THE EUROPEAN
PHYSICAL JOURNAL C

Induced fermionic current in AdS spacetime in the presence of a cosmic string and a compactified dimension

S. Bellucci[1,a] , **W. Oliveira dos Santos**[1,2,b], **E. R. Bezerra de Mello**[2,c]

[1] INFN, Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy
[2] Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, João Pessoa, PB 58059-900, Brazil

Abstract In this paper, we consider a massive charged fermionic quantum field and investigate the current densities induced by a magnetic flux running along the core of an idealized cosmic string in the background geometry of a 5-dimensional anti-de Sitter spacetime, assuming that an extra dimension is compactified. Along the compact dimension quasi-periodicity condition is imposed on the field with a general phase. Moreover, we admit the presence of a magnetic flux enclosed by the compactified axis. The latter gives rise to Ahanorov–Bohm-like effect on the vacuum expectation value of the currents. In this setup, only azimuthal and axial current densities take place. The former presents two contributions, with the first one due to the cosmic string in a 5-dimensional AdS spacetime without compact dimension, and the second one being induced by the compactification itself. The latter is an odd function of the magnetic flux along the cosmic string and an even function of the magnetic flux enclosed by the compactified axis with period equal to the quantum flux. As to the induced axial current, it is an even function of the magnetic flux along the string's core and an odd function of the magnetic flux enclosed by the compactification perimeter. For untwisted and twisted field along compact dimension, the axial current vanishes. The massless field case is presented as well some asymptotic limits for the parameters of the model.

1 Introduction

In the context of grand unified theories, a consequence of the vacuum symmetry breaking phase transitions is the formation of different types of topological defects in the early Universe [1,2]. In particular, cosmic strings have been receiv-

[a] e-mail: bellucci@lnf.infn.it (corresponding author)

[b] e-mail: wagner.physics@gmail.com

[c] e-mail: emello@fisica.ufpb.br

ing considerable attention. Although the recent observational data on the cosmic microwave background (CMB) have ruled out cosmic strings as a primary source for primordial density perturbations, they are still possible candidates for the generation of a variety of interesting physical effects such as gamma ray bursts [3], high energy cosmic rays [4] and gravitational waves [5]. In addition, cosmic strings have also been considered in scenarios beyond the standard model of particles, like in supersymmetry and in string theory approach [6,7]. More recently, the proposal of a formation mechanism for cosmic strings in the framework of brane-inflation has been attracting renewed interest on this type of topological defects [8].

The simplest theoretical model describes a cosmic string as an infinitely long and straight object which locally presents no curvature, except on its core, where it has a delta shaped curvature tensor [9]. Moreover, a cosmic string in this model is mainly characterized by a deficit angle on the two-dimensional sub-space perpendicular to its core. The corresponding nontrivial topology gives rise to a number of interesting physical phenomena. One of these concerns the effect of a cosmic string on the vacuum structure.

The presence of a cosmic string in anti-de Sitter (AdS) spacetime provides an even more interesting framework, given that the combined geometry allows to distinguish the source of each part in the vacuum expectation values (VEV) of some observables. An interesting feature of the negative cosmological constant solution of the General Relativity field equations, is the fact that it makes possible to solve several problems exactly as a consequence of its high symmetry, allowing for the quantization of the fields more easily, besides providing deep insights into the quantization of the fields in other curved spacetimes. Furthermore, the AdS spacetime appears as a ground-state solution of string and supergravity theories and plays a fundamental role in the context of the AdS/CFT correspondence, a framework that realizes the holographic principle, relating the aforementioned funda-

mental theories with conformal field theories living on their boundaries [10]. Moreover, the AdS space also appears in braneworlds scenarios with large extra dimensions, providing a way to overcome the hierarchy problem between the gravitational and electroweak mass scales [11].

Another typical ingredient coming from the context of the above-mentioned fundamental theories, in which AdS spacetime plays a relevant role, is given by compact extra dimensions. The latter induce non-vanishing contributions to the expectation values of physical observables, such as the energy–momentum tensor, which has not only the energy density component, but also the stress components (see [12] and references therein). In this case, for instance, the vacuum energy density induced by the extra compact dimensions offers an explanation for the observed and still unexplained accelerated expansion of the Universe at recent epoch. In Kaluza-Klein-type models and in braneworld scenarios, on the other hand, the dependence of the size of the compact extra dimension by the vacuum energy density serves as a mechanism to stabilize fields known as moduli fields [13].

All the above-mentioned ingredients can provide modifications in the vacuum quantum fluctuations of relativistic fields in curved spacetime. In particular, the induced VEV of the current density by curved backgrounds with a cosmic string has been investigated in Refs. [14–19]. In these papers the authors have shown that the presence of a magnetic flux running along the cosmic string induces vacuum azimuthal current densities if the ratio of the magnetic flux by the quantum one has a non-vanishing fractional part. This is an Aharonov–Bohm-like effect. Moreover, the importance of investigating the effects of cosmic string on the vacuum current is because its conical topology increases the magnitude of the induced current. In AdS spacetime with compact dimension, investigations of the vacuum polarizations as well as induced vacuum currents have been carried out in [20–25] providing strong motivations to study it.

In the present paper, we are mainly interested in investigating the effects on the vacuum current densities associated with a charged massive Dirac field arising from the geometry and topology of a 5-dimensional AdS spacetime, in the presence of a cosmic string carrying a magnetic flux. Also we will admit the compactification of the extra dimension and the existence of a magnetic flux enclosed by the compactified axis. The non-trivial topology of this compactified extra dimension, as well as the magnetic fluxes, will give rise to additional contributions to the VEV of the current densities.

The paper is organized as follows. In the next section we obtain the mode-functions for a massive charged fermionic quantum field in the setup considered. In Sect. 3 these mode-functions are used for the evaluation of the induced azimuthal current density. By making use of the Abel-Plana summation formula, the latter is decomposed as the sum of contributions induced by the cosmic string without compactification, and

a second one induced by compactification itself. For these contributions we investigate some asymptotic limits of the model parameters. Toward to the end of this section, we show that the charge density as well as the currents along the radial and Poincaré coordinates are zero. In Sect. 4 we calculate the axial current density and investigate its behavior for some specific asymptotic limits. Section 5 is devoted for the draw of the main conclusions about our results. Throughout the paper, we use natural units $G = \hbar = c = 1$.

2 Dirac equation in (1+4)

In this section we will obtain the fermionic wave-function associated to a massive charged quantum field, $\psi(x)$, in the background geometry of a $(1 + 4)$-dimensional AdS spacetime in the presence of a cosmic string and a compactified extra dimension. The mode-summation approach is used and the mode-functions will be needed in the calculation of the current densities.

2.1 Setup

By using cylindrical coordinates, the geometry associated with a cosmic string in a $(3 + 1)$-dimensional AdS spacetime is given by the line element below:

$$ds^2 = e^{-2y/a}[dt^2 - dr^2 - r^2 d\phi^2] - dy^2 , \qquad (2.1)$$

where $r \geqslant 0$ and $\phi \in [0, 2\pi/q]$ define the coordinates on the conical geometry, $(t, y) \in (-\infty, \infty)$, and the parameter a determines the curvature scale of the background spacetime. In the above coordinate system the string is along the $y-$axis. The parameter $q \geq 1$ defines the planar angle deficit on the two-dimensional surface orthogonal to the string. Using the *Poincaré* coordinate defined by $w = ae^{y/a}$, the line element above can be conformally related to the line element associated with a cosmic string in Minkowski spacetime:

$$ds^2 = \left(\frac{a}{w}\right)^2 [dt^2 - dr^2 - r^2 d\phi^2 - dw^2] . \qquad (2.2)$$

For the new coordinate one has $w \in [0, \infty)$. Specific values for this coordinate deserve to be mentioned: $w = 0$ and $w = \infty$ correspond to the AdS boundary and horizon, respectively.

In order to write the line element (2.2) in a five-dimensional AdS spacetime, we adopt the standard procedure, by adding an extra spatial coordinate [21]:

$$ds^2 = \left(\frac{a}{w}\right)^2 \left[dt^2 - dr^2 - r^2 d\phi^2 - dw^2 - dz^2 \right]. \qquad (2.3)$$

The cosmological constant, Λ, and the Ricci scalar, R, are related with the scale a by the formulas

$$\Lambda = -\frac{6}{a^2}, \quad R = -\frac{20}{a^2}. \tag{2.4}$$

The dynamics of a spinorial quantum field in curved spacetime coupled with a gauge field, A_μ, is given by the following equation

$$i\gamma^\mu(\nabla_\mu + ieA_\mu)\psi - \tilde{s}m\psi = 0, \quad \nabla_\mu = \partial_\mu + \Gamma_\mu, \quad \tilde{s} = \pm 1, \tag{2.5}$$

where γ^μ are the Dirac matrices in curved spacetime and Γ_μ is the spin connection. Moreover, the two possible values of $\tilde{s}$ correspond to the two irreducible representations of Dirac matrices in spacetimes with odd number of dimensions. Both sets of matrices in curved spacetime are related to the flat ones, $\gamma^{(a)}$, by the relations,

$$\gamma^\mu = e^\mu_{(a)}\gamma^{(a)}, \quad \Gamma_\mu = \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^\nu_{(a)}e_{(b)\nu;\mu}. \tag{2.6}$$

The tetrad basis, $e^\mu_{(a)}$, satisfies the relation $e^\mu_{(a)}e^\nu_{(b)}\eta^{ab} = g^{\mu\nu}$, with η^{ab} being the Minkowski spacetime metric tensor.

Note that we will also consider the presence of a constant vector potential along the extra compact dimension. This compactification is implemented by assuming that $z \in [0, L]$, and the matter field obeys the quasiperiodicity condition below,

$$\psi(t, r, \phi, w, z + L) = e^{2\pi i\beta}\psi(t, r, \phi, w, z), \tag{2.7}$$

where $0 \leq \beta \leq 1$. The special cases $\beta = 0$ and $\beta = 1/2$ correspond to the untwisted and twisted fields, respectively, along the z-direction.

The set of Dirac matrices in flat spacetime assumed in this paper is the following:

$$\gamma^{(0)} = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^{(a)} = -i\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix},$$

$$\text{with } a = 1, 2, 3, \quad \gamma^{(4)} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

where σ_1, σ_2, σ_3 are the Pauli matrices. It is easy to verify that the above matrices obey the Clifford algebra, $\{\gamma^{(a)}, \gamma^{(b)}\} = 2\eta^{ab}$. We take the tetrad basis as follows:

$$e^\mu_{(a)} = \frac{w}{a}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(q\phi) & -\sin(q\phi)/r & 0 & 0 \\ 0 & \sin(q\phi) & \cos(q\phi)/r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

where the index a identifies the rows of the matrix. With this choice, the gamma matrices take the form

$$\gamma^0 = \frac{w}{a}\gamma^{(0)}, \quad \gamma^l = -i\frac{w}{a}\begin{pmatrix} \sigma^l & 0 \\ 0 & -\sigma^l \end{pmatrix}, \quad \gamma^4 = i\frac{w}{a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.10}$$

where we have introduced the 2×2 matrices for $l = (r, \phi, w)$:

$$\sigma^r = \begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \quad \sigma^\phi = -\frac{i}{r}\begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix},$$

$$\sigma^w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.11}$$

For the spin connection we have:

$$\Gamma_\mu = \frac{1}{2a}\gamma^{(3)}\gamma_\mu + \frac{(1-q)}{2}\gamma^{(1)}\gamma^{(2)}\delta^\phi_\mu, \quad \Gamma_w = 0, \tag{2.12}$$

and its contraction with the Dirac matrices in curved spacetime is given by,

$$\gamma^\mu\Gamma_\mu = -\frac{2}{w}\gamma^w + \frac{1-q}{2r}\gamma^r. \tag{2.13}$$

Thus the Dirac equation takes the following form:

$$\left(\gamma^\mu(\partial_\mu + ieA_\mu) - \frac{2}{w}\gamma^w + \frac{1-q}{2r}\gamma^r + i\tilde{s}m\right)\psi = 0. \tag{2.14}$$

As we have already mentioned, in this paper we want to consider the presence of a vector potential along the string's core and enclosed by the compact dimension. In this case we assume

$$A_\mu = (0, 0, A_\phi, 0, A_z). \tag{2.15}$$

For positive energy solutions, assuming the time-dependence of the eigenfunction in the form e^{-iEt} and the boost symmetry in the coordinate z represented by eigenfunctions in the form $e^{ik_z z}$, we may decompose the spinor field, ψ, as

$$\left(\sigma^l\partial_l - \frac{2}{w}\sigma^w + \frac{1-q}{2r}\sigma^r + ieA_\phi\sigma^\phi - \frac{\tilde{s}ma}{w}\right)\varphi$$
$$-i[E + (k_z + eA_z)]\chi = 0,$$
$$\left(\sigma^l\partial_l - \frac{2}{w}\sigma^w + \frac{1-q}{2r}\sigma^r + ieA_\phi\sigma^\phi + \frac{\tilde{s}ma}{w}\right)\chi$$
$$+i[(k_z + eA_z) - E]\varphi = 0, \tag{2.16}$$

where φ and χ are the upper and lower components of the spinor field, respectively. Now taking the function χ from the second equation into the first one, we get the following second order differential equation for the φ:

$$\left\{ \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\left[\partial_\phi + ieA_\phi - \frac{i(1-q)}{2}\sigma_w\right]^2 + \partial_w^2 \right.$$

$$-\frac{4}{w}\partial_w + \frac{6 - (\tilde{s}ma)^2 - (-1)^l \tilde{s}ma}{w^2}$$

$$\left. + \left[E^2 - \tilde{k}_z^2\right]\right\}\varphi = 0\,, \qquad (2.17)$$

where we have introduced the notation $\tilde{k}_z = k_z + eA_z$. In order to find the solution for (2.17), we may use the following Ansatz, compatible with the cylindrical symmetry of the problem:

$$\varphi = \begin{pmatrix} C_1 R_1(r) W_1(w) e^{iqn_1\phi} \\ C_2 R_2(r) W_2(w) e^{iqn_2\phi} \end{pmatrix}\,, \qquad (2.18)$$

with C_1 and C_2 being two arbitrary constants. Plugging this Ansatz into (2.17), we can see that the solution of the radial equation which is regular on the string is expressed in terms of the Bessel function of the first kind, $R_l = J_{\beta_l}(\lambda r)$, where its order is given by

$$\beta_1 = |q(n_1+\alpha)-(1-q)/2|\,, \quad \beta_2 = |q(n_2+\alpha)+(1-q)/2|\,, \qquad (2.19)$$

with $n_l = 0, \pm 1, \pm 2, \ldots$ Note that we have introduced the notation $\alpha = eA_\phi/q = -\Phi_\phi/\Phi_0$, where $\Phi_0 = 2\pi/e$ is the quantum flux. As to the differential equation associated with the Poincaré coordinate, the general solution is given in terms of a linear combination of the functions $w^{5/2}J_{\nu_l}(pw)$ and $w^{5/2}Y_{\nu_l}(pw)$, where $Y_\nu(x)$ is the Neumann function and

$$\nu_l = |\tilde{s}ma + (-1)^l/2|\,. \qquad (2.20)$$

Considering $ma \geq 1/2$, the Neumann function must be excluded according to the normalizability conditions of the mode functions, and therefore, we shall adopt the solution

$$W_l(w) = w^{5/2}J_{\nu_l}(pw)\,. \qquad (2.21)$$

The energy associated to the modes is given by

$$E = \sqrt{\lambda^2 + p^2 + \tilde{k}_z^2}\,. \qquad (2.22)$$

For sake of simplicity, from now on we shall assume the representation of the Clifford algebra corresponding to $\tilde{s} = 1$.

In short notation, the solution for the upper component is written as

$$\varphi_l = C_l w^{5/2}J_{\beta_l}(\lambda r)J_{\nu_l}(pw)e^{iqn_l\phi}\,. \qquad (2.23)$$

Taking (2.23) into the first equation in (2.16), after some intermediate steps, we obtain

$$\chi = w^{5/2}\begin{pmatrix} B_1 J_{\beta_1}(\lambda r)J_{\nu_1}(pw)e^{iqn_1\phi} \\ B_2 J_{\beta_2}(\lambda r)J_{\nu_2}(pw)e^{iqn_2\phi} \end{pmatrix} \qquad (2.24)$$

with the relations

$$n_2 = n_1 + 1\,, \quad \beta_2 = \beta_1 + \epsilon_{n_1}, \qquad (2.25)$$

where $\epsilon_n = 1$ for $n > -\alpha$ and $\epsilon_n = -1$ for $n < -\alpha$. The coefficients $B_{1,2}$ in (2.24) are given by

$$B_1 = \frac{i}{E + \tilde{k}_z}(pC_1 - \epsilon_n\lambda C_2)\,, \quad B_2 = \frac{i}{E + \tilde{k}_z}(\epsilon_n\lambda C_1 + pC_2)\,. \qquad (2.26)$$

We can see from the upper and lower spinor components given by (2.23) and (2.24), respectively, that the wave function, ψ, is an eigenfunction of the total angular momentum projected along the direction of the string:

$$\hat{J}_w\psi = \left(-i\partial_\phi + \frac{q}{2}\Sigma^w\right)\psi\,, \quad \Sigma^w = \begin{pmatrix} \sigma^w & 0 \\ 0 & \sigma^w \end{pmatrix}\,, \qquad (2.27)$$

where

$$j = n_1 + 1/2\,, \quad j = \pm 1/2, \pm 3/2, \ldots\,. \qquad (2.28)$$

Note that the components of the fermionic wave function obtained have four coefficients and two equations relating them, given in (2.26). The normalization condition on the wave function yields an additional relation. Thus, one of the coefficients remains arbitrary and in order to determine this coefficient some additional condition should be imposed on the coefficients. The imposition of this condition comes from the fact that the quantum numbers $\sigma = (\lambda, p, j, k_z)$ do not specify the fermionic wave function uniquely and some additional quantum number is required.

In order to specify the second constant we will require the following relation between the upper and lower components [26]:

$$\chi_1 = \kappa\varphi_1\,, \quad \chi_2 = -\varphi_2/\kappa\,. \qquad (2.29)$$

From the expressions for the spinor components we find the eigenvalues of the parameter κ,

$$\kappa = \kappa_s = \frac{-\tilde{k}_z + s\sqrt{\tilde{k}_z^2 + p^2}}{p}\,, \quad s = \pm 1, \qquad (2.30)$$

and the relation

$$C_2 = -\frac{\epsilon_n\kappa_s}{\lambda}\left(E - s\sqrt{\tilde{k}_z^2 + p^2}\right)C_1, \qquad (2.31)$$

for the coefficients in (2.23). Note that now the fermionic wave function is uniquely specified by the set of quantum numbers $\sigma = (\lambda, p, j, k_z, s)$. The eigenvalues of the quantum number k_z are determined from the quasi-periodicity condition (2.7):

$$k_z = k_l = 2\pi(l + \beta)/L \ , \ \text{with} \ l = 0, \pm 1, \pm 2, \ldots . \quad (2.32)$$

On the basis of all considerations above, the positive-energy fermionic wave function can be expressed as

$$\psi_\sigma^{(+)}(x) = C_\sigma^{(+)} e^{-iEt + ik_l z} w^{5/2}$$
$$\times \begin{pmatrix} J_{\beta_j}(\lambda r) J_{\nu_1}(pw) \\ -\epsilon_j \kappa_s b_s^{(+)} J_{\beta_j + \epsilon_j}(\lambda r) J_{\nu_2}(pw) e^{iq\phi} \\ i\kappa_s J_{\beta_j}(\lambda r) J_{\nu_2}(pw) \\ i\epsilon_j b_s^{(+)} J_{\beta_j + \epsilon_j}(\lambda r) J_{\nu_1}(pw) e^{iq\phi} \end{pmatrix} e^{iq(j-1/2)\phi} , \quad (2.33)$$

where $\epsilon_j = 1$ for $j > -\alpha$ and $\epsilon_j = -1$ for $j < -\alpha$, and the order of the Bessel function is defined as:

$$\beta_j = q|j + \alpha| - \epsilon_j/2. \quad (2.34)$$

Note that $\epsilon_j = \epsilon_n$. The energy is expressed in terms of λ, p and $\tilde{k}_l$ by the relation

$$E = \sqrt{\lambda^2 + p^2 + \tilde{k}_l^2} , \quad (2.35)$$

where $\tilde{k}_l = 2\pi(l + \tilde{\beta})/L$, with

$$\tilde{\beta} = \beta + eA_z L/(2\pi) = \beta - \Phi_z/\Phi_0. \quad (2.36)$$

In (2.33) and in what follows below, we use the notation

$$b_s^{(\pm)} = \frac{E \mp s\sqrt{\tilde{k}_l^2 + p^2}}{\lambda}. \quad (2.37)$$

Note that one has the relation $b_s^{(+)} b_s^{(-)} = 1$.

The coefficient $C_\sigma^{(+)}$ in (2.33) is determined from the normalization condition

$$\int d^3x \sqrt{\gamma} (\psi_\sigma^{(+)})^\dagger \psi_{\sigma'}^{(+)} = \delta_{\sigma,\sigma'} , \quad (2.38)$$

where γ is the determinant of the spatial metric. The delta symbol on the right-hand side is understood as the Dirac delta function for continuous quantum numbers (λ, p) and the Kronecker delta for discrete ones (j, s, l). Taking the eigenspinor in (2.33) into (2.38) and using the value of the

standard integral involving the products of the Bessel functions [27], we find

$$|C_\sigma^{(+)}|^2 = \frac{sqp^2\lambda^2}{8\pi a^4 LE\kappa_s b_s^{(+)} \sqrt{\tilde{k}_l^2 + p^2}} . \quad (2.39)$$

The negative-energy fermionic mode function can be obtained in a similar way. The corresponding result is given by the expression:

$$\psi_\sigma^{(-)}(x) = C_\sigma^{(-)} e^{iEt + ik_l z} w^{5/2}$$
$$\times \begin{pmatrix} J_{\beta_j}(\lambda r) J_{\nu_1}(pw) \\ \epsilon_j \kappa_s b_s^{(-)} J_{\beta_j + \epsilon_j}(\lambda r) J_{\nu_2}(pw) e^{iq\phi} \\ i\kappa_s J_{\beta_j}(\lambda r) J_{\nu_2}(pw) \\ -i\epsilon_j b_s^{(-)} J_{\beta_j + \epsilon_j}(\lambda r) J_{\nu_1}(pw) e^{iq\phi} \end{pmatrix} e^{iq(j-1/2)\phi} , \quad (2.40)$$

and the normalization constant is given by the relation

$$|C_\sigma^{(-)}|^2 = \frac{sqp^2\lambda^2}{8\pi a^4 LE\kappa_s b_s^{(-)} \sqrt{\tilde{k}_l^2 + p^2}} . \quad (2.41)$$

3 Fermionic current density

The VEV of the fermionic current density, $\langle j^\mu \rangle = e\bar{\psi}\gamma^\mu\psi$, can be evaluated by using the mode sum formula,

$$\langle j^\mu \rangle = \frac{e}{2} \sum_\sigma [\bar{\psi}_\sigma^{(-)} \gamma^\mu \psi_\sigma^{(-)} - \bar{\psi}_\sigma^{(+)} \gamma^\mu \psi_\sigma^{(+)}] . \quad (3.1)$$

where $\bar{\psi}_\sigma^{(\pm)} = \psi_\sigma^{(\pm)\dagger} \gamma^{(0)}$ is the Dirac adjoint, and the summation over σ is a compact notation defined below,

$$\sum_\sigma = \int_0^\infty d\lambda \int_0^\infty dp \sum_{s=\pm 1} \sum_{j=\pm 1/2, \ldots} \sum_{l=-\infty}^\infty . \quad (3.2)$$

As we will see, this VEV is a periodic function of the magnetic fluxes Φ_ϕ and Φ_z with the period equal to the quantum flux. Thus, it is convenient to express the parameter α as

$$\alpha = n_0 + \alpha_0, \quad (3.3)$$

where n_0 is an integer number and $|\alpha_0| < 1/2$.

3.1 Charge density, current densities in the radial and w-directions

Let us compute the VEV of the charge density

$$\rho(x) = \langle j^0(x) \rangle = \frac{ew}{2a} \sum_\sigma [\psi_\sigma^{(-)\dagger} \psi_\sigma^{(-)} - \psi_\sigma^{(+)\dagger} \psi_\sigma^{(+)}]. \quad (3.4)$$

Taking (2.33) and (2.40) into (3.4), we arrive directly to a vanishing charge density. Thus, there is no charge density induced.

As to the VEV of the radial current, substituting the mode functions for positive and negative-energy as well the Dirac matrices given in (2.10) and (2.11) into (3.1), we can see that there appears a cancellation between the terms. Thus, there is no induced radial current. A similar analysis also leads to a vanishing induced current density along the coordinate w.

3.2 Azimuthal current

The VEV of the azimuthal current is given by

$$\langle j^\phi \rangle = \frac{e}{2} \sum_\sigma [\bar{\psi}_\sigma^{(-)} \gamma^\phi \psi_\sigma^{(-)} - \bar{\psi}_\sigma^{(+)} \gamma^\phi \psi_\sigma^{(+)}] . \qquad (3.5)$$

Substituting the positive and negative wave functions into the above expression, we get

$$\langle j^\phi \rangle = -\frac{qew^6}{2\pi L a^5 r} \int_0^\infty d\lambda \int_0^\infty dp$$
$$\times \sum_j \epsilon_j p \lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j + \epsilon_j}(\lambda r)[J_{\nu_2}^2(pw) + J_{\nu_1}^2(pw)]$$
$$\times \sum_{l=-\infty}^\infty \frac{1}{\sqrt{\lambda^2 + p^2 + [2\pi(l + \tilde{\beta})/L]^2}} , \qquad (3.6)$$

where the sum over s has been already performed and it has yielded an overall factor of 2. In order to sum over the quantum number l, we shall apply the Abel-Plana formula in the form [28]

$$\sum_{l=-\infty}^\infty g(l + \tilde{\beta}) f(|l + \tilde{\beta}|) = \int_0^\infty du[g(u) + g(-u)]f(u)$$
$$+ i \int_0^\infty du[f(iu) - f(-iu)] \sum_{n=\pm 1} \frac{g(inu)}{e^{2\pi(u+in\tilde{\beta})} - 1} , \qquad (3.7)$$

taking $g(u) = 1$ and

$$f(u) = \frac{1}{\sqrt{(2\pi u/L)^2 + \lambda^2 + p^2}} . \qquad (3.8)$$

As a result we may decompose the induced azimuthal current as

$$\langle j^\phi \rangle = \langle j^\phi \rangle_{cs} + \langle j^\phi \rangle_c \qquad (3.9)$$

where the first term, $\langle j^\phi \rangle_{cs}$, comes from the first integral on the right-hand side of (3.7) and corresponds to the induced azimuthal current density in the geometry of a cosmic string without compactification. The second contribution, $\langle j^\phi \rangle_c$, is

induced by the compactification. As we shall see the latter vanishes in the limit $L \to \infty$.

Combining (3.6) and (3.7), we get

$$\langle j^\phi \rangle_{cs} = -\frac{qew^6}{2\pi^2 a^5 r} \int_0^\infty d\lambda \int_0^\infty dp$$
$$\times \sum_j \epsilon_j p \lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j + \epsilon_j}(\lambda r)[J_{\nu_2}^2(pw) + J_{\nu_1}^2(pw)]$$
$$\times \int_0^\infty \frac{dx}{\sqrt{\lambda^2 + p^2 + x^2}} , \qquad (3.10)$$

where we have introduced the variable $x = 2\pi u/L$. Using the following integral representation

$$\frac{1}{\sqrt{\lambda^2 + p^2 + x^2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty ds e^{-s^2(\lambda^2 + p^2 + x^2)} , \qquad (3.11)$$

we can carry out the integrations over all variables except over s, obtaining

$$\langle j^\phi \rangle_{cs} = -\frac{qew^6}{4\pi^2 a^5 r^6} \int_0^\infty dy y^2 e^{-(1+w^2/r^2)y}$$
$$\times \left[I_{\nu_2}\left(\frac{w^2}{r^2}y\right) + I_{\nu_1}\left(\frac{w^2}{r^2}y\right) \right]$$
$$\times [\mathcal{I}(q, \alpha_0, y) - \mathcal{I}(q, -\alpha_0, y)] , \qquad (3.12)$$

where we have introduced a new variable $y = r^2/2s^2$ and the function $\mathcal{I}(q, \alpha_0, y)$ is given by [29]

$$\mathcal{I}(q, \alpha_0, y) = \frac{e^y}{q} - \frac{1}{\pi} \int_0^\infty dz \frac{e^{-y \cosh z} f(q, \alpha_0, z)}{\cosh(qz) - \cos(q\pi)}$$
$$+ \frac{2}{q} \sum_{k=1}^p (-1)^k \cos[2\pi k(\alpha_0 - 1/2q)] e^{y \cos(2\pi k/q)} , \qquad (3.13)$$

with $2p < q < 2p + 2$ and with the notation

$$f(q, \alpha_0, z) = \cos[q\pi(1/2 - \alpha_0)] \cosh[(q\alpha_0 + q/2 - 1/2)z]$$
$$- \cos[q\pi(1/2 + \alpha_0)] \cosh[(q\alpha_0 - q/2 - 1/2)z] . \qquad (3.14)$$

Using the formula (3.13), after the integration over y with the help of [27], (3.12) can be presented in the form

$$\langle j^\phi \rangle_{cs} = -\frac{8e}{(2\pi)^{5/2} a^5} \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \sin(\pi k/q) \sin(2\pi \alpha_0 k) \mathcal{Z}(u_k) \right.$$
$$\left. + \frac{q}{\pi} \int_0^\infty dz \frac{\cosh(z) g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} \mathcal{Z}(u_z) \right] , \qquad (3.15)$$

where $[q/2]$ represents the integer part of $q/2$ and the prime on the sign of the summation means that in the case $q = 2p$ the term $k = q/2$ should be taken with the coefficient 1/2. Moreover, the function $g(q, \alpha_0, z)$ is defined as
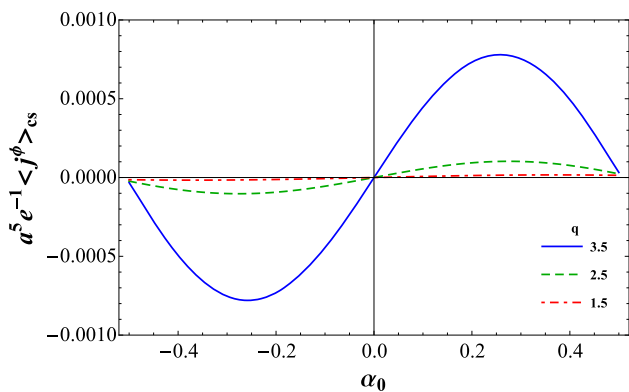
Fig. 1 The VEV of the induced azimuthal current without compactification in (3.15) is plotted, in units of $a^5 e^{-1}$, as a function of α_0 for $r/w = 1$, $ma = 5$, $q = 1.5$, 2.5 and 3.5

$$g(q, \alpha_0, z) = \cos[q\pi(1/2 + \alpha_0)]\cosh[q(1/2 - \alpha_0)z] \\ - \cos[q\pi(1/2 - \alpha_0)]\cosh[q(1/2 + \alpha_0)z] . \tag{3.16}$$

In (3.15) we have introduced the function

$$\mathcal{Z}(u) = \sum_{i=1,2} F_{\nu_i}(u) , \text{ with } F_{\nu_i} = \frac{e^{-i5\pi/2} Q_{\nu_i - 1/2}^{5/2}(u)}{(u^2 - 1)^{5/4}} , \tag{3.17}$$

where $Q_\mu^\nu(z)$ represents the Legendre Associated Functions of second kind [27]; moreover, we have defined the new variables

$$u_k = 1 + 2(r/w)^2 \sin^2(\pi k/q) , \\ u_z = 1 + 2(r/w)^2 \cosh^2(z) . \tag{3.18}$$

Let us now consider some special cases of the function $\mathcal{Z}(u)$ defined in (3.17). For values of $ma \geq 1/2$, according to (2.20), we have the relation $\nu_2 = \nu_1 + 2$. In this case, for large values of the argument, $u \gg 1$, the function $\mathcal{Z}(u)$ is simplified and to the leading term takes the form:

$$\mathcal{Z}(u) \approx \frac{\sqrt{2\pi}(1 + \nu_2)\nu_2}{2^{\nu_2} u^{\nu_2 + 2}} . \tag{3.19}$$

Now considering $m = 0$, it implies from (2.20) that $\nu_1 = \nu_2 = 1/2$. In this case, simplifications can be carried out and Eq. (3.17) may be presented as

$$\mathcal{Z}(u) = \frac{3\sqrt{\pi}}{4} \frac{1}{(u - 1)^{5/2}} . \tag{3.20}$$

In Fig. 1 we plot the string contribution to the azimuthal current as function of the magnetic flux along the string for $q = 1.5$, 2.5 and 3.5. As we can see, $\langle j^\phi \rangle_{cs}$ is an odd function of α_0 and has its amplitude depending on the deficit angle caused by the presence of the string.
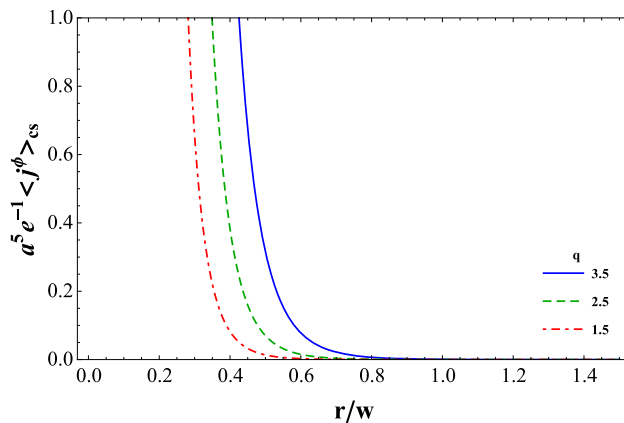

Fig. 2 The VEV of the string contribution to the induced azimuthal current in Eq. (3.15) is plotted, in units of $a^5 e^{-1}$, as a function of r/w for $\alpha_0 = 0.25$, $ma = 5$, $q = 1.5$, 2.5 and 3.5

Now we turn our attention to the investigation of (3.15) for small and large values of the ratio r/w by using the asymptotic expressions from Ref. [12]. For points close to the string, $r/w \ll 1$, we have the following expression:

$$\langle j^\phi \rangle_{cs} \approx -\frac{3e}{(4\pi)^2}\left(\frac{w}{ar}\right)^5 \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \frac{\sin(2\pi\alpha_0 k)}{\sin^4(\pi k/q)} \right. \\ \left. + \frac{q}{\pi}\int_0^\infty dz \frac{\cosh^{-4}(z)g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} \right] . \tag{3.21}$$

From the above expression we can see that this VEV diverges as $(w/r)^5$. On the other hand, for points far away from the string, $r/w \gg 1$, and values of $ma \geq 1/2$, we find by plugging (3.19) into (3.15), the following asymptotic expression

$$\langle j^\phi \rangle_{cs} \approx -\frac{\nu_2(1 + \nu_2)e}{2^{2\nu_2 + 1}\pi^2 a^5}\left(\frac{w}{r}\right)^{2\nu_2 + 4} \\ \times \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \frac{\sin(2\pi\alpha_0 k)}{\sin^{2\nu_2 + 3}(\pi k/q)} \right. \\ \left. + \frac{q}{\pi}\int_0^\infty dz \frac{\cosh^{-(2\nu_2 + 3)}(z)g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} \right] . \tag{3.22}$$

We can see from this expression that in this regime the cosmic string influence on the intensity of the azimuthal current starts to fade and the decay behavior depends on the mass of the particle. In Fig. 2 is displayed the string contribution without compactification as function of the proper distance in units of a, r/w, for different values of q. This plot shows that this contribution diverges on the string, $r = 0$, as well as it vanishes for distant points with its decay behavior depending on q.

Now let us develop the contribution induced by the compactification:

$$
\langle j^\phi \rangle_c = -\frac{ieqw^6}{2\pi^2 a^5 r} \int_0^\infty d\lambda \int_0^\infty dp
$$
$$
\times \sum_j \epsilon_j p\lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j+\epsilon_j}(\lambda r)[J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)]
$$
$$
\times \int_{\sqrt{\lambda^2+p^2}}^\infty \frac{dx}{\sqrt{x^2-\lambda^2-p^2}}
$$
$$
\times \left(\frac{1}{e^{Lx+2\pi i\tilde\beta}-1} + \frac{1}{e^{Lx-2\pi i\tilde\beta}-1} \right). \tag{3.23}
$$

In order to proceed with the calculation, we shall use the series expansion, $(e^u - 1)^{-1} = \sum_{l=1}^\infty e^{-lu}$ in the above expression. Taking this expansion into (3.23), we can perform the integral over x with the help of [27]. After some intermediate steps, we obtain

$$
\langle j^\phi \rangle_c = -\frac{eqw^6}{\pi^2 a^5 r} \sum_{l=1}^\infty \cos(2\pi l\tilde\beta)
$$
$$
\times \sum_j \epsilon_j \int_0^\infty d\lambda \lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j+\epsilon_j}(\lambda r)
$$
$$
\times \int_0^\infty dpp[J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)]K_0(lL\sqrt{\lambda^2+p^2}). \tag{3.24}
$$

Using the following integral representation for the Macdonald function [30],

$$
K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty ds \frac{e^{-s-x^2/4s}}{s^{\nu+1}}, \tag{3.25}
$$

we may write (3.24) as

$$
\langle j^\phi \rangle_c = -\frac{eqw^6}{2\pi^2 a^5 r} \sum_{l=1}^\infty \cos(2\pi l\tilde\beta) \int_0^\infty ds \frac{e^{-s}}{s} \sum_j \epsilon_j
$$
$$
\times \int_0^\infty d\lambda \lambda^2 J_{\beta_j}(\lambda r) J_{\beta_j+\epsilon_j}(\lambda r)e^{-l^2L^2\lambda^2/(4s)}
$$
$$
\times \int_0^\infty dpp[J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)]e^{-l^2L^2p^2/(4s)}. \tag{3.26}
$$

Performing the integrations over λ and p with the help of [27], we get

$$
\langle j^\phi \rangle_c = -\frac{eqw^6}{2\pi^2 a^5 r^6} \sum_{l=1}^\infty \cos(2\pi l\tilde\beta)
$$
$$
\times \int_0^\infty dy y^2 e^{-[1+w^2/r^2+(lL)^2/2r^2]y}
$$
$$
\times \left[I_{\nu_1}\left(\frac{w^2}{r^2}y\right) + I_{\nu_2}\left(\frac{w^2}{r^2}y\right) \right]
$$
$$
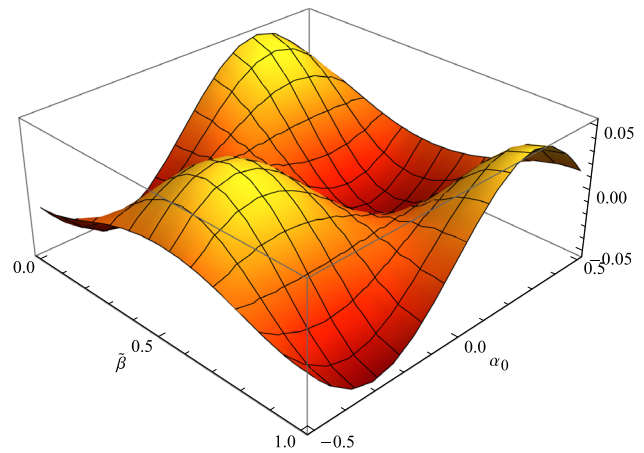\times [\mathcal{I}(q,\alpha_0,y) - \mathcal{I}(q,-\alpha_0,y)], \tag{3.27}
$$

Fig. 3 The azimuthal current induced by compactification given in (3.28) is plotted in units of $a^5 e^{-1}$ as function of $\tilde\beta$ and α_0 for fixed $r/w = 1$, $ma = 5$ and $q = 2.5$

where we have introduced the variable $y = 2sr^2/(lL)^2$. Using the representation given in (3.13) for the function $\mathcal{I}(q,\pm\alpha_0,y)$, we can perform the integral over y, arriving to the expression

$$
\langle j^\phi \rangle_c = -\frac{16e}{(2\pi)^{5/2}a^5} \sum_{l=1}^\infty \cos(2\pi l\tilde\beta)
$$
$$
\times \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \sin(\pi k/q) \sin(2\pi\alpha_0 k)\mathcal{Z}(u_{lk}) \right.
$$
$$
\left. + \frac{q}{\pi} \int_0^\infty dz \frac{\cosh(z)g(q,\alpha_0,2z)}{\cosh(2qz)-\cos(q\pi)}\mathcal{Z}(u_{lz}) \right], \tag{3.28}
$$

where we have introduced the variables

$$
u_{lk} = 1 + \frac{(lL)^2 + 4r^2\sin^2(\pi k/q)}{2w^2},
$$
$$
u_{lz} = 1 + \frac{(lL)^2 + 4r^2\cosh^2(z)}{2w^2}. \tag{3.29}
$$

We can see that (3.15) and (3.28) are odd functions of α_0. This means that if we change $\alpha_0 \to -\alpha_0$ the azimuthal current density changes its sign. In fact this changing corresponds to invert the direction of the magnetic flux along the string, implying also in a reversion of the induced azimuthal current. Moreover, (3.28) is an even function of $\tilde\beta$. The reason for this behavior is because it is not expected to invert the direction of the azimuthal current due to a changing of the magnetic flux enclosed by the compactified dimension.

In Fig. 3, $\langle j^\phi \rangle_c$ is plotted in units of $a^5 e^{-1}$ as function of α_0 and $\tilde\beta$, for $ma = 5$ and $q = 2.5$. This plot exhibits the above mentioned characteristics of the induced azimuthal current density.

On the string's core, $r = 0$, $\langle j^\phi \rangle_c$ is finite and takes the form:

$$\langle j^\phi \rangle_c|_{r=0} = -\frac{16e}{(2\pi)^{5/2}a^5} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta}) \mathcal{Z}\left(1 + (lL)^2/2w^2\right)$$
$$\times \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \sin(\pi k/q) \sin(2\pi\alpha_0 k) \right.$$
$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\cosh(z) g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} \right]. \quad (3.30)$$

Combining (3.15) and (3.28), we may write the total azimuthal current density as

$$\langle j^\phi \rangle = -\frac{16e}{(2\pi)^{5/2}a^5} \sum_{l=0}^{\infty}{}_* \cos(2\pi l \tilde{\beta})$$
$$\times \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \sin(\pi k/q) \sin(2\pi\alpha_0 k) \mathcal{Z}(u_{lk}) \right.$$
$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\cosh(z) g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} \mathcal{Z}(u_{lz}) \right], \quad (3.31)$$

where the asterisk sign in the summation over l in (3.31) indicates that the term $l = 0$ must be halved.

For a massless charged fermion field we have $\nu_1 = \nu_2 = 1/2$. Thus, taking (3.20) into (3.31), the total azimuthal current is simplified and takes the form

$$\langle j^\phi \rangle = -\frac{12e}{\pi^2 L^5} \left(\frac{w}{a}\right)^5 \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \sin(\pi k/q) \sin(2\pi\alpha_0 k) V(\tilde{\beta}, \rho_k) \right.$$
$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\cosh(z) g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} V(\tilde{\beta}, \tau(z)) \right], \quad (3.32)$$

where we have introduced the function

$$V(\tilde{\beta}, x) = \sum_{l=0}^{\infty}{}_* \frac{\cos(2\pi\tilde{\beta}l)}{(l^2 + x^2)^{5/2}}, \quad (3.33)$$

in the integrand of (3.32), with the corresponding arguments defined as

$$\rho_k = \frac{2r \sin(\pi k/q)}{L}, \quad \tau(z) = \frac{2r \cosh(z)}{L}. \quad (3.34)$$

The summation over l can be developed with the help of [27]. So, after some intermediate steps, we obtain:

$$V(\tilde{\beta}, x) = \frac{\pi^2 \cosh(2\pi\tilde{\beta}x)}{4x^2 \sinh^2(\pi x)}$$
$$+ \pi \frac{\cosh[\pi(1 - 2\tilde{\beta})x] + 2\pi\tilde{\beta}x \sinh[\pi(1 - 2\tilde{\beta})x]}{4x^3 \sinh(\pi x)}, \quad (3.35)$$

for $0 \leqslant \tilde{\beta} \leqslant 1$.

At large lengths of the compact extra dimension, $L/w \gg 1$, and for $ma \geq 1$ we have by plugging (3.19) into (3.28), the following expression

$$\langle j^\phi \rangle_c \approx -\frac{16\nu_2(\nu_2 + 1)e}{\pi^2 a^5} \left(\frac{w}{L}\right)^{2\nu_2+4} \sum_{l=1}^{\infty} \cos(2\pi l \tilde{\beta})$$
$$\times \left\{ \sum_{k=1}^{[q/2]}{}' \frac{(-1)^k \sin(\pi k/q) \sin(2\pi\alpha_0 k)}{(l^2 + \rho_k^2)^{\nu_2+2}} \right.$$
$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\cosh(z) g(q, \alpha_0, 2z)}{[\cosh(2qz) - \cos(q\pi)][l^2 + \tau^2(z)]^{\nu_2+2}} \right\}. \quad (3.36)$$

From the above equation we note that the azimuthal current induced by the compactness starts to fade for large lengths of the compact dimension, and therefore, in this regime, the total azimuthal current is dominated by the string induced contribution.

To conclude this section, we turn our focus to the consideration of the Minkowskian asymptotic limit. In this case, we consider $a \to \infty$ for fixed value of the coordinate y. As a consequence $ma \gg 1$ and the coordinate w goes like $w \approx a+y$, such that the function $\mathcal{Z}(u)$ assumes the following form: [25]

$$\mathcal{Z}(u) = 2m^{5/2}a^5 \frac{K_{5/2}(mu)}{u^{5/2}}, \quad (3.37)$$

where $K_\nu(x)$ is the Macdonald function. Taking this result into (3.31), after a few simplifications, we get the following expression:

$$\langle j^\phi \rangle_M \approx -e\left(\frac{2m}{\pi L}\right)^{5/2} \sum_{l=0}^{\infty}{}_* \cos(2\pi l \tilde{\beta})$$
$$\times \left\{ \sum_{k=1}^{[q/2]}{}' (-1)^k \sin(\pi k/q) \sin(2\pi\alpha_0 k) \frac{K_{5/2}(mL\sqrt{l^2 + \rho_k^2})}{(l^2 + \rho_k^2)^{5/4}} \right.$$
$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\cosh(z) g(q, \alpha_0, 2z)}{\cosh(2qz) - \cos(q\pi)} \frac{K_{5/2}(mL\sqrt{l^2 + \tau^2(z)})}{[l^2 + \tau^2(z)]^{5/4}} \right\}. \quad (3.38)$$

4 Induced current along the compactified extra dimension

In this section, we want to analyze the VEV of the induced current density along the compactified axis, namely axial current. As we are going to see, this VEV goes to zero in the limit $L \to 0$.

The current density induced in the compactified extra dimension is calculated by

$$\langle j^z \rangle = \frac{e}{2} \sum_\sigma [\bar{\psi}_\sigma^{(-)} \gamma^z \psi_\sigma^{(-)} - \bar{\psi}_\sigma^{(+)} \gamma^z \psi_\sigma^{(+)}]. \tag{4.1}$$

Plugging (2.33) and (2.40) into the above expression, we obtain

$$\langle j^z \rangle = \frac{qew^6}{8\pi La^5} \int_0^\infty d\lambda \int_0^\infty dp \sum_{s=\pm 1} \sum_j$$
$$\times \sum_{l=-\infty}^\infty \frac{\lambda^2 p^2}{E\kappa_s \sqrt{\tilde{k}_l^2 + p^2}} s[J_{\nu_1}^2(pw) - \kappa_s^2 J_{\nu_2}^2(pw)]$$
$$\times [b_s^{(+)} J_{\beta_j + \epsilon_j}^2(\lambda r) - b_s^{(-)} J_{\beta_j}^2(\lambda r)]. \tag{4.2}$$

Summing over s, we obtain

$$\langle j^z \rangle = -\frac{qew^6}{4\pi La^5} \int_0^\infty d\lambda \int_0^\infty dp \sum_j$$
$$\times \sum_{l=-\infty}^\infty \lambda p \left\{ \frac{\tilde{k}_l}{E} [J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)] \right.$$
$$\times [J_{\beta_j}^2(\lambda r) + J_{\beta_j + \epsilon_j}^2(\lambda r)]$$
$$+ [J_{\nu_1}^2(pw) - J_{\nu_2}^2(pw)][J_{\beta_j}^2(\lambda r) - J_{\beta_j + \epsilon_j}^2(\lambda r)] \right\}. \tag{4.3}$$

Note that this expression is divergent due to the second term inside the integrand, and therefore some regularization procedure is necessary. We shall assume that a cutoff function is introduced in the aforementioned divergent term without explicitly writing it, since the explicit form of this function is not relevant for this discussion. Moreover, using the Abel-Plana formula given in (3.7), we can see that this term vanishes.

For the remaining term, on the other hand, we take $g(u) = 2\pi u/L$ and the expression given in (3.8) for the function $f(u)$. We can see that the first integral on right-hand side of (3.7) is zero, because $g(u)$ is an odd function. Thus, the only contribution for the axial current comes from the second integral in (3.7):

$$\langle j^z \rangle = -\frac{iqew^6}{4\pi^2 a^5} \int_0^\infty dp p [J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)]$$
$$\times \sum_j \int_0^\infty d\lambda \lambda [J_{\beta_j}^2(\lambda r) + J_{\beta_j + \epsilon_j}^2(\lambda r)]$$
$$\times \int_{\sqrt{\lambda^2 + p^2}}^\infty \frac{dx x}{\sqrt{x^2 - \lambda^2 - p^2}}$$
$$\times \left(\frac{1}{e^{Lx + 2\pi i\tilde{\beta}} - 1} - \frac{1}{e^{Lx - 2\pi i\tilde{\beta}} - 1} \right), \tag{4.4}$$

where we have introduced the variable $x = 2\pi u/L$. As before, the next step is to use the expansion $(e^u - 1)^{-1} = \sum_{l=0}^\infty e^{-lu}$ in the above expression, and with the help of [27] the integral over x can be performed, with the result given in terms of the Macdonald function, $K_1(z)$,

$$\langle j^z \rangle = -\frac{qew^6}{2\pi^2 a^5} \sum_{l=1}^\infty \sin(2\pi l\tilde{\beta})$$
$$\times \int_0^\infty dp p [J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)]$$
$$\times \sum_j \int_0^\infty d\lambda \lambda [J_{\beta_j}^2(\lambda r)$$
$$+ J_{\beta_j + \epsilon_j}^2(\lambda r)] \sqrt{\lambda^2 + p^2} K_1(lL\sqrt{\lambda^2 + p^2}). \tag{4.5}$$

Using the integral representation for the Macdonald function already given in (3.25) and the fact that $K_\nu(z) = K_{-\nu}(z)$, we can write

$$\langle j^z \rangle = -\frac{qew^6}{2\pi^2 La^5} \sum_{l=1}^\infty \frac{\sin(2\pi l\tilde{\beta})}{l}$$
$$\times \int_0^\infty ds e^{-s} \int_0^\infty dp p e^{-(lL)^2 p^2/4s} [J_{\nu_1}^2(pw) + J_{\nu_2}^2(pw)]$$
$$\times \sum_j \int_0^\infty d\lambda \lambda e^{-(lL)^2 \lambda^2/4s} [J_{\beta_j}^2(\lambda r) + J_{\beta_j + \epsilon_j}^2(\lambda r)]. \tag{4.6}$$

Performing the integrals over λ and p with the help of [27], we obtain

$$\langle j^z \rangle = -\frac{qew^6 L}{4\pi^2 a^5 r^6} \sum_{l=1}^\infty l \sin(2\pi l\tilde{\beta})$$
$$\times \int_0^\infty dy y^2 e^{-[1 + (lL)^2/2r^2 + w^2/r^2)]y}$$
$$\times \left[I_{\nu_1}\left(\frac{w^2}{r^2} y\right) + I_{\nu_2}\left(\frac{w^2}{r^2} y\right) \right]$$
$$\times [\mathcal{I}(q, \alpha_0, y) + \mathcal{I}(q, -\alpha_0, y)], \tag{4.7}$$

where we have introduced the new variable $y = 2sr^2/(lL)^2$. Using again (3.13) for the function $\mathcal{I}(q, \alpha_0, y)$, we can eval-

uate the integral over y, obtaining

$$\langle j^z \rangle = -\frac{8eL}{(2\pi)^{5/2}a^5} \sum_{l=1}^{\infty} l \sin(2\pi l\tilde{\beta})$$

$$\times \left[\sum_{k=0}^{[q/2]}{}' (-1)^k \cos(\pi k/q) \cos(2\pi\alpha_0 k) \mathcal{Z}(u_{lk}) \right.$$

$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\sinh(z)h(q,\alpha_0,2z)}{\cosh(2qz) - \cos(q\pi)} \mathcal{Z}(u_{lz}) \right], \quad (4.8)$$

where we have introduced the function

$$h(q,\alpha_0,z) = \cos[q\pi(1/2 + \alpha_0)]\sinh[q(1/2 - \alpha_0)z]$$

$$+ \cos[q\pi(1/2 - \alpha_0)]\sinh[q(1/2 + \alpha_0)z]. \quad (4.9)$$

Note that the prime on the sum sign means that the term with $k = 0$ should be halved. The VEV in (4.8) may be also represented as

$$\langle j^z \rangle = \langle j^z \rangle^{(0)} + \langle j^z \rangle^{(q,\alpha_0)}, \quad (4.10)$$

The first term, $\langle j^z \rangle^{(0)}$, does not depend on α_0 and q, and it corresponds to the current density induced only by compactification in AdS spacetime. For this contribution one has

$$\langle j^z \rangle^{(0)} = -\frac{4qeL}{(2\pi)^{5/2}a^5} \sum_{l=1}^{\infty} l \sin(2\pi l\tilde{\beta}) \mathcal{Z}(u_{l0}), \quad (4.11)$$

where u_{l0} is given in the first relation of (3.18) for $k = 0$.= The second term in (4.10) is given by

$$\langle j^z \rangle^{(q,\alpha_0)} = -\frac{8eL}{(2\pi)^{5/2}a^5} \sum_{l=1}^{\infty} l \sin(2\pi l\tilde{\beta})$$

$$\times \left[\sum_{k=1}^{[q/2]}{}' (-1)^k \cos(\pi k/q) \cos(2\pi\alpha_0 k) \mathcal{Z}(u_{lk}) \right.$$

$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\sinh(z)h(q,\alpha_0,2z)}{\cosh(2qz) - \cos(q\pi)} \mathcal{Z}(u_{lz}) \right]. \quad (4.12)$$

From (4.12) one can see that this contribution for the axial current density vanishes, for the special cases of untwisted and twisted field configuration, in the absence of magnetic flux enclosed by the compactified extra dimension, and more in general for half-integer values of $\tilde{\beta}$. Another feature of this current density is that it is an odd function of $\tilde{\beta}$. It means that for the case $\beta = 0$ (untwisted field), for example, the axial current density changes its sign if the magnetic flux enclosed by the compacitifed dimension has its direction inverted.
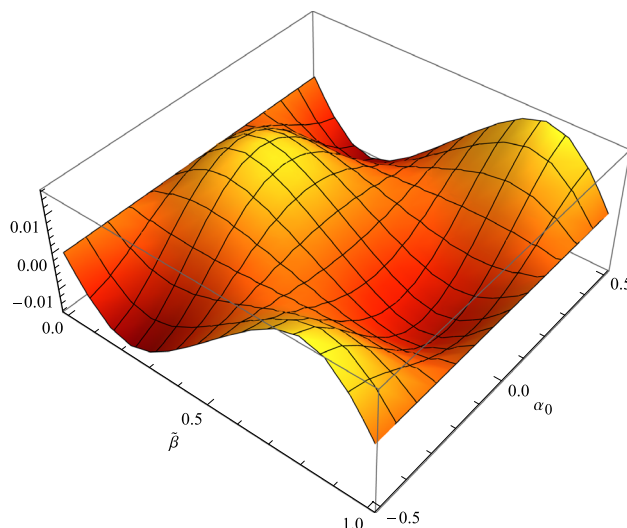

Fig. 4 The VEV of the axial current induced by the compactification in Eq. (4.12) is plotted, in units of $a^5 e^{-1}$, for $q = 2.5$, $r/w = 1$ and $ma = 5$

Besides, we note that the axial current is an even function of α_0. In fact, as an opposite situation of the azimuthal current, it is not expect that reversing the magnetic flux along the string, the induced axial current will be affected.

In Fig. 4 is presented a plot of (4.12) in terms of the α_0 and $\tilde{\beta}$, for values of $r/w = 1$, $ma = 5$ and $q = 2.5$. In Fig. 5 we plot (4.12) for $\alpha_0 = 0$ (left panel) and $\alpha_0 = 0.35$ (right panel) as function of $\tilde{\beta}$ for different values of q. As we can observe, the magnetic flux along the cosmic string can invert the direction of the current density along the axial axis.

For a massless spinor field, we have $\nu_1 = \nu_2 = 1/2$. Thus, by plugging (3.20) into (4.8), we find

$$\langle j^z \rangle = -\frac{6e}{\pi^2 L^4}\left(\frac{w}{a}\right)^5 \left[\sum_{k=0}^{[q/2]}{}' (-1)^k \cos(\pi k/q) \cos(2\pi\alpha_0 k) G(\tilde{\beta}, \rho_k) \right.$$

$$\left. + \frac{q}{\pi} \int_0^{\infty} dz \frac{\sinh(z)h(q,\alpha_0,2z)}{\cosh(2qz) - \cos(q\pi)} G(\tilde{\beta}, \tau(z)) \right], \quad (4.13)$$

where we have defined a new function

$$G(\tilde{\beta}, x) = \sum_{l=1}^{\infty} \frac{l \sin(2\pi l\tilde{\beta})}{(l^2 + x^2)^{5/2}}. \quad (4.14)$$

The summation on the quantum number l can be developed with the help of [27]. The result is:

$$G(\tilde{\beta}, x) = -\frac{\pi^2}{4x} \frac{[\sinh(2\pi\tilde{\beta}x) - 2\tilde{\beta}\sinh(\pi x)\cosh[\pi x(1 - 2\tilde{\beta})]]}{\sinh^2(\pi x)}. \quad (4.15)$$

Clearly we can see that $G(\tilde{\beta}, x)$ vanishes for $\tilde{\beta} = 0$, $1/2$, 1.

Now let us turn our attention to some special asymptotic limits of (4.8). Taking w fixed and large values of lengths for
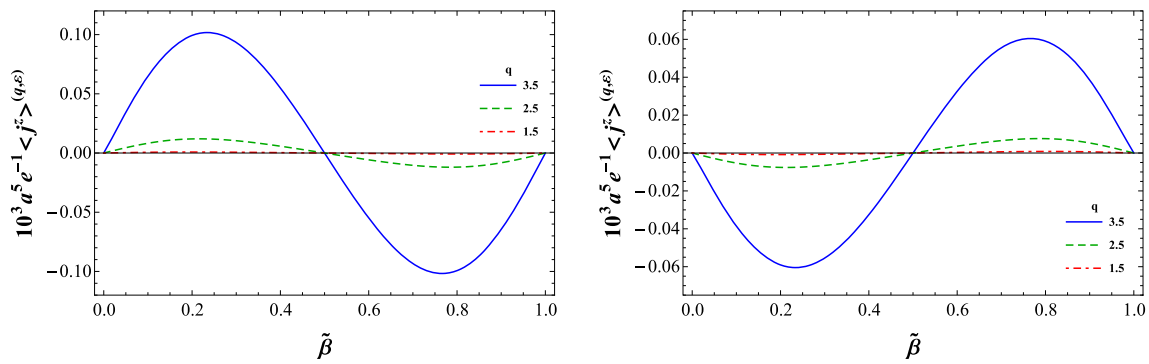
Fig. 5 The VEV of the axial current induced by the compactification in Eq. (4.12) is plotted, in units of $a^5 e^{-1}$, as function of $\tilde{\beta}$ for $q = 2.5$, $r/w = 1$ and $ma = 5$. In the left plot, we consider $\alpha_0 = 0$, while in the right plot, we take $\alpha_0 = 0.35$

the compact dimension, $L/w \gg 1$, the total axial current is expressed as

$$
\langle j^z \rangle \approx -\frac{8v_2(v_2+1)eL}{\pi^2 a^5}\left(\frac{w}{L}\right)^{2v_2+4}
$$
$$
\times \sum_{l=1}^{\infty} l\sin(2\pi l\tilde{\beta})\left\{\sum_{k=0}^{[q/2]}{}'(-1)^k\frac{\cos(\pi k/q)\cos(2\pi\alpha_0 k)}{(l^2+\rho_k^2)^{v_2+2}}\right.
$$
$$
\left.+\frac{q}{\pi}\int_0^{\infty}dz\frac{\sinh(z)h(q,\alpha_0,2z)}{[\cosh(2qz)-\cos(q\pi)][l^2+\tau^2(z)]^{v_2+2}}\right\},
$$
$$(4.16)$$

which is valid only for values of $ma \geq 1/2$.

Another very interesting asymptotic behavior comes from the Minkowskian limit. Making use again of the corresponding $\mathcal{Z}(u)$ function in this limit given by (3.37), and taking it into (4.8), the total axial current density is expressed as

$$
\langle j^z \rangle_{\mathrm{M}} \approx -\frac{2^{3/2}m^{5/2}e}{\pi^{5/2}L^{3/2}}\sum_{l=1}^{\infty} l\sin(2\pi l\tilde{\beta})
$$
$$
\times \left\{\sum_{k=0}^{[q/2]}{}'(-1)^k\cos(\pi k/q)\cos(2\pi\alpha_0 k)\frac{K_{5/2}\left(mL\sqrt{l^2+\rho_k^2}\right)}{(l^2+\rho_k^2)^{5/4}}\right.
$$
$$
\left.+\frac{q}{\pi}\int_0^{\infty}dz\frac{\sinh(z)h(q,\alpha_0,2z)}{\cosh(2qz)-\cos(q\pi)}\frac{K_{5/2}\left(mL\sqrt{l^2+\tau^2(z)}\right)}{[l^2+\tau^2(z)]^{5/4}}\right\}.
$$
$$(4.17)$$

5 Conclusion

In this paper, we have considered a massive charged fermionic field and investigated the induced current densities in a 5-dimensional AdS space-time, assuming the presence of a cosmic string having a magnetic flux running along its axis. Additionally, we admit that an extra dimension is compactified to a circle of perimeter L and that, in addition, a magnetic flux enclosed by the compactified axis exists. In order to develop this analysis, we calculated the complete set of

fermionic wave functions of positive and negative frequency obeying quasi-periodicity condition, with an arbitrary phase, β, along the compact dimension. To calculate the VEV of the fermionic induced current, we have made use of the sum mode formula given in (3.1) and the positive and negative fermionic modes given in (2.33) and (2.40), respectively. In our analysis we proved that only azimuthal and axial current densities take place. These VEVs are periodic functions of the magnetic flux running along the string, α_0, and the parameter $\tilde{\beta}$. This is an Aharonov–Bohm-like effect.

By using the Abel-Plana summation formula given in (3.7), we were able to decompose the VEV of the total azimuthal current density given in (3.31) into two parts: one purely associated to the cosmic string (corresponding to $l = 0$) and another related to the compactification ($l \neq 0$). The string contribution is given in (3.15) and it is an odd function of the magnetic flux, α_0, with its periodic behavior displayed in Fig. 1. The analysis of the asymptotic behaviors was made for points close and far away from the string, and it is presented in (3.21) and (3.22), respectively. In the former, $r/w \ll 1$, $\langle j^\phi \rangle_{\mathrm{cs}}$ diverges on the string as $(w/r)^5$; in the latter, $r/w \gg 1$, the current density starts to fade away with its decay strength depending on the mass of the fermion. These behaviors are graphically exhibited in Fig. 2.

As to the azimuthal current induced by compactification given in Eq. (3.28), we note that it is an odd function of the magnetic flux along the string's core, α_0, and it is an even function of the parameter, $\tilde{\beta}$, with a period equal to the quantum flux Φ_0. This periodic behavior can be observed in Fig. 3. In particular, for $\beta = 0$, $\langle j^\phi \rangle_{\mathrm{c}}$ becomes an even function of the magnetic flux enclosed by the compact dimension. Besides, it is finite on the string's core, and its asymptotic expression for large lengths of the compactified dimension is presented in (3.36). The total azimuthal current density is calculated for a massless spinor field, and it is given in (3.32). In (3.38) we present the Minkowskian asymptotic limit for the total induced azimuthal current density.

The presence of compactification also gives rise to an induced axial current density, and it is presented in closed form in (4.8). This contribution can be separated into two components. The first one, given in (4.11), comes from the term $k = 0$ and does not depend on the radial distance r, neither on q or α_0. The second one, given by (4.12), however, depends on the aforementioned parameters, and it is an even function of α_0 and an odd function of the parameter $\tilde{\beta}$, with this periodic behavior displayed in Fig. 4. For $\alpha_0 = 0$ and $\alpha_0 = 0.35$ we have plotted Eq. (4.12) in Fig. 5. The plots show that the magnetic flux along the string's core has the effect of inverting the direction of the current. The expression for a massless field is presented in (4.13). For large lengths of the compact dimension, an asymptotic expression is given in (4.16) and shows that the induced axial current vanishes when $L \to \infty$. Finally, the Minkowskian asymptotic limit is calculated for the total axial current density, and it is given in (4.17).

In conclusion, we have showed in this paper that the fermionic current densities induced in the vacuum are not only generated by the magnetic fluxes, but also from the nontrivial topology of the compactified extra dimension introduced by the arbitrary phase in the quasi-periodic condition. As it was showed graphically, the angular deficit produced on the background geometry by the cosmic string plays an important role on the intensity and behavior of the VEVs analysed.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and there are no additional data to be deposited.]

References

1. T.W. Kibble, Topology of cosmic domains and strings. J. Phys. A **9**, 1387 (1976)

2. A. Vilenkin, E.P.S. Shellard, *Cosmic strings and other topological defects* (Cambridge University Press, Cambridge, England, 1994)

3. V. Berezinski, B. Hnatyk, A. Vilenkin, Gamma ray bursts from superconducting cosmic strings. Phys. Rev. D **64**, 043004 (2001)

4. P. Bhattacharjee, G. Sigl, Origin and propagation of extremely high energy cosmic rays. Phys. Rep. **327**, 109 (2000)

5. T. Damour, A. Vilenkin, Gravitational wave bursts from cosmic strings. Phys. Rev. Lett. **85**, 3761 (2000)

6. Mark Hindmarsh, Signals of inflationary models with cosmic strings. Prog. Theor. Phys. Suppl. **190**, 197–228 (2011)

7. Edmund J. Copeland, Levon Pogosian, Tanmay Vachaspati, Seeking string theoryin the cosmos. Class. Quant. Grav. **28**, 204009 (2011)

8. S. Sarangi, S.-H. Henry Tye, Cosmic string production towards the end of brane inflation. Phys. Lett. B **536**, 185 (2002)

9. A. Vilenkin, Gravitational field of vacuum domain walls and strings. Phys. Rev. D **23**, 852 (1981)

10. O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, Large N field theories, string theory and gravity. Phys. Rep. **323**, 183 (2000)

11. P. Brax, C. van de Bruck, Cosmology and brane worlds: a review. Classical Quantum Gravity **20**, R201 (2003)

12. E. R. Bezerra de Mello, A. A. Saharian, V. Vardanyan, *Induced vacuum currents in anti-de Sitter space with toral dimensions*, *Phys. Lett. B*. **741**, 155–162 (2015). https://doi.org/10.1016/j.physletb.2014.12.036. arxiv:1410.2860

13. Panagiota Kanti, Keith A. Olive, Maxim Pospelov, On the stabilization of the size of extra dimensions. Phys. Lett. B **538**, 146 (2002)

14. E.A.F. Bragança, H.F. Santana Mota, E.R. Bezerra de Mello, *Induced vacuum bosoniccurrent by magnetic flux in a higher dimensional compactified cosmic string spacetime*. Int. J. Mod. Phys. D **24**, 1550055 (2015). arxiv:1410.1511

15. E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, H.H. Harutyunyan, Vacuum currents induced by a magnetic flux around a cosmic string with finite core. Phys. Rev. D **91**, 064034 (2015). arxiv:1411.1258

16. E.R. Bezerra de Mello, A.A. Saharian, Fermionic current induced by magnetic flux in compactified cosmic string spacetime. Eur. Phys. J. C **73**, 2532 (2013). arxiv:1305.6902

17. E.R. Bezerra de Mello, Induced fermionic current densities by magnetic flux in higher dimensional cosmic string spacetime. Class. Quant. Grav. **27**, 095017 (2010). arxiv:0907.4139

18. E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, V.M. Bardeghyan, Fermionic current densities induced by magnetic flux in a conical space with a circular boundary. Phys. Rev. D **82**, 085033 (2010). arxiv:1008.1743

19. M.S. Maior de Sousa, R.F. Ribeiro, E.R. Bezerra de Mello, Induced fermionic current by a magnetic tube in the cosmic string spacetime. Phys. Rev. D **93**, 043545 (2016). arxiv:1511.02745

20. W. Oliveira dos Santos, H.F. Mota, E.R. Bezerra de Mello, Induced current in high-dimensional AdS spacetime in the presence of a cosmic string and a compactified extra dimension. Phys. Rev. D **99**, 045005 (2019). arXiv:1809.00702 [hep-th]

21. E. R. Bezerra de Mello, A.A. Saharian, Vacuum polarization induced by a cosmic string in anti-de Sitter spacetime. J. Phys. A **45**, 115402 (2012). arxiv:1110.2129

22. S. Bellucci, A.A. Saharian, V. Vardanyan, Vacuum currents in braneworlds on AdS bulkwith compact dimensions. JHEP **11**, 092 (2015). arxiv:1508.07255

23. W. Oliveira dos Santos, E.R. Bezerra de Mello, H.F. Mota, Vacuum polarization in high-dimensional AdS space-time in the presence of a cosmic string and a compactified extra dimension. Eur. Phys. J. Plus **135**, 27 (2020). arXiv:1904.08419 [hep-th]

24. S. Bellucci, A.A. Saharian, and V. Vardanyan, *Hadamard function and the vacuum currentsin braneworlds with compact dimen-*

sions: Two-brane geometry. Phys. Rev. D **93**, 084011 (2016). arxiv:1512.06569

25. S. Bellucci, A.A. Saharian, V. Vardanyan, Fermionic currents in AdS spacetime with compact dimensions. Phys. Rev. D **96**, 065025 (2017)

26. M. Bordag, N. Khusnutdinov, A remark on bound states in conical spacetime. Class. Quantum Grav. **13**, L41 (1996)

27. I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series and products* (Academic Press, New York, 1980)

28. S. Bellucci, A. Saharian, V. Bardeghyan, Induced fermionic current in toroidally compactified spacetimes with applications to cylindrical and toroidal nanotubes. Phys. Rev. D **82**, 065011 (2010). arXiv:1002.1391

29. E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, V.M. Bardeghyan, Fermionic current densities induced by magnetic flux in a conical space with a circular boundary. Phys. Rev. D **82**, 085033 (2010)

30. M. Abramowitz, I.A. Stegun (eds.), Handbook of Mathematical Functions (Dover, New York, 1972)