauge fields (or metric), it is natural to guess that the boundary condition will also be transformed non-trivially under the $T\bar{T}$ deformation. Let us suppose the new boundary is at $r=r_c$ with a large enough r_c. If we identify $r_c^2\lambda=1$, (61) leads to $2e_\theta^\pm=r_c$, which is consistency with the metric (45) at finite cutoff $r=r_c$. In Sect. 3.4, we will check the identification $r_c^2\lambda=1$ in more details by calculating the boundary stress–energy tensor.

3.3 One-loop torus partition function

The partition function in the undeformed theory was obtained and shown to be one-loop exact in [35]. We now compute the one-loop torus partition function in the deformed theory. Let us Wick-rotate to the Euclidean time $t=-iy$ and put the boundary theory on a torus of complex structure τ. The Euclidean action is $S_E=-iS$. On the torus, the coordinate $z=\theta+iy$ has the identifications $z\sim z+2\pi$ and $z\sim z+2\pi\tau$. We first focus on the ${\rm Diff}(S^1)/PSL(2,\mathbb{R})$ case. The fields ϕ and $\bar{\phi}$ satisfy the boundary condition

$$\begin{aligned}&f(\theta+2\pi,y)=f(\theta,y)+2\pi,\\&f(\theta,y)=f(\theta+2\pi{\rm Re}(\tau),y+2\pi{\rm Im}(\tau)).\end{aligned} \tag{74}$$

We consider the saddle point of the Euclidean Lagrangian

$$\phi_0=\bar{\phi}_0=\theta-\frac{\tau_1}{\tau_2}y,\quad p_0=\bar{p}_0=\frac{1}{2\gamma}, \tag{75}$$

where τ_1 and τ_2 are real and imaginary part of τ respectively and we will use

$$\gamma = \frac{\sqrt{\lambda+1}-1}{\lambda}, \tag{76}$$

instead of λ to avoid square root.

Expanding ϕ and $\bar{\phi}$ in fluctuations around the saddle

$$\phi = \phi_0 + \delta\phi, \quad \bar{\phi} = \bar{\phi}_0 + \delta\bar{\phi}, \tag{77}$$

the fluctuations of p and $\bar{p}$ depend on $\delta\phi$ and $\delta\bar{\phi}$ via the constraints. We have

$$p = p_0 + p_1 + p_2 + \cdots, \quad \bar{p} = \bar{p}_0 + \bar{p}_1 + \bar{p}_2 + \cdots \tag{78}$$

where p_1 ($\bar{p}_1$) and p_2 ($\bar{p}_2$) are linear and quadratic terms in the fluctuation fields $\delta\phi$ and $\delta\bar{\phi}$ respectively. We then expand the Lagrangian and the constraints around the saddle, and express every term by using $\delta\phi$ and $\delta\bar{\phi}$. On the torus, the fluctuation fields $\delta\phi$ and $\delta\bar{\phi}$ can be expand as

$$\delta\phi = \sum_{\substack{m,n\\ n\neq -1,0,1}} \epsilon_{m,n} f_{m,n}, \quad \delta\bar{\phi} = \sum_{\substack{m,n\\ n\neq -1,0,1}} \bar{\epsilon}_{m,n} f_{m,n}, \tag{79}$$

where we have set the zero modes to zero and the functions

$$f_{m,n} = \frac{1}{2\pi}\exp\left(i\frac{my}{\tau_2} + in\left(\theta - \frac{\tau_1}{\tau_2}y\right)\right), \tag{80}$$

satisfy

$$\int_{T^2} d^2x f_{m_1,n_1} f_{m_2,n_2} = \tau_2 \delta_{m_1,-m_2}\delta_{n_1,-n_2}. \tag{81}$$

Then p_1 and $\bar{p}_1$ are solved by

$$\begin{aligned} p_1 &= \sum_{\substack{m,n\\ n\neq -1,0,1}} q_n\big((1-2\gamma)(2n^2-1)\epsilon_{m,n} + \bar{\epsilon}_{m,n}\big) f_{m,n},\\ \bar{p}_1 &= \sum_{\substack{m,n\\ n\neq -1,0,1}} q_n\big((1-2\gamma)(2n^2-1)\bar{\epsilon}_{m,n} + \epsilon_{m,n}\big) f_{m,n}. \end{aligned} \tag{82}$$

where

$$q_n = -\frac{i(2\gamma-1)n\left(n^2-1\right)}{2\left(-2\gamma n^2+\gamma+n^2-1\right)\left(-2\gamma n^2+\gamma+n^2\right)}. \tag{83}$$

Finally, the quadratic action is given by

$$-\frac{\pi}{k}S_E = \frac{\gamma\tau_2}{2}(2\pi)^2 + \sum_{\substack{m,n\\ n\neq -1,0,1}} (\epsilon_{m,n}, \bar{\epsilon}_{m,n}) M_{m,n} \begin{pmatrix}\epsilon_{-m,-n}\\ \bar{\epsilon}_{-m,-n}\end{pmatrix}, \tag{84}$$

where $M_{m,n}$ is a 2×2 matrix

$$M_{m,n} = \frac{n(n^2-1)}{16\left(n^2(\chi-1)+1\right)^2\left(\chi - n^2(\chi-1)\right)^2} \times \begin{pmatrix} A_{m,n}(\tau_1,\tau_2) & B_{m,n}(\tau_1,\tau_2)\\ B_{m,n}(\tau_1,\tau_2) & -A_{m,n}(\tau_1,-\tau_2)\end{pmatrix}, \tag{85}$$

with $\chi = \frac{\gamma}{1-\gamma} = \frac{1}{\sqrt{\lambda+1}}$ and

$$\begin{aligned} &A_{m,n}(\tau_1,\tau_2)\\ &\quad= -2i\chi(n^2(n^2-1)(\chi-1)^2-\chi)(m+in\tau_2\chi-n\tau_1)\\ &\qquad - n(n^2-1)^2\tau_2\chi(\chi^2-1)^2, \end{aligned} \tag{86}$$

$$\begin{aligned} &B_{m,n}(\tau_1,\tau_2)\\ &\quad= -n(n^2-1)\tau_2\chi(\chi^2-1)(n^2(\chi-1)^2-\chi^2-1). \end{aligned} \tag{87}$$

The determinant of $M_{m,n}$ is

$$\begin{aligned} &\det M_{m,n}\\ &\quad= \frac{n^2(n^2-1)^2\chi^2\left(m-in\tau_2\chi-n\tau_1\right)\left(m+in\tau_2\chi-n\tau_1\right)}{64\left(n^2(\chi-1)+1\right)^2\left(\chi-n^2(\chi-1)\right)^2}. \end{aligned} \tag{88}$$

Then following the procedure in [35], we obtain the classical partition function

$$Z_c = \exp\left(2\pi C\tau_2\frac{\sqrt{\lambda+1}-1}{6\lambda}\right), \quad C = 6k, \tag{89}$$

and the one-loop torus partition function

$$\begin{aligned} Z_{1-\text{loop}} &= \exp\left(2\pi\tau_2\left(\frac{C\left(\sqrt{\lambda+1}-1\right)}{6\lambda} + \frac{13}{12\sqrt{\lambda+1}}\right)\right)\\ &\quad\times\left|\prod_{n=2}^{\infty}\frac{1}{1-\exp(2\pi in(\tau_1+i\frac{\tau_2}{\sqrt{\lambda+1}}))}\right|^2. \end{aligned} \tag{90}$$

Note that the partition function is not modular invariant even in the undeformed theory. The spectrum should become complex when $\lambda > 0$. In this case the one-loop torus partition function should be understood as an analytic continuation from the regime of $\lambda < 0$. The singularity at $\lambda = -1$ is related to the Hagedorn divergence. It is easy to check that the classical partition satisfies the flow equation on the torus [7]

$$\begin{aligned} &-\frac{\pi C}{6}\partial_\lambda Z_c\\ &\quad= \left(\frac{\tau_2}{4}(\partial_{\tau_2}^2+\partial_{\tau_1}^2) + \frac{\lambda}{2}(\partial_{\tau_2}-\tau_2^{-1})\partial_\lambda\right) Z_c, \end{aligned} \tag{91}$$

while the one-loop partition function satisfies the flow equation up to the one-loop

$$-\frac{1}{Z_c}\frac{\pi}{6}\partial_\lambda Z_{1-\text{loop}} = \frac{1}{CZ_c}\left(\frac{\tau_2}{4}(\partial_{\tau_2}^2+\partial_{\tau_1}^2) +\frac{\lambda}{2}(\partial_{\tau_2}-\tau_2^{-1})\partial_\lambda\right)Z_{1-\text{loop}} + O(C^{-1}), \quad (92)$$

where the first term on the right hand side is order $O(C)$. The one-loop torus function satisfy the flow equation up to the one loop order $O(C^0)$. This suggests that the $T\bar{T}$ deformed partition function should not be one-loop exact as in the undeformed theory. It is worth to note that the flow equations (91) and (92) are satisfied for all order in λ, which provide evidence for our all-order $T\bar{T}$ deformed Lagrangian.

One can also compute the partition function of the $\text{Diff}(\mathbb{S}^1)/U(1)$ case, where

$$F = \tan\frac{\alpha\phi}{2}, \quad \bar{F} = \tan\frac{\alpha\bar{\phi}}{2}, \quad (93)$$

with $\alpha \neq n, n \in \mathbb{Z}$. One can repeat the same steps and finally get

$$Z_{1-\text{loop}} = \exp\left(2\pi\tau_2\left(\frac{C\left(\sqrt{\alpha^2\lambda+1}-1\right)}{6\lambda}+\frac{13}{12\sqrt{\alpha^2\lambda+1}}\right)\right) \times\left|\prod_{n=1}^{\infty}\frac{1}{1-\exp(2\pi i n(\tau_1+i\frac{\tau_2}{\sqrt{\alpha^2\lambda+1}}))}\right|^2. \quad (94)$$

3.4 $T\bar{T}$ deformation and BTZ black hole

Following the same procedure in Sect. 3.1, one can describe the BTZ black hole in the formalism of the Chern–Simons theory. In this subsection, we compute the stress–energy tensor of the $T\bar{T}$ deformed boundary theory of the BTZ background. For simplicity, we will focus on the classical solution of the BTZ Chern–Simons theory. We also compare the associated stress–energy tensor with the “boundary stress–energy tensor” of the BTZ gravity with a finite cutoff.

The BTZ black hole is described by the metric

$$ds^2 = -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2(d\theta-\omega(r)dt)^2,$$
$$f^2(r) = r^2 - 8GM + \frac{16G^2J^2}{r^2}, \quad \omega(r) = \frac{4GJ}{r^2}. \quad (95)$$

To describe the BTZ black hole in the Chern–Simons formulation, it is convenient to define

$$J = \frac{b^2-\bar{b}^2}{4G}, \quad M = \frac{-b^2-\bar{b}^2}{4G},$$
$$r = \sqrt{\frac{(1-z^2b^2)(1-z^2\bar{b}^2)}{z^2}}. \quad (96)$$

Then the metric can be written as

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}((1-\bar{b}^2z^2)d\theta + (1+\bar{b}^2z^2)dt) \times((1-b^2z^2)d\theta - (1+b^2z^2)dt). \quad (97)$$

The associated classical gauge fields are

$$A^{(0)} = \begin{pmatrix} \frac{dz}{2z} & -zb^2(d\theta+dt) \\ z^{-1}(d\theta+dt) & -\frac{dz}{2z}\end{pmatrix},$$
$$\bar{A}^{(0)} = \begin{pmatrix} \frac{-dz}{2z} & -z^{-1}(d\theta-dt) \\ z\bar{b}^2(d\theta-dt) & \frac{dz}{2z}\end{pmatrix}, \quad (98)$$

and the group elements are

$$g^{(0)} = \begin{pmatrix} \frac{\cos(bx^+)}{\sqrt{b}\sqrt{z}} & -\sqrt{b}\sqrt{z}\sin(bx^+) \\ \frac{\sin(bx^+)}{\sqrt{b}\sqrt{z}} & \sqrt{b}\sqrt{z}\cos(bx^+)\end{pmatrix},$$
$$\bar{g}^{(0)} = \begin{pmatrix} \sqrt{\bar{b}}\sqrt{z}\cos(\bar{b}x^-) & -\frac{\sin(\bar{b}x^-)}{\sqrt{\bar{b}}\sqrt{z}} \\ \sqrt{\bar{b}}\sqrt{z}\sin(\bar{b}x^-) & \frac{\cos(\bar{b}x^-)}{\sqrt{\bar{b}}\sqrt{z}}\end{pmatrix}, \quad (99)$$

where $A^{(0)} = (g^{(0)})^{-1}dg^{(0)}$ and $\bar{A}^{(0)} = (\bar{g}^{(0)})^{-1}d\bar{g}^{(0)}$. The BTZ metric leads to same boundary condition (3.1) of gauge fields at boundary. In the same way as in Sect. 3.1, we could derive the boundary action and the constrains of the BTZ black hole, which have the same form as (52) and (55) respectively. However, instead of (56), the fields in the BTZ black hole are

$$F = \tan(b(\theta+t)), \quad \bar{F} = \tan(\bar{b}(\theta-t)), \quad (100)$$

which provides the orbit $\text{Diff}(S^1)/U(1)$. To describe an BTZ black hole, we require $b^2 < 0$ and $\bar{b}^2 < 0$. When $b = \bar{b} \in (0, 1/2)$ we have a conical defect rather than a BTZ black hole. See [35] for more discussions.

In Appendix B, we show that the solutions to the EOM of the deformed theory can be obtained from the ones of original theory. The deformed solution associated with $g^{(0)}$ and $\bar{g}^{(0)}$ is

$$g = g^{(0)}(\tilde{x}^+)|_{b\to b_\lambda}, \quad \bar{g} = \bar{g}^{(0)}(\tilde{x}^-)|_{\bar{b}\to\bar{b}_\lambda}, \quad (101)$$

where

$$\tilde{x}^+ = \frac{x^+ + \lambda\bar{b}_\lambda^2 x^-}{1-b_\lambda^2\bar{b}_\lambda^2\lambda^2}, \quad \tilde{x}^- = \frac{x^- + \lambda b_\lambda^2 x^+}{1-b_\lambda^2\bar{b}_\lambda^2\lambda^2},$$
$$b_\lambda = \frac{\sqrt{\lambda^2\left(b^2-\bar{b}^2\right)^2+2\lambda\left(b^2+\bar{b}^2\right)+1}+b^2\lambda-\bar{b}^2\lambda-1}{2b\lambda},$$
$$\bar{b}_\lambda = \frac{\sqrt{\lambda^2\left(b^2-\bar{b}^2\right)^2+2\lambda\left(b^2+\bar{b}^2\right)+1}-b^2\lambda+\bar{b}^2\lambda-1}{2\bar{b}\lambda}. \quad (102)$$

Here we introduce b_λ and $\bar{b}_\lambda$ such that the boundary condition

$$\arctan F|_{\theta=0}^{\theta=2\pi} = 2\pi b, \quad \arctan\bar{F}|_{\theta=0}^{\theta=2\pi} = 2\pi\bar{b}, \quad (103)$$

are undeformed. The deformed stress–energy tensor in terms of the classical solution is

$$
\begin{aligned}
(T_\lambda)^\mu_{\ \nu} &= \frac{k}{\pi}\frac{1}{2-2b_\lambda^2\bar{b}_\lambda^2\lambda^2}\\
&\times\begin{pmatrix} -2\lambda\bar{b}_\lambda^2 b_\lambda^2 - b_\lambda^2 - \bar{b}_\lambda^2 & \bar{b}_\lambda^2 - b_\lambda^2 \\ b_\lambda^2 - \bar{b}_\lambda^2 & -2\lambda\bar{b}_\lambda^2 b_\lambda^2 + b_\lambda^2 + \bar{b}_\lambda^2 \end{pmatrix},\\
&= \frac{1}{8\pi G}\begin{pmatrix} \frac{(1-\mathcal{S})}{\lambda} & \bar{b}^2 - b^2 \\ b^2 - \bar{b}^2 & \frac{1}{\lambda}\left(1+\mathcal{S}-\frac{\mathcal{S}}{\lambda(b-\bar{b})^2+1}-\frac{S}{\lambda(b+\bar{b})^2+1}\right)\end{pmatrix},
\end{aligned}
\tag{104}
$$

where

$$
\mathcal{S}=\sqrt{1+2\lambda\left(b^2+\bar{b}^2\right)+\lambda^2\left(b^2-\bar{b}^2\right)^2}. \tag{105}
$$

In the following of this subsection, we compare (104) with the boundary stress–energy tensor of the BTZ black hole at a cutoff surface $r=r_c$ [43] in our convention. Here we mainly follow the derivation in [14]. The boundary stress–energy tensor is define as

$$
T_{ij}=\frac{1}{4G}(K_{ij}-Kg_{ij}+g_{ij}), \tag{106}
$$

where g_{ij} is the boundary metric and K_{ij} the extrinsic curvature. On a surface at a finite radial location

$$
z\to z_c=\left(\frac{\bar{b}^2+\bar{b}^2+r_c^2-\sqrt{r_c^4+2(b^2+\bar{b}^2)r_c^2+(b^2-\bar{b}^2)^2}}{2\bar{b}^2\bar{b}^2}\right)^{\frac{1}{2}}, \tag{107}
$$

we have

$$
g_{ij}=\begin{pmatrix} -\frac{(b^2z_c^2+1)(\bar{b}^2z_c^2+1)}{z_c^2} & \bar{b}^2-b^2 \\ \bar{b}^2-b^2 & \frac{(b^2z_c^2-1)(\bar{b}^2z_c^2-1)}{z_c^2}\end{pmatrix}, \tag{108}
$$

$$
K_{ij}=-z\partial_z g_{ij}|_{z\to z_c}=\begin{pmatrix} b^2\bar{b}^2z_c^2-\frac{1}{z_c^2} & 0 \\ 0 & \frac{1}{z_c^2}-b^2\bar{b}^2z_c^2\end{pmatrix}, \tag{109}
$$

from which we find

$$
\begin{aligned}
T^i_{\ j} &= -\frac{z_c^2}{4G-4z_c^4b^2\bar{b}^2G}\\
&\times\begin{pmatrix} 2z_c^2\bar{b}^2b^2+b^2+\bar{b}^2 & b^2-\bar{b}^2 \\ \bar{b}^2-b^2 & 2z_c^2\bar{b}^2b^2-b^2-\bar{b}^2\end{pmatrix}.
\end{aligned}
\tag{110}
$$

We define basis vectors

$$
v_0^i=\left(\frac{z_c\sqrt{\left(b^2z_c^2-1\right)\left(\bar{b}^2z_c^2-1\right)}}{1-b^2\bar{b}^2z_c^4},\ \frac{z_c^3\left(b^2-\bar{b}^2\right)}{\left(1-b^2\bar{b}^2z_c^4\right)\sqrt{\left(b^2z_c^2-1\right)\left(\bar{b}^2z_c^2-1\right)}}\right), \tag{111}
$$

$$
v_1^j=\left(0,\ \frac{z_c}{\sqrt{\left(b^2z_c^2-1\right)\left(\bar{b}^2z_c^2-1\right)}}\right), \tag{112}
$$

where v_0 is a unit vector normal to a constant t slice of the boundary and v_1 is a unit vector normal to v_0. In the new basis $\{v_I\}$, the stress–energy tensor becomes

$$
\begin{aligned}
T^I_{\ J} &= \frac{z_c^2}{4G\left(b^2z_c^2-1\right)\left(\bar{b}^2z_c^2-1\right)}\\
&\times\begin{pmatrix} 2b^2\bar{b}^2z_c^2-b^2-\bar{b}^2 & \bar{b}^2-b^2 \\ b^2-\bar{b}^2 & \frac{2b^4\bar{b}^4z_c^6-3b^4\bar{b}^2z_c^4-3b^2\bar{b}^4z_c^4+6b^2\bar{b}^2z_c^2-b^2-\bar{b}^2}{b^2\bar{b}^2z_c^4-1}\end{pmatrix},
\end{aligned}
\tag{113}
$$

or

$$
\begin{aligned}
T^I_{\ J} &= \frac{1}{4Gr_c^2}\\
&\times\begin{pmatrix} r_c^2(1-\mathcal{S}_c) & \bar{b}^2-b^2 \\ b^2-\bar{b}^2 & r_c^2\left(-\frac{r_c^2\mathcal{S}_c}{(b-\bar{b})^2+r_c^2}-\frac{r_c^2S}{(b+\bar{b})^2+r_c^2}+1+\mathcal{S}_c\right)\end{pmatrix},
\end{aligned}
\tag{114}
$$

$$
\mathcal{S}_c=r_c^{-2}\sqrt{r_c^4+2(b^2+\bar{b}^2)r_c^2+(b^2-\bar{b}^2)^2}, \tag{115}
$$

which matches (104) up to a factor under the identification $r_c^2\lambda=1$. As in [13,14], to compare with the energy obtained on the QFT side one should multiply the energy by the circumference of the circle $L=2\pi r_c$ to get a dimensionless "proper energy"

$$
\mathcal{E}=\frac{L}{2\pi}\int_0^{2\pi}\sqrt{g_{\theta\theta}}T^0_{\ 0}d\theta=2\pi r_c^2T^0_{\ 0}. \tag{116}
$$

When r_c is large, we have $\mathcal{E}=2\pi M+O(r_c^{-1})$.

The same result can be derived in the Chern–Simons formulation. We assume that the boundary term on a finite cutoff surface has the same form as that at infinity

$$
\mathcal{L}_{\text{bdy}}=-\frac{1}{4}\text{Tr}(A_\theta A_\theta)-\frac{1}{4}\text{Tr}(\bar{A}_\theta\bar{A}_\theta). \tag{117}
$$

The boundary conditions consistent with the variational principle are

$$
A_\theta^3=\bar{A}_\theta^3=0,\quad A_\theta^+=z^{-1},\quad A_\theta^-=-z^{-1}. \tag{118}
$$

To obtain the boundary stress–energy tensor we need to insert back the zweibein. The zweibein on the cutoff surface are

$$E_c^+ = \frac{1}{\sqrt{2}}((1-\bar{b}^2z^2)d\theta + (1+\bar{b}^2z^2)dt),$$
$$E_c^- = \frac{1}{\sqrt{2}}((1-b^2z^2)d\theta - (1+b^2z^2)dt). \tag{119}$$

Therefore the on-shell boundary term should be interpreted as

$$\begin{aligned}\mathcal{L}_{\text{bdy}} &= -\frac{1}{4}\left(\frac{\sqrt{2}E_t^+ + z\bar{A}_t^{(0)+}}{\sqrt{2}E_\theta^+ + z\bar{A}_\theta^{(0)+}}\text{tr}(A_\theta^{(0)}A_\theta^{(0)}) - \frac{\sqrt{2}E_t^- - zA_t^{(0)-}}{\sqrt{2}E_\theta^- - zA_\theta^{(0)-}}\text{tr}(\bar{A}_\theta^{(0)}\bar{A}_\theta^{(0)})\right) \\ &= -\frac{1}{2}\left(\frac{\sqrt{2}E_t^+ - z^2\bar{b}^2}{\sqrt{2}E_\theta^+ + z^2\bar{b}^2}b^2 - \frac{\sqrt{2}E_t^- + z^2b^2}{\sqrt{2}E_\theta^- + z^2b^2}\bar{b}^2\right).\end{aligned} \tag{120}$$

Then we get the boundary stress–energy tensor,

$$\begin{aligned}T^i{}_j &= -\frac{k}{\pi}\frac{1}{\det E}E_j^A\frac{\partial\mathcal{L}_{\text{bdy}}}{\partial E_i^A}|_{E^\pm\to E_c^\pm} \\ &= \frac{k}{\pi}\frac{1}{2-2z^4b^2\bar{b}^2}\begin{pmatrix}2z^2\bar{b}^2b^2+b^2+\bar{b}^2 & b^2-\bar{b}^2 \\ \bar{b}^2-b^2 & 2z^2\bar{b}^2b^2-b^2-\bar{b}^2\end{pmatrix},\end{aligned} \tag{121}$$

which equals (110) up to a factor.

4 Conclusions and discussions

In this paper, we have studied the $T\bar{T}$ deformation of chiral bosons. In particular, the $T\bar{T}$ deformation of two chiral bosons of opposite chiralities is equivalent to that of a non-chiral free scalar theory at the Hamiltonian level. Furthermore, we have obtained the all-order $T\bar{T}$ deformed Lagrangian of more general theories which contain an arbitrary number of chiral bosons with potentials. Based on these results, we study the $T\bar{T}$ deformation of the boundary theory in Chern–Simons AdS_3 gravity which is a constrained chiral WZW model. We have derived the all-order $T\bar{T}$ deformed Lagrangian and computed the one-loop torus partition function of the deformed theory, which satisfies the flow equation of general $T\bar{T}$ torus partition function up to one-loop order. Our result suggests that the one-loop torus partition function is not one-loop exact under the $T\bar{T}$ deformation, which is unlike the situation in the undeformed theory [35]. Moreover, we have computed the stress–energy tensor of the solution associated with a BTZ black hole in the deformed theory, which matches the boundary stress–energy tensor of the BTZ black hole at a finite radial location on the bulk side.

Let us comment on future research directions. It would be interesting to start with the Chern–Simons theory describing the AdS_3 gravity with a finite cutoff to derive the $T\bar{T}$ deformed boundary action. This will help us to realize the holography under $T\bar{T}$ deformation more explicitly. Moreover, the original exact boundary action of Chern–Simons AdS_3 gravity can be applied to compute the four-point functions in the light-light and heavy-light limit [35]. Recently, many studies have been devoted to the correlators in general $T\bar{T}$ deformed CFTs [44–48]. It would be interesting to compute correlators in our deformed model and compare them with these results. It would also be interesting to generalize our analysis to higher spin theories of gravity formulated in terms of $SL(N,\mathbb{R})$ Chern–Simons theory [49].

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Appendix A: $J\bar{J}$ and $T\bar{J}$ deformation of two chiral bosons

In this Appendix, we consider the $J\bar{J}$ and $T\bar{J}$ deformation of the chiral bosons.

A.1 $J\bar{J}$ deformation

Consider the Lagrangian of two chiral bosons of opposite chiralities

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_t\phi\partial_\theta\phi - \partial_\theta\phi\partial_\theta\phi - \partial_t\bar{\phi}\partial_\theta\bar{\phi} - \partial_\theta\bar{\phi}\partial_\theta\bar{\phi}\right). \tag{122}$$

To define currents J and $\bar{J}$, we couple the chiral bosons to gauge fields

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}\left(\partial_t\phi\partial_\theta\phi - \partial_\theta\phi\partial_\theta\phi - (A_\theta - A_t)(2\partial_\theta\phi + A_\theta)\right) \\ &+ \frac{1}{2}\left(-\partial_t\bar{\phi}\partial_\theta\bar{\phi} - \partial_\theta\bar{\phi}\partial_\theta\bar{\phi} - (\bar{A}_\theta + \bar{A}_t)(2\partial_\theta\bar{\phi} + \bar{A}_\theta)\right).\end{aligned} \tag{123}$$

We define the currents as

$$J^i = \frac{\partial \mathcal{L}}{\partial A^i}, \quad \bar{J}^i = \frac{\partial \mathcal{L}}{\partial \bar{A}^i}. \tag{124}$$

In the undeformed theory

$$\partial_i J_0^i = \frac{1}{2}\partial_t A_\theta - \frac{1}{2}\partial_\theta A_t, \quad \partial_i \bar{J}_0^i = \frac{1}{2}\partial_\theta \bar{A}_t - \frac{1}{2}\partial_t \bar{A}_\theta. \tag{125}$$

When A and $\bar{A}$ are closed, J and $\bar{J}$ are conserved. The $J\bar{J}$ operator in the deformed theory is defined as

$$(J\bar{J})_\lambda = 2J_\lambda^t \bar{J}_\lambda^\theta - 2\bar{J}_\lambda^t J_\lambda^\theta. \tag{126}$$

Solving the flow equation

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = (J\bar{J})_\lambda. \tag{127}$$

We get

$$\begin{aligned}\mathcal{L}_\lambda = \mathcal{L}_0 &- \frac{4\lambda(\partial_\theta\phi^2\lambda + \partial_\theta\bar{\phi}^2\lambda + \partial_\theta\phi\partial_\theta\bar{\phi}(1+\lambda^2))}{(1-\lambda^2)^2} \\ &- \frac{\lambda}{\lambda^2-1}\left(A_t\partial_\theta\bar{\phi} - \bar{A}_t\partial_\theta\phi + \frac{A_t\bar{A}_\theta - A_\theta\bar{A}_t}{2}\right) \\ &- \frac{\lambda^2}{\lambda^2-1}\left(A_t\partial_\theta\phi - \bar{A}_t\partial_\theta\bar{\phi} + \frac{A_tA_\theta - \bar{A}_\theta\bar{A}_t}{2}\right) \\ &- \frac{2\lambda}{(\lambda^2-1)^2}A_\theta\bar{A}_\theta + \frac{\lambda^2\left(\lambda^2-3\right)}{2\left(\lambda^2-1\right)^2}(A_\theta^2 + \bar{A}_\theta^2) \\ &+ \frac{\lambda^2\left(\lambda^2-5\right)}{\left(\lambda^2-1\right)^2}(A_\theta\partial_\theta\phi + \bar{A}_\theta\partial_\theta\bar{\phi}) \\ &- \frac{\lambda\left(\lambda^2+3\right)}{\left(\lambda^2-1\right)^2}(\bar{A}_\theta\partial_\theta\phi + A_\theta\partial_\theta\bar{\phi}).\end{aligned} \tag{128}$$

Finally setting $A = \bar{A} = 0$, we get

$$\begin{aligned}\mathcal{L}_\lambda &= \mathcal{L}_0 - \frac{4\lambda(\partial_\theta\phi^2\lambda + \partial_\theta\bar{\phi}^2\lambda + \partial_\theta\phi\partial_\theta\bar{\phi}(1+\lambda^2))}{(1-\lambda^2)^2} \\ &= \mathcal{L}_0 - 4\lambda\partial_\theta\phi\partial_\theta\bar{\phi} - 4\lambda^2(\partial_\theta\phi^2 + \partial_\theta\bar{\phi}^2) + O(\lambda^3).\end{aligned} \tag{129}$$

A.2 $T\bar{J}$ deformation

We couple the left chiral boson to the zweibein and the left chiral boson to a gauge field

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{2}\left(-\partial_t\phi\partial_\theta\phi + \frac{E_t^+}{E_\theta^+}\partial_\theta\phi\partial_\theta\phi\right) \\ &+\frac{1}{2}\left(-\partial_t\bar{\phi}\partial_\theta\bar{\phi} - \partial_\theta\bar{\phi}\partial_\theta\bar{\phi} - (\bar{A}_\theta + \bar{A}_t)(2\partial_\theta\bar{\phi} + \bar{A}_\theta)\right).\end{aligned} \tag{130}$$

We define the currents as

$$T_+^i = \frac{\partial \mathcal{L}}{\partial E_i^+}, \quad \bar{J}^i = \frac{\partial \mathcal{L}}{\partial \bar{A}^i}. \tag{131}$$

The $T\bar{J}$ operator in the deformed theory is defined as

$$(T\bar{J})_\lambda = 2T_{+\lambda}^t \bar{J}_\lambda^\theta - 2\bar{J}_\lambda^t T_{+\lambda}^\theta. \tag{132}$$

Solving the flow equation

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = (T\bar{J})_\lambda. \tag{133}$$

We get

$$\begin{aligned}\mathcal{L}_\lambda = &-\frac{1}{2}\left(-\partial_t\phi\partial_\theta\phi + \partial_\theta\phi\partial_\theta\phi\right) \\ &+\frac{1}{2}\left(-\partial_t\bar{\phi}\partial_\theta\bar{\phi} - \partial_\theta\bar{\phi}\partial_\theta\bar{\phi} - (\bar{A}_\theta + \bar{A}_t)(2\partial_\theta\bar{\phi} + \bar{A}_\theta)\right) \\ &+\frac{E_\theta^+(2(E_\theta^+ + E_t^+) - \lambda(\bar{A}_\theta + \bar{A}_t))}{2\lambda^2} \\ &\times\left(\sqrt{1 + \frac{\lambda}{E_\theta^+}(\bar{A}_\theta + 2\partial_\theta\bar{\phi}) + \frac{\lambda^2}{E_\theta^{+2}}((\partial_\theta\bar{\phi} + \frac{\bar{A}_\theta}{2})^2) - \partial_\theta\phi^2} - 1 - \frac{\lambda}{E_\theta^+}(\partial_\theta\bar{\phi} + \frac{\bar{A}_\theta}{2})\right).\end{aligned} \tag{134}$$

Finally we set $E_\theta^+ - 1 = E_t^+ - 1 = \bar{A} = 0$ and obtain

$$\begin{aligned}\mathcal{L}_\lambda = &\frac{1}{2}\left(\partial_t\phi\partial_\theta\phi - \partial_\theta\phi\partial_\theta\phi - \partial_t\bar{\phi}\partial_\theta\bar{\phi} - \partial_\theta\bar{\phi}\partial_\theta\bar{\phi}\right) \\ &+\frac{2\left(\sqrt{\lambda^2\left(\partial_\theta\bar{\phi}^2 - \partial_\theta\phi^2\right) + 2\partial_\theta\bar{\phi}\lambda + 1} - \partial_\theta\bar{\phi}\lambda - 1\right)}{\lambda^2}.\end{aligned} \tag{135}$$

Appendix B: $T\bar{T}$ deformed chiral WZW model

We consider the sum of a left and a right chiral WZW model

$$S = S_-[g] + S_+[\bar{g}], \tag{136}$$

with

$$\begin{aligned}S_\pm[g] = &\frac{k}{2\pi}\left(\int d^2x\,\mathrm{Tr}\big((g^{-1})'\partial_\pm g\big)\right. \\ &\left.\mp\frac{1}{6}\int_B \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)\right),\end{aligned} \tag{137}$$

where g and $\bar{g}$ are group elements of group G and $\bar{G}$ respectively. We define

$$A_i = g^{-1}\partial_i g, \quad \bar{A}_i = \bar{g}^{-1}\partial_i \bar{g}. \tag{138}$$

The equations of motions are

$$\partial_- A_\theta = \partial_+ \bar{A}_\theta = 0. \tag{139}$$

Consider the $T\bar{T}$ deformed chiral WZW model

$$\frac{2\pi}{k}S_\lambda = \frac{1}{2}\int d^2x \left((\mathrm{tr}(A_\theta A_t) - \mathrm{tr}(\bar{A}_\theta \bar{A}_t) + \frac{1}{\lambda}(\mathcal{S}-1)\right) + \frac{1}{6}\int_B \Big(\mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) - \mathrm{tr}(\bar{g}^{-1}d\bar{g} \wedge \bar{g}^{-1}d\bar{g} \wedge \bar{g}^{-1}d\bar{g})\Big), \tag{140}$$

with

$$\mathcal{S} = \sqrt{1 - 2(\mathrm{tr}(A_\theta A_\theta) + \mathrm{tr}(\bar{A}_\theta \bar{A}_\theta))\lambda + (\mathrm{tr}(A_\theta A_\theta) - \mathrm{tr}(\bar{A}_\theta \bar{A}_\theta))^2\lambda^2}. \tag{141}$$

The equations of motions are

$$\partial_\theta \left(\frac{1-\lambda(\mathrm{tr}(A_\theta A_\theta) - \mathrm{tr}(\bar{A}_\theta \bar{A}_\theta))}{\mathcal{S}} A_\theta\right) - \partial_t A_\theta = 0, \tag{142}$$

$$\partial_\theta \left(\frac{1+\lambda(\mathrm{tr}(A_\theta A_\theta) - \mathrm{tr}(\bar{A}_\theta \bar{A}_\theta))}{\mathcal{S}} \bar{A}_\theta\right) + \partial_t \bar{A}_\theta = 0. \tag{143}$$

When $\lambda = 0$, a solution to the equation of motion is

$$g = h(t)g_0(x^+), \quad \bar{g} = \bar{h}(t)\bar{g}_0(x^-). \tag{144}$$

We introduce a new set of coordinate $(\tilde{t}, \tilde{\theta})$ and define a field dependent coordinate transformation with the Jacobian

$$\begin{pmatrix} \partial_{\tilde{t}}t & \partial_{\tilde{t}}\theta \\ \partial_{\tilde{\theta}}t & \partial_{\tilde{\theta}}\theta \end{pmatrix} = \begin{pmatrix} -\frac{F_{\tilde{\theta}\tilde{\theta}}\lambda}{2} - \frac{\bar{F}_{\tilde{\theta}\tilde{\theta}}\lambda}{2} + 1 & \frac{F_{\tilde{\theta}\tilde{\theta}}\lambda}{2} - \frac{\bar{F}_{\tilde{\theta}\tilde{\theta}}\lambda}{2} \\ \frac{F_{\tilde{\theta}\tilde{\theta}}\lambda}{2} - \frac{\bar{F}_{\tilde{\theta}\tilde{\theta}}\lambda}{2} & \frac{F_{\tilde{\theta}\tilde{\theta}}\lambda}{2} + \frac{\bar{F}_{\tilde{\theta}\tilde{\theta}}\lambda}{2} + 1 \end{pmatrix}, \tag{145}$$

where

$$F_{\tilde{\theta}\tilde{\theta}} = \mathrm{tr}(g_0^{-1}(\tilde{x}^+)\partial_{\tilde{\theta}}g_0(\tilde{x}^+)g_0^{-1}(\tilde{x}^+)\partial_{\tilde{\theta}}g_0(\tilde{x}^+)), \tag{146}$$

$$\bar{F}_{\tilde{\theta}\tilde{\theta}} = \mathrm{tr}(\bar{g}_0^{-1}(\tilde{x}^-)\partial_{\tilde{\theta}}\bar{g}_0(\tilde{x}^-)\bar{g}_0^{-1}(\tilde{x}^-)\partial_{\tilde{\theta}}\bar{g}_0(\tilde{x}^-)), \tag{147}$$

$$\tilde{x}^\pm = \tilde{\theta} \pm \tilde{t}. \tag{148}$$

Then the solution

$$g(t,\theta) = h(t)g_0(\tilde{x}^+(t,\theta)), \quad \bar{g}(t,\theta) = \bar{h}(t)\bar{g}_0(\tilde{x}^-(t,\theta)). \tag{149}$$

satisfies the equation of motion for the deformed theory. Using

$$\partial_\theta + \partial_t = 2\frac{\partial_{\tilde{x}^+} - \lambda F_{\tilde{\theta}\tilde{\theta}}\partial_{\tilde{x}^-}}{1-\lambda^2 F_{\tilde{\theta}\tilde{\theta}}\bar{F}_{\tilde{\theta}\tilde{\theta}}}, \tag{150}$$

$$\partial_\theta - \partial_t = 2\frac{\partial_{\tilde{x}^-} - \lambda \bar{F}_{\tilde{\theta}\tilde{\theta}}\partial_{\tilde{x}^+}}{1-\lambda^2 F_{\tilde{\theta}\tilde{\theta}}\bar{F}_{\tilde{\theta}\tilde{\theta}}}, \tag{151}$$

$$A_\theta = g_0^{-1}(\tilde{x}^+)\partial_{\tilde{\theta}}g_0(\tilde{x}^+)\frac{1-\lambda\bar{F}_{\tilde{\theta}\tilde{\theta}}}{1-\lambda^2 F_{\tilde{\theta}\tilde{\theta}}\bar{F}_{\tilde{\theta}\tilde{\theta}}}, \tag{152}$$

$$\bar{A}_\theta = \bar{g}_0^{-1}(\tilde{x}^-)\partial_{\tilde{\theta}}\bar{g}_0(\tilde{x}^-)\frac{1-\lambda F_{\tilde{\theta}\tilde{\theta}}}{1-\lambda^2 F_{\tilde{\theta}\tilde{\theta}}\bar{F}_{\tilde{\theta}\tilde{\theta}}}, \tag{153}$$

$$\frac{1-\lambda(\mathrm{tr}(A_\theta A_\theta) - \mathrm{tr}(\bar{A}_\theta \bar{A}_\theta))}{\mathcal{S}} A_\theta = g_0^{-1}(\tilde{x}^+)\partial_{\tilde{\theta}}g_0(\tilde{x}^+)\frac{1+\lambda\bar{F}_{\tilde{\theta}\tilde{\theta}}}{1-\lambda^2 F_{\tilde{\theta}\tilde{\theta}}\bar{F}_{\tilde{\theta}\tilde{\theta}}}, \tag{154}$$

$$\frac{1-\lambda(\mathrm{tr}(A_\theta A_\theta) - \mathrm{tr}(\bar{A}_\theta \bar{A}_\theta))}{\mathcal{S}} A_\theta = \bar{g}_0^{-1}(\tilde{x}^-)\partial_{\tilde{\theta}}\bar{g}_0(\tilde{x}^-)\frac{1+\lambda F_{\tilde{\theta}\tilde{\theta}}}{1-\lambda^2 F_{\tilde{\theta}\tilde{\theta}}\bar{F}_{\tilde{\theta}\tilde{\theta}}}, \tag{155}$$

one can check that (142) and (143) are satisfied. One should also consider boundary condition so in general g_0 and $\bar{g}_0$ should depend on λ.

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