



Revisiting the minimum length in the Schwinger–Keldysh formalism

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Abstract The existence of a minimum length in quantum gravity is investigated by computing the in-in expectation value of the proper distance in the Schwinger–Keldysh formalism. No minimum geometrical length is found for arbitrary gravitational theories to all orders in perturbation theory. Using non-perturbative techniques, we also show that neither the conformal sector of general relativity nor higher-derivative gravity features a minimum length. A minimum length scale, on the other hand, seems to always be present when one considers in-out amplitudes, from which one could extract the energy scale of scattering processes.

1 Introduction

As we live in a world where all of our daily observations take place at scales such as the meter, the second and the kilogram, it is not easy for modern human minds to grasp the possibility that there exists fundamental upper or lower bounds on physical quantities that could otherwise become evident at much smaller or larger scales. Our experience and the convenience of describing it with continuum mathematics therefore make us think that it is natural for physical quantities to admit an infinite range of possible values. In fact, nothing in classical mechanics forbids us from speeding to infinity or dismantling the spacetime into infinitesimally small distances. Yet it is a fact of nature that there exists a limiting speed, which special relativity incorporates and allows us to describe its kinemat-

ical consequences. Naturally, this raises the similar question of whether it is possible to probe decreasingly small lengths or if there is a limiting factor that keep us from accessing some fundamental length scales.

The notion of a minimum length (see Ref. [1] for a in-depth review) dates back to the early days of quantum field theory, when physicists were desperately attempting to get rid of the troubling ultraviolet divergences, but it soon became unattractive with the advent of the more sophisticated methods of renormalization. It only regained notoriety with the increasing interest in trans-Planckian effects. Currently, many models of quantum gravity exhibit some notion of minimum length, including string theory, loop quantum gravity, asymptotically safe gravity and the conformal sector of general relativity. However, some works have established the possibility of a minimum geometrical length by employing the standard Feynman path integral for the calculation of time-ordered in-out amplitudes [2]. These amplitudes are the correct ingredients for obtaining S -matrix elements from the LSZ formula, but are otherwise acausal and complex, being subjected to Feynman boundary conditions. Taken literally, an observable minimum length in quantum gravity should be real to all loop orders and bare the statistical properties of an expectation value. In this respect, it is therefore very important to distinguish between the use of in-out amplitudes and in-in amplitudes, the latter being the objects which admit a proper statistical interpretation. These requirements lead us to study the minimum length using the in-in expectation value, which can be obtained in the Schwinger–Keldysh path

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integral formalism [3] and whose evolution is subjected to retarded boundary conditions [4].

The main goal of this paper is to investigate the distinct properties of the in-in proper distance, which can be directly interpreted as a geometrical length, and the in-out proper “length”, which cannot be interpreted as a physical distance but sets the length scale of the underlying scattering process. As we will see, the former vanishes quite generally at the coincidence limit, suggesting that a geometrical minimum length is most likely absent. On the other hand, when the latter is evaluated at the coincidence limit, it acquires a finite value of the order of the Planck scale under very general assumptions, indicating that a minimum length *scale* is very likely to exist. The implication of these results is that nothing prevents one from going through vanishingly small distances *in principle*, but scattering experiments cannot reliably distinguish between events taking place at the Planck scale, since any two processes differing only at trans-Planckian scales would produce the same scattering amplitudes.

This paper is organized as follows: in Sect. 2, we briefly review some aspects of the Schwinger–Keldysh formalism used for the calculation of in-in amplitudes; in Sect. 3, we show that a minimum length cannot exist to second order in the metric perturbation for any metric theory of gravity whose gravitational propagator can be written as the sum of partial fractions of the form $(q^2 - m^2)^{-1}$, but a minimum length scale is always present. The absence of interactions allows the extension of this result to all orders in perturbation theory, although interacting theories would require the evaluation of higher-order amplitudes; Sect. 4 is devoted to the study of a minimum length in higher-derivative gravity. Without resorting to perturbation theory, we show that higher-derivative gravity does not exhibit any obstruction to the continuous shrinkage of the quantum proper length to zero; in Sect. 5, we revisit the conformal degree of freedom in gravity, which had previously been shown to yield a ground-state length in the in-out approach. The Schwinger–Keldysh formalism allows us to show that the minimum length is again absent in this theory; we finally draw our conclusions and briefly compare with other approaches in Sect. 6.

2 Schwinger–Keldysh formalism

Before elaborating on the minimum length, we need to clarify an important point that has been largely ignored in the literature. In all calculations of the expectation value of the proper length $\langle ds^2 \rangle$, the in-out formalism has been implicitly employed with no proper justification, which makes $\langle ds^2 \rangle$ a short-hand notation for $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$, namely some sort of transition amplitude from an in-vacuum state to an out-vacuum state. This is the standard kind of amplitude obtained from functional derivatives of the generating functional $Z[J]$ which results from Feynman path integrals and satisfies Feynman boundary conditions. Transition amplitudes are in general acausal and complex (even for Hermitian operators) distributions, thus they cannot make up the list of observables of a quantum field theory. This is usually not an issue because they only show up in intermediate steps of the calculation of S -matrix components, eventually yielding cross sections, which are the ultimate object of interest in scattering experiments.

Although the in-out formalism is the standard approach for the calculation of scattering amplitudes, its use obscures the physical interpretation of the quantum proper length. For the above reasons, the fact that $|0_{\text{in}}\rangle \neq |0_{\text{out}}\rangle$ makes it impossible to interpret $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$ as an expectation value or to attribute to it any statistical meaning. It appears hard to accept that a length which is neither real nor respects causality can bare any physical reality. In order to talk of a minimum length, we need to calculate $\langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle$ instead, namely the quantum proper length evaluated on one and the same quantum state $|0_{\text{in}}\rangle$. It is important to remark that the in-in mean field $\langle 0_{\text{in}} | \phi | 0_{\text{in}} \rangle$ not only is real for Hermitian operators ϕ , it also evolve causally, which is particularly important for time-dependent settings in which ones does not know, or is not interested in, the final state $|0_{\text{out}}\rangle$ of the system.

The calculation of in-in amplitudes does not follow directly from the usual Feynman path integral, but it can be performed using the slightly different Schwinger–Keldysh formalism (or closed-time path integral) [3,4]. The idea is to double each degree of freedom ϕ and commonly denote the two peers with ϕ_+ and ϕ_- . The field ϕ_+ is generated by an external source J_+ and is responsible for the transition between $|0_{\text{in}}\rangle$ and an intermediate state $|\Sigma_\alpha\rangle$ belonging to a future Cauchy surface Σ , while ϕ_- is generated

by J_- and takes care of the transition from $|\Sigma_\alpha\rangle$ back to $|0_{in}\rangle$. Assuming $\{|\Sigma_\alpha\rangle\}$ form a complete set of states, the functional generator of connected in-in correlation functions is then obtained by summing over all possible intermediate states $|\Sigma_\alpha\rangle$, to wit

$$e^{iW[J_+, J_-]} = \sum_\alpha \langle 0_{in} | \Sigma_\alpha \rangle_{J_-} \langle \Sigma_\alpha | 0_{in} \rangle_{J_+}. \tag{1}$$

If we further assume that $\{|\Sigma_\alpha\rangle\}$ are eigenstates of ϕ on Σ , we can write Eq. (1) in terms of Feynman path integrals as

$$e^{iW[J_+, J_-]} = \int \mathcal{D}\phi_+ \mathcal{D}\phi_- e^{\frac{i}{\hbar}\{S[\phi_+] + S[\phi_-] + J_+ \phi_+ - J_- \phi_-\}}, \tag{2}$$

where the integration variables are subjected to vacuum boundary conditions in the remote past (corresponding to the state $|0_{in}\rangle$) and $\phi_+ = \phi_-$ on Σ . The various in-in correlation functions are obtained by functionally differentiating $W[J_+, J_-]$ with respect to the sources and setting $J_+ = J_- = 0$ in the end. Because there are now two types of fields and two types of sources, there will be two kinds of vertices and four kinds of propagators involved in Feynman diagrams, namely

$$G_{ab}(x, x') = \frac{\hbar \delta}{\text{sign}(a) i \delta J_a(x)} \frac{\hbar \delta}{\text{sign}(b) i \delta J_b(x')} e^{iW[J_+, J_-]} \Big|_{J_+ = J_- = 0}, \tag{3}$$

where

$$\text{sign}(a) = \begin{cases} +1 & \text{for } a = + \\ -1 & \text{for } a = -. \end{cases} \tag{4}$$

The diagonal components of G_{ab} correspond to the Feynman and anti-Feynman propagators,

$$G_{++}(x, x') = \langle 0_{in} | T \phi(x) \phi(x') | 0_{in} \rangle \tag{5}$$

$$G_{--}(x, x') = \langle 0_{in} | \bar{T} \phi(x) \phi(x') | 0_{in} \rangle, \tag{6}$$

where T and \bar{T} denote the time-ordered and anti time-ordered operators, respectively. The off-diagonal components correspond to Wightman correlation functions,

$$G_{+-}(x, x') = \langle 0_{in} | \phi(x') \phi(x) | 0_{in} \rangle \tag{7}$$

$$G_{-+}(x, x') = \langle 0_{in} | \phi(x) \phi(x') | 0_{in} \rangle. \tag{8}$$

Apart from the additional vertices and propagators, the in-in Feynman rules are identical to the standard ones.

For our purposes, the most important features of the Schwinger–Keldysh formalism are the reality and causality of the in-in mean field $\langle 0_{in} | g_{\mu\nu} | 0_{in} \rangle$, and consequently of $\langle 0_{in} | ds^2 | 0_{in} \rangle$. These properties can be verified at every loop order by using the effective equations derived from the in-in effective action $\Gamma[\phi_+, \phi_-]$, which is in turn given by the Legendre transform of the in-in generating functional $W[J_+, J_-]$ with respect to the sources J_\pm . The reality of the mean field is crucial for the interpretation of $\langle 0_{in} | ds^2 | 0_{in} \rangle$ as a physical length, whereas its causality uniquely determines the retarded Green’s function $G^{\text{ret}} = G_{++} - G_{+-}$ as the correct propagator to be used for the calculation of the minimum length in the next section. We refer the reader to Refs. [4,5] for the detailed proof of the reality and causality of the mean field.

3 Absence of a minimum length, presence of a minimum length scale

In the present section, we use the results of Sect. 2 to elaborate a model-independent argument for the absence of a minimum geometrical distance to all orders of perturbation theory. We only assume that the gravitational field is described by a metric tensor $g_{\mu\nu}$ for which a background value $\bar{g}_{\mu\nu}$ exists in the vacuum $|0_{in}\rangle$, and on which its quantum fluctuations are free of interactions. While the latter is obviously unrealistic, it should be enough for grasping the idea of a minimum length. In fact, we would expect that a minimum length could exist as a consequence of quantum fluctuations, which promote uncertainties in the proper length regardless of whether they are interacting or not.

Instead of parameterizing the quantum field by the usual linear perturbation $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, we shall use the exponential parameterization previously considered in Refs. [6–8], that is ¹

¹ With this parameterization, the quantum fluctuation $h_{\mu\nu}$ has the dimensions of a canonical scalar field, that is $\sqrt{\text{mass}/\text{length}}$.

$$\begin{aligned}
 g_{\mu\nu} &= \bar{g}_{\mu\rho} \left(e^{\sqrt{\frac{32\pi\ell_p}{m_p}} h} \right)^\rho_{\nu} \\
 &= \bar{g}_{\mu\nu} + \sqrt{\frac{32\pi\ell_p}{m_p}} h_{\mu\nu} + \frac{16\pi\ell_p}{m_p} h_{\mu\rho} h^\rho_{\nu} \\
 &\quad + O\left((\ell_p/m_p)^{3/2}\right), \tag{9}
 \end{aligned}$$

where $\ell_p = \sqrt{G_N \hbar}$ and $m_p = \sqrt{\hbar/G_N}$ denote the Planck length and mass, respectively. The exponential parameterization has the advantage of transforming the problem of calculating the expectation value of ds^2 into the problem of computing correlation functions of the quantum field $h_{\mu\nu}$. Note that, classically, there is nothing that prevents the proper distance between two spacetime points of coordinates x^μ and y^μ from going to zero in the limit in which $dx^\mu = y^\mu - x^\mu$ vanish and the points coincide. We thus expect

$$\lim_{x \rightarrow y} ds^2 = \lim_{x \rightarrow y} (\bar{g}_{\mu\nu} dx^\mu dx^\nu) \equiv \lim_{x \rightarrow y} [\ell^2(x, y)] = 0, \tag{10}$$

for any classical metric $\bar{g}_{\mu\nu}$. Nonetheless, since the expectation value of quadratic and higher-order quantities evaluated at the same spacetime event, such as $\langle 0_{\text{in}} | h_{\mu\rho}(x) h^\rho_\nu(x) | 0_{\text{in}} \rangle$, are divergent in quantum field theory, the coincidence limit of the quantum proper length must be computed with care. In fact, we must first regularize the divergences as there might be occasional cancelations leading to a minimal length. Because we are interested only in the coincidence limit, it is natural to isolate the divergences with the covariant point-splitting, namely

$$\begin{aligned}
 &\langle 0_{\text{in}} | h_{\mu\rho}(x) h^\rho_\nu(x) | 0_{\text{in}} \rangle \\
 &= \lim_{x \rightarrow y} \langle 0_{\text{in}} | h_{\mu\rho}(x) h^\rho_\nu(y) | 0_{\text{in}} \rangle, \tag{11}
 \end{aligned}$$

with similar expressions for higher-order correlators. This allows us to write the quantum proper length in terms of correlation functions, which at second order in $h_{\mu\nu}$ reads

$$\begin{aligned}
 \lim_{x \rightarrow y} \langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle &= \lim_{x \rightarrow y} (\langle 0_{\text{in}} | g_{\mu\nu} | 0_{\text{in}} \rangle dx^\mu dx^\nu) \\
 &= \frac{16\pi\ell_p}{m_p} \lim_{x \rightarrow y} [\langle 0_{\text{in}} | h_{\mu\rho}(x) h^\rho_\nu(y) | 0_{\text{in}} \rangle dx^\mu dx^\nu] \\
 &\equiv \frac{16\pi\ell_p}{m_p} \lim_{x \rightarrow y} [G_{\mu\rho}^\rho{}_\mu(x, y) dx^\mu dx^\nu] \tag{12}
 \end{aligned}$$

where we used the expansion in Eq. (9) together with the fact that the contribution at zero separation vanishes according to Eq. (10), as well as does the first order $\langle 0_{\text{in}} | h_{\mu\nu} | 0_{\text{in}} \rangle = 0$.

The question of a minimum length is thus translated into the calculation of the in-in gravitational propagator $G_{\mu\rho}^\rho{}_\mu$. But as we saw in Sect. 2, there are four different types of propagators associated to in-in processes and, furthermore, they can be combined into other propagators, such as the retarded and the advanced ones. The immediate consequence is that $\langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle$ appears ambiguous as there is a priori no reason to choose one propagator over the others. In our case, the way we determine the relevant propagator should depend on how one measures distances between two points at such (expectedly Planckian) small scales. Such a measurement can take place via scattering processes (e.g. to determine the mean free path), which requires the Feynman propagator, or via the observation of a certain signal at different times along its evolution, which would require the retarded Green’s function. Thus, the requirement of causality in the evolution of $\langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle$ entails the use of the retarded Green’s function [4,5]. Note that mid-step calculations will involve all four types of Green’s functions, but the final result must necessarily depend solely on the retarded Green’s function due to causality. In fact, as shown in Refs. [5] (see also [9] for a detailed review), the in-in correlation functions (to any loop order) are obtained by replacing form factors by retarded Green’s functions. In the asymptotically flat and empty spacetime, this agrees with the boundary conditions in the remote past of the mean field.

The calculation of propagators for an arbitrary curved background $\bar{g}_{\mu\nu}$ only add unnecessary complication, thus we shall take $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ as the Minkowski spacetime in the rest of this paper. Our argument can then be generalised to curved spaces with the aid of the Schwinger proper-time representation for propagators. We shall also treat $h_{\mu\nu}$ as a free field and assume the gravitational propagator in momentum space to take the simplest form of a sum over the number of simple poles m_i^2 in the q^2 -plane, that is

$$\Delta_{\mu\nu\rho\sigma}(q^2) = \sum_i \frac{\hbar P_{\mu\nu\rho\sigma}^i}{q^2 - m_i^2}, \tag{13}$$

where

$$P_{\mu\nu\rho\sigma}^i = \alpha^i \eta_{\mu\rho} \eta_{\nu\sigma} + \beta^i \eta_{\mu\sigma} \eta_{\nu\rho} + \gamma^i \eta_{\mu\nu} \eta_{\rho\sigma} \tag{14}$$

is the most general tensorial structure that can be combined into a tensor of fourth rank and which is symmetric in $\{\mu\nu\}$

and $\{\rho\sigma\}$. The coefficients α^i, β^i and γ^i take different values according to the theory at hand. The propagator in position space is obtained from the ϵ -prescription or, equivalently, the integration contour corresponding to the retarded boundary condition and reads

$$G_{\mu\nu\rho\sigma}^{\text{ret}}(x, y) = \sum_i \left[-\frac{\theta(x^0 - y^0)}{2\pi} \delta(\ell^2) + \theta(x^0 - y^0) \theta(\ell^2) \frac{m_i J_1(m_i \ell)}{4\pi \ell} \right] \hbar P_{\mu\nu\rho\sigma}^i, \tag{15}$$

where $\ell^2 \equiv \ell^2(x, y) = \eta_{\mu\nu} dx^\mu dx^\nu$ is the background proper distance between x and $y = x + dx$. The contraction $P_{\mu\rho}^i{}^\rho{}_\nu dx^\mu dx^\nu$ will always result in a factor of ℓ^2 in the numerator that can potentially be canceled by a divergence ℓ^{-2} of the propagator, leaving a non-zero minimum length behind. Note, however, that the first term above only contains a Dirac delta divergence that cannot be canceled by ℓ^2 and actually vanishes on integration, whereas the second term diverges as ℓ^{-1} and cannot prevent ℓ^2 from going to zero. Putting this all together, gives

$$\lim_{x \rightarrow y} \langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle = \frac{16\pi \ell_p}{m_p} \lim_{x \rightarrow y} [G_{\mu\rho}^{\text{ret}\rho}{}_\nu(x, y) dx^\mu dx^\nu] = 0 \tag{16}$$

and we conclude that there is no minimum length to second order in $h_{\mu\nu}$. For an interacting theory, this does not imply the absence of a minimum length to all orders in perturbation theory. In the free theory, however, Wick’s theorem can be used to reduce higher-order vacuum correlation functions into a sum over products of the propagator, leading to

$$\langle h^{n+2} \rangle \sim \frac{1}{\ell^{n+2}}, \quad n = 1, 2, \dots \tag{17}$$

which suggests that there is no other relevant correlation function (in addition to the one for $n = 0$) that could possibly cancel the vanishing length ℓ^2 to produce a non-zero minimum length, thus extending Eq. (16) to all orders in $h_{\mu\nu}$. This is in fact confirmed by the following non-perturbative calculation. From Eq. (9) we have,

$$\begin{aligned} \lim_{x \rightarrow y} \langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle &= \lim_{x \rightarrow y} \left[\eta_{\mu\rho} \langle 0_{\text{in}} | \left(e^{\frac{1}{2} \sqrt{\frac{32\pi\ell_p}{m_p}} h(x)} e^{\frac{1}{2} \sqrt{\frac{32\pi\ell_p}{m_p}} h(y)} \right)^\rho | 0_{\text{in}} \rangle dx^\mu dx^\nu \right] \\ &= \lim_{x \rightarrow y} \left[\eta_{\mu\rho} \left(e^{\frac{8\pi\ell_p}{m_p} \langle 0_{\text{in}} | h(x) h(y) | 0_{\text{in}} \rangle} \right)^\rho dx^\mu dx^\nu \right] \end{aligned}$$

$$\begin{aligned} &= \lim_{x \rightarrow y} \left[\ell^2 e^{-4\ell_p^2 \theta(x^0 - y^0) \delta(\ell^2) \sum_i (\alpha^i + 4\beta^i + \gamma^i)} \right] \\ &= 0, \end{aligned} \tag{18}$$

where we used point-splitting in the first line, applying normal ordering in both exponential operators separately, and the Baker–Campbell–Hausdorff formula together with Wick’s theorem in the second equality. The third equality is obtained by manipulating the exponential as an infinite series and resumming back to the exponential form.² Free gravitational fluctuations are thus not prone to minimum length. Even when interactions are switched on, loop corrections to the free propagator cannot change this picture at second order. In fact, the dressed propagator can be written in the Källén–Lehmann spectral representation in terms of the free propagator itself as

$$G_{\mu\nu\rho\sigma}^{\text{dressed}}(x, y) = \int_0^\infty d\mu^2 \rho(\mu^2) G_{\mu\nu\rho\sigma}^{\text{ret}}(x - y; \mu^2) \tag{19}$$

where $\rho(\mu^2)$ is the spectral density. Therefore, replacing G^{ret} with G^{dressed} in Eq. (16) would still give zero. However, in the interacting theory one can no longer rely on Wick’s theorem to express higher-order correlation functions as products of the two-point function. The vanishing of $\langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle$ at second order does therefore not allow us to come to any definite conclusion about the existence of a minimum length in an interacting theory.

Before continuing, let us comment on the in-out proper “length” $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$. Although we have emphasized that it cannot be interpreted as a physical length or a statistical quantity, it might suggest the existence of a minimum length *scale*. If we repeat the above argument for the in-out amplitude, we find

$$\begin{aligned} \lim_{x \rightarrow y} \langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle &= \mathcal{N} \lim_{x \rightarrow y} \left[\ell^2 e^{\frac{2\ell_p^2}{\pi} \sum_i (\alpha^i + 4\beta^i + \gamma^i)} \right] \\ &= \begin{cases} \frac{2\ell_p^2}{\pi} \sum_i (\alpha^i + 4\beta^i + \gamma^i) \sim \ell_p^2 & \text{for } \sum_i (\alpha^i + 4\beta^i + \gamma^i) > 0 \\ 0 & \text{for } \sum_i (\alpha^i + 4\beta^i + \gamma^i) \leq 0, \end{cases} \end{aligned} \tag{20}$$

where $\mathcal{N} = \langle 0_{\text{out}} | 0_{\text{in}} \rangle$ is a normalization factor chosen to cancel divergences at $\ell = 0$. We used the Feynman propagator for small distances,

² We defined the product of Dirac deltas as a convolution $\delta^2 \rightarrow \delta * \delta = \delta$.

$$G_{\mu\nu\rho\sigma}^F(x, y) = \sum_i \frac{\hbar P_{\mu\nu\rho\sigma}^i}{4\pi^2(x-y)^2} + \mathcal{O}(|x-y|), \tag{21}$$

which is obtained by Fourier transforming Eq. (13) with Feynman boundary conditions.

Note that the exponential is divergent, but one can isolate the divergences from the finite part by expanding the exponential function as a Taylor series. The zeroth order term is simply ℓ^2 which goes smoothly to zero. The first order term in the expansion is also finite (but non-zero) because of the cancelation of ℓ^2 in the numerator and in the denominator. Higher-order terms contain the divergences. Nonetheless, one can use the arbitrariness of the normalization factor \mathcal{N} to cancel the divergent part so that the limit $\ell \rightarrow 0$ is finite. Eq. (20) points at the Planck scale as a potential limiting factor that screens everything that goes beyond it. This is not to say that physical distances cannot vanish, but it suggests that scattering experiments cannot tell apart trans-Planckian effects. In the foreseeable future, astrophysics and cosmology seem to be the only hope to probe quantum gravity experimentally.

We kept the argument completely general, without the need of specifying the gravitational theory, thus the conclusions above are quite general with the only restriction that the gravitational field be described solely in terms of the metric. Different theories will differ by their propagators with different values for the coefficients α^i , β^i and γ^i , but they will all produce vanishing minimum lengths and non-zero minimum length scales of Planckian order unless

$$\sum_i (\alpha^i + 4\beta^i + \gamma^i) \leq 0. \tag{22}$$

In general relativity, for example, the massless spin-2 field (graviton) is the only degree of freedom,

$$\hbar^{-1} \Delta_{\mu\nu\rho\sigma} = \frac{\eta_{\rho\mu} \eta_{\sigma\nu} + \eta_{\sigma\mu} \eta_{\rho\nu} - \eta_{\mu\nu} \eta_{\rho\sigma}}{q^2}. \tag{23}$$

The above considerations imply that no minimum length exists for general relativity, but a minimum length scale is again inferred from

$$\lim_{x \rightarrow y} \langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle = \frac{8 \ell_p^2}{\pi}. \tag{24}$$

More general theories of gravity are expected to contain other degrees of freedom in addition to the graviton. This is evident in higher-derivative theories where new degrees of freedom are essential for the renormalizability of the theory. For example, the propagator of Stelle’s theory reads [10, 11]

$$\hbar^{-1} \Delta_{\mu\nu\rho\sigma} = \frac{2 P_{\mu\nu\rho\sigma}^{(2)} - P_{\mu\nu\rho\sigma}^{(0)}}{q^2} - \frac{2 P_{\mu\nu\rho\sigma}^{(2)}}{q^2 - m_2^2} + \frac{P_{\mu\nu\rho\sigma}^{(0)}}{q^2 - m_0^2}, \tag{25}$$

where $P_{\mu\nu\rho\sigma}^{(i)}$ are spin-projection operators, and one can see the additional massive degrees of freedom, namely a scalar excitation of mass $\hbar m_0$ and a spin-2 particle of mass $\hbar m_2$, that turn out to make the theory renormalizable. The minimum length scale in this case vanishes

$$\lim_{x \rightarrow y} \langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle = 0. \tag{26}$$

due to accidental cancelations of the coefficients in the numerator $\sum_i (\alpha^i + 4\beta^i + \gamma^i) = 0$. When self-interactions are considered for $h_{\mu\nu}$, all the three degrees of freedom will couple to each other, making the whole analysis much more difficult. In this scenario, Wick’s theorem is of no help to us and nothing can be said about the contributions from higher-order correlation functions, thus a non-perturbative treatment is certainly desirable. This is the subject of the following section.

4 A non-perturbative example: higher-derivative gravity

In this section, we compute the quantum proper length $\langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle$ non-perturbatively for higher-derivative gravity without resorting on the exponential parameterization used in the last Section. The idea is to perform field redefinitions in the action in order to make the additional degrees of freedom explicit from the outset.

The action of higher-derivative gravity reads

$$S = \frac{m_p}{16\pi \ell_p} \int d^4x \sqrt{-g} \times \left(R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right), \tag{27}$$

where R , $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the Ricci scalar, Ricci tensor and Riemann tensor of the metric $g_{\mu\nu}$, respectively,³ and c_i are dimensionful coupling constants. The above action contains massive particles of spin-0 and spin-2 in addition to the usual graviton which corresponds to the massless spin-2 excitation. All these degrees of freedom can be made explicit in the action via a Legendre transform [12] followed by a field redefinition of the form [13]

$$g_{\mu\nu} = e^{-\sqrt{\frac{16\pi \ell_p}{3m_p}} \chi} \bar{g}_{\mu\nu}, \tag{28}$$

resulting in the action [13]

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{m_p}{16\pi \ell_p} \bar{R} - \frac{1}{2} \bar{\nabla}^\mu \chi \bar{\nabla}_\mu \chi - \frac{3m_p}{32\pi \ell_p} m_0^2 \left(1 - e^{-\sqrt{\frac{16\pi \ell_p}{3m_p}} \chi} \right)^2 - \frac{m_p}{16\pi \ell_p} \bar{G}_{\mu\nu} \pi^{\mu\nu} + \frac{m_p}{64\pi \ell_p} m_2^2 (\pi_{\mu\nu} \pi^{\mu\nu} - \pi^2) \right], \tag{29}$$

where $\pi \equiv \bar{g}^{\mu\nu} \pi_{\mu\nu}$, $m_0 = (6c_1 + 2c_2 + 2c_3)^{-1/2}$ is the inverse Compton length of the scalar field χ and $m_2 = (-c_2 - 4c_3)^{-1/2}$ that of the massive spin-2 particle $\pi_{\mu\nu}$. Note that the action for $\pi_{\mu\nu}$ is not in canonical form (it does not even contain a kinetic term). Canonicalizing $\pi_{\mu\nu}$ requires an additional field redefinition (see Ref. [13]). This additional field redefinition gives rise to the kinetic term of $\pi_{\mu\nu}$ as well as it makes explicit the coupling between $\pi_{\mu\nu}$ and χ . Nonetheless, the frame with a canonical $\pi_{\mu\nu}$ is no better than any other frame. We chose to work in the frame (29) because it simplifies the calculation of the minimum length.

We interpret $\bar{g}_{\mu\nu}$ as a classical background⁴ where the quantum fields χ and $\pi_{\mu\nu}$ live on and, as before, we consider the Minkowski background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Since there is no explicit interaction of χ with $\pi_{\mu\nu}$ in the action (29), we can focus solely on the spin-0 sector. From the translational symmetry of the path integral measure, we can shift $\chi \rightarrow \chi + \chi_0$

³ Note that the square of the Riemann tensor is usually eliminated in favour of the other two curvature invariants by invoking Gauss-Bonnet theorem. Here we choose to leave it explicit in the action just to follow the same notations commonly used in the literature.

⁴ Fluctuations $h_{\mu\nu}$ of $\bar{g}_{\mu\nu}$ would only contribute to higher-order terms $h \chi \square \chi \sim \mathcal{O}(E^5/m_p^5)$, which are negligible to leading order.

and take $\chi_0 \rightarrow \infty$, which simplifies the spin-0 action to [14]

$$S_\chi = \frac{1}{2} \int d^4x \chi \square \chi, \tag{30}$$

where we discarded a constant term as it does not contribute to the equations of motion. The retarded propagator for χ is thus simply given by the propagator of a massless scalar field [14]

$$\langle 0_{\text{in}} | \chi(x) \chi(y) | 0_{\text{in}} \rangle = -\hbar \frac{\theta(x^0 - y^0)}{2\pi} \delta(\ell^2). \tag{31}$$

From Eqs. (28) and (31), the quantum proper length in the in-vacuum state vanishes in the coincidence limit as

$$\begin{aligned} & \lim_{x \rightarrow y} \langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle \\ &= \lim_{x \rightarrow y} \left[\left\langle 0_{\text{in}} \left| e^{-\frac{1}{2} \sqrt{\frac{16\pi \ell_p}{3m_p}} \chi(x)} e^{-\frac{1}{2} \sqrt{\frac{16\pi \ell_p}{3m_p}} \chi(y)} \right| 0_{\text{in}} \right\rangle, \eta_{\mu\nu} dx^\mu dx^\nu \right] \\ &= \lim_{x \rightarrow y} \left[\ell^2 e^{\frac{4\pi \ell_p}{3m_p}} \langle 0_{\text{in}} | \chi(x) \chi(y) | 0_{\text{in}} \rangle \right] \\ &= 0. \end{aligned} \tag{32}$$

As before, we performed a point-splitting in the first line, imposing normal ordering in each of the exponential operators separately. The second equality follows from the Baker–Campbell–Hausdorff formula in combination with Wick’s theorem.⁵ Therefore, the finding (32) confirms that the vanishing of the quantum proper length observed in Eq. (16) for non-interacting fluctuations $h_{\mu\nu}$ actually extends to the interacting case as well. Similarly, the in-out proper “length” reads

$$\begin{aligned} \lim_{x \rightarrow y} \langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle &= \mathcal{N} \ell^2 \lim_{x \rightarrow y} \left[e^{\frac{4\pi \ell_p}{3m_p}} \langle 0_{\text{out}} | \chi(x) \chi(y) | 0_{\text{in}} \rangle \right] \\ &= \frac{\ell_p^2}{3\pi}, \end{aligned} \tag{33}$$

⁵ Notice that we started with the full interacting theory Eq. (29), but we managed to reduce the scalar sector to that of a free scalar field (30), which permits the application of the Wick’s theorem. That is not to say that χ is physically free of interactions as the action we started with does contain interaction terms among all degrees of freedom. In fact, choosing a background other than Minkowski would invalidate our argument, since the path integral measure would no longer have translational symmetry. Non-trivial path integral contributions would then come into play, making the calculation very difficult at the non-perturbative level. As long as we make the simplifying assumption that the background is Minkowski, non-perturbative calculations are possible. The same applies to the example of Sect. 5

where we again chose the normalization factor \mathcal{N} to absorb the divergence and we used

$$\langle 0_{\text{out}} | \chi(x) \chi(y) | 0_{\text{in}} \rangle = \frac{\hbar}{4\pi^2(x-y)^2}. \tag{34}$$

This shows that the finite part of $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$ is not zero and indicates the existence of a minimum length scale. It is important to stress that Eq. (33) is a non-perturbative result which takes into account all interactions between the degrees of freedom present in the theory. This explains the difference with respect to the non-interacting case in Eq. (26).

5 Revisiting the conformal degree of freedom

In Ref. [2], it was argued that a Planckian minimum length exists when one quantizes the conformal degree of freedom of general relativity on a classical background. This was performed by first parameterizing the metric as⁶

$$g_{\mu\nu} = (1 + \phi)^2 \bar{g}_{\mu\nu}, \tag{35}$$

which separates the conformal degree of freedom ϕ from the other degrees of freedom present in the classical background $\bar{g}_{\mu\nu}$. In the parameterization (35), the Einstein–Hilbert action becomes

$$S = \frac{m_p}{16\pi\ell_p} \int d^4x \sqrt{-\bar{g}} \times \left[\bar{R} (1 + \phi)^2 - 2\Lambda (1 + \phi)^4 - 6\partial_\mu\phi\partial^\mu\phi \right]. \tag{36}$$

In a Minkowski background, namely $\bar{R} = \Lambda = 0$, the action effectively becomes that of a free and massless scalar field. Because of the simplicity of the action when $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, one is able to perform non-perturbative calculations. Upon quantizing the conformal degree of freedom ϕ , its Feynman propagator can be easily obtained as⁷

$$\langle 0_{\text{out}} | \phi(x) \phi(y) | 0_{\text{in}} \rangle = \frac{\hbar\ell_p^2}{3\pi m_p(x-y)^2}. \tag{37}$$

⁶ We keep the field ϕ dimensionless here, instead of choosing the canonical normalization of previous sections, in order to ease the comparison with the original work [2].

⁷ The non-standard numerical factor appears because of the non-canonical normalization of ϕ .

The quantum proper distance $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$ in the in-out formalism was then calculated with the aid of the point-splitting regularization as in Section 3. One therefore obtains

$$\begin{aligned} & \lim_{x \rightarrow y} \langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle \\ &= \lim_{x \rightarrow y} \left[\langle 0_{\text{out}} | \phi(x) \phi(y) | 0_{\text{in}} \rangle \eta_{\mu\nu} dx^\mu dx^\nu \right], \\ &= \frac{\ell_p^2}{3\pi}, \end{aligned} \tag{38}$$

which precisely equals the result (33).

However, as we stressed previously, $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$ should not be interpreted as a physical distance because it is a complex number in general. Eq. (38) only gives a real result because it was computed at the tree level, but when loop corrections are taken into account, an imaginary part shows up in Eq. (38). The correct way of computing geometrical distances at the quantum level is via in-in amplitudes, in which case we must replace the Feynman propagator (37) with the retarded propagator (31) (with ϕ in place of χ and taking into account the field normalizations), which yields

$$\begin{aligned} & \lim_{x \rightarrow y} \langle 0_{\text{in}} | ds^2 | 0_{\text{in}} \rangle \\ &= \lim_{x \rightarrow y} \left[(1 + \langle 0_{\text{in}} | \phi(x) \phi(y) | 0_{\text{in}} \rangle) \ell^2 \right] \\ &= 0, \end{aligned} \tag{39}$$

showing, once again, the absence of a minimum length.

6 Conclusions

In this paper, we have reconsidered the idea of a minimum geometrical length in quantum gravity through the lens of the Schwinger–Keldysh formalism, from which in-in amplitudes can be derived. Because the in-in quantum proper distance is calculated from a single state, one is able to interpret it as a truly geometrical length that happens to be real at all loop orders and satisfies a causal equation of motion, which is manifested via retarded Green’s functions. When the in-in proper length is evaluated at coinciding points, we used perturbative arguments to show it vanishes at second order for any metric theory of gravity. In the absence of interactions, this result can be extended to all orders of perturbation theory. Under suitable reparametrizations of the metric, we also showed non-perturbatively that a minimum length cannot exist in higher-derivative gravity or in the conformal sec-

tor of general relativity. Whereas the requirement of reality should be obvious for the notion of a geometrical distance, one might argue why causality is also a welcome property. The use of the retarded propagator demanded by the in-in formalism implies that quantum corrections to the distance between two spacetime points will always vanish when the points lie outside the respective light cones in the background metric. This result therefore appears as a consistency condition for the very existence of a background metric and the geometrical description of gravity.⁸ Moreover, and indeed equivalently, this result implies that the free propagation of physical signals of any frequency will not be affected by a fundamental length scale. Their dispersion relation will be simply determined by the background metric and quantum gravity effects cannot be probed by detecting the way signals travel through spacetime.

While a geometrical minimum length seems to be unlikely, we made the case for a minimum length scale, namely the scale extracted from the in-out amplitude $\langle 0_{\text{out}} | ds^2 | 0_{\text{in}} \rangle$ at the coincidence limit. By following the same reasoning as for the in-in length, we found theoretical evidence that points at the Planck length as a universal scale beyond which scattering experiments become useless as, even in principle, they cannot distinguish between physical effects taking place at energies $E \gtrsim m_p$. This only reinforces the need for a change of paradigm in quantum field theory from scattering experiments to time-dependent evolutions, which signifies the importance of in-in amplitudes in physics. Of course, one could further argue that most physical processes involve scatterings at some level. For instance, the physical signals we can detect will have been produced by interactions, whose field theoretic description is given in terms of an S -matrix involving Feynman propagators. Here is where the minimum length *scale* seems to enter the picture again, opening up the possibility of probing quantum gravity indirectly from the imprints left in the signals at lower energies.⁹

⁸ Note that the background metric could still be determined self-consistently by solving effective field equations which include loop corrections without affecting our argument.

⁹ We also mention that scatterings at trans-Planckian energies are expected to produce (quantum or semiclassical) black holes and the issue of a minimum length becomes intertwined with the features of such non-perturbative configurations, which generalised uncertainty principles were envisioned to account for [1, 15].

We would like to conclude by remarking once more that the basic assumption in our analysis is the existence of a background metric (irrespective of what that metric actually is). Approaches which lead to the appearance of a minimum geometric length must somehow violate this requirement. For instance, the resemblance of general relativity to thermodynamics [16] suggests that the classical geometry of spacetime is an emergent phenomenon, very much like the notion of thermodynamics for a classical fluid emerges from the statistical mechanics of a more fundamental microscopic theory. Waves in such a fluid can be produced and freely propagate only if their wavelength is significantly larger than the scale of the underlying microscopic structure. This brings forth the questions of what is the fundamental dynamics of gravity at the Planck scale and, not less important, what is the quantum state $|0_{\text{in}}\rangle$, which describe the Universe as we see it. Results from effective field theoretic descriptions at experimentally accessible scales can hopefully serve as a guideline in this quest.

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