Noether symmetries of Einstein-aether scalar field cosmology

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Abstract In this paper, we explore an Einstein-aether cosmological model by adding the scalar field in which it has an interaction with the aether field. For the cosmological implications of the model, we consider that the universe can be described by the spatially flat FRW metric together with the matter dominated universe. Applying Noether symmetry approach to the point-like Lagrangian we determine the explicit forms of unknown functions i.e. the potential and coupling function. We solve the analytical cosmological solutions of the field equations admitting the Noether symmetry, basically divided into two parts. Our results show that the obtained solutions lead to an accelerated expansion of the universe. We also discuss the tensor perturbations within the framework of this model in order to get information about the mass of gravitational waves.

1 Introduction

Motivations for quantum gravitational theories has led to a growing focus in recent years that Lorentz symmetry may not be the main precise symmetry of nature. Lorentz symmetry can be violated by using a vector field in gravitational theories. This vector field creates a preferred direction in space-time. That is why, recently, the study of Einstein-aether's theory has received much attention. In fact, this theory is a modification of Einstein's general theory of relativity, in which the space-time is described by the metric and a time-like vector field, which is called aether. This theory violates Lorentz symmetry since it has a preferred reference. In 2004, in [1], the theoretical and phenomenological aspects of this theory were examined. The existence of a dynamical time-like vector field can break the Lorentz symmetry. On the other hand, from particle physics, we know that breaking the symmetry can cause new phenomena. In this case, like the

Higgs mechanism, the breaking of the Lorentz symmetry by this vector field can cause graviton to be massive. Various aspects of Einstein-aether's theory have been studied in [2–4]. These articles show that in this theory, by redefining the metric and the aether field, one of the coupling coefficients can be assumed to be zero, which makes it very easy to solve this model. In [5], the Noether's conserved charge related to this model as well as the black hole mechanism in the framework of this model is studied. In this paper, although a general term for variation of energy and angular momentum has been obtained on the black hole horizon, no general term for black hole entropy has been found. The spherical solution of the Einstein-aether theory has been investigated in [6–9]. Where they show that a three-parameter family of spherical solutions is available in such a way that these spherical solutions to the state of a perfect fluid star accurately determine the amplitude of stability for stars with a constant density. In [10–23], the properties of different black holes in this theory are discussed in detail; these include the gravitational spectrum of black holes, the study of cosmological perturbations, and numerical solutions for black holes in the framework of Einstein-aether gravity. Also, the polarization of cosmic microwave background radiation and weak gravitational lensing has been studied in the context of this model [24, 25].

On the other hand, one of the key issues in modern cosmology is related to the existence of the dark energy problem that still maintains its mystery. This issue is linked to the accelerated expansion of the Universe and is a clear problem for scientists. The study of cosmology related to Einstein-aether's theory has received a lot of attention, in [26–36], the theoretical and phenomenological aspects of the cosmology of this theory have been studied, and it can be shown that this model is a good candidate to describe dark energy. Moreover, a model called Lorentz-violated inflation has been proposed that a scalar field contributes additionally to the dynamics of Einstein-aether's theory where there is an interaction between the aether field and the scalar field in [37].

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Interestingly, it has been shown that inflation can occur without the need for inflation potential. Some exact and analytical solutions of homogeneous and isotropic spacetimes for such models was performed in [38–45]. These solutions allow describing both the inflation behavior in the early universe and the late-time cosmic acceleration. In [46], it is shown that Einstein-aether theory including the scalar field results in phantom divide line crossing. Recently, attempts have even been made to examine this theory in the presence of non-homogeneous and homogeneous metrics such as the Kantowski-Sachs metric or the Bianchi metric [47–51].

There are several ways to study cosmological theories. However, since symmetry plays an essential role in physical systems, the best way is to solve the problem using the symmetries in the system. Existing symmetries can lead to invariant quantities over time. Noether's approach has long been used to solve cosmological problems, and so this approach has been widely used in the literature [52]. Using this approach, one can determine the function of potential and the coupling coefficients so that the model remains invariant under continuous symmetry. The paper [53] examines the cosmology of phantom-quintessence using Noether's approach. In this paper, it is shown that the potential function obtained using Noether symmetry is consistent with the type Ia supernova observations, and this model could be a good candidate for dark energy. In [54] a model-independent study is performed and the scalar field dark energy models are classified utilizing the Noether symmetries related to the equations of motion. In general, Noether symmetries play an important role in physics because they can be used to simplify a given system of differential equations as well as to determine system integrability. Therefore, it can be seen that Noether symmetry has been used to simplify the equations related to generalized gravity [55]. For example, in solving the teleparallel gravity not only with the FRW metric [56] but also with the Schwarzschild black hole [57] or Kantowski-Sachs metric [58], the Noether symmetry method can be used. Since the Noether approach not only allows us to obtain accurate answers but also determines the arbitrary functions in the action, this method can be considered as a geometric criterion for selecting gravitational theories [59–67]. Due to the importance of this approach, in examining the answers of various scalar-tensor models such as the Tachyonic model [68], Brans-Dicke theory [69], Fermion fields with non-minimum coupling in the framework of both general relativity [70, 71] and teleparallel gravity [72, 73], teleparallel dark energy model [74, 75] and with the boundary term [76], Gauss-Bonnet gravity [77], The theories of $f(R)$ or $f(R, T)$ gravities [78] and etc, this method has been widely used. It can even be seen that higher-order corrections to Einstein-Hilbert action were examined by imposing the existence of Noether symmetry [79] and accurate cosmological solutions, or at least the simplification of dynamical equa-

tions, can also be achieved by using the conserved Noether charge. Therefore, motivated by the above discussions, we study Einstein-aether's theory using Noether approach and obtain an accurate answer for this model using the invariant quantities and symmetries in the system. Applying the Noether's approach to the theory, we obtain the potential function and other coupling functions and show that this model can be a good candidate for dark energy.

After obtaining the answers of the Einstein-aether model and examining the cosmological aspects of this model, we explore the tensor perturbations within the framework of this model and use it to calculate the mass of gravitational waves. We show that the potential function and the coupling functions play an essential role in the mass relation of gravitational waves.

This article is organized as follows. In the second section, we examine the action and the field equations related to Einstein-aether's theory. In the third section, using the Noether symmetry approach, we solve the Einstein-aether model and then investigate the cosmological aspects of the model. In this section, by obtaining the cosmological parameters within the framework of the model, we show that this model can be a good candidate for dark energy. In the fourth section, tensor perturbations in this model are studied, and by using these disorders, we find the mass of gravitational waves. The fifth section of the article is devoted to a conclusion.

2 Action and field equations

Gravitational action integral we focus on the Einstein-aether theory with a scalar field coupled to the aether field, is given by [37, 38]

$$S = \int dx^4 \sqrt{-g} \left(\frac{R}{2} - \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + V(\phi) + L_m \right) - S_{Aether}, \quad (1)$$

where L_m is a standard matter Lagrangian and the action S_{Aether} can be written in term of the aether field u^μ

$$S_{Aether} = \int dx^4 \sqrt{-g} [\beta_1(\phi) \nabla^\mu u^\nu \nabla_\mu u_\nu + \beta_2(\phi) \nabla^\mu u^\nu \nabla_\nu u_\mu + \beta_3(\phi) (\nabla_\mu u^\mu)^2 + \beta_4(\phi) u^\mu u^\nu \nabla_\mu u^\alpha \nabla_\nu u_\alpha - \lambda(u^\mu u_\mu + 1)]. \quad (2)$$

In Einstein-aether theory, while β_i , ($i = 1 \dots 4$) quantities are constants and describe the interaction between the aether field and the gravitational field, in the above theory these quantities depend on the scalar field and describe the interacting between the aether field and the scalar field. The function λ is a Lagrange multiplier which yields a constraint on the aether field u^μ as $u^\mu u_\mu = -1$.

We consider an isotropic and homogeneous Universe which described by the spatially flat Friedman-Robertson-Walker (FRW) space-time with the line element

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \quad (3)$$

where $a(t)$ donates the scale factor of the Universe. Choosing a time-like unit vector field as the aether field $u^\mu = (1, 0, 0, 0)$ and considering the line element (3), it is possible to obtain a point-like Lagrangian from the action (1) [38]

$$L = -3B(\phi)a\dot{a}^2 + a^3\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right) - \rho_{m0}, \quad (4)$$

where a dot denotes derivative with respect to the cosmic time t , $B(\phi)$ is defined by $B(\phi) = \beta_1(\phi) + 3\beta_2(\phi) + \beta_3(\phi) - 1$, and the standard matter Lagrangian is assumed to be $L_m = -\rho_{m0}a^{-3}$ for matter-dominated Universe. It should be noted that the function $\beta_4(\phi)$ has no effect on the dynamical equations. Although this Lagrangian seems to be similar to scalar tensor theories, it has quite different dynamics. Quintessence cosmological model can be recovered if we select the function $B(\phi)$ as $B(\phi) = 1$ or $\beta_i = 0$.

The equations of motion for variables can be obtained, so that the equations of motion for the variables a and ϕ are obtained as follows, respectively

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -k_{eff}P_{eff}, \quad (5)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + 3B'\frac{\dot{a}^2}{a^2} + V' = 0, \quad (6)$$

where a prime indicates differentiation with respect to the scalar field ϕ and k_{eff} is defined by $k_{eff} = \frac{1}{B(\phi)}$ which can be interpreted as an effective gravitational constant. Also, it should mention here that the most important requirement for the model to be physically acceptable is $k_{eff} > 0$ since gravity must be attractive. From point-like Lagrangian (4), one can calculate the total energy and if we set it to zero we will have

$$3\frac{\dot{a}^2}{a^2} = k_{eff}\rho_{eff}. \quad (7)$$

In Eqs. (5) and (7), one can define the energy density ρ_{eff} and the pressure P_{eff} of the effective fluid as follows, respectively

$$\rho_{eff} = \frac{\dot{\phi}^2}{2} + V + \rho_m, \quad (8)$$

$$P_{eff} = \frac{\dot{\phi}^2}{2} - V + 2B'\frac{\dot{a}}{a}\dot{\phi}. \quad (9)$$

3 Noether symmetry approach and cosmological solutions

In this section, we consider the existence of the Noether symmetry for the Lagrangian (4). The configuration space of point-like canonical Lagrangian is $Q = (a, \phi)$ and the related tangent space is $TQ = (a, \phi, \dot{a}, \dot{\phi})$. The existence of Noether symmetry implies the presence of a vector field $\mathbf{X}$ such that

$$\mathbf{X} = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}}, \quad (10)$$

where α and β depend on a and ϕ . The Noether symmetry condition requires the vanishing of Lie derivative of the Lagrangian with respect to the vector field $\mathbf{X}$, i.e.

$$\mathcal{L}_{\mathbf{X}}L = 0, \quad (11)$$

If $\mathbf{X}$ is a symmetry for the dynamics then it will be generated the following constant of motion

$$I_0 = \alpha \frac{\partial L}{\partial \dot{a}} + \beta \frac{\partial L}{\partial \dot{\phi}}. \quad (12)$$

The equation (11) yields a second-degree expression in terms of $\dot{a}^2$, $\dot{a}\dot{\phi}$ and $\dot{\phi}^2$, whose coefficients are functions of a and ϕ and their partial derivatives. Thus, the resulting expression should be zero, separately. Accordingly, one gets a set of following two equations

$$\alpha + \frac{B'}{B}a\beta + 2a\frac{\partial \alpha}{\partial a} = 0, \quad (13)$$

$$6B\frac{\partial \alpha}{\partial \phi} - a^2\frac{\partial \beta}{\partial a} = 0, \quad (14)$$

$$3\alpha + 2a\frac{\partial \beta}{\partial \phi} = 0, \quad (15)$$

$$3V\alpha + aV'\beta = 0, \quad (16)$$

With the choice of $B(\phi) = 1$, Lagrangian (4) is reduced to minimally coupled to scalar field theory so that Noether symmetry analysis for such models has already been discussed and obtained the potential in the exponential form [80]. Therefore we will exclude this case. In order to solve the aforementioned Noether symmetry equations we assume that the potential has in the form of power law i.e.

$$V(\phi) = V_0\phi^n, \quad (17)$$

where V_0 and n are arbitrary constants. By placing the Eq. (17) in Eq. (16), we can conclude

$$\alpha = -\frac{na}{3\phi}\beta. \quad (18)$$

Table 1 Set of solutions of the Noether symmetry equations

Case	α	β	$V(\phi)$	$B(\phi)$
1	$-\frac{n\alpha_0}{3}a^{-\frac{n-6}{2n}}\phi^{\frac{n-2}{2}}, \alpha_0 \equiv \alpha_1\alpha_2$	$\alpha_0a^{-\frac{3(n-2)}{2n}}\phi^{\frac{n}{2}}$	$V_0\phi^n$	$\frac{3}{2n^2}\phi^2$
2	$-\frac{2\alpha_0}{3}a$	$\alpha_0\phi$	$V_0\phi^2$	$\frac{3}{8}\phi^2$

Using this expression in the equation (13), and limiting ourselves to the method of separation of variables, i.e. $\beta(a, \phi) = A(a)\Phi(\phi)$, we can write the Eq. (13) as follows

$$\frac{3}{2} \left(\frac{\phi}{n} \frac{B'}{B} - 1 \right) = \frac{a}{A} \frac{dA}{da}, \quad (19)$$

here $n \neq 0$. Since this expression is a function of a only on the left-hand side and a function of ϕ only on the right-hand side, they must be equal to the same constant that we say m . Therefore, Eqs. (19) yield the following solutions

$$B(\phi) = B_0\phi^{\frac{n(2m+3)}{3}}, \quad (20)$$

$$A(a) = \alpha_1 a^m, \quad (21)$$

where B_0 and α_1 are integration constants. From the Eq. (15) we can write the following ordinary differential equation for $\Phi(\phi)$

$$2\phi \frac{d\Phi}{d\phi} - n\Phi = 0, \quad (22)$$

whose solution reads

$$\Phi(\phi) = \alpha_2 \phi^{\frac{n}{2}}. \quad (23)$$

Above, α_2 is an integration constant. From Eq. (14) we obtain

$$n(n-2)B_0\phi^{\frac{n(4m+9)}{6}} + m\phi^{\frac{n+4}{2}} = 0. \quad (24)$$

In order for this expression to be identically valid, it must provide the constraints $m = -\frac{3(n-2)}{2n}$ and $(n-2)(B_0 - \frac{3}{2n^2}) = 0$. As a result, we can summarize the solutions of the Noether symmetry equations in Table 1 as two cases.

3.1 Case 1

Since the existing dynamical equations are not linear, it is very difficult to find a solution, so we need new variables in Lagrangian (4), which will make it easier to find solutions. The presence of the Noether symmetry allows us to perform a coordinate transformation where one of the variables is cyclic. Such a transformation provides the following partial differential equations:

$$\alpha \frac{\partial z}{\partial a} + \beta \frac{\partial z}{\partial \phi} = 1, \quad (25)$$

$$\alpha \frac{\partial u}{\partial a} + \beta \frac{\partial u}{\partial \phi} = 0, \quad (26)$$

in which z and u are new variables related to old ones, which are functions of old variables, a and ϕ . A favorable solution of these equations for case 1, the general values of n , of Table 1 is

$$z = -\frac{1}{(n-2)\alpha_0} a^{\frac{3(n-2)}{2n}} \phi^{\frac{2-n}{2}}, \quad u = a^{\frac{3}{n}} \phi, \quad (27)$$

where $n \neq 2$. When a and ϕ are expressed in terms of new variables, we have

$$a = [(2-n)\alpha_0 z]^{\frac{n}{3(n-2)}} u^{\frac{n}{6}}, \quad \phi = [(2-n)\alpha_0 z]^{\frac{1}{2-n}} u^{\frac{1}{2}}. \quad (28)$$

By mean of this transformation, the Lagrangian (4) can be written in an extremely useful form as follows

$$L = \alpha_0 u^{\frac{n}{2}} \dot{u} \dot{z} - V_0 u^n - \rho_{m0}. \quad (29)$$

We note that because the Lagrangian (29) does not contain the variable z , it is a cyclic variable. Dynamical equations associated to the Lagrangian (29) in the new variables take the simple form as

$$\alpha_0 u^{\frac{n}{2}} \dot{u} = I_0, \quad (30)$$

$$\alpha_0 \ddot{z} + n V_0 u^{\frac{n-2}{n}} = 0, \quad (31)$$

$$\alpha_0 \dot{u} \dot{z} + V_0 u^{\frac{n}{2}} + \rho_{m0} u^{-\frac{n}{2}} = 0, \quad (32)$$

where I_0 is a constant of motion related to the Noether symmetry. After integrating the Eq. (30), we obtain $u(t)$ as

$$u(t) = (kt + u_1)^{\frac{2}{n+2}}, \quad (33)$$

where u_1 is an integration constant, $n \neq -2$ and we define $k = \frac{(n+2)I_0}{2\alpha_0}$. By substituting the solution (33) into the Eq.(31), we finally find $z(t)$ as

$$z(t) = -\frac{2\alpha_0 V_0}{(3n+2)I_0^2} (kt + u_1)^{\frac{3n+2}{n+2}} + z_0 t + z_1, \quad (34)$$

where z_0 and z_1 are an integration constant, $n \neq -\frac{2}{3}$ and $I_0 \neq 0$. From the Eq. (32) we have a constraint as $\rho_{m0} + I_0 z_0 = 0$. From this constraint, it is seen that in the absence of the standard matter, z_0 must be zero. Utilizing these expressions

obtained for u and z in the Eq. (28) we reach the cosmological solutions as follows

$$a(t) = \left[\alpha_0(2-n)(kt+u_1)^{\frac{n-2}{n+2}} \times \left(-\frac{2\alpha_0 V_0}{(3n+2)I_0^2} (kt+u_1)^{\frac{3n+2}{n+2}} + z_0 t + z_1 \right) \right]^{\frac{n}{3(n-2)}},$$

$$\phi(t) = \left[\alpha_0(2-n)(kt+u_1)^{\frac{2-n}{n+2}} \times \left(-\frac{2\alpha_0 V_0}{(3n+2)I_0^2} (kt+u_1)^{\frac{3n+2}{n+2}} + z_0 t + z_1 \right) \right]^{\frac{1}{2-n}} \quad (35)$$

We note that this solution does not valid for $n = -\frac{2}{3}$ and $n = -2$ and such cases have to be examined separately. From the Eqs. (35) and (35) we have, asymptotically,

$$a(t) \sim t^{\frac{4n^2}{3(n^2-4)}}, \quad \phi(t) \sim t^{-\frac{2}{n-2}}, \quad (36)$$

for $n < -2$ and $n > 0$. Then, the scale factor leads to the power-law behavior for $n < -2$ and $n > 2$. For the large values of n we get $a(t) \sim t^{4/3}$. On the other hand, while the scalar field converges for the values of $n > 2$ at $t \rightarrow \infty$, it diverges for $n < -2$.

3.2 Case 2

In this section, we consider the case of the value $n = 2$ in which the potential and function B have the forms $V(\phi) = V_0 \phi^2$ and $B(\phi) = \frac{3}{8} \phi^2$ (see Case 2 of Table 1). In this case, the Eqs. (25) and (26) produce the coordinate transformations as

$$z = -\frac{3}{2\alpha_0} \ln a, \quad u = a^{\frac{3}{2}} \phi, \quad (37)$$

whose inverse transformations are

$$a = e^{-\frac{2\alpha_0}{3}z}, \quad \phi = u e^{\alpha_0 z}. \quad (38)$$

We can rewrite the Lagrangian (4) in terms of new variables z and u using the inverse transformations (38) as follows

$$L = \alpha_0 u \dot{u} \dot{z} + \frac{1}{2} \dot{u}^2 - V_0 u^2 - \rho_{m0}. \quad (39)$$

This Lagrangian gives us the following the equations of motion

$$\alpha_0 u \ddot{u} = I_0, \quad (40)$$

$$\ddot{u} + \alpha_0 u \ddot{z} + 2V_0 u = 0, \quad (41)$$

$$\alpha_0 u \dot{z} + \frac{1}{2} \dot{u}^2 + V_0 u^2 + \rho_{m0} = 0. \quad (42)$$

The above differential equations can be easily solved to give

$$u(t) = \left(\frac{2I_0}{\alpha_0} t + u_1 \right)^{\frac{1}{2}}, \quad (43)$$

$$z(t) = -\frac{V_0}{\alpha_0} t^2 + \left(z_0 - \frac{2V_0 u_1}{I_0} \right) t - \frac{1}{4\alpha_0} \ln(I_0 t + \alpha_0 u_1) + z_1, \quad (44)$$

with a constraint as $\rho_{m0} + I_0 z_0 = 0$. Substituting the above expressions into Eq. (38), we obtain the exact solutions as follows

$$a(t) = (I_0 t + \alpha_0 u_1)^{\frac{1}{6}} e^{\frac{2}{3} \left[V_0 t^2 - \alpha_0 \left(z_0 - \frac{2V_0 u_1}{I_0} \right) t - \alpha_0 z_1 \right]}, \quad (45)$$

$$\phi(t) = \sqrt{\frac{2}{\alpha_0}} (I_0 t + \alpha_0 u_1)^{\frac{1}{4}} e^{\left[-V_0 t^2 + \alpha_0 \left(z_0 - \frac{2V_0 u_1}{I_0} \right) t + \alpha_0 z_1 \right]}. \quad (46)$$

The solutions we obtained above include three integration constants u_1 , z_0 and z_1 . Some restrictions can be made on these constants in terms of other constants appearing in the solutions (45) and (46) by adopting the procedure given in Ref. [80]. First of all, we consider the condition $a(0) = 0$ such that the origin of time is fixed. This condition should be interpreted as an arbitrary choice of the origin of time. By applying this condition to the scale factor (45) we obtain $u_1 = 0$. Then we set the present time $t_0 = 1$. That is, the whole history of the universe is squeezed into the time interval $[0, 1]$. So we can assume the condition $a(t_0 = 1) = 1$ as a standard. This condition yields an expression as $z_1 = \frac{4V_0 - 4z_0\alpha_0 + \ln(I_0)}{4\alpha_0}$. Finally, the last condition is to assume $H(t_0 = 1) = h_0$ where $H(t)$ is Hubble parameter. It should be noted that the parameter h_0 does not give any information about the value obtained from the observations for the standard Hubble constant H_0 . (See also detailed discussion of this procedure Ref. [81].) This last condition gives us $z_0 = \frac{1+8V_0-6h_0}{4\alpha_0}$. By using these restrictions, the scale factor (45) and the scalar field (46) reduce to the following form, respectively

$$a(t) = t^{\frac{1}{6}} e^{\frac{(t-1)}{6} [4V_0(t-1) + 6h_0 - 1]}, \quad (47)$$

$$\phi(t) = \sqrt{\frac{2I_0}{\alpha_0}} t^{\frac{1}{4}} e^{-\frac{(t-1)}{4} [4V_0(t-1) + 6h_0 - 1]}. \quad (48)$$

The Hubble parameter evolves with time for this model given by

$$H(t) = \frac{1}{6} \left[8V_0(t-1) + \frac{1}{t} + 6h_0 - 1 \right]. \quad (49)$$

Other relevant physical quantities such as the deceleration parameter, effective equation of state parameter, fractional energy densities of the matter and scalar field can be easily calculated as

Fig. 1 The evolution of scale factor versus the time t for the values of $h_0 = 1$ and $V_0 = 0.28$. The vertical line shows the present time $t_0 = 1$

$$q(t) = -1 - \frac{48V_0t^2 - 6}{[(8V_0(t-1) + 6h_0 - 1)t + 1]^2}, \quad (50)$$

$$\omega_{eff}(t) = -1 - \frac{2}{3} \frac{48V_0t^2 - 6}{[(8V_0(t-1) + 6h_0 - 1)t + 1]^2}, \quad (51)$$

$$\Omega_m(t) = \frac{4t(6h_0 - 8V_0 - 1)}{[(8V_0(t-1) + 6h_0 - 1)t + 1]^2} = 1 - \Omega_\phi(t). \quad (52)$$

From the above cosmological quantities, we see that this model is only parameterized by V_0 and h_0 . For example, if we choose $h_0 = 1$ and $V_0 = 0.28$, then $\Omega_{m0} \approx 0.3$, $\Omega_{\phi0} \approx 0.7$ and $\omega_{eff} \approx -1.1$ for the present time $t_0 = 1$. These values are acceptable and are in good agreement with cosmological data [82]. In Figs. 1, 2, 3 and 4, we also present the qualitative behavior of some cosmological parameter in terms of time. Figure 1 shows the scale factor with monotonically increasing with time describing an expansionary phase of the universe. Figure 2 relates to the Hubble parameter, which decreases for a short time but then increases again. The deceleration parameter shown in Fig. 3 indicates the existence of a transition from a decelerating expansion of the universe to an accelerating expansion one. Observational data result in that ω_{eff} is in a very close range to $\omega_{eff} = -1$. Finally, we plot the evolution of the effective equation of state parameter as a function of time in Fig. 4. It can be clearly seen from this figure that the state equation parameter leads to the crossing of the phantom dividing line $\omega_{eff} = -1$, that is, a model with a transition from the quintessence phase i.e. $\omega_{eff} = > -1$ to the phantom phase $\omega_{eff} = < -1$.

Fig. 2 The variations of the Hubble parameter with respect to the time t for the values of $h_0 = 1$ and $V_0 = 0.28$. The vertical line shows the present time $t_0 = 1$

Fig. 3 Plot of the deceleration parameter versus the time t for the values of $h_0 = 1$ and $V_0 = 0.28$. The vertical line shows the present time $t_0 = 1$

Fig. 4 The behavior of the effective equation of state parameter with respect to the time t for the values of $h_0 = 1$ and $V_0 = 0.28$. The vertical line shows the present time $t_0 = 1$

4 Tensor perturbation

In many cosmological theories, it is important to look at tensor perturbations in the formation of large structures. The effects of tensor perturbations in metrics are noticeable because they cause significant distortions in cosmic background radiation on large scales. In this section, we introduce tensor perturbations in the metric $g_{\mu\nu}$ as $\delta g_{ij} = a^2 h_{ij}$, where the perturbations h_{ij} are defined to be transverse and traceless, i.e. $\partial^i h_{ij} = 0$ and $g^{ij} h_{ij} = 0$. Here, we lower and raise spatial indices on the perturbations with δ_{ij} . While we should always raise and lower indices with the metric, using the Kronecker delta does not affect the second-order action, so the transverse and traceless conditions can be rewritten as $\delta^{ik} \partial_k h_{ij} = 0$ and $\delta^{ij} h_{ij} = 0$. Also, note that the perturbed metric $\delta g_{\mu\nu}$ is always defined to be first order, however the inverse metric $g^{\mu\nu}$ can be higher order. Applying tensor perturbations to the gravitational part of the action, we obtain:

$$S_{GR}^2 = \frac{1}{8} M_{pl}^2 \int d^3 k dt a^3 B(\phi) \times \left[\dot{h}^{ij} \dot{h}_{ij} - \left(\frac{k^2}{a^2} + 4\dot{H} + 6H^2 \right) h^{ij} h_{ij} \right] \quad (53)$$

Here we have used the Fourier transform convention $h_{ij}(\mathbf{x}, t) = \int \frac{d^3 k}{(2\pi)^{3/2}} h_{ij}(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}$ and we have suppressed the arguments of h_{ij} for simplicity. Let us now calculate the second order perturbations arising from the scalar sector:

$$S_{scalar}^{(2)} = \frac{1}{8} M_{pl}^2 \int d^3 k dt a^3 B(\phi) \left[\frac{V(\phi) - \dot{\phi}^2}{B(\phi)} \right] h^{ij} h_{ij} \quad (54)$$

Combining the second-order actions $S_{GR}^2, S_{scalar}^{(2)}$, gives us the final second-order action for tensor perturbations

$$S^{(2)} = \frac{1}{8} M_{pl}^2 \int d^3 k dt a^3 B(\phi) \times \left[\dot{h}^{ij} \dot{h}_{ij} - \left(\frac{k^2}{a^2} + M_{GW}^2 \right) h^{ij} h_{ij} \right] \quad (55)$$

where M_{GW} is the mass of gravitational wave:

$$M_{GW}^2 = \frac{V(\phi) - \dot{\phi}^2}{B(\phi)} + 4\dot{H} + 6H^2 \quad (56)$$

It can also be written as follows under effective pressure:

$$M_{GW}^2 = \kappa_{eff} \left(V(\phi) - \dot{\phi}^2 - 2P_{eff} \right) \quad (57)$$

In Fig. 1, the case $n = 2$, the mass of gravitational waves is drawn. As you can see, the stability of long-wavelength gravitational waves is guaranteed by the condition $M_{GW}^2 > 0$. If this condition is violated, we will be in the state of a tachyonic mass that will cause instability and the instability would take the age of the universe to develop (Fig. 5).

Gravitational-wave mass from a phenomenological point of view and compatibility with the results of various observa-

Fig. 5 The behavior of the mass of gravitational wave versus the time t for the values of $h_0 = 1$ and $V_0 = 0.28$

tions such as LIGO, VIRGO, Planck and etc, are very important. If we express the mass of gravitational waves in Hertz, we can see that, it is marvelous if the present value $M_{GW,0}$ is within the frequency range to which gravitational wave observatories are sensitive. The sensitivity range of the laser interferometer space antenna (LISA) is in the range of 10^{-4} Hz to 10^{-5} Hz [83], as well as square kilometre array (SKA) [84,85] and the Parkes Pulsar Timing Array (PPRA) are in the range of 10^{-8} Hz [86,87]. Finally, with use of the orbital decays of binary pulsars that range is estimated [88,89]

$$10^{-8} \text{ Hz} < M_{GW,0} < 10^{-5} \text{ Hz} \quad (58)$$

The bottom line is that most observations about the amount of the mass of gravitational waves related to the present time are limiting and cannot be conclusive about varying in the past. Therefore, if the function $B(\phi)$ has a very small value for small redshifts, the mass of the gravitational waves obtained will be consistent with the observational results. For this purpose, the initial conditions and coefficients can be determined, for $n = 2$, if we consider the value of V_0 to be smaller than 10^{-9} , the mass of gravitational waves will be compatible with observational limits.

5 Summary and conclusion

In this paper, we have used the Noether symmetry approach to study Einstein-aether theory, in which a scalar field is coupled with an aether field. This model is sometimes referred to as Lorentz-violated inflation. In order to write the point-like Lagrangian, considering that $B(\phi)$ is a coupling function between the scalar field with aether field, we choose $B(\phi) = \beta_1(\phi) + 3\beta_2(\phi) + \beta_3(\phi) - 1$ in terms of aether parameters $\beta_i(\phi)$, ($i = 1 \dots 4$). The obtained point-like Lagrangian, although similar to scalar-tensor theories, shows theories, shows different dynamics. This theory is reduced to the

quintessence model at $B(\phi) = 1$. In the second part of the paper, after obtaining the point-like Lagrangian of the model, we obtain the equation of motions related to this model. Variation from the action (1) with respect to the scalar field, the Klein–Gordon equation (6) is obtained, which can be seen that if we assume that $B(\phi)$ is not dependent on the scalar field, this equation is the same as the Klein–Gordon equation of the quintessence field. In the third part of the paper, we considered the proposed model using the Noether approach and showed that with appropriate choices for the $V(\phi)$ and $B(\phi)$ functions, the answers are consistent with the Noether symmetry. One of these choices can be $B(\phi) = \frac{3}{2n^2}\phi^2$ and $V(\phi) = V_0\phi^n$, where $n \neq -2, -2/3$ and V_0 is a constant. In this case, we conclude that for $n < -2$ and $n > 0$, the answers asymptotically are $a(t) \sim t^{\frac{4n^2}{3(n^2-4)}}$ and $\phi(t) \sim t^{-\frac{2}{n-2}}$. Similarly, for $n < -2$ and $n > 2$, we get the power-law answer for the scale factor, and for very large n , we can conclude that $a(t) \sim t^{\frac{4}{3}}$. According to the Noether symmetry approach, another choice for potential function and coupling coefficient can be $V(\phi) = V_0\phi^2$ and $B(\phi) = \frac{8}{3}\phi^2$. The solution of the field equations are given in Eqs. (47) and (48) for the scale factor and scalar field, respectively. For this model, we have presented the evolution of some important cosmological parameters over time in Figs. 1, 2, 3 and 4. It should be a note that the model shows that crossing of the phantom divide line can be realized (see Fig. 4). It is concluded that the model can explain the accelerated expansion of the universe, and as a result, the answer obtained from Noether's approach for the model can be a good candidate to explain dark energy.

Observational cosmology is now in its golden years. Physicists and astronomers study the large scale structure of the universe using a variety of observational techniques. Cosmic microwave background radiation (CMB) is the most widely observed observation window in recent years. Anisotropies in CMB are now detected on a wide range of angular scales and provide us with a wealth of information since the last cosmic photons were scattered. For this purpose, in the last section, we examined the tensor disorders in this model. The quantification of gravitational waves is parallel to the quantification of scalar disorders but is simpler because there is no gauge indeterminacy. Note that at the level of linear fluctuations, scalar fluctuations and gravitational waves are independent. Both can be measured in the same cosmological context determined by the background scale factor and the background matter. However, unlike in the case of scalar fluctuations, tensor states are also present in the absence of matter. In the fourth part of the paper, we examined the tensor perturbations in the Einstein-aether model and obtained the mass of gravitational waves. We also concluded that the state of a tachyonic mass arises in the con-

dition $V(\phi) - \dot{\phi}^2 < 2P_{eff}$. This state causes instability that would take the age of the universe to develop.

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