



Junction conditions and local spacetimes in general relativity

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Abstract In the present work, a theoretical framework focussing on local geometric deformations is introduced in order to cope with the problem of how to join spacetimes with different geometries and physical properties. This framework is used to show that two Lorentzian manifolds can be matched by considering local deformations of the associated spacetime metrics. Based on the fact that metrics can be suitably matched in this way, it is shown that the underlying geometric approach allows the characterization of local spacetimes in general relativity. Furthermore, it is shown that said approach not only extends the conventional thin shell formalism, but also allows the treatment of geometric problems that cannot be treated with standard gluing techniques.

Introduction

The general theory of relativity, like most of its countless generalizations, is a nonlinear theory of gravity. For this reason, it allows the existence of different types of solutions of Einstein's field equations and thus the coexistence and co-evolution of different types of gravitational fields.

However, the fact that the theory allows for a variety of different solutions, among which many lead to predictions that are in complete agreement with observation, leads to the challenging mathematical problem of how pairs of geometrically distinct spacetimes can be joined with each other along a hypersurface separating the corresponding Lorentzian manifolds and thus combined to form a single geometric field. This very problem is usually addressed in general relativity by considering local junction conditions, the fulfilment of which ensures that the intrinsic geometric properties of spacetime at the boundary are such that the local geometries fit together and can therefore be joined with each other. In this respect, two cases have to be distinguished:

On the one hand, there is the case in which the boundary hypersurface is either space- or timelike. In this case, the well-established Darmois–Israel formalism can be used, which requires the first and second fundamental forms of given spacetimes to match with each other across the space- or timelike boundary hypersurface [1, 2]. This matching must be given in such a way that the first fundamental forms of the corresponding geometric fields are continuous and coincide at the boundary, whereas the second fundamental forms do not have to coincide and are allowed to be discontinuous across the hypersurface separating the Lorentzian manifolds [2–4]. The condition for this to be the case, however, is the existence of a concentrated, singular matter distribution – a so-called thin shell of matter – that happens to form a joint boundary layer for both spacetimes.

On the the hand, there is the case in which the boundary is lightlike [4, 5], which is more sophisticated from a geometrical point of view. In this case, different types of junction conditions have to be considered, which allow the pairwise identification of (projections of) gradients of the corresponding null vector fields across the boundary. The method used then makes it possible to glue together spacetimes that are separated by a lightlike boundary hyperface. However, this is not the only advantage offered by that approach: As it turns out, the geometric framework used is more general than the Darmois–Israel framework, since it allows one to combine the null and non-null formalism into a single formalism called general thin shell formalism, and also to formulate associated junction conditions that are always valid regardless of the causal structure of the boundary hyperface [4, 6, 7].

The main problem that arises in this context, however, is that it often proves difficult to actually meet said junction conditions, especially for spacetimes with different causal structures and symmetry properties. For this reason, it is a rather common case that spacetime pairs with excessively different geometrical properties cannot be glued together. This is, moreover, also one of the main reasons why the above-mentioned conditions have so far been successfully applied

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mainly to spacetimes with a comparatively high degree of symmetry, i.e. to spacetimes with either spherical, cylindrical or plane symmetry, while the treatment of the problem of how to glue less symmetric spacetimes, such as stationary, axisymmetric or even non-stationary ones, has so far received far less attention in the literature.

A primary cause of this shortcoming is the fact that the methods being used to cope with the junction conditions, in spite of leading to appropriate local discontinuities in the curvature of respective gravitational fields (at most ‘delta-like’-singularities), in different cases, fail to deliver physically feasible predictions. The usual reason for this drawback is the fact that the corresponding methods lead to concentrated, singular gravitational source terms that often do not obey the energy conditions of the theory, which significantly reduces their phenomenological and physical relevance. An even greater problem occurs, moreover, when the two spacetimes to be joined have metrics of low regularity; a case that may lead to undefined products of distributions when calculating the associated curvature fields, such as in the case of gravitational shock wave spacetimes, where said fields contain ‘squares’ of the delta distribution. In this case, the thin shell formalism suffers from severe mathematical problems.

In conclusion, however, there appears to be a need for a more reliable geometric framework that does not lead to unphysical gravitational source terms and, in contrast to traditional spacetime gluing approaches, also allows the gluing of distributional metrics that contain terms proportional to Dirac’s delta distribution.

In response to that fact, the aim of a major part of this work is to provide a simple geometric framework that endeavors to avoid the aforementioned technical and conceptual difficulties, while at the same time guaranteeing that the junction conditions of the theory are met.

The key difference between the model to be developed and former approaches to the subject is the fact that said model considers the transition of one spacetime to another as a dynamical deformation process. This idea is formally realized by deforming the metric of a given background geometry and showing that the effect of the deformation completely subsides if appropriate boundary conditions, which follow from the addressed junction conditions, are imposed.

Taking advantage of the fact that the geometric structure of any spacetime metric can be arbitrarily modified by specifying a suitable deformation term, and that it is also possible to confine oneself only to those deformations that have compact supports in an embedded subregion of a given Lorentzian manifold (or go to zero in a suitable limit), the geometric framework to be presented ensures that the junction conditions of the theory are met, thereby allowing a rigorous characterization of local spacetime geometries in general relativity.

This is demonstrated in Sect. 3 of this work, where it is shown that the standard thin shell formalism (which focuses on the gluing of spacetime metrics that are at least C^2) can be extended by using the metric deformation formalism in combination with Colombeau’s theory of generalized functions to allow the gluing of spacetime metrics with low regularity. In particular, it is shown that spacetime metrics containing a Dirac delta distribution term can be glued together in a mathematically rigorous way, whereas it turns out that certain types of metric deformations are more suitable for such an endeavor from a physical point of view than others. In the process, it is shown that thin shell formalism appears as a special case of the geometric framework presented. This is clarified using concrete geometric examples, that is, gravitational shock wave spacetimes, whose curvature cannot be calculated using standard gluing techniques. To further improve the spacetime gluing approach, it is shown that deformation formalism – based on the use of smooth transition functions – allows the smooth gluing of arbitrary spacetimes and thus the treatment of cases that cannot be treated with the standard technical machinery of the general thin shell formalism.

1 Junction conditions and gluings of spacetimes

In Einstein’s General Theory of Relativity, the situation quite often occurs that two spacetime partitions (\mathcal{M}^\pm, g^\pm) with two associated Lorentzian manifolds $\mathcal{M}^\pm = M^\pm \cup \partial M^\pm$ are given, which are bounded by a hypersurface Σ that forms a part of the boundary of both spacetimes, so that $\Sigma \subset \partial M^\pm$ applies. Given this situation, the question arises as to whether or not both spacetimes can be ‘combined’ into an ambient spacetime (\mathcal{M}, g) , whose manifold is the union of the manifolds of the individual parts such that $\mathcal{M} \equiv \mathcal{M}^- \cup \mathcal{M}^+$.

A relatively straightforward method that allows one to deal with this question and thus solve the underlying geometric problem is the method of gluing spacetimes together across a boundary hypersurface $\Sigma \equiv \partial M^+ \cap \partial M^-$, using the so-called thin shell formalism [2–9].

According to this method, which has a rich history and important applications in general relativity, it is usually assumed that an ambient spacetime (\mathcal{M}, g) with above-mentioned properties is given, i.e. a spacetime with Lorentzian manifold $M = M^+ \cup \Sigma \cup M^-$ and metric g_{ab} , which reduces to the metrics g_{ab}^\pm in M^\pm . As a basis for this, it is further assumed that there is a restricted C^2 -metric $g_{ab}^+ = g_{ab}|_{M^+}$ associated with the part (\mathcal{M}^+, g^+) and another C^2 -metric $g_{ab}^- = g_{ab}|_{M^-}$ associated with (\mathcal{M}^-, g^-) , respectively; parts, in relation to which the metric g_{ab} of the ambient spacetime (\mathcal{M}, g) can be decomposed in the form

$$g_{ab} = \theta g_{ab}^+ + (1 - \theta) g_{ab}^-, \quad (1)$$

where θ is the Heavyside step function. This step function is usually assumed to take a value of one half for points lying on Σ , a value of one for points lying in M^+ and a value of zero for points lying in M^- . This makes sense as long as it is ensured that the ambient metric is continuous across the layer, which implies that (in appropriate coordinates) it must apply that

$$[g_{ab}] = 0, \tag{2}$$

where $[g_{ab}] \equiv \lim_{x \rightarrow x_0}^{M^+} g_{ab}^+(x) - \lim_{x \rightarrow x_0}^{M^-} g_{ab}^-(x)$ applies for all $x_0 \in \Sigma$.

To provide a coordinate independent description in this context, it is only natural to use a formalism that is compatible with the intrinsic geometric structure of the boundary portion Σ . However, since such a description cannot be independent of the causal structure of Σ , it seems convenient to first closely pursue the most essential ideas of the so-called general (or mixed) thin shell formalism developed in [4], not least because said formalism, unlike previous approaches to the subject, allows the treatment of the current physical problem of joining pairs of spacetimes with different geometries without having to fix the geometric character of the boundary portion Σ . Rather, Σ can very well be null somewhere in spacetime and non-null elsewhere.

To enable such treatment of the problem, the formalism takes advantage of the fact that a pair of normal vector fields ζ_\pm^a exists on each side of the layer Σ such that ζ_+^a corresponds to M^+ and ζ_-^a corresponds to M^- . In addition, the fact is exploited that – regardless of the causal structure of the boundary portion – bases of vector fields $\{E_\rho^a\}$ can be chosen in $T(\Sigma)$ with $\rho = 1, 2, 3$ as well as associated co-bases $\{e_\rho^a\}$ in $T^*(\Sigma)$ such that $e_\rho^a E_\sigma^a = \delta_\rho^\sigma$, $\zeta_\pm^a e_\rho^a = 0$ and $g_{ab}^\pm|_\Sigma E_\rho^a E_\sigma^a = g_{ab}^\mp|_\Sigma E_\rho^a E_\sigma^a$. Furthermore, it is observed that there is a pair of vector fields ξ_\pm^a , usually called rigging vector fields, and an associated pair of co-vector fields ξ_a^\pm such that $\xi_\pm^a \zeta_\pm^a = -1$ and $\xi_a^\pm E_\rho^a = 0$. The corresponding rigging vector fields are fixed in this context by demanding $g_{ab}|_\Sigma \xi_+^a E_\rho^b = g_{ab}|_\Sigma \xi_-^a E_\rho^b \equiv g_{ab}|_\Sigma \xi^a E_\rho^b$ and $g_{ab}|_\Sigma \xi_+^a \xi_+^a = g_{ab}|_\Sigma \xi_-^a \xi_-^a$, so that the two bases on the tangent spaces $\{\xi_\pm^a, E_\rho^a\} \equiv \{\xi^a, E_\rho^a\}$ are identified and the (\pm) can be dropped. The two one-forms ζ_a^\pm are automatically identified as well, so that $\{\zeta_a^\pm, e_\rho^a\} \equiv \{\zeta_a, e_\rho^a\}$. Consequently, it then turns out to be possible to construct a projector of the type $o_a^c = \delta_a^c + \zeta^c \xi_a$ with the properties $o_a^c o_b^c = o_b^c$ and $o_a^c \xi^a = o_a^c \zeta_c = 0$, which can be used as a projector onto Σ .

With these definitions at hand, the difference (or jump) of any object from the + or the – sides of Σ can be specified. In particular, any (m, n) -tensor field with definite limits on Σ from M^\pm (regardless of whether it is discontinuous across Σ or not) can be split up in a +-part and a --part, so that

$$T_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_m} = \theta T_{b_1 b_2 \dots b_n}^{+a_1 a_2 \dots a_m} + (1 - \theta) T_{b_1 b_2 \dots b_n}^{-a_1 a_2 \dots a_m}. \tag{3}$$

The covariant derivative of the same object then reads

$$\nabla_c T_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_m} = \theta \nabla_c^+ T_{b_1 b_2 \dots b_n}^{+a_1 a_2 \dots a_m} + (1 - \theta) \nabla_c^- T_{b_1 b_2 \dots b_n}^{-a_1 a_2 \dots a_m} + \delta_c [T_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_m}], \tag{4}$$

where $[T_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_m}] \equiv \lim_{x \rightarrow x_0}^{M^+} T_{b_1 b_2 \dots b_n}^{+a_1 a_2 \dots a_m}(x) - \lim_{x \rightarrow x_0}^{M^-} T_{b_1 b_2 \dots b_n}^{-a_1 a_2 \dots a_m}(x)$ applies for all $x_0 \in \Sigma$ and $\delta_c \equiv \zeta_c \delta$ is a vector-valued distribution constructed from Dirac’s delta distribution $\delta = \delta(x)$.

As already mentioned above, given a suitable pair of coordinate charts $(x_\pm^0, x_\pm^1, x_\pm^2, x_\pm^3)$, the metric is continuous across Σ . However, its derivatives, and thus the corresponding connections, are discontinuous. In fact, it is found in this context that

$$[\partial_c g_{ab}] = 2 \cdot \zeta_c \gamma_{ab} \tag{5}$$

and therefore

$$[\Gamma_{bc}^a] = \gamma_b^a \zeta_c + \gamma_c^a \zeta_b - \gamma_{bc} \zeta^a, \tag{6}$$

where γ_{ab} is a symmetric tensor field defining the properties of the shell.

The associated Riemann tensor is of the form

$$R_{bcd}^a = \theta R_{bcd}^{+a} + (1 - \theta) R_{bcd}^{-a} + \delta H_{bcd}^a, \tag{7}$$

where δ is the Dirac delta distribution and H_{bcd}^a represents the singular part of the curvature tensor distribution, which is explicitly given by $H_{bcd}^a = \frac{1}{2}(\gamma_d^a \zeta_b \zeta_c - \gamma_c^a \zeta_b \zeta_d + \gamma_{bc} \zeta^a \zeta_d - \gamma_{bd} \zeta^a \zeta_c)$.

Given this definition, the said approach allows for a generalized formulation of Einstein’s field equations in a distributional sense, which leads to a distributional Einstein tensor of the form

$$G_b^a = \theta G_b^{+a} + (1 - \theta) G_b^{-a} + \delta \cdot \rho_b^a, \tag{8}$$

where $\rho_b^a = H_b^a - \frac{1}{2} \delta_b^a H$ (with $H \equiv g^{ab}|_\Sigma H_{ab}$) is a symmetric covariant tensor field defined only at points of the hypersurface Σ . The associated stress-energy tensor, on the other hand, has to possess the form

$$T_b^a = \theta T_b^{+a} + (1 - \theta) T_b^{-a} + \delta \cdot \tau_b^a, \tag{9}$$

which is why, of course, it is required that $G_b^{\pm a} = 8\pi T_b^{\pm a}$ and $\rho_b^a = 8\pi \tau_b^a$.

On the basis of these relations (and some others which will not be specifically relevant for this work), the underlying thin-shell formalism can be used to join different partitions of spacetime, including those where parts of Σ are either null or non-null, that is, either null or spacelike or timelike.

In the latter cases, where portions of the thin shell are allowed to be non-null, the above general formalism traces back to the Darmois–Israel method, which is based on a $3 + 1$ -decomposition of spacetime. The geometric setting used in this method is essentially the same as that of the general formalism mentioned above, with the only exception being that it is additionally required a priori that the ambient spacetime (\mathcal{M}, g) admits a foliation in either spacelike or timelike hypersurfaces and a boundary portion Σ (with fixed causal structure) that can be embedded in said foliation of spacetime. This additional restriction of the geometry of spacetime allows the consideration of a congruence of curves generated by a normalized vector field n^a that is necessarily orthogonal to Σ and therefore fulfills $n_a n^a = \epsilon$ and $\nabla_{[c} n_{a]} = 0$ with $\epsilon = \pm 1$. This normal vector field (and its associated co-normal) can then be used to define the first and second fundamental forms $h_{ab} = g_{ab} + \epsilon n_a n_b$ and $K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)}$.

In order to find the corresponding shell equations and to guarantee that the spacetime partitions (\mathcal{M}^\pm, g^\pm) can be ‘combined’ to the ambient spacetime (\mathcal{M}, g) in the given case, it must be ensured that the said pair of spacetimes exhibits spacelike or timelike foliations compatible to that of the ambient spacetime (\mathcal{M}, g) . Essentially, this means that pairs of either timelike or spacelike generating vector fields n_\pm^a with the properties $n_\pm^a n_\pm^a = \epsilon$ and $\nabla_{[c} n_{a]}^\pm = 0$ must exist, which can be appropriately identified across the shell in a manner similar to the general formalism. By definition, these vector fields have to be orthogonal to the respective first and second fundamental forms $h_{ab}^\pm = g_{ab}^\pm + \epsilon n_a^\pm n_b^\pm$ and $K_{ab}^\pm = h_a^c h_b^d \nabla_{(c} n_{d)}^\pm$. In addition, it needs to be assumed that the three-metrics of the spacetime partitions are, at least, continuous across Σ , so that $[h_{ab}] = 0$ is valid; although $[K_{ab}] = 0$ does not necessarily have to apply in this context. The corresponding shell equations then yield conditions for matching the Cauchy data (h_{ab}^\pm, K_{ab}^\pm) of the bounded spacetimes (\mathcal{M}^\pm, g^\pm) in such a way across Σ that they are consistent with the Cauchy data (h_{ab}, K_{ab}) of the ambient spacetime (\mathcal{M}, g) . These shell equations result directly from the general formalism discussed above if one sets $\xi_a \equiv \epsilon n_a$, $\zeta^a \equiv n^a$, $o_a^c \equiv h_a^c = \delta_a^c + \epsilon n^c n_a$ and $\gamma_{ab} = \epsilon [K_{ab}]$ in relations (5)–(9), thus proving the fact that said formalism actually contains the Darmois–Israel framework as a special case.

Accordingly, in order to avoid the existence of ill-defined singular contributions to the field equations, it must be required that the pairs of first and the second fundamental forms associated with pairs of spacetimes (\mathcal{M}^\pm, g^\pm) satisfy the junction conditions

$$[h_{ab}] = 0 \tag{10}$$

and

$$[K_{ab}] = 8\pi \epsilon \left(\tau_{ab} - \frac{1}{2} h_{ab} \tau \right); \tag{11}$$

conditions which represent, in a quite generic way, geometrically necessary requirements for the identification of the corresponding Lorentzian manifolds.

It is worth noting that even though the given approach is formulated in a coordinate-independent manner, it still leads back to alternative formulations of junction conditions, for example those given by Lichnerowicz or O’Brian and Sygne in case of the special choice of so-called admissible coordinates on both sides of the layer [2].

Furthermore, there is also the case that Σ is locally null; a case that seems to require a more deliberate approach than the traditional non-null description of the problem not least due to the fact that the first fundamental form is degenerate on a null hypersurface and the associated null normal is not only orthogonal but also tangential to it. Nevertheless, assuming the existence of two pairs of null congruences generated by a pair of lightlike vector fields l^a and k^a that are orthogonal to a spacelike two-slice $S \subset \Sigma$, it turns out that the general formalism is versatile enough to include said special case as well; giving rise to the same shell equations previously found in [5]. These equations can be obtained from expressions (5)–(9) if the choice $\xi_a \equiv l_a$, $\zeta^a \equiv k^a$ and $o_a^c = \delta_a^c + k^c l_a$ is made in this context and a covariant symmetric two-form $\mathcal{H}_{ab} = 2o_a^c o_b^d \nabla_{(c} l_{d)}$ is specified, which (in the continuous null limit of the Darmois–Israel framework) has the same jump discontinuity features as the extrinsic curvature across Σ . More precisely, while in the non-null case generically one has $\gamma_{bc} \equiv \epsilon [K_{bc}]$, in the lightlike case, where there is a different geometric setting, one has $\gamma_{ab} \equiv [\mathcal{H}_{ab}]$. With regard to this specific quantity, which shall be called Mars-Senovilla two-form from now on, the above series of junction relations turns into

$$[\mathcal{H}_{ab}] = 0 \tag{12}$$

and therefore becomes, in contrast to the non-null case in which the right hand side is non-vanishing, a trivial set of relations; relations that guarantee that the Einstein tensor of the geometry contains no singular part proportional to Dirac’s delta distribution (and thus no surface layer).

However, as shown in [4], these junction relations can actually be relaxed in the given null case by requiring that the respective Mars-Senovilla two-forms meet the conditions

$$[\mathcal{H}_{ab}] k^b = [\mathcal{H}] = 0, \tag{13}$$

which also guarantee that the singular part of the curvature tensor distribution vanishes identically.

Anyhow, junction conditions do not necessarily have to be based on a 3 + 1-decomposition of spacetime; they also have been formulated in the 2 + 2-framework toward general relativity or in space-time approaches that are based on a 1 + 1 + 2-decomposition of spacetime.

In particular, as shown by Penrose in [10], junction conditions can be formulated (in a coordinate dependent manner) which are based on a dual null foliation of spacetime in space-like surfaces. These conditions form the basis of Penrose’s now infamous cut-and-paste method, which provides the formal basis for the description of gravitational shock wave spacetimes in general relativity and turns out to be closely related to the thin shell framework in specific applications.

Besides that, in order to characterize boundary portions that possess a ‘corner’ or a ‘sharp edge’, another set of junction relations has been formulated in the literature in the past [11]. For the purpose of formulating said conditions, a timelike generating vector field n^a and a spacelike one u^a associated with respective timelike and spacelike congruences have been considered, which yield spacelike and timelike foliations and thus a 1 + 1 + 2-decomposition of spacetime. Based on the existence of said foliations, the fact was exploited that the spacetime metric g_{ab} can be decomposed in the form $g_{ab} = -n_a n_b + h_{ab} = u_a u_b + \gamma_{ab}$ with $h_{ab} = q_{ab} + s_a s_b$ and $\gamma_{ab} = q_{ab} - v_a v_b$, respectively, where s^a and v^a are spacelike and timelike unit normals orthogonal to n^a and u^a and q_{ab} is the induced Riemannian metric on a spacelike two-slice $S \subset \Sigma$. Due to the fact that the given vector fields n^a and u^a are not assumed to be normalized with respect to each other in this context, it then typically turns out that one has to deal (in the case of a spacelike joint) with a non-vanishing edge ‘angle’ $\Theta = \cos^{-1}((n, u))$ in such approaches. This non-vanishing quantity has been shown to lead to jump discontinuities and therefore to an additional set of junction conditions given by

$$[\Theta]q_{ab} = \mathcal{T}_{ab}, \tag{14}$$

where \mathcal{T}_{ab} is the stress-energy tensor restricted to S . This particular set of conditions completes the list of junction conditions discussed in this work. In the following, however, only conditions (10) and (11) or conditions (12) and (13) will prove to be really relevant, since the validity of these conditions will be used as a basis for a geometric extension of the general thin shell formalism.

The reason why such a geometric extension proves useful (or even necessary) is the following: When using the thin shell formalism one must expect that the boundaries of the spacetimes to be glued together can be singular hypersurfaces. But that means that the curvature along these hypersurfaces can grow infinitely, which is problematic as long as no plausible physical reason for the occurrence of such infinities is given, such as possibly the occurrence of a relativistic shock

wave at the boundary of the spacetimes or something similar. Besides that, the general thin shell formalism suffers from the problem that the singular parts of the energy-momentum tensor occurring in (9) may fail to obey relevant energy conditions and, what is even worse, it turns out that said conditions cannot even be formulated in all cases of relevance, such as, in particular, in the case of the dominant energy condition. The main reason for this drawback is that in order to set up the dominant energy condition, one would have to deal ill-defined products of distributions, that is, with ‘squares’ of the delta distribution.

The exact same problem occurs if the metrics g_{ab}^{\pm} considered in (1) are not C^2 -metrics, but have lower regularity. The situation becomes particularly alarming if one of the metrics of the two spacetime partitions $(\mathcal{M}^{\pm}, g^{\pm})$ contains a part that is proportional to the Dirac delta distribution. To illustrate this, the specific case shall be considered in which $g_{ab}^+ = g_{ab}^0 + \delta e_{ab}$ and $g_{ab}^- = g_{ab}^0$, where g_{ab}^0 is some smooth C^2 -metric and e_{ab} and is a smooth tensor field. In this case, taking advantage of the fact $\theta \delta \approx A \delta$, where A is a constant and \approx means association in the sense of distributions [12], splitting (1) yields $g_{ab} \approx g_{ab}^0 + A \delta e_{ab}$ and Eqs. (7)–(9) become (similar as in the case of the generic energy condition) relations that contain ‘squares’ of the delta distribution. Consequently, in this particular case not even the field equations of the theory can be defined meaningfully.

In response to these deficiencies of the general thin shell formalism, the remainder of the present work will address the problem of joining different spacetimes from a slightly different angle, namely by means of a geometric approach based on the use of metric deformations. This approach generalizes the thin shell formalism in such a way that in important special cases the treatment of the above mentioned problems becomes possible in full accordance with the junction conditions of the theory. To make this possible, the following two steps are taken: First, the junction conditions of the general thin shell formalism are reformulated in the language of the geometric deformation approach, and second, the support properties of the corresponding deformation fields are restricted so that all types of junction conditions discussed in this section are fulfilled. In this way, as will be explained, the concept of local spacetime geometry is introduced in relation to a fixed ambient geometry of spacetime.

2 Local geometries and deformations of spacetime

In order to approach now the subject of joining spacetimes from a different angle, namely by using special metric deformations that allow one to meet the junction conditions discussed in the previous section, two different spacetime partitions $(\mathcal{M}^{\pm}, g^{\pm})$ of an ambient spacetime (\mathcal{M}, g) shall once more be considered. These partitions, as before, shall be

assumed to be bounded by a hypersurface Σ which forms a part of the boundary of both spacetimes such that $\Sigma \subset \partial M^\pm$ applies.

Without any further assumptions about the geometric structures of both of the spacetimes (\mathcal{M}^\pm, g^\pm) , both partitions are allowed to exhibit totally different geometric properties anywhere except for their boundary, where, by the introduced junction conditions, both spacetimes have to possess identical induced geometries. As a necessary prerequisite for obtaining a spacetime (\mathcal{M}, g) with connected manifold $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, it must therefore be ensured that said spacetime partitions can be identified along the boundary portion Σ in such a way that the junction conditions discussed in the previous section are met.

In order to ensure this, the same approach as in the previous section shall be followed, namely different bases $\{\xi_\pm^a, E_\rho^a\}$ and co-bases $\{\zeta_a^\pm, e_a^\rho\}$ shall be considered at each side of the boundary, which can be identified along Σ in the same way as in Sect. 1 of this work.

Against this background, the main task now is to construct a spacetime (\mathcal{M}, g) with connected Lorentzian manifold $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, with local spacetime partitions (\mathcal{M}^\pm, g^\pm) that may have different geometric structures everywhere except along the boundary hyperface Σ .

To face this task, the following observation proves useful: A change of one spacetime geometry with respect to another can be characterized by considering a deformation of the associated metrics, where deformation in this context means any backreaction that changes the geometric properties of a given spacetime metric with respect to a given background metric and a given class or group of deformation fields propagating on the corresponding background spacetime.

To make this final statement precise, let g_{ab} be a fundamental metric field associated with the ambient spacetime (\mathcal{M}, g) with the manifold structure $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$. Considering the tensor deformations

$$g_{ab} = g_{ab}^\pm + e_{ab}^\pm \tag{15}$$

and

$$g^{ab} = g^{\pm ab} + f^{\pm ab}, \tag{16}$$

one can define the non-vanishing tensor fields $g_{ab}^\pm = g_{ab} - e_{ab}^\pm$ and $g^{\pm ab} = g^{ab} - f^{\pm ab}$, which shall be required to be at least C^1 , although they may actually turn out to be C_0^∞ (locally) in various cases of interest. These tensor fields a priori do not represent a metric and an inverse metric, respectively. Instead, the objects in question are special tensor fields whose properties depend on the choice of the deformation tensor fields e_{ab}^\pm and $f^{\pm ab}$.

Using these deformation tensor fields, one can re-write various geometric relations involving the metric g_{ab} of

(\mathcal{M}, g) and its inverse g^{ab} . Specifically, provided that the tensor fields $e_a^{\pm b} := e_{ac}^\pm g^{\pm cb} = e_{ac}^\pm (g^{cb} - f^{\pm cb})$ and $f_a^{\pm b} := g_{ac}^\pm f^{\pm cb} = (g_{ab} - e_{ab}^\pm) f^{\pm cb}$ are identified as tensor fields on (\mathcal{M}, g) , the relation

$$g_{ab} g^{bc} = \delta_a^c, \tag{17}$$

can be brought into the form

$$e_a^{\pm b} + f_a^{\pm b} + e_a^{\pm c} f_c^{\pm b} = 0. \tag{18}$$

In addition, one can define the difference tensors $C_{bc}^{\pm a} = \frac{1}{2}(g^{\pm ad} + f^{\pm ad})(\nabla_b^\pm e_{dc}^\pm + \nabla_c^\pm e_{bd}^\pm - \nabla_d^\pm e_{bc}^\pm)$ in relation to the unique Levi-Civita connection defined on (\mathcal{M}, g) , which then allows one to decompose the Riemann tensor of the geometry in the following form

$$R_{bcd}^a = R_{bcd}^{\pm a} + E_{bcd}^{\pm a}, \tag{19}$$

where $E_{bcd}^{\pm a} = 2\nabla_{[c}^\pm C_{d]b}^{\pm a} + 2C_{e[c}^{\pm a} C_{d]b}^{\pm e}$ shall apply by definition. By contracting indices, one finds that the Ricci tensor takes the form

$$R_{bd} = R_{bd}^\pm + E_{bd}^\pm, \tag{20}$$

where $E_{bd}^\pm = E_{bad}^{\pm a} = E_{bd}^\pm = 2\nabla_{[a}^\pm C_{d]b}^{\pm a} + 2C_{e[a}^{\pm a} C_{d]b}^{\pm e}$ follows from the foregoing definition. By repeating that procedure, the decomposition of the Ricci scalar

$$R = R^\pm + g^{\pm bd} E_{bd}^\pm + f^{\pm bd} R_{bd}^\pm + f^{\pm bd} E_{bd}^\pm \tag{21}$$

can also be obtained. However, as a direct consequence, Einstein's equations

$$G_{ab} = 8\pi T_{ab} \tag{22}$$

can be re-written in the form

$$G_{ab}^\pm + \rho_{ab}^\pm = 8\pi T_{ab}, \tag{23}$$

provided that $\rho_{ab}^\pm = \psi_{ab}^\pm - \frac{1}{2}g_{ab}^\pm (f^{\pm cd} R_{cd}^\pm + f^{\pm cd} E_{cd}^\pm) - \frac{1}{2}e_{ab}^\pm (R^\pm + f^{\pm cd} R_{cd}^\pm + g^{\pm cd} E_{cd}^\pm + f^{\pm cd} E_{cd}^\pm)$ with $\psi_{ab}^\pm = E_{ab}^\pm - \frac{1}{2}g_{ab}^\pm (g^{\pm cd} E_{cd}^\pm)$ holds in the given context.

Having obtained these relations, the following observation can be made: By requiring that the deformation tensor fields e_{ab}^\pm and $f^{\pm ab}$ defined above vanish somewhere in local subregions of the manifold \mathcal{M} , the tensor fields g_{ab}^\pm coincide locally with the metric g_{ab} of the spacetime (\mathcal{M}, g) . Therefore, the following can be concluded: as long as the tensor fields e_{ab}^\pm and $f^{\pm ab}$ are defined in such a way that they vanish globally in $\mathcal{M}^\pm \subseteq \mathcal{M}$, which is certainly the case if the components of said fields are C_0^∞ functions or distributions with compact supports lying in the complements $\mathcal{M}_C^\pm \equiv \mathcal{M} \setminus \mathcal{M}^\pm$

of the Lorentzian manifolds \mathcal{M}^\pm , the tensor fields g_{ab}^\pm represent well-defined continuous (possibly even smooth) metric fields, but only within the local regions \mathcal{M}^\pm . 'Outside' these regions, however, they are just specific tensor fields, so that it can be concluded that the pairs (\mathcal{M}^\pm, g^\pm) define pairs of *local spacetimes*, i.e. pairs of spacetimes whose metrics g_{ab}^\pm represent well-defined tensor fields on (\mathcal{M}, g) , which coincide locally with the 'correct' metric of spacetime, which is g_{ab} .¹

Probably the simplest way to construct deformation fields e_{ab}^\pm and $f^{\pm ab}$ with the required properties is to make an ansatz of the form $e_{ab}^\pm = \chi_{\mathcal{M}_C^\pm} e_{ab}^\pm$, where the $\chi_{\mathcal{M}_C^\pm}$ are indicator functions (also called characteristic functions) with compact support in $\mathcal{M}_C^\pm \equiv \mathcal{M} \setminus \mathcal{M}^\pm$ and e_{ab}^\pm are continuous, at least twice differentiable tensor fields. Since the Heaviside step function is a special indicator function, which is at the same time a generalized function, one can always choose said functions and the corresponding deformation fields in such a way that the distributional splitting (1) results as a special case of the given construction.

However, as it turns out, there is no need in general to require that $\chi_{\mathcal{M}_C^\pm}$ are indicator functions. Rather, it suffices to choose said functions as smooth transition functions (or a sequence of such functions), which provide a smooth transition from zero to one in the unit interval $[0, 1]$. A transition (alias cut-off) function with these particular properties can be obtained by considering the non-analytic smooth function

$$\psi(x) := \begin{cases} e^{-\frac{1}{x}} & x > 0, \\ 0 & x \leq 0, \end{cases} \tag{24}$$

which meets the conditions $0 \leq \psi \leq 1$ and $\psi(x) > 0$ if and only if $x > 0$. This function can be used to define the transition function

$$\chi(x) = \frac{\psi(\frac{x_0}{x})}{\psi(\frac{x_0}{x}) - \psi(1 - \frac{x_0}{x})}, \tag{25}$$

which contains a constant x_0 that ensures that the exponent in (24) is dimensionless and therefore takes a value of zero for $x < 0$, a value of one for $x \geq x_0$ and is strictly increasing in the interval $[0, 1]$. This can then be used to give the further

¹ In this context, it is important to note that neither e_{ab}^\pm nor $f^{\pm ab}$ are assumed to be small and that both pairs of objects define a whole class of tensor deformations which in principle can become arbitrarily large, so that for a given vector field w^a , which is causal with regard to one of the local metrics g_{ab}^\pm , there is no need for it to be causal with regard to the other local metric g_{ab}^\mp or the ambient metric g_{ab} in the complement $\mathcal{M} \setminus \mathcal{M}^\pm$ of \mathcal{M}^\pm . However, in the opposite case, the deformations may also become arbitrarily small, so that they become the subject of relativistic perturbation theory.

definition

$$1 - \chi(x) = \frac{\psi(1 - \frac{x_0}{x})}{\psi(\frac{x_0}{x}) - \psi(1 - \frac{x_0}{x})}, \tag{26}$$

which has the same properties as $\chi(x)$, but is strictly decreasing.

By using one of these transition functions instead of the Heaviside step function, a smooth analogon of the distributional splitting (1) can then be obtained. It is worth noting that similar approaches can, of course, be given by considering bump functions or other smooth functions with similar support properties, which may be constructed from convolutions of smooth functions with mollifiers.

Since in all these approaches, the full Einstein equations (22) reduce to the restricted local Einstein equations $G_{ab}^\pm = 8\pi T_{ab}^\pm$ on (\mathcal{M}^\pm, g^\pm) , it becomes clear that the remaining equations

$$\rho_{ab}^\pm = 8\pi \tau_{ab}^\pm, \tag{27}$$

can be determined independently in agreement with the introduced junction conditions, where, of course, $\tau_{ab}^\pm := T_{ab} - T_{ab}^\pm$ applies in the given context.

However, it must be stressed that there is a price to be paid in this context: By introducing transition functions of the form (25) and (26), the manifold structure is no longer $\mathcal{M} = \mathcal{M}^- \cup \mathcal{M}^+$, but rather $\mathcal{M} = \mathcal{M}^- \cup \mathcal{O} \cup \mathcal{M}^+$, where \mathcal{O} is some transition region in which $\chi(x)$ continuously increases until it reaches a value of one. Consequently, by considering transition function of the above form in order to make sure that there is a smooth geometric transition between the pairs of local spacetimes (\mathcal{M}^\pm, g^\pm) , one ends up in a situation where has to deal with three spacetime partitions (\mathcal{M}^\pm, g^\pm) and (\mathcal{O}, g) . This implies, however, that one suddenly has to deal with a completely new geometric setting, which is slightly different from the one usually considered by general thin-shell formalism.

As a direct consequence, however, the question arises which junction conditions need to be fulfilled at the boundaries $\Sigma_\pm = \partial\mathcal{M}_\pm \cap \partial\mathcal{O}$. Beyond that, generally speaking, the question arises of how the junction conditions of general thin shell formalism can be formulated to describe the first case mentioned above and under which circumstances these conditions can be fulfilled in the given setting.

To face these questions, one may take a closer look at conditions (10)–(13) of the previous section. Sure enough, these conditions, if they were to be fulfilled, will lead to constraints on the deformation fields e_{ab}^\pm and $f^{\pm ab}$. More specifically, considering the case in which the manifold structure is $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, said conditions lead to the requirement that the fields e_{ab}^\pm and $f^{\pm ab}$ match exactly at Σ and have compact

support in $\mathcal{M} \setminus \{\mathcal{M}^\pm \setminus \Sigma\}$, thereby ensuring that the spacetime partitions (\mathcal{M}^\pm, g^\pm) can pointwise be joined along Σ .

To see this, one may consider that in the given setting, the junction conditions (10) and (11) of the Darmois–Israel framework require

$$[e_{ab}] = 0 \tag{28}$$

and

$$n_a h_b^e h_c^f [C_{ef}^a] = 8\pi \epsilon \left(\tau_{bc} - \frac{1}{2} h_{cb} \tau \right) \tag{29}$$

to hold in a suitable coordinate chart. In the lightlike case, on the other hand, junction condition (12) requires

$$l_a o_b^e o_c^f [C_{ef}^a] = 0 \tag{30}$$

to be fulfilled. Consequently, however, it can be concluded that the spacetime partitions (\mathcal{M}^\pm, g^\pm) can always be smoothly joined – regardless of the causal structure of the boundary hypersurface – if

$$[e_{ab}] = [f^{ab}] = 0, [C_{bc}^a] = 0 \tag{31}$$

applies in a suitable coordinate chart. In the smooth case mentioned, however, the case may very well occur that (30) is valid instead of $[C_{bc}^a] = 0$ if Σ is null, or that the right side of (29) is zero if Σ is not null, since the condition $[C_{bc}^a] = 0$ can only be met in special cases. Moreover, since not all spacetimes can be smoothly joined, it may be required in cases where it is not possible to fulfill condition (31) that

$$[e_{ab}] = [f^{ab}] = 0, [C_{bc}^a] \neq 0 \tag{32}$$

applies and conditions (29) or (30) are met as well. However, these are exactly the conditions of the thin shell formalism simply transferred to the given geometric setting; conditions that are known to produce reasonable results when the metrics to be glued are C^2 -metrics.

The reformulation of these conditions in the context of the geometrical deformation approach developed in this section can be vindicated by the fact that said approach – in combination with Colombeau’s theory of generalized functions [13, 14] - allows an extension of the ‘classic’ thin shell formalism. In particular, as shall be substantiated by concrete examples in the next section, it becomes possible to glue spacetime metrics that differ by deformation terms that are proportional to Dirac’s delta distribution, but are nevertheless of such a form that the condition (32) and the conditions (29) or (30) can still be fulfilled. The reason for this is that the geometric deformation approach provides direct information on certain problematic terms and expressions that require careful treatment or, in other words, need to be studied in more

detail using Colombeau’s theory. In this way, the approach enables the treatment of problems that would go beyond the usual scope of the formalism due to the low regularity of the spacetime metrics to be glued. Concrete examples, however, will only be given later – in the next section of this work.

Anyway, the situation is completely different when the metrics of the spacetime partitions (\mathcal{M}^\pm, g^\pm) are not glued together directly, but rather joined via using smooth transition functions of the form (25) or (26). In this particular case, the manifold structure is $\mathcal{M} = \mathcal{M}^- \cup \mathcal{O} \cup \mathcal{M}^+$, where \mathcal{O} is a transition region with boundary hypersurfaces $\Sigma_\pm = \partial\mathcal{O} \cup \partial\mathcal{M}^\pm$, and condition (31) takes the form

$$e_{ab}^\pm = f^{\pm ab} = 0, C_{ef}^{\pm a} = 0. \tag{33}$$

As may be noticed, this condition is fulfilled on Σ_\pm due to the fact that the deformation fields e_{ab}^\pm and $f^{\pm ab}$ have been chosen to be local tensor fields with the property that all their components possess compact supports in $\mathcal{M} \setminus \mathcal{M}^\pm$. To be more precise, based on the fact that e.g. the choice $e_{ab}^\pm = \chi^\pm e_{ab}^\pm$ can always be made in the given context, where χ^\pm are smooth transition functions of the form (25) in which the constant x_0 is replaced by constants x_\pm and e_{ab}^\pm are continuous, at least twice differentiable tensor fields, it is clear from the very outset that condition (33) and therefore either condition (29) or (30) are met as well on Σ_\pm .

The same line of argument can be used to handle a variety of smooth geometric transitions, i.e. to include partitions $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{O}_{1,2} \cup \mathcal{M}_2 \cup \dots \mathcal{M}_{n-1} \cup \mathcal{O}_{n-1,n} \cup \mathcal{M}_n$ of the ambient manifold \mathcal{M} , where the $\mathcal{O}_{k,k+1}$ are transition regions connecting the four-dimensional Lorentzian manifolds \mathcal{M}_k and \mathcal{M}_{k+1} with $k = 1, 2, \dots, n - 1$. This can be achieved by considering the deformation relations

$$g_{ab} = g_{ab}^1 + e_{ab}^1 = g_{ab}^2 + e_{ab}^2 = \dots = g_{ab}^n + e_{ab}^n \tag{34}$$

and

$$g^{ab} = g_1^{ab} + f_1^{ab} = g_2^{ab} + f_2^{ab} = \dots = g_n^{ab} + f_n^{ab}, \tag{35}$$

which are given with respect to associated sequences of deformation fields $e_{ab}^1, e_{ab}^2, \dots, e_{ab}^n$ and $f_1^{ab}, f_2^{ab}, \dots, f_n^{ab}$ that are chosen in such a way that $e_{ab}^{(k)} = \chi_k e_{ab}^{(k)}$ for $k = 1, 2, \dots, n$, where each χ_k is a transition function of the form (25) with x_0 replaced by x_k and $e_{ab}^{(k)}$ is a continuous, at least twice differentiable tensor field. The corresponding deformation fields must be given such that they obey consistency relation (18) and also

$$e_{ab}^{(k)} = f_{(k)}^{ab} = 0, C_{ef}^{(k)a} = 0. \tag{36}$$

By requiring this, however, it becomes clear that the field equations of the theory will have a completely different form

locally than globally. Therefore, from a local point of view, the structure of the said equations will change over time, just like that of the Einstein–Hilbert action. This proves to be relevant for the action principle of the theory.

In the case that the ambient spacetime (\mathcal{M}, g) exhibits a boundary ∂M without edges or corners, this action is given by

$$S[g] = \int_M R\omega_g + \int_{\Sigma'} K\omega_h, \tag{37}$$

where Σ and Σ' are spacelike hypersurfaces and $\omega_g \equiv \sqrt{-g}d^4x$ is the four-volume element and $\omega_h \equiv \sqrt{hd^3x}$ is the three-volume element of spacetime. If the same boundary ∂M of the ambient spacetime, on the other hand, does indeed contain a sharp edge or corner, its action alternatively can be specified by Hayward’s action [11, 15]

$$S[g] = \int_M R\omega_g + \int_{\Sigma} K\omega_h + \int_B \tilde{K}\omega_\gamma + \int_{\Omega'} \sinh^{-1} \eta\omega_q, \tag{38}$$

where $\omega_h \equiv \sqrt{hd^3x}$, $\omega_\gamma \equiv \sqrt{-\gamma}dt d^2x$ and $\omega_q \equiv \sqrt{q}d^2x$ are volume forms associated with the individual parts of the boundary ∂M of (\mathcal{M}, g) , which consists of two spacelike hypersurfaces Σ and Σ' and a timelike hypersurface B , which intersects the spacelike hypersurfaces Σ and Σ' in Ω and Ω' . Here, the quantities K and \tilde{K} are extrinsic curvature scalars and $\eta := n_a u^a$ is a generally non-zero scalar parameter originating from the fact that the boundary normals n^a and u^a are usually non-orthogonal in the given case.

Regardless of whether one or the other type of action is considered in this context, it may happen that, in the course of a geometric transition, the structure of the metric does not change momentarily over time, whereas that of the Einstein equations does very well. In such a case, the local metric and the ambient metric do coincide, but neither the corresponding field equations nor the corresponding action functionals (37) or (38) do so as well.

Considering the simplest case of two local spacetimes (\mathcal{M}^\pm, g^\pm) , the changes in the field equations can be described by Eqs. (23) and (27). The reason why these changes have to be taken into account here are the following: The tensor fields e_{ab}^\pm may be vanishing in $\mathcal{M} \setminus \mathcal{M}^\pm$, so that it may happen that $e_{ab}^\pm \rightarrow 0$ due to the fact that $e_{ab}^\pm \rightarrow 0$ in $\mathcal{M} \setminus \mathcal{M}^\pm$. Therefore, it must be expected that $C_{ef}^{\pm a} \rightarrow 0$ applies in the event that $\chi^\pm \rightarrow 0$, but not in the event that $e_{ab}^\pm \rightarrow 0$, in which case $C_{ef}^{\pm a} \neq 0$ rather applies in general. Therefore, it may occur that the local metric and the ambient metric coincide, but not the corresponding curvature fields.

As a result, the structures of the action and the field equations of the theory may change, but those of the local metrics

may not. In the case that an ambient spacetime (\mathcal{M}, g) with Lorentzian manifold $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{O}_{1,2} \cup \mathcal{M}_2 \cup \dots \cup \mathcal{M}_{n-1} \cup \mathcal{O}_{n-1,n} \cup \mathcal{M}_n$ is given, which exhibits a boundary ∂M without edges or corners, the change of the field equations in the course of the k -th geometric transition can straightforwardly be determined to be

$$G_{ab}^{(k)} + \rho_{ab}^{(k)} = 8\pi T_{ab}, \tag{39}$$

where $\rho_{ab}^{(k)} = \psi_{ab}^{(k)} - \frac{1}{2}g_{ab}^{(k)}(f^{(k)cd}R_{cd}^{(k)} + f^{(k)cd}E_{cd}^{(k)}) - \frac{1}{2}e_{ab}^{(k)}(R^{(k)} + f^{(k)cd}R_{cd}^{(k)} + g^{(k)cd}E_{cd}^{(k)} + f^{(k)cd}E_{cd}^{(k)})$ with $\psi_{ab}^{(k)} = E_{ab}^{(k)} - \frac{1}{2}g_{ab}^{(k)}(g^{(k)cd}E_{cd}^{(k)})$ applies in the given context. Consequently, using the definition $\tau_{ab}^{(k)} := T_{ab} - T_{ab}^{(k)}$, one obtains the deformed field equations

$$\rho_{ab}^{(k)} = 8\pi \tau_{ab}^{(k)}. \tag{40}$$

Also the change of the Einstein–Hilbert action can be determined step by step in such a case. Assuming for this purpose that $\mathbf{1} + e^i$ is the matrix representation of the object $\delta_b^a + e_b^{(i)a}$ for the i -th spacetime partition (\mathcal{M}_i, g^i) , the relation $|X| = e^{\ln|X|} = e^{tr \ln X} = 1 + \sum_{m=1}^{\infty} \frac{(tr \ln X)^m}{m!}$ between the determinant and the trace of a matrix X can be set up in order to obtain the identity $\sqrt{-g} = (1 + \varphi^i)\sqrt{-g^i}$, where $\varphi^i = \sum_{m=1}^{\infty} \frac{(tr \ln(\mathbf{1} + e^{(i)}))^m}{2^m m!}$ applies by definition. Moreover, using the fact that one can always decompose the lapse function N of the ambient spacetime with respect to the lapse function N^i of the i -th local background such that $N = N^i + e_{00}^{(i)}$, the result obtained implies that $\sqrt{h} = (1 + \varphi^i) \frac{N^i}{N^i + e_{00}^{(i)}} \sqrt{h^i} = (1 + \varphi^i) \frac{1}{1 + \frac{e_{00}^{(i)}}{N^i}} \sqrt{h^i} =: (1 + \varphi^i)(1 + \psi^i)\sqrt{h^i}$, where $\psi^i = \sum_{n=1}^{\infty} \left(-\frac{e_{00}^{(i)}}{N^i}\right)^n$ holds by definition. In addition, the extrinsic curvature tensor K_{ab} of (\mathcal{M}, g) can be decomposed with respect to the i -th extrinsic curvature tensor K_{ab}^i of the i -th local spacetime (\mathcal{M}_i, g^i) in the form $K_{ab} = K_{ab}^i + \kappa_{bd}^i$.

Consequently, in the case of a boundary ∂M without edges or corners, the corresponding action decomposes according to the rule

$$S[g] \equiv S[g^1] + \Sigma[g^1, e^1, f^1] = \dots = S[g^n] + \Sigma[g^n, e^n, f^n] \\ = \frac{1}{n} \left\{ \sum_{i=1}^n S[g^i] + \sum_{i=1}^n \Sigma[g^i, e^i, f^i] \right\}, \tag{41}$$

where

$$S[g^i] = \int_{M_i} R_i \omega_{g^i} + \int_{\Sigma_i'} K_i \omega_{h^i} \tag{42}$$

and

$$\begin{aligned}
 & \Sigma[g^i, e^i, f^i] \\
 &= \int_{M_i} \chi^i R_i \omega_{g^i} + \int_{\Sigma_i} (\varphi^i + \psi^i + \varphi^i \psi^i) K_i \omega_{h^i} \\
 &+ \int_{M_i} (1 + \varphi^i) (g_i^{bd} E_{bd}^i + f_i^{bd} R_{bd}^i + f_i^{bd} E_{bd}^i) \omega_{g^i} \\
 &+ \int_{\Sigma_i} (1 + \varphi^i)(1 + \psi^i)(g_i^{bd} \kappa_{bd}^i + f_i^{bd} K_{bd}^i + f_i^{bd} \kappa_{bd}^i) \omega_{h^i} \\
 &+ \int_{M \setminus M_i} (1 + \varphi^i) (R_i + g_i^{bd} E_{bd}^i + f_i^{bd} R_{bd}^i + f_i^{bd} E_{bd}^i) \omega_{g^i} \\
 &+ \int_{\Sigma \setminus \Sigma_i} (1 + \varphi^i)(1 + \psi^i)(K_i + g_i^{bd} \kappa_{bd}^i + f_i^{bd} K_{bd}^i + f_i^{bd} \kappa_{bd}^i) \omega_{h^i}
 \end{aligned} \tag{43}$$

applies for the i -th part of the action. Thus, it can be seen that the action of the ambient spacetime (\mathcal{M}, g) can be decomposed into a system of ‘subactions’ $S[g^i] + \Sigma[g^i, e^i, f^i]$, whose variation with respect to g_i^{ab} possibly leads to modifications of the ‘standard’ field equations obtained from a variation of $S[g^i]$ with respect to g_i^{ab} .

A similar, but slightly more complicated decomposition relation is also obtained in the case of a boundary ∂M with edges or corners, which is consistent not least due to the fact that Hayward’s action has been shown to be additive in a generalized sense [16]. The associated formalism therefore allows one to add up the Einstein–Hilbert actions of spacetimes with non-smooth boundaries and different topologies and causal structures.

However, it is important to note in this context that all modifications to the Einstein’s field equations and the Einstein–Hilbert action need to be consistent with the geometric structure of the ambient spacetime (\mathcal{M}, g) . This point marks an important difference to alternative multi-metric theories of gravity treated in literature for which a priori no such correspondence is required [17–20].

This proves to be a very important point in that deformations of spacetime metrics do not always have to lead to physically meaningful results. Consequently, it is important to ensure that the respective fields are chosen in a meaningful way. In order to ensure this and to treat models of physical interest, it generally proves to be useful to consider only deformation fields, which allow the fulfillment of suitable energy conditions [21] on (\mathcal{M}, g) . By requiring this, it is ensured that the resulting confined stress-energy tensor is

well-defined from a physical point of view. Moreover, it is ensured that the same conditions locally hold on (\mathcal{M}^\pm, g^\pm) .

Anyway, after this has now been clarified, it remains to be discussed what advantages working with the deformation approach presented in this section has over working with the thin shell formalism presented in the previous section.

One of the main advantages of working with the deformation approach is more general and versatile than the thin shell formalism and other closely related approaches to the subject, such as, in particular, Penrose’s cut-and-paste method. This is not least because it allows the metrics and curvature fields of pairs of local spacetimes to be smoothly deformed into each other via introducing smooth transition functions instead of step functions. In this context, the main advantage compared to the thin shell formalism is that the geometric deformation approach cannot fail in the sense that, in principle, a smooth geometric transition always exists for arbitrary spacetime pairs. In contrast, the gluing of arbitrary spacetime pairs is not always possible.

In addition, as shall be explained in greater detail in the next section on the basis of concrete geometric examples, the deformation approach allows for a more careful handling of the subject in the sense that it allows the treatment of problems where the thin shell method is expected to lead to distributionally ill-defined terms, which cannot be properly treated from a mathematical point of view. It turns out, however, that the thin shell formalism can be extended using Colombeau’s theory of generalized functions in order to enable a mathematically rigorous treatment of the problematic terms mentioned, whereas all extensions of the formalism mentioned prove to be completely consistent with the geometric deformation approach.

Furthermore, in contrast to the thin shell formalism, the geometric deformation approach allows new solutions of the field equations to be constructed using transformations that leave the geometric character of the background metric unchanged, but lead to a new ambient spacetime or classes of ambient spacetimes.

Last but not least the deformation formalism includes the perturbative approach to general relativity as a special case. Not least for this reason, it allows one to weaken conditions (28)–(33), which are designed to meet the previously discussed junction conditions in various cases, in a perturbative sense, so that they are no longer exact, but only approximately valid, i. e. up to higher orders in a fixed parameter or systems of parameters.

All this will be explained in the next section using concrete geometric models. For the sake of simplicity, however, only simple models will be discussed, which can be obtained by specifying a suitable deformation of a (usually highly symmetric) background geometry.

3 Geometric deformations, thin shells and distributional metrics of spacetime

In the previous section, it was argued that the geometric deformation framework is more general than the thin shell formalism and, moreover, can be used (in combination with Colombeau’s theory of generalized functions) to extend said formalism to such an extent that the gluing of spacetime metrics with low regularity becomes possible. Specifically, it was stressed that spacetime metrics with components containing delta functions can be glued together using the geometric deformation framework. This shall now be demonstrated by some concrete geometric examples, where – on the basis of the metric deformation formalism discussed in the previous section – it will be made clear that certain classes of distributional spacetimes are better suited to be glued together than others. In due course, it will be made clear that the thin shell formalism not only yields the exact same results as the deformation approach, but rather emerges as a special case from this approach. Furthermore, it will be made clear that deformation formalism, in contrast to thin shell formalism, allows the gluing of arbitrary pairs of local spacetimes by using suitable transition functions.

To deal with the points mentioned step by step, different classes of spacetimes with deformed metrics shall be considered. The very first of these classes will be the so-called generalized Gordon class [22–24]; a class of spacetimes $(\bar{\mathcal{M}}, \bar{g})$ with metrics of the type

$$\bar{g}_{ab} = g_{ab} + f n_a n_b. \tag{44}$$

This special class of spacetimes can be obtained directly by deforming the metric g_{ab} of a given a background spacetime (\mathcal{M}, g) , using only two different objects: Some function f , whose form is either known in advance or must be determined by solving the field equations of the theory, and a smooth non-vanishing co-vector field $n_a = g_{ab} n^b$, which shall be assumed to be timelike in relation to the background metric g_{ab} , i.e. $g_{ab} n^b n^b < 0$. For the sake of simplicity, said vector field shall even be assumed to be normalized with respect to the background metric, so that $g_{ab} n^a n^b = -1$ applies in the present context.

A well known example of a spacetime metric, which lies in the resulting generalized Gordon class of metrics, is the so-called acoustic metric, which plays an important role in describing deflections of light or sound in bodies with different optical densities or acoustic properties in both special and general relativity. There are, however, also other important representatives of this class, many of which lie in a closely related class of metrics, the so-called generalized conformal Gordon class, which is a subclass of the generalized Gordon

class with metrics of the form

$$\bar{g}_{ab} = \Omega^2 (g_{ab} + f n_a n_b). \tag{45}$$

Important representatives of this subclass have been studied within the theory of so-called acoustic black holes and in the context of analogue gravity; theories that aim to explain, among other things, the geometric structure of acoustic black holes as well as electromagnetic phenomena in linear media and the behavior of condensed matter models in general relativity or even more general theories of gravity [24–28].

Given this special class of metrics and associated spacetimes, one may now ask the question of how two spacetimes of this class can be joined with each other. To address this question, one may consider a thin shell splitting and therefore make the specific choice $f = \theta f_+ + (1 - \theta) f_-$ for the function f in (44), where for the time being it shall be assumed that f_{\pm} are C^2 -functions. In addition, it may be required that $[f] = 0$ holds on a spacelike hypersurface Σ in spacetime. The resulting form of metric can then straightforwardly be brought into a form of type (1) by adding and subtracting a term term of the form θg_{ab} , provided that the definitions $g_{ab}^{\pm} \equiv g_{ab} + f_{\pm} n_a n_b$ are used in the present context. But this makes clear that by this special splitting of the function f , a splitting of the metric in the sense of the thin shell formalism results. In a similar way, however, a decomposition of the metric in the sense of the geometric deformation approach, i.e. a splitting of the form (15), can be obtained. To obtain such a decomposition of the metric, one may simply add and subtract the term $f_{\pm} n_a n_b$ in (44), which yields $\bar{g}_{ab} = g_{ab}^{\pm} + e_{ab}^{\pm}$, where the definitions $e_{ab}^+ \equiv (1 - \theta)(f_- - f_+) n_a n_b$ and $e_{ab}^- \equiv \theta(f_+ - f_-) n_a n_b$ are used. Consequently, as can be seen, both decompositions are equal, so that it becomes clear – due to the fact that it is known that the thin-shell formalism for C^2 -metrics yields mathematically meaningful results – that also the deformation approach must yield mathematically meaningful results in the given case.

More precisely, in view of the fact that condition (18) can readily be satisfied in a distributional sense by making an ansatz of the form $f^{+ab} \equiv (1 - \theta) \frac{f_+ - f_-}{1 + f_+ - f_-} n^a n^b$ and $f^{-ab} \equiv \theta \frac{f_- - f_+}{1 + f_- - f_+} n^a n^b$, it can be seen that conditions (28) and (29) on Σ are automatically fulfilled due to the that $[f] = 0$ must hold on that very hypersurface. To additionally meet the stricter conditions listed in (31), it must be additionally required that $[\partial_a f] = 0$ applies in the given context. However, due to the fact that there is a great number of suitable choices for the functions f_{\pm} , namely all, in relation to which either $[f] = 0$ or $[f] = 0$ and $[\partial_a f] = 0$ applies. Concretely, the functions f_{\pm} may be selected as C^∞ -functions with compact supports in M_{\mp} , so that it becomes possible to construct local spacetimes in the sense of Sect. 2

of this work. As can be seen, the fulfillment of the junction conditions of the theory does not cause any problems in this context.

A problem that arises on the other hand is that it can happen that a stress-energy tensor of the form (9) does not allow the fulfillment of relevant energy conditions of the theory or in special cases (as in the case of the dominant energy condition) does not even allow a mathematically meaningful formulation of the mentioned conditions at all. But apart from this particular drawback, the formalism produces mathematically and physically meaningful results.

However, the situation changes drastically if the requirement that $f_{\pm}(x)$ are (at least) C^2 -functions is dropped, and it becomes particularly problematic when the choice $f_{\pm} = f_{\pm}^0 \delta$ is made, where $f_{\pm}^0(x)$ are C^2 -functions and $\delta(x)$ is the Dirac delta distribution. In this particular case, the association relation $\theta \delta \approx A \delta$, previously considered in Sect. 1 of this work, can be used to show that relation (44) takes the form $\bar{g}_{ab} = g_{ab} + f_0 \delta n_a n_b$; at least provided that the definition $f_0 = A f_+^0 + (1 - A) f_-^0$ is used in the present context. The problem, which then arises in this context, is the following: Given this special distributional form of the metric, one is confronted, as already indicated in Sect. 1, with the serious problem that the curvature of spacetime and all associated quantities cannot simply be calculated by considering products of the delta distribution. The reason for this is that such an approach would lead to undefined 'squares' of the Dirac delta distribution, which are ill-defined from a mathematical point of view. For this reason, the thin shell formalism cannot lead to meaningful results in this particular case.

However, there is a feasible way to deal with this issue, which is to use Colombeau's theory of algebras of generalized functions. Given a paracompact manifold X , the center of attention of this theory is the so-called Colombeau algebra $\mathcal{G}(X)$ (or rather an entire system of Colombeau algebras, as there are many of such algebras), which is a commutative, associative and unital differential algebra of manifold-valued generalized functions. As such, it is an algebra consisting of one-parameter families of C^∞ -functions $(f_\varepsilon(x))_{\varepsilon \in (0,1]}$, which have to meet certain growth conditions in the so-called regularization parameter ε . To be more precise, $\mathcal{G}(X)$ results from forming the quotient algebra $\mathcal{E}_m(X)/\mathcal{N}(X)$ of the algebra of nets of moderate functions $\mathcal{E}_m(X) = \{(f_\varepsilon)_\varepsilon \in C^\infty(X)^{(0,1]} : \forall K \subset\subset X \forall P \in \mathcal{P}(M) \exists l \sup_{x \in K} |P f_\varepsilon(x)| = O(\varepsilon^{-l})\}$ by the ideal of nets of so-called negligible functions $\mathcal{N}(X) = \{(f_\varepsilon)_\varepsilon \in C^\infty(X)^{(0,1]} : \forall K \subset\subset X \forall m \forall P \in \mathcal{P}(M) \sup_{x \in K} |P f_\varepsilon(x)| = O(\varepsilon^m)\}$, where, in this context, $\mathcal{P}(X)$ denotes the space of all linear differential operators on the manifold X . It may be noted that $\mathcal{G}(X)$ contains the vector space of Schwartz distributions as a linear subspace, and the space of smooth functions as a faithful subalgebra.

Working with Colombeau algebras of generalized functions has a decisive advantage over working directly with distributions: It allows one to perform nonlinear operations on generalized functions, which result in well-defined expressions coinciding with distributions in the limit $\varepsilon \rightarrow 0$. Therefore, by considering Colombeau algebras, it becomes possible to treat mathematical problems that cannot be treated in the standard theory of Schwartz distributions. In particular, it becomes possible to perform nonlinear operations on a so-called strict delta net $(\delta_\varepsilon)_{\varepsilon \in (0,1]} \in C^\infty(\bar{M})^{(0,1]}$, which converges to the delta distribution in the limit $\varepsilon \rightarrow 0$ and thus allows for a regularization of the delta distribution [12–14]. This offers the possibility to work with a delta sequence δ_ε instead of directly with the delta distribution and thus calculate undefined products of the delta distribution in a mathematically rigorous way. However, the situation is subtle: Different regularizations of the delta distribution can lead to different results, which may lead to different physical interpretations of the subject. To ensure that a physically meaningful result is obtained in the end, one must therefore be careful when selecting a preferred regularization by hand, as can be illustrated very well by the example of the distributional Gordon metric discussed above.

To illustrate this, it is sufficient to consider a simple geometric example. As a basis for considering such a simple example, the simplifying assumption shall be made that a covariantly constant timelike normal vector field n^a exists on the background spacetime (\mathcal{M}, g) , i.e. a vector field with the properties $\nabla_a n^b = 0$ and $g_{ab} n^a n^b = -1$. Using this vector field to set up (in somewhat sloppy notation) a relation of the form

$$\bar{g}_{ab}^\varepsilon = g_{ab} + f_0 \delta_\varepsilon \cdot n_a n_b, \tag{46}$$

where f_0 is a continuous, at least twice differentiable function and $(\delta_\varepsilon)_\varepsilon$ is a delta sequence that coincides with the delta distribution in the limit $\varepsilon \rightarrow 0$. To treat a particularly easy example, it shall be assumed that the delta sequence δ_ε and the function f_0 are specified in such a way that $\nabla_a \delta_\varepsilon = -n_a \dot{\delta}_\varepsilon$ and $\nabla_a f_0 = -n_a \dot{f}_0$ applies. From (46) then follows directly

$$\bar{g}_{ab} = \lim_{\varepsilon \rightarrow 0} \bar{g}_{ab}^\varepsilon = g_{ab} + f_0 \delta \cdot n_a n_b. \tag{47}$$

Given this form of the metric, the inverse metric can be obtained by making an ansatz of the form

$$\bar{g}_\varepsilon^{ab} = g^{ab} + h_0 \delta_\varepsilon \cdot n^a n^b, \tag{48}$$

where h_0 is some function that has to be determined with respect to f_0 via solving

$$\lim_{\varepsilon \rightarrow 0} \bar{g}_{ab}^\varepsilon \bar{g}_\varepsilon^{bc} = \delta_a^c. \tag{49}$$

Relation (48) then yields an inverse metric of the type

$$\bar{g}^{ab} = \lim_{\epsilon \rightarrow 0} \bar{g}_\epsilon^{ab} = g^{ab} + h_0 \delta \cdot n_a n_b, \tag{50}$$

whereas it turns out that the value of h_0 will depend on the choice of regularization of the delta distribution. For the possibility of selecting a particular regularization by hand leads in this context to ambiguities or, to put it more precisely, to different solutions for relations (49) and (50). For example, there are regularizations of the delta distribution, which lead to the result $\delta^2 \approx 0$, while there are also other types of regularizations, which instead lead to the result $\delta^2 \approx c\delta$ [12,29–32]. Both of these types of regularizations can be used to handle the nonlinear operations on distributions,² which are necessary to solve relation (49), where both relations are equally correct from a purely mathematical point of view. This is why, from a physics standpoint, the question arises as to which of the two choices of regularization is the most reasonable and whether there are other useful regularizations that allow to solve the problem at hand or not. Ultimately, from the point of view of Colombeau’s theory of generalized functions, there is no silver bullet to solve relations like (49) and to derive from them, by means of solving (50), the form of the inverse metric \bar{g}^{ab} ; in order get then in the position to determine the form of the Levi–Civita connection $\bar{\Gamma}_{bc}^a$ and the field equations of the theory, which, in the given context, take the form

$$\bar{G}_{ab} \approx 8\pi \bar{T}_{ab}, \tag{51}$$

or equivalently

$$\bar{R}_{ab} \approx 8\pi \left(\bar{T}_{ab} - \frac{1}{2} \bar{g}_{ab} \bar{T} \right), \tag{52}$$

where the LHS of (52) is given by $\bar{R}_{ab} = R_{ab} + E_{ab}$ with a deformed Ricci tensor distribution of the form $E_{ab} = 2\nabla_{[c} C_{b]a}^c + 2C_{d[c}^c C_{b]a}^d$.

However, as may be realized, making an optimal choice of delta regularization is also important from a mathematical point of view, since such a choice plays an essential role in the solution of the generalized field equations of the theory for the given type of metric deformation. For in order to find a solution of the mentioned equations, different problematic distributional products must be determined, i.e. different powers of the delta distribution as well as different powers of its derivative. However, the calculation of such products is, if at all, only possible for a suitable choice of regularization

of the delta distribution. Regardless of the concrete choice to be made, however, one finds in the given case that

$$C_{d[c}^c C_{b]a}^d \approx 0 \tag{53}$$

is valid, which implies that

$$E_{ab} \approx 2\nabla_{[c} C_{b]a}^c. \tag{54}$$

Unfortunately, despite the validity of this identity, it generally proves to be difficult to solve the remaining part of the field equations even when using Colombeau’s framework of generalized functions. The problem here is not only that it is difficult to find solutions of (51) and (52), respectively, but that it must be expected that there are several solutions for the problem under consideration, whereby it is unclear which one is the most interesting and most suitable from a physical point of view. As shown in the rear part of this section, there are, however, also cases in which the above mentioned regularizations allow one to join pairs of distributional generalized Gordon spacetimes in a mathematically rigorous way.

Notwithstanding that, from a pragmatic point of view, it generally proves to be advantageous to continue to consider generalized Gordon metrics, which are C^2 , when gluing them together. For even with the simplest types of distributional metrics of this form, it may occur that the thin shell formalism reaches its limits – even when using Colombeau’s theory of generalized functions. On the other hand, this cannot happen in the C^2 -case. Here Colombeau’s theory is only needed to set up problematic energy conditions (such as the dominant energy condition) for the stress-energy tensors of the form (9), which are undefined in the Schwartz theory of distributions.

But when it comes to demonstrating that spacetimes with metrics containing a delta term can be unambiguously glued together, it is advisable to consider another class of deformed spacetimes whose representatives lead to simpler curvature expressions and therefore to a simpler structure of the field equations.

One class of local geometric deformations, for which this is actually the case, is the class of so-called generalized Kerr–Schild deformations; a class, whose representatives are metrics of the type

$$\bar{g}_{ab} = g_{ab} + f l_a l_b, \tag{55}$$

which are given with respect to a lightlike geodesic covector field $l_a = g_{ab} l^b$ that meets the conditions $g_{ab} l^a l^b = 0$, $(l\nabla) l^a = 0$ and $\bar{g}_{ab} l^a l^b = 0$, $(l\bar{\nabla}) l^a = 0$.

The generalized Kerr–Schild class is a family of solutions that encompass a considerably large class of geometric models that are of interest to general relativity, such as all

² Note that there are many ways of modeling δ^2 in Colombeau’s theory, but these will not be of further relevance at this point, because they are not suitable for solving the problem at hand.

stationary geometries that are in the Kerr–Newman family of spacetimes and, in addition, all dynamical radiation fluid spacetimes lying in the even more general Bonnor–Vaidya family. Moreover, it includes various models with cosmological horizons, like for instance Kottler alias Schwarzschild–de Sitter spacetime and its generalizations.

As in the case of the generalized Gordon class of metrics, the problem of gluing pairs of generalized Kerr–Schild spacetimes can be treated in the given case by considering a thin shell splitting of the form $f = \theta f_+ + (1 - \theta) f_-$ in (55) and requiring that $[f] = 0$ on a lightlike hypersurface Σ . In this case, the metric \bar{g}_{ab} of the ambient spacetime can straightforwardly be decomposed in such a way that it takes the form (1). Alternatively, one may add and subtract a term of the form $f_{\pm} l_a l_b$ in (55) to obtain a splitting of the form (15), which yields $\bar{g}_{ab} = g_{ab}^{\pm} + e_{ab}^{\pm}$, where the definitions $e_{ab}^+ \equiv (1 - \theta)(f_- - f_+) l_a l_b$ and $e_{ab}^- \equiv \theta(f_+ - f_-) l_a l_b$ are used. Making then an ansatz of the form $f_{\pm} = f_{\pm}^0 \delta$, where $f_{\pm}^0(x)$ are C^2 -functions and $\delta(x)$ is the Dirac delta distribution, one obtains the result

$$\bar{g}_{ab} = g_{ab} + f_0 \cdot \delta l_a l_b, \tag{56}$$

provided that the definition $f_0 = A f_+^0 + (1 - A) f_-^0$ is used in the present context. In order to be able to perform those nonlinear operations on \bar{g}_{ab} needed in order to set up the inverse metric, the Levi–Civita connection $\bar{\Gamma}_{bc}^a$ and Einstein’s field equations, which have again the form (51) and (52), respectively, also in the given case a strict delta net $(\delta_{\varepsilon})_{\varepsilon \in (0,1]} \in C^{\infty}(\bar{M})^{(0,1]}$ shall be considered, which converges to the delta distribution in the limit $\varepsilon \rightarrow 0$. The associated regularized generalized Kerr–Schild metric

$$\bar{g}_{ab}^{\varepsilon} = g_{ab} + f_0 \delta_{\varepsilon} \cdot l_a l_b \tag{57}$$

can then be used to obtain the form of the inverse metric via solving relation (49) with respect to the ansatz

$$\bar{g}^{ab} = g^{ab} - f_0 \delta_{\varepsilon} \cdot l^a l^b \tag{58}$$

regularized generalized inverse Kerr–Schild metric. This yields the result

$$\bar{g}^{ab} = \lim_{\varepsilon \rightarrow 0} \bar{g}_{\varepsilon}^{ab} = g^{ab} - f_0 \delta \cdot n_a n_b, \tag{59}$$

where, in contrast to the previous case of the Gordon class of metrics, it turns out that the form of (59) is independent of the choice of regularization of the delta distribution.

As shown in [33], the geometric structure of the deformed field equations is particularly simple in the given case of the generalized Kerr–Schild class. More specifically, the mixed Einstein tensor \bar{G}_b^a of the ambient metric \bar{g}_{ab} turns out to be linear in the profile function f . Moreover, as shown in [34],

also the Einstein tensor with lowered and raised indices are linear in $f \equiv f_0 \cdot \delta$ if the geometric constraints

$$\bar{\nabla}_{[a} l_{b]} = \nabla_{[a} l_{b]} = 0, \quad (l \bar{\nabla}) f = (l \nabla) f = 0 \tag{60}$$

are met. To see this, one may use the fact that there holds

$$C_{bc}^a \approx \frac{1}{2} \nabla_b (f l^a l_c) + \frac{1}{2} \nabla_c (f l^a l_b) - \frac{1}{2} \nabla^a (f l_b l_c), \quad C_{ab}^b \approx 0 \tag{61}$$

for the corresponding deformation tensor of the generalized Kerr–Schild class, which relates the pair of covariant derivative operators $\bar{\nabla}_a$ associated with \bar{g}_{ab} and ∇_a and associated with g_{ab} . Using this result, one finds that the deformed Ricci tensor with lowered indices reads

$$\bar{R}_{ab} = R_{ab} + E_{ab}, \tag{62}$$

where $E_{ab} = \nabla_c C_{ab}^c + C_{ad}^c C_{cb}^d$ applies. As it then turns out in this context, the conditions

$$C_{ad}^c C_{cb}^d \approx 0 \tag{63}$$

and

$$\nabla_c C_{ab}^c l^a \approx 0, \quad \nabla_c C_{ab}^c l^b \approx 0, \quad E_b^a l_a l^b \approx 0 \tag{64}$$

are met regardless of the choice of regularization of the delta distribution, where the conditions listed in (64) result from the consistency condition $\bar{R}_{ab} = \bar{g}_{ac} \bar{R}_b^c$. Thus, using Colombeau’s theory of generalized functions in combination with the geometric framework of local metric deformations, one finds that the field equations of the theory are linear in $f = f_0 \cdot \delta$.

From these results, it can be concluded that the generalized Kerr–Schild framework is well suited to address the problem of gluing spacetimes of low regularity. In particular, it is found that it is much more straightforward to glue together distributional metrics belonging to the generalized Kerr–Schild class than distributional metrics belonging to the generalized Gordon class. From this, however, it can be concluded that the choice of the type of local geometric deformation determines how well the problem of gluing spacetimes with low regularity can be treated in practice, which in turn is the reason why the methods discussed in Sect. 2 are useful for gluing spacetimes with low regularity.

In order to highlight this point, a few concrete geometric examples shall now be considered, whereas the main focus shall be placed directly on representatives of the generalized Kerr–Schild class. This makes sense not least because this class provides the best known examples of pairs of distributional spacetimes that can be glued together. The best

known examples are gravitational shock wave geometries, whose importance for the thin shell formalism was recognized long ago [9] (although the treatment of mathematically problematic expressions has not received the necessary attention at the time). The focus in this context shall be on gravitational shock wave geometries in black holes and cosmological backgrounds. These geometries, which all were found by using Penrose’s cut-and-paste alias scissors-and-paste procedure, that is, a method for gluing spacetimes along lightlike hyperfaces, characterize the fields of spherical shock waves caused by a massless particle moving at the speed of light along the corresponding event or cosmological horizons. The most famous representatives of this class are the geometries of Dray and ’t Hooft [35], Sfetsos [36] and Lousto and Sanchez [37], which characterize the gravitational fields of spherical shock waves in Schwarzschild, Reissner–Nordström and Kottler alias Schwarzschild–de Sitter backgrounds.

All of these geometries have in common that their line elements can be written down in the form

$$ds^2 = 2B^2 f_0 \delta dU^2 - 2B^2 dU dV + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{65}$$

where $\delta = \delta(U)$ is Dirac’s delta distribution and $B = B(UV)$ is a function whose explicit form depends of whether background spacetime is Schwarzschild, Reissner–Nordström or Kottler. Thus, it can be concluded that the metrics

$$\bar{g}_{ab} = g_{ab} + 2B^2 f_0 \delta l_a l_b, \tag{66}$$

corresponding to these line elements belong to the generalized Kerr–Schild classes of the respective backgrounds, where in each case one has $l_a = g_{ab} l^b = -dU_a$. Accordingly, given the fact that one can always choose $f_0 = f_0(\theta, \phi)$ with $f_0 = Af_+^0 + (1 - A)f_-^0$ in this context, it becomes clear that the junction conditions (32) are met if it is required that $\partial_V B|_{U=0} = \partial_V r|_{U=0} = 0$. As a result, the Einstein tensor of the corresponding classes of geometries takes the form $G^a_b = (\Delta - c) f_0 \delta l^a l_b$ and thus characterizes the geometric field of a null fluid source.

The validity of this result cannot be deduced from thin shell formalism alone; it requires geometric deformation theory to make it possible. This can be concluded from the fact that in the past, on the basis of careless application of Penrose’s method, which according to [35] gives the same results as the thin shell formalism (except for a single not particularly relevant term), the authors of the above-mentioned works came to the erroneous conclusion that despite the validity of $\partial_V B|_{U=0} = \partial_V r|_{U=0} = 0$, the field equations should contain ill-defined ‘delta-square’ terms. As it turns out, however, the deformed field equations of the generalized Kerr–Schild class do not contain such terms after all, but lead to a single

differential equation for the reduced profile function of the form

$$(\Delta - c) f_0 = 2\pi b \delta, \tag{67}$$

where $\delta \equiv \delta(\cos \theta - 1)$ is Dirac’s delta distribution and b and c are constants, whereas c is given by $c = 2r_+(\kappa - \Lambda r_+)$ in the Schwarzschild–de Sitter case, $c = 2r_+\kappa$ in the Reissner–Nordström case and by $c = 1$ in the Schwarzschild case.

The resulting equation can be solved by expanding the reduced profile function on the left hand side and the delta function on the right hand side simultaneously in Legendre polynomials. Using here the fact that $\delta(x) = \sum_{l=0}^{\infty} (l + \frac{1}{2}) P_l(x)$, one obtains the solution

$$f_0(\theta) = -b \sum_{l=0}^{\infty} \frac{l + \frac{1}{2}}{l(l + 1) + c} P_l(\cos \theta) \tag{68}$$

by solving the corresponding eigenvalue problem. As was shown in [36], however, it is quite possible to find another representation for the function f , which is fully consistent with the thin shell formalism discussed in Sect. 1 of this work.

Other examples of Kerr–Schild deformed local spacetimes with deformation fields that have compact support in a single null hypersurface of the geometry are pp-wave spacetimes. The perhaps most well-known models in this regard are the spacetimes of Aichelburg and Sexl [38] and Lousto and Sanchez [37, 39], which have in common that they are specified by a Brinkmann form that contains a delta distribution and therefore has support only on a single lightlike hypersurface of spacetime. For that reason, they determine a local background geometry that coincides everywhere with that of a spherically symmetric black hole spacetime except for one single null hypersurface.

Consequently, as it turns out, the deformation approach is not only fully compatible with the thin shell formalism, but also shows the treatability of problems that cannot be treated by the naive application of the standard methods of said formalism. But this does not only concern the gluing of distributional metrics: In contrast to the thin shell formalism, the geometric deformation approach allows the smooth gluing of arbitrary pairs of local spacetimes by using a suitable set of transition functions.

This is because the thin shell formalism requires that only pairs of spacetimes can be glued together for which the corresponding junction conditions are fulfilled. On the other hand, the use of suitable transition functions within the geometric deformation approach ensures that pairs of local spacetimes can be glued together smoothly, i.e., without the possibility of the existence of a singular confined stress-energy tensor with a delta shock at the joint boundary hyperface of the spacetimes.

This can be easily made clear by considering a splitting of the form $f = \chi_+ f_+ + \chi_- f_-$ either in (44) or in (55), where the χ_{\pm} are functions of the form (25) which are zero in M_{\pm} and f_{\pm} are essentially arbitrary functions, which, however, shall be assumed to be at least C^2 . Such a choice makes it possible to model local spacetimes whose geometry changes continuously (and not instantly from one moment to the next, as in the thin shell formalism) in such a way that a given initial geometry transitions smoothly into a certain final geometry of spacetime. Thus, in other words, choosing the function f in this way allows a smooth geometric transition between pairs of local spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$.

To demonstrate this, the special case $f = \chi f_0$ shall be considered for a generalized Kerr–Schild metric in (55). More specifically, the Bonnor–Vaidya family of spacetimes [40] shall be considered, whose metric, in the general rotating case, can be read off the line element

$$d\bar{s}^2 = -dv^2 + 2(dv - a \sin^2 \theta d\phi)dr + \Sigma d\theta^2 + \frac{(r^2 + a^2) \sin^2 \theta}{\Sigma} d\phi^2 + \frac{2Mr - e^2}{\Sigma} (dv - a \sin^2 \theta d\phi)^2, \tag{69}$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $M = M(v)$, $e = e(v)$. The energy–momentum tensor of this geometry consists of a null fluid part and an additional part, i.e. $\bar{T}_{ab} = \varepsilon l_a l_b + 2\vartheta (l_a k_b + m_a \bar{m}_b) + 2\zeta l_a \bar{m}_b + 2\bar{\zeta} l_a m_b$, where $\varepsilon = -\frac{2r(\dot{M} - \dot{e}) + a^2 \sin \theta (r\dot{M} - \dot{e}\dot{e} - e\ddot{e})}{8\pi \Sigma^2}$, $\vartheta = \frac{e^2}{8\pi \Sigma^2}$ and $\zeta = \frac{-ia \sin \theta}{\sqrt{28\pi} \Sigma^2} \{ \Sigma \dot{M} - 2e\dot{e} \}$ with $\dot{M} := \frac{dM}{dv}$ and $\dot{e} := \frac{de}{dv}$.

By considering a splitting of the mass and the charge functions into constant and non-constant parts, i.e. $M(v) = M_0 + m(v)$ and $e(v) = e_0 + e(v)$ with $M_0 = const.$ and $e_0 = const.$, the metric associated with line element (69) can be written in the form

$$\bar{g}_{ab} = g_{ab} + f l_a l_b, \tag{70}$$

where $f = \frac{m + 2e_0 e + e^2}{\Sigma}$, provided that $l_a = -dv_a + a \sin^2 \theta d\phi_a$. Due to the fact that the mixed Einstein tensor is linear in the profile function, the energy-momentum tensor decomposes according to the rule $\bar{T}_b^a = T_b^a + \tau_b^a$, where T_b^a is the energy-momentum tensor of the Kerr–Newman black hole background spacetime. The resulting deformed geometry of spacetime may therefore be physically interpreted as the gravitational field of a Kerr–Newman black hole that accretes null radiation.

Although $m(v)$ and $e(v)$ may in principle be chosen arbitrarily, one may choose them to be of the form $m(v) = \psi(\frac{v}{v_0})m_0$ and $e(v) = \psi(\frac{v}{v_0})e_0$, where $m_0 = const.$ and $e_0 = const.$ The transition function $\psi(\frac{v}{v_0})$ takes a value of zero for $v < 0$, a value of one for $v \geq 1$ and is strictly increasing in the interval $[0, 1]$, so that the set of conditions given in

(33) is met and it can therefore be concluded that the metric of spacetime coincides locally with that of Kerr–Newman spacetime; a spacetime that describes the electrovac gravitational field of a stationary axially symmetric charged rotating black hole, which has, by necessity, a completely different physical interpretation from the metric of a rotating Bonnor–Vaidya spacetime. More specifically, the spacetime geometry at hand describes how an initially given Kerr–Newman black hole geometry with ‘degrees of freedom’ (M_0, e_0, a) transitions smoothly into one with different parameter values $(M_0 + m_0, e_0 + e_0, a)$, so that it can be concluded that Bonnor–Vaidya spacetime characterizes the gravitational field of an charged rotating black hole that accretes null radiation over a finite period of time.

As a consequence, it is found that the Bonnor–Vaidya model can always be set up to predict the collapse of a null radiation field and its absorption by a charged rotating black hole, which could even result in the complete discharge of the black hole, whereas it is worth mentioning that these results are in complete agreement with the famous black hole uniqueness theorems [41–43].

Of course, one could also try to make another choice for the function f in (70), which is in better agreement with the thin shell formalism. In particular, one could choose $m(v) = \theta(v - v_0)m_0$ and $e(v) = \theta(v - v_0)e_0$ where $\theta(v - v_0)$ represents the Heaviside step function $\theta(v - v_0) := \begin{cases} 0 & v - v_0 < 0 \\ \frac{1}{2} & v - v_0 = 0 \\ 1 & v - v_0 > 0 \end{cases}$. However, from a purely physical point of view, this would actually be a very poor choice, since the resulting geometry would describe the very unphysical case of a black hole that accretes material of mass m_0 and charge e_0 within an infinitesimally small instant of time, which is why it is more sensible to stick to the above smooth description of the problem. Nevertheless, the given choice also provides a well-defined example of a local geometry in the previously introduced sense and the resulting geometric model reveals the structure of the gravitational field of a black hole that absorbs null radiation.

Now that this has been clarified, the next thing to be noted is that the junction conditions of the thin shell formalism result as a special case of the discussed smooth geometric framework if the limit is considered where the size of the smooth transition region goes to zero. To see this, one may consider a transition region \mathcal{O} with length (or time) scale L and the generalized function

$$\chi_L(x - x_0) = \frac{1}{2} \left(1 + \tanh \frac{x - x_0}{L} \right) = \frac{1}{1 + e^{-\frac{x-x_0}{L}}} \tag{71}$$

which provides a smooth, analytic approximation of the step function, converging exactly to said function in the limit $L \rightarrow 0$, i.e. $\theta(x - x_0) = \lim_{L \rightarrow 0} \chi_L(x - x_0)$. Using this definition

as a basis for setting up the decomposition relation

$$g_{ab} = \lim_{L \rightarrow 0} \chi_L g_{ab}^+ + (1 - \chi_L) g_{ab}^- \tag{72}$$

and, moreover, the fact that

$$\frac{d\theta(x - x_0)}{dx} = \lim_{L \rightarrow 0} \frac{1}{L} \frac{1}{2 + e^{\frac{x-x_0}{L}} + e^{\frac{x_0-x}{L}}} \tag{73}$$

turns out to be singular for $x = x_0$ and zero otherwise, one finds that condition (2) must be met in order to join pairs of local spacetimes (\mathcal{M}^\pm, g^\pm) . In addition, using the fact that the generalized function $\chi_L(x)$ provides a smooth approximation of the Heaviside step function and its derivative with respect to x a smooth approximation of the delta distribution, steps analogous to those described in Sect. 1 lead first to relations (5) and (6) and then to relations (7)–(9). By distinguishing the non-lightlike and lightlike cases one then finds that junction conditions (10) and (11) must hold in the first case, whereas junction conditions (12) or (13) must hold in the latter case. Consequently, it follows that the junction conditions of the theory can be derived via using the smooth metric deformation framework presented in Sect. 2, but under the premise that the size of the transition region approaches zero in a suitable limit. However, since there is a wide range of generalized functions that are associated with the step function and whose first derivatives are associated with the delta distribution in a distributional sense, it becomes clear that the local geometric deformation approach discussed in Sect. 2 of this work generalizes the thin-shell formalism discussed in Sect. 1.

The next point to note is that by considering suitable transition functions – as already mentioned earlier at the beginning of Sect. 3 – the unambiguous gluing of generalized Gordon class spacetimes becomes possible. To see this, one may consider a generalized Gordon class metric of the type

$$\bar{g}_{ab} = g_{ab} + \lim_{\varepsilon \rightarrow 0} f_0 \delta_\varepsilon \cdot n_a n_b, \tag{74}$$

which is defined with respect to a timelike vector field $n^a = \frac{1}{\sqrt{2}}(l^a + \chi k^a)$, where χ is a smooth transition function of the form (25) whose support does not intersect that of the delta distribution resulting from the limit $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon$. Generalized Gordon spacetimes of this type can always be glued together, since the field equations coincide with those of the generalized Kerr–Schild class in those local spacetime domains in which the delta distribution becomes singular. The reason for this is that the timelike vector field n^a becomes locally lightlike by the given choice, so that the gravitational shock wave geometries considered above can be generalized in a mathematically rigorous way. However, this makes it clear that in principle it is possible to glue generalized Gordon spacetimes together without having to calculate

problematic powers of the delta distribution and its derivatives, which is a challenge despite the use of Colombeau’s theory of generalized functions.

In any case, the results obtained so far can certainly be generalized in many ways. This can be demonstrated, for example, by generalizing the metric associated with the line element (70). This can be accomplished by performing a null rescaling of the form $l_a \rightarrow \lambda l_a, k_a \rightarrow \lambda^{-1} k_a$, which leaves the geometric structure of the background metric g_{ab} invariant, but changes the geometric structure of the metric \bar{g}_{ab} of the ambient spacetime $(\bar{\mathcal{M}}, \bar{g})$. This yields a more general class of solutions to Einstein’s field equations with a metric of the form

$$\bar{g}_{ab} = g_{ab} + \lambda^2 f l_a l_b, \tag{75}$$

where the only condition that one may wish to impose in this context is that $(l \nabla) \lambda = 0$ and therefore $\lambda = \lambda(v, \theta, \phi)$ holds by definition, so that the resulting class of geometries still belongs to the generalized Kerr–Schild class of Kerr–Newman spacetime and the corresponding mixed field equations remain linear in f . But, of course, that restriction is not a must by any means.

Another possibility to construct a new class of models from the one given above is to use the fact that the null geodesic vector field can be extended to an associated null geodesic frame $(l^a, k^a, m^a, \bar{m}^a)$ and then to perform a null rotation of the form $k_a \rightarrow k_a, m_a \rightarrow m_a + \xi k_a, l_a \rightarrow l_a + \xi \bar{m}_a + \bar{\xi} m_a + |\xi|^2 k_a$, which again leaves the geometric structure of the background metric g_{ab} invariant, but changes the geometric structure of the metric \bar{g}_{ab} of the ambient spacetime $(\bar{\mathcal{M}}, \bar{g})$. In this way, once again a more general class of solutions to Einstein’s field equations is obtained, whose metric is of the form

$$\begin{aligned} \bar{g}_{ab} = & g_{ab} + f l_a l_b + 2 f \xi l_{(a} \bar{m}_{b)} + 2 f \bar{\xi} l_{(a} m_{b)} \\ & + 2 f |\xi|^2 (l_{(a} k_{b)} + m_{(a} \bar{m}_{b)}) + f \xi^2 \bar{m}_a \bar{m}_b + f \bar{\xi}^2 m_a m_b \\ & + 2 f \xi \bar{\xi} |\xi|^2 k_{(a} \bar{m}_{b)} + 2 f \bar{\xi} \xi |\xi|^2 k_{(a} m_{b)} + f |\xi|^4 k_a k_b. \end{aligned} \tag{76}$$

Consequently, models of any complexity can be constructed by repeatedly applying a combination of null rescalings and null rotations. Therefore, it is actually quite straightforward to construct another, more general type of ambient spacetime $(\bar{\mathcal{M}}, \bar{g})$ with metric

$$\bar{g}_{ab} = g_{ab} + e_{ab} \tag{77}$$

from an ambient spacetime with metric form (70), whose geometry coincides locally with that of Kerr–Newman spacetime (\mathcal{M}, g) . It may be noted that the concrete choice of the local background metric g_{ab} is, of course, irrelevant in this context and that the metric in question therefore certainly

does not need to specifically match the Kerr–Newman metric or any other metric considered in this section.

On the other hand, one must be a little more careful when generalizing metrics of type (75). In this case, it is favorable to consider only rescalings $l_a \rightarrow \lambda l_a$, $k_a \rightarrow \lambda^{-1} k_a$ and null rotations $l_a \rightarrow l_a$, $m_a \rightarrow m_a + \xi l_a$, $k_a \rightarrow k_a + \xi \bar{m}_a + \bar{\xi} m_a + |\xi|^2 l_a$, because otherwise the resulting local geometry of spacetime would no longer be a Kerr–Schild geometry and thus similar problems could occur as in the case of spacetimes of the generalized Gordon class. In this way, very general metrics of type (77) can be obtained, which allow a simple generalization of shock wave geometries of type (74), which in turn can be constructed from known shock wave geometries of type (66).

In any case, there are also other useful ways to construct interesting models for ambient spacetimes $(\bar{\mathcal{M}}, \bar{g})$, e.g. by considering other more complex types of metric deformations. As a specific example example, superimposed generalized Kerr–Schild deformations may be mentioned, not least because these types of metric deformations allow an extension of thin shell formalism (although one must be very careful with the calculation of curvature expressions). These types of metric deformations are of the form

$$\bar{g}_{ab} = g_{ab} + \sum_{A=1}^N f_{(A)} l_a^{(A)} l_b^{(A)}, \tag{78}$$

where each $l_a^{(A)} = a^{(A)} l_a + b^{(A)} k_a + c^{(A)} \bar{m}_a + \bar{c}^{(A)} m_a$ must meet the conditions $l_a^{(A)} l_{(A)}^a = 0$ and $(l_{(A)} \nabla) l_{(A)}^a = 0$. Representatives of this class lead to a whole series of nested Kerr–Schild spacetimes, i.e.

$$\begin{aligned} \bar{g}_{ab} &= g_{ab}^{(1)} + \sum_{A=2}^N f_{(A)} l_a^{(A)} l_b^{(A)} \\ &= g_{ab}^{(2)} + \sum_{A=3}^N f_{(A)} l_a^{(A)} l_b^{(A)} = \dots = g_{ab}^{(N-1)} + f_{(N)} l_a^{(N)} l_b^{(N)}, \end{aligned} \tag{79}$$

where $g_{ab}^{(1)} = g_{ab} + f_{(1)} l_a^{(1)} l_b^{(1)}$, $g_{ab}^{(2)} = g_{ab} + \sum_{A=1}^2 f_{(A)} l_a^{(A)} l_b^{(A)}$, ..., $g_{ab}^{(N-1)} = g_{ab} + \sum_{A=1}^{N-1} f_{(A)} l_a^{(A)} l_b^{(A)}$ applies by definition. As highlighted in several works on the subject [44–48], the corresponding metric deformations can be used to provide initial data for the construction of solutions to Einstein’s equations that characterize multiple black holes in general relativity. On a non-numerical level, these types of metric deformations have been used to construct generalizations of the gravitational shock wave spacetimes previously considered in this section [49].

However, there are many other classes of both exact and approximate metric deformations besides the geometric mod-

els mentioned above, which can be used to construct local spacetimes. The deformation of the background metric can also be applied perturbatively, which can lead to physically interesting examples. This is not least due to the fact that the deformation approach allows the geometric transition of arbitrary spacetimes and thus the gluing of perturbative metrics.

Because of all these points, the geometric deformation approach represents a useful extension of the well-established thin shell formalism, which allows the treatment of a larger class of problems than previously possible.

Summary

In the present work, a specific approach to the construction of local spacetimes in general relativity was presented. This approach is based on the idea of using local deformations of the metric to join spacetimes with different geometries and physical properties. As it turned out in this context, the approach presented allows the calculation of the curvature fields of spacetimes with metrics of low regularity, such as gravitational shock wave spacetimes, which, from the point of view of standard gluing techniques, does not seem feasible (or even possible). Furthermore, it was found that smooth gluings of arbitrary spacetime pairs can be carried out by using suitable transition functions and that complex types of ambient metrics and associated spacetimes can be constructed by transformations that leave the local geometric structure of spacetime invariant. Not least because the above mentioned results do not only apply to exact deformations, but also to local metric perturbations, it can be plausibly concluded from the observations made in this work that there may be valuable extensions of already known geometric models of Einstein–Hilbert gravity or more general gravitational theories that have remained undiscovered so far. It might therefore be worthwhile to use the geometric deformation approach in the analysis of complex geometric transitions in spacetime, especially if these transitions cannot be treated on the basis of the traditional thin shell formalism.

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