DC conductivities and Stokes flows in Dirac semimetals influenced by *hidden sector*

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Abstract In the holographic model of Dirac semimetals, the Einstein–Maxwell scalar gravity with the auxiliary $U(1)$ -gauge field, coupled to the ordinary Maxwell one by a *kinetic mixing* term, the black brane response to the electric fields and temperature gradient has been elaborated. Using the foliation by hypersurfaces of constant radial coordinate we derive the exact form of the Hamiltonian and equations of motion in the phase space considered. Examination of the Hamiltonian constraints enables us, to the leading order expansion of the linearised perturbations at the black brane event horizon, to derive the Stokes equations for an incompressible doubly charged fluid. Solving the aforementioned equations, one arrives at the DC conductivities for the holographic Dirac semimetals.

1 Introduction

There has been observed a major resurgence of theoretical interest in the connections among theories of gravity and Navier–Stokes equations governing fluid dynamics, namely the development of the idea of the so-called black hole membrane paradigm [1]. Approximate solutions of the gravity theory relations were obtained by solving relativistic hydrodynamics equations, by the derivative expansion. Moreover, the Navier–Stokes equations has begun to attract more attention in strongly correlated systems, via the AdS/CFT correspondence [3, 4]. In [5] the equations of relativistic hydrodynamics were reduced to the incompressible Navier–Stokes equation in a particular scaling limit. The implementation of this scaling limit to holographically induced fluid dynamics enabled one to find gravity dual descriptions of an arbitrary solution of the forced non-relativistic incompressible Navier–Stokes equation. In [6] the hydrodynamics of a relativistic conformal fluid at finite temperature was studied. It

was revealed that, for viscous hydrodynamics in the limit of slow motion, the equations could be cast in the form of non-relativistic, incompressible Navier–Stokes theory. On the other hand, it was shown [7] that such a kind of relations could be derived from black hole membrane dynamics.

In [8] it was shown how the solutions of Navier–Stokes equations on a hypersurface in Minkowski spacetime could lead to the solutions of Einstein theory of gravity.

In the near-horizon and non-relativistic limits it was found that the perturbation effects in massive gravity in the bulk could be governed by an incompressible Navier–Stokes equation [9]. The implications of the certain modes for the vacuum solutions being consistent with the hydrodynamic scaling, were elaborated in [10]. It happens that the inclusion in question corresponds to the solutions with certain types of matter. This fact reveals that gravity has a description not only on null surfaces but also on time-like ones. The new setting of the Navier–Stokes equations was presented in [11], where a metric constructed with the help of the scaling and symmetry properties of the Navier–Stokes relations was proposed.

Recently it has been also revealed in Einstein–Maxwell scalar gravity with a potential, holographically dual to the conformal field theory, in an asymptotically AdS spacetime, that DC conductivities can be found by solving the Stokes-like equation on the black hole event horizon [12, 13]. The non-vanishing magnetic field in the theory in question was treated in [14].

The obtained results revive the ones obtained by the concept of holographic Q-lattice used in studies of DC-transport coefficients. Axion-like fields enable one to break the translation invariance, delivering the mechanism for momentum dissipation and leading to finite values of the DC coefficients. Interesting results using this technique for various models and strengths of dissipation were obtained [15–20, 23]. The studies in question were also elaborated in a higher derivative gravity background [24] and in Gauss–Bonnet–Maxwell scalar theory with momentum relaxation [25].

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Our work will justify the generalisation of the studies presented in [12, 13] to describe CD thermoelectric conductivities in holographic Dirac semimetals, described on the gravity side by Einstein–Maxwell scalar gravity with an auxiliary $U(1)$ -gauge field coupled to the Maxwell one by the so-called *kinetic mixing* term. Using a foliation of the spacetime with hypersurfaces of constant r , we find the momentum constraints and derive the Stokes equations in terms of the linearised perturbation data on the black brane event horizon. Solving these equations in the next step and finding the form of DC thermoelectric conductivity currents, we read off the adequate thermoelectric coefficients for the holographic Dirac semimetals. The principal objective of our studies is to reveal the influence of the auxiliary field, from the *hidden sector* and the α -coupling constant on the thermoelectric conductivities in Dirac semimetals.

The paper is organised as follows. In Sect. 2 we present the model of AdS Einstein–Maxwell scalar gravity with an additional $U(1)$ -gauge field coupled to the ordinary one via the so-called *kinetic mixing term*. In Sect. 3 the Arnowitt–Deser–Misner (ADM) formalism for the gravity model in question is elaborated. We study the case of the spacetime foliation by hypersurfaces of constant r . Finding the momenta connected with the fields in the theory, by Legendre transformation, we build the Hamiltonian and obtain the Hamilton equations on motion. Section 4 will be devoted to linear perturbations of the black brane spacetime and the construction of gauge and heat currents, on the event horizon of the black object. In Sect. 5 we derive the Stokes equations on the black brane event horizon, for an incompressible doubly charged fluid. Next, we study their main features. In Sect. 6 we present the example of finding the CD thermoelectric conductivities obtained by solving the linearised, time-independent, forced Stokes equations, in the case of a one-dimensional lattice. We end with a summary and our conclusions.

2 Background holographic model

We deal with the generalisation of the previously studied models [12, 13], by adding two interacting $U(1)$ -gauge fields, in order to find the influence of them on the DC thermoelectric transport coefficients and to compare with the existing results. In our model the gravitational action in $(3+1)$ dimensions is taken in the form

$$S = \int \sqrt{-g} d^4x \left(R + \frac{6}{L^2} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{\alpha}{4} F_{\mu\nu} B^{\mu\nu} \right), \quad (1)$$

where ϕ is the scalar field, $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ stands for the ordinary Maxwell field strength tensor, while the second $U(1)$ -gauge field $B_{\mu\nu}$ is given by $B_{\mu\nu} = 2\nabla_{[\mu} B_{\nu]}$. α is a

coupling constant between the two gauge fields. L is the radius of AdS spacetime. The coupling constant is denoted by α .

The justifications of such kind of gravity with electromagnetism coupled to the other gauge field exonerate from the top–down perspective [26]. Namely, starting from the string/M-theory the reduction to lower dimensional gravity is performed. It is relevant for the holographic correspondence presumption, because the theory in question is a fully consistent quantum theory (string/M-theory), and this fact guarantees that any predicted phenomenon by the top–down theory will be physical.

In the considered action (1) we have to do with a second gauge field coupled to the ordinary Maxwell one. This field is connected with the so-called *hidden sector* [26]. The term describing the interaction of the visible (Maxwell field) sector and the *hidden* $U(1)$ -gauge field is called the *kinetic mixing term*. For the first time such types of terms were revealed in [27], in order to describe the existence and subsequent integrating out of heavy bi-fundamental fields charged under the $U(1)$ -gauge groups. The predicted values of the α -coupling constant, being the kinetic mixing parameter between the two $U(1)$ -gauge fields, for realistic string compactifications range between 10^{-2} and 10^{-16} [28–31].

It happens that in string phenomenology [29] the dimensionless *kinetic mixing term* parameter α can be produced at an arbitrary high energy scale and due to this fact its measurement can provide some interesting features of high energy physics beyond the range of the contemporary colliders; this fact is of great significance for future experiments.

On the other hand, the idea that *hidden sector* (*dark matter* fields) and the *visible* one (the ordinary Maxwell field) are coupled has found firm support due to the recent astronomical observations of 511 eV gamma rays, experiments detecting the electron positron excess in galaxy, and the potential explanation of the muon anomalous magnetic moment [32–35]. On cosmological scales collisions among galaxy clusters can also be useful in testing non-gravitational forces acting on *dark matter* [36]. On the other hand, the correctness of the proposed model could be justified in the recent experiments aimed at gamma ray emissions from dwarf galaxies [37], dilaton-like coupling to photons caused by ultra-light *dark matter* [38], oscillations of the fine structure constant [39], revisions of the constraints on *dark photon* 1987A supernova emission [40], measurements of excitations of electrons in CCD-like detector [41], and examinations in e^+e^- Earth colliders [42]. The ongoing and future planned experiments may provide the mass constraints on the *hidden sector* particles, especially for *dark photons*, which enable us to find this elusive ingredient of the mass of our Universe.

One also should remark that the model with two coupled vector fields was used in a generalisation of p-wave supercon-

ductivity, for the holographic model of ferromagnetic superconductivity [43].

It has been argued that near charge neutrality, a strong interacting plasma (Dirac fluid) can form. The evidence of such a fluid was revealed in experiments on the violation of the Wiedemann–Franz law in extremely clean graphene near the charge neutral point [44]. This fact triggered the appearance of the holographic generalisation of the hydrodynamical approach to the problem in question [45, 46]. Namely in [47] the holographic model of strongly coupled plasma with two non-interacting $U(1)$ -gauge currents was introduced, in order to explain the existing experimental data. Two currents envisage the electron and hole currents present in the system with Fermi energy turned out to coincide with Dirac point. However, there still exists room for improvements. In [48] the model with two interacting $U(1)$ -gauge currents ($\alpha \neq 0$) was proposed, where the thermoelectric transport properties, Hall effect geometry with the magnetic field perpendicular to the graphene plane and with the electric field and temperature gradients in the plane but being perpendicular to each other, were studied. Among all, the DC-transport coefficients were found by the introduction of the axionic field which on the condensed matter side provides a momentum relaxation mechanism related to the mobility of the material. The *hidden sector* of the $U(1)$ -gauge field taken into account in the action affects the kinetic and transport coefficients *via* the parameter α .

To be compared with [47], a model with two interacting gauge currents predicts that the increase of the α -coupling constant value leads to an increase of the width of normalised thermal conductivity with the parameter which bounds the charges of *visible* and *hidden* sectors. On the other hand, the dependence of α -coupling constant on the Wiedemann–Franz ratio (WFR) is related to the changes of the width of curves and their heights. The general tendency is visible in the fact that WFR decreases as the α -coupling constant increases. The aforementioned dependence is valid for all charge densities. The dependence of the Seebeck coefficient on the charge concentration for the different values of mobilities was studied and one obtained very good agreement with the experimental data.

In Ref. [49], the extension of the previous analysis [47, 48] of transport in graphene in three directions was provided. The interaction between the two currents and the application of an external magnetic field enable one to calculate the transport coefficients in the model under consideration. Among all the obtained results we find the fact that WFR depends on the parameter characterising the contribution of the adequate charge from *visible* and *hidden* sectors, as well as the α -coupling constant. They change the maximal values of the transport coefficients. The influence of the aforementioned factors is not large but may play a crucial role in the detailed description of the future experiments.

The hydrodynamical model of a Dirac semimetal having two Dirac nodes separated in momentum space along the rotation axis was considered in a system with two interacting gauge fields, with implementation of the chiral anomaly and the Z_2 topological charge [50]. The topological charge was bounded with the anomaly term in the *hidden sector*. The addition of the Z_2 topological charge modifies the equations and leads to the appearance of new kinetic coefficients connected with vorticity and the magnetic field of the *hidden sector*. The Chern–Simons terms in the five-dimensional generalisation of the holographic model presented enables one to model the holographic Weyl semimetals with Z_2 symmetry. In [51] the magneto-transport coefficients were found for the holographic model in question.

Various other settings of the model in question were described in [52–61], where was mimicked that the *dark matter* sector influences the properties of superconductors and superfluids.

Variation of the action S with respect to the metric, the scalar and gauge fields yields the following equations of motion:

$$G_{\mu\nu} - g_{\mu\nu} \frac{3}{L^2} = T_{\mu\nu}(\phi) + T_{\mu\nu}(F) + T_{\mu\nu}(B) + \alpha T_{\mu\nu}(F, B), \quad (2)$$

$$\nabla_\mu F^{\mu\nu} + \frac{\alpha}{2} \nabla_\mu B^{\mu\nu} = 0, \quad (3)$$

$$\nabla_\mu B^{\mu\nu} + \frac{\alpha}{2} \nabla_\mu F^{\mu\nu} = 0, \quad (4)$$

$$\nabla_\mu \nabla^\mu \phi = 0, \quad (5)$$

where we have denoted by $G_{\mu\nu}$ the Einstein tensor, while the energy momentum tensors for the fields in the theory are given by

$$T_{\mu\nu}(\phi) = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4} g_{\mu\nu} \nabla_\delta \phi \nabla^\delta \phi, \quad (6)$$

$$T_{\mu\nu}(F) = \frac{1}{2} F_{\mu\delta} F_\nu^\delta - \frac{1}{8} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \quad (7)$$

$$T_{\mu\nu}(B) = \frac{1}{2} B_{\mu\delta} B_\nu^\delta - \frac{1}{8} g_{\mu\nu} B_{\alpha\beta} B^{\alpha\beta}, \quad (8)$$

$$T_{\mu\nu}(F, B) = \frac{1}{2} F_{\mu\delta} B_\nu^\delta - \frac{1}{8} g_{\mu\nu} F_{\alpha\beta} B^{\alpha\beta}. \quad (9)$$

For the gauge fields in the considered theory we assume the following components:

$$A_\mu dx^\mu = a_t dt, \quad B_\mu dx^\mu = b_t dt. \quad (10)$$

3 Arnott–Desser–Misner formalism

In this section we present the basic idea for the $(3+1)$ formalism for the Einstein–Maxwell scalar theory with an additional $U(1)$ -gauge field, coupled to the ordinary Maxwell one. The

formalism in question considers the four-dimensional spacetime foliated by three-geometries of constant r -coordinate. On three-dimensional hypersurfaces the induced metric may be written as $h_{ab} = g_{ab} - n_a n_b$. The line element is of the form

$$ds^2 = N^2 dr^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \quad (11)$$

where N is the lapse function while N^a stands for the shift vector for the constant r -coordinate hypersurface in the underlying manifold. It turns out that the spacetime geometry can be described in terms of the intrinsic metric and the extrinsic curvature of a three-dimensional hypersurface; N and N^a relate the intrinsic coordinate on one hypersurface to the intrinsic coordinates on a nearby hypersurface. The general covariance allows for a great arbitrariness in the choice of the aforementioned functions, $N^\mu = (N, N^a)$.

In the canonical formulation of the Einstein–Maxwell scalar gravity with additional gauge field, the point in the phase space corresponds to the specification of the fields $(h_{ab}, \pi^{ab}, \tilde{A}_i, \tilde{B}_i, \phi, E_i, B_i, E)$ on a three-dimensional Σ_r manifold, where h_{ab} denotes induced Riemannian metric on Σ_r , $\tilde{A}_i$ and $\tilde{B}_i$ are $U(1)$ -gauge fields on the three-dimensional manifold, while E_i and B_i are, respectively, Maxwell, auxiliary gauge *electric* fields in the evolved spacetime. They constitute tensor densities and imply the following relations:

$$E_k = \sqrt{-h} F_{\mu k} n^\mu, \quad (12)$$

for the Maxwell *electric* field, while for the *dark matter* sector *electric* field B_i , one obtains

$$B_k = \sqrt{-h} B_{\mu k} n^\mu. \quad (13)$$

ON the other hand, for the scalar field one has

$$E = \sqrt{-h} \nabla_\mu \phi n^\mu, \quad (14)$$

where in all the cases n^μ constitutes the unit normal vector to the hypersurface Σ_r of constant r , in the underlying spacetime.

The Lagrangian density of Einstein double $U(1)$ -gauge scalar gravity is subject to the relation

$$\begin{aligned} \mathcal{L} = N\sqrt{-h} & \left[{}^{(3)}R - K_{ab}K^{ab} + K^2 \right. \\ & - \frac{1}{4} \left(F_{ab}F^{ab} + \frac{2}{(-h)} E_i E^i \right) \\ & - \frac{1}{4} \left(B_{ab}B^{ab} + \frac{2}{(-h)} B_i B^i \right) - \frac{\alpha}{4} \left(F_{ab}B^{ab} \right. \\ & \left. \left. + \frac{2\alpha}{(-h)} E_i B^i \right) \right], \end{aligned} \quad (15)$$

where K_{ab} denotes the extrinsic curvature, while ${}^{(3)}R$ is three-dimensional Ricci scalar.

In order to obtain the corresponding field momenta, one ought to perform a variation of the underlying Lagrangian with respect to $\nabla_r h_{ab}$, $\nabla_r \pi_{ab}$, $\nabla_r \tilde{A}_i$, $\nabla_r \tilde{B}_i$, where ∇_r denotes the derivative with respect to r -coordinate.

It leads to the relation for the gravitational momentum

$$\pi^{ab} = -\sqrt{-h} \left(K^{ab} - h^{ab} K \right). \quad (16)$$

Consequently, for the momentum responsible for $U(1)$ -gauge fields one obtains

$$\begin{aligned} \pi_{(F)}^k &= \frac{\delta \mathcal{L}}{\delta (\nabla_r \tilde{A}_k)} = -E^k - \frac{\alpha}{2} B^k, & \pi_{(B)}^k &= \frac{\delta \mathcal{L}}{\delta (\nabla_r \tilde{B}_i)} \\ &= -B^k - \frac{\alpha}{2} E^k, \end{aligned} \quad (17)$$

while, for the scalar field ϕ , it yields

$$\pi_{(\phi)} = \frac{\delta \mathcal{L}}{\delta (\nabla_r \phi)} = -E. \quad (18)$$

The Hamiltonian for the considered theory with auxiliary gauge field will be defined by the Legendre transform. It is given by

$$\begin{aligned} \mathcal{H} &= \pi^{ij} \nabla_0 h_{ij} + \pi_{(F)}^i \nabla_0 \tilde{A}_i \pi_{(B)}^i \nabla_0 \tilde{B}_i - \mathcal{L} \\ &= N^\mu C_\mu + \tilde{A}_r \tilde{\tilde{A}} + \tilde{B}_r \tilde{\tilde{B}} + \mathcal{H}_{div}, \end{aligned} \quad (19)$$

where $\mathcal{H}_{div}$ is the total derivative and has the form

$$\mathcal{H}_{div} = D_k \left(\tilde{A}_r \pi_{(F)}^k + \tilde{B}_r \pi_{(B)}^k \right) + 2D_a \left(\frac{N_b \pi^{ab}}{\sqrt{-h}} \right), \quad (20)$$

where D_m denotes the covariant derivative with respect to the metric h_{ab} , on the hypersurface Σ .

In our considerations we shall deal with the so-called *asymptotically flat initial data*, i.e., one has to do with an asymptotic region of hypersurface Σ_r , which is diffeomorphic to $\mathbf{R}^3 - B$, where B is compact. Moreover, the fall-off conditions for the fields in the phase space imply

$$h_{ab} \approx \delta_{ab} + \mathcal{O}\left(\frac{1}{r}\right), \quad (21)$$

$$\pi_{ab} \approx \mathcal{O}\left(\frac{1}{r^2}\right), \quad (22)$$

$$\tilde{A}_i \approx \mathcal{O}\left(\frac{1}{r}\right), \quad \tilde{B}_i \approx \mathcal{O}\left(\frac{1}{r}\right), \quad (23)$$

$$E_i \approx \mathcal{O}\left(\frac{1}{r^2}\right), \quad B_i \approx \mathcal{O}\left(\frac{1}{r^2}\right), \quad (24)$$

$$\phi \approx \mathcal{O}\left(\frac{1}{r^2}\right), \quad E \approx \mathcal{O}\left(\frac{1}{r^2}\right). \quad (25)$$

The standard asymptotic behaviour of the lapse function is provided by $N \approx 1 + \mathcal{O}(1/r)$, while the shift function

behaves as $N^a \approx \mathcal{O}(1/r)$. Because of the fact that the r-components of the $U(1)$ gauge fields $\tilde{A}_r$, $\tilde{B}_r$ do not possess an associated kinetic terms, we can consider them as Lagrange multipliers, being subject to the generalised Gauss laws provided by

$$\tilde{A} = -D_k \pi_{(F)}^k = 0, \quad \tilde{B} = -D_k \pi_{(B)}^k = 0. \quad (26)$$

On the other hand, the components C^μ imply the relations

$$C_0 = \frac{\sqrt{-h}}{4} (F_{ab} F^{ab} + B_{ab} B^{ab} + \alpha F_{ab} B^{ab} + 2D_a \phi D^a \phi) - \frac{1}{2\sqrt{-h}} (E_k E^k + B_k B^k + \alpha E_k B^k + E^2) + \sqrt{-h} \left(-{}^{(3)}R + \frac{6}{L^2} - \left(\frac{1}{(-h)} \pi_{ij} \pi^{ij} - \frac{1}{2(-h)} \pi^2 \right) \right), \quad (27)$$

$$C_a = -\left(B_{ak} B^k + \frac{\alpha}{2} B_{ak} E^k + F_{ak} E^k + \frac{\alpha}{2} F_{ak} B^k + D_a \phi E \right) - 2\sqrt{-h} D_a \left(\frac{N_b \pi^{ab}}{\sqrt{-h}} \right). \quad (28)$$

The evolution equations for the theory in question may be formally derived by considering the *volume integral contribution* to the Hamiltonian [62], denoted by $\mathcal{H}_v$, which has a pure constraint form, provided by

$$\mathcal{H}_v = \int_\Sigma d\Sigma N^\mu C_\mu. \quad (29)$$

Finding arbitrary infinitesimal variations $(\delta h_{ab}, \delta \pi_{ab}, \delta \tilde{A}_i, \delta \tilde{B}_i, \delta \phi, \delta E_i, \delta B_i, \delta E)$, after integration by parts, we obtain the change of the Hamiltonian $\mathcal{H}_v$ caused by the variations in question, given by

$$\delta \mathcal{H}_v = \int_\Sigma d\Sigma \left(P^{ab} \delta h_{ab} + Q^{ab} \delta \pi_{ab} + R^i \delta \tilde{A}_i + P^i \delta \tilde{B}_i + S^i \delta E_i + Q^i \delta B_i \right). \quad (30)$$

The evolution equations for the considered system yield

$$\dot{h}_{ab} = \frac{\delta \mathcal{H}_v}{\delta \pi^{ab}} = Q_{ab}, \quad \dot{\pi}_{ab} = -\frac{\delta \mathcal{H}_v}{\delta h^{ab}} = -P_{ab}, \quad (31)$$

$$\dot{E}_k = -\frac{\delta \mathcal{H}_v}{\delta \tilde{A}^k} = -R_k, \quad \dot{\tilde{A}}_k = \frac{\delta \mathcal{H}_v}{\delta E^k} = S_k, \quad (32)$$

$$\dot{B}_k = -\frac{\delta \mathcal{H}_v}{\delta \tilde{B}^k} = -P_k, \quad \dot{\tilde{B}}_k = \frac{\delta \mathcal{H}_v}{\delta B^k} = Q_k, \quad (33)$$

$$\dot{E} = -\frac{\delta \mathcal{H}_v}{\delta \phi} = -Z, \quad \dot{\phi} = \frac{\delta \mathcal{H}_v}{\delta E} = W, \quad (34)$$

where we have denoted

$$P^{ab} = N\sqrt{-h} w^{ab} - N^m \left(B_m^a B^b + \frac{\alpha}{2} B_m^a E^b + F_m^a E^b + \frac{\alpha}{2} F_m^a B^b \right)$$

$$+ N^a D^b \phi + \sqrt{-h} (h^{ab} D^m D_m N - D^a D^b N) - \mathcal{L}_{Ni} \pi^{ab}, \quad (35)$$

$$Q_{ab} = -\frac{N}{\sqrt{-h}} (2\pi_{ab} - \pi_k^k h_{ab}) + \mathcal{L}_{Ni} h_{ab}, \quad (36)$$

$$R^i = -\sqrt{-h} D_a \left[N \left(F^{ai} + \frac{\alpha}{2} B^{ai} \right) \right] + \mathcal{L}_{Ni} \left(E^i + \frac{\alpha}{2} B^i \right), \quad (37)$$

$$P^i = -\sqrt{-h} D_a \left[N \left(B^{ai} + \frac{\alpha}{2} F^{ai} \right) \right] + \mathcal{L}_{Ni} \left(B^i + \frac{\alpha}{2} E^i \right), \quad (38)$$

$$S_k = \frac{-N}{\sqrt{-h}} \left(E_k + \frac{\alpha}{2} B_k \right) - \left(\mathcal{L}_{Ni} \tilde{A}_k + \frac{\alpha}{2} \mathcal{L}_{Ni} \tilde{B}_k \right) - D_k (N \tilde{A}_r) - \frac{\alpha}{2} D_k (N \tilde{B}_r), \quad (39)$$

$$Q_k = \frac{-N}{\sqrt{h}} \left(B_k + \frac{\alpha}{2} E_k \right) - \left(\mathcal{L}_{Ni} \tilde{B}_k + \frac{\alpha}{2} \mathcal{L}_{Ni} \tilde{A}_k \right) - D_k (N \tilde{B}_r) - \frac{\alpha}{2} D_k (N \tilde{A}_r), \quad (40)$$

$$W = N^a D_a \phi, \quad (41)$$

$$Z = D_m (N^m E). \quad (42)$$

On the other hand, for the quantity w^{ab} , entering (35), we have the following relation:

$$w^{ab} = -\frac{1}{(-h)} \left(2\pi^a_j \pi^{bj} - \pi_m^m \pi^{ab} \right) + \frac{1}{2} h^{ab} \left(\pi_{ij} \pi^{ij} - \frac{1}{2} \pi_m^m \pi_k^k \right) + {}^{(3)}R^{ab} - \frac{1}{2} h^{ab} {}^{(3)}R + \frac{3}{L^2} + \frac{h^{ab}}{8} \left(F_{ij} F^{ij} + B_{ij} B^{ij} + \alpha F_{ij} B^{ij} + 2D_m \phi D^m \phi \right) + \frac{1}{2} \left(F^{aj} B_j^b + B^{aj} B_j^b + \alpha F^{aj} B_j^b + D^a \phi D^b \phi \right) + \frac{h^{ab}}{4(-h)} \left(E_k E^k + B_k B^k + \alpha E_k B^k + E^2 \right) - \frac{N}{2(-h)} \left(E^a E^b + B^a B^b + \alpha E^a B^b \right). \quad (43)$$

In the above formulae, $\mathcal{L}_{Ni} E^k$, and $\mathcal{L}_{Ni} B^k$ and $\mathcal{L}_{Ni} \pi^{ab}$ denote the Lie derivatives of tensor densities and take the forms

$$\mathcal{L}_{Ni} E^k = \sqrt{-h} N^c D_c \left(\frac{E^k}{\sqrt{-h}} \right) - E^c D_c N^k + E^k D_c N^c, \quad (44)$$

$$\mathcal{L}_{Ni} B^k = \sqrt{-h} N^c D_c \left(\frac{B^k}{\sqrt{-h}} \right)$$

$$-B^c D_c N^k + B^k D_c N^c, \quad (45)$$

$$\begin{aligned} \mathcal{L}_{N^i} \pi^{ab} = & \sqrt{-h} N^c D_c \left(\frac{\pi^{ab}}{\sqrt{-h}} \right) - 2\pi^{c(a} D_c N^{b)} \\ & + \pi^{ab} D_c N^c, \end{aligned} \quad (46)$$

while $\mathcal{L}_{N^i} \tilde{A}_k$, $\mathcal{L}_{N^i} \tilde{B}_k$ and $\mathcal{L}_{N^i} h_{ab}$ correspond to the ordinary Lie derivatives.

The quantities N^μ , $\tilde{A}_0$, $\tilde{B}_0$ are viewed as non-dynamical variables which are not represented in the phase space of Einstein–Maxwell scalar auxiliary $U(1)$ -gauge theory. It enables us to designate them arbitrarily. The choice of N^μ is caused by the evolution of the system one looks for. On the other hand, for $\tilde{A}_r$ and $\tilde{B}_r$ one restricts oneself to the case when the r -coordinates tend to infinity.

4 Black brane spacetime

In the our analysis we consider the line element of a charged under two $U(1)$ -gauge groups, static black brane

$$ds^2 = -U(r)G(r, x_i)dt^2 + \frac{F(r, x_i)dr^2}{U(r)} + ds^2(\Sigma_2), \quad (47)$$

where Σ_2 stands for the two-dimensional hypersurface at the chosen r -coordinate. As in [12], the line element at $r \rightarrow \infty$ approaches the AdS boundary with the following conditions:

$$U \rightarrow r^2, \quad F \rightarrow 1, \quad G \rightarrow G(x), \quad g_{ij} \rightarrow r^2 \tilde{g}_{ij} \quad (48)$$

$$\begin{aligned} a_t(r, x_i) &\rightarrow \mu(x), \quad b_t(r, x_i) \rightarrow \mu_d(x), \quad \phi(r, x_i) \\ &\rightarrow r^{\Delta-3} \tilde{\phi}(x_i), \end{aligned} \quad (49)$$

where $\mu(x)$ and $\mu_d(x)$ are the spatially dependent chemical potentials bounded with the adequate $U(1)$ -gauge field. $\tilde{\phi}(x_i)$ is a spatially dependent source accompanied by it dual operator. The operator has a dimensional scaling Δ . For brevity of the subsequent notation we set the radius of AdS spacetime L equal to 1.

The black brane event horizon Σ_2 , of the topology defined, is situated at $r = 0$. Having in mind the in-going coordinates

$$v = t + \frac{\ln r}{4\pi T} + \dots, \quad (50)$$

the near-horizon expansions of the metric tensor components and fields are given by [13] the following relations:

$$U(r) = r(4\pi T + U^{(1)}r + \dots), \quad (51)$$

$$G(r, x_i) = G^{(0)}(x) + G^{(1)}(x)r + \dots, \quad (52)$$

$$F(r, x_i) = F^{(0)}(x) + F^{(1)}r + \dots, \quad (53)$$

$$g_{ij} = g_{ij}^{(0)} + g_{ij}^{(1)}r + \dots, \quad (54)$$

$$a_t(r, x_i) = r(a_t^{(0)} G^{(0)}(x) + a_t^{(1)}(x)r + \dots), \quad (55)$$

$$b_t(r, x_i) = r(b_t^{(0)} G^{(0)}(x) + b_t^{(1)}(x)r + \dots), \quad (56)$$

$$\phi(r, x_i) = \phi^{(0)}(x) + \phi^{(1)}(x)r + \dots, \quad (57)$$

with the auxiliary condition that $G^{(0)}(x) = F^{(0)}(x)$.

4.1 Perturbed black brane

In the next step we turn on the *electric* Maxwell E_a and auxiliary $U(1)$ -gauge B_a fields, as well as temperature gradient ξ , in the spacetime under inspection, at fixed r -coordinate. The black brane will react to our action. We shall restrict our attention to the linear appropriate perturbations $\delta g_{\mu\nu}$, δa_μ , δb_μ , and $\delta\phi$. The linear perturbations of the metric and fields are given by

$$\delta(ds^2) = \delta g_{\alpha\beta} dx^\alpha dx^\beta - 2t M \xi_a dt dx^a, \quad (58)$$

$$\delta A = \delta a_\beta dx^\beta - t E_a dx^a + t N \xi_b dx^b, \quad (59)$$

$$\delta B = \delta b_\beta dx^\beta - t B_a dx^a + t N_d \xi_b dx^b, \quad (60)$$

and we have a perturbation of the scalar field $\delta\phi$. In what follows one supposes that $\delta g_{\mu\nu}$, δa_μ , δb_μ , and $\delta\phi$ are functions of the (r, x_m) -coordinates. On the other hand, E_a , B_a , ξ_i are functions of x_a . On the submanifold Σ_2 we require that E , B , ξ are closed one-forms, i.e.,

$$d(E_m dx^m) = d(B_m dx^m) = d(\xi_m dx^m) = 0. \quad (61)$$

These assumptions are of great importance when the submanifold has a torus topology or when we consider n -dimensional black objects. For instance studying five-dimensional spacetime, we have to consider the torus topology of the event horizon for the so-called black ring, being a stationary axisymmetric black object solution. The closed form assumption enables us to define potential and charges, i.e., calculating the Noether charges over the event horizon we obtain a constant value (potential) multiplied by the adequate charge [63–66]. Namely, using the Hodge theorem, the closed p -form, in n -dimensions, can be rewritten on the event horizon as a sum of an exact and harmonic forms. An exact form does not contribute to the equations because the equations of motion are satisfied. The harmonic part concerns the only contribution. From the duality between homology and cohomology one concludes that there is a harmonic dual form to the $n - p - 1$ cycle S , in the sense of the equality of the adequate surface integrals. Then it follows that the surface term will be of the form of a constant multiplied by the local charge.

In order to establish the form of the linearised perturbations, one has to take into account that the t -coordinate is no longer a good one at the black brane event horizon. The regularity of the perturbations near $r \rightarrow 0$ requires that some restrictions should be imposed on them. Consequently, at the black brane near-horizon area, when $r \rightarrow 0$, for the leading

order, we obtain

$$\delta g_{tt} = U(r) \left(\delta g_{tt}^{(0)}(x_i) + \mathcal{O}(r) \right),$$

$$\delta g_{tr} = \delta g_{tr}^{(0)}(x_i) + \mathcal{O}(r), \quad (62)$$

$$\delta g_{rr} = \frac{1}{U(r)} \left(\delta g_{rr}^{(0)}(x_i) + \mathcal{O}(r) \right),$$

$$\delta g_{ij} = \delta g_{ij}^{(0)}(x_i) + \mathcal{O}(r), \quad (63)$$

$$\delta g_{ti} = \delta g_{ti}^{(0)}(x_i) - GU \xi_i \frac{\ln r}{4\pi T} + \mathcal{O}(r),$$

$$\delta g_{ri} = \frac{1}{U(r)} \left(\delta g_{ri}^{(0)}(x_i) + \mathcal{O}(r) \right), \quad (64)$$

$$\delta a_t = \delta a_t^{(0)}(x_i) + \mathcal{O}(r), \quad \delta a_i = \frac{\ln r}{4\pi T} \left(-E_i + N \xi_i \right) + \mathcal{O}(r), \quad (65)$$

$$\delta a_r = \frac{1}{U(r)} \left(\delta a_r^{(0)}(x_i) + \mathcal{O}(r) \right), \quad (66)$$

$$\delta b_t = \delta b_t^{(0)}(x_i) + \mathcal{O}(r), \quad \delta b_i = \frac{\ln r}{4\pi T} \left(-B_i + N_d \xi_i \right) + \mathcal{O}(r), \quad (67)$$

$$\delta b_r = \frac{1}{U(r)} \left(\delta b_r^{(0)}(x_i) + \mathcal{O}(r) \right). \quad (68)$$

Moreover, it turns out that constraints on the leading order have to be imposed. Namely one has

$$\delta g_{tt}^{(0)} + \delta g_{rr}^{(0)} - 2\delta g_{ri}^{(0)} = 0, \quad \delta g_{ri}^{(0)} = \delta g_{ti}^{(0)}, \quad (69)$$

$$\delta a_r^{(0)} = \delta a_t^{(0)}, \quad \delta b_r^{(0)} = \delta b_t^{(0)}. \quad (70)$$

4.2 Electric currents

In order to define the thermoelectric currents and heat conductivity one needs to find quantities in the bulk, which are identified with boundary currents. We have to pay attention to the suitable Killing vector fields, as well as the equations of motion to find the two-forms, being subject to the divergence of the adequate coordinate equal to 0.

The *electric currents* will be associated with the radially independent components of Eqs. (3) and (4), which in turn can be calculated everywhere in the bulk. Because of the form of the underlying equations they will constitute a mixture of the two $U(1)$ -gauge fields. Consequently for the current connected with Maxwell gauge field we define

$$J_{(F)}^i(r) = \sqrt{-g} \left(F^{ir} + \frac{\alpha}{2} B^{ir} \right), \quad (71)$$

while the current bounded with the auxiliary $U(1)$ -gauge field becomes

$$J_{(B)}^i(r) = \sqrt{-g} \left(B^{ir} + \frac{\alpha}{2} F^{ir} \right). \quad (72)$$

On this account, having in mind the relations for the background metric and components of the gauge fields, it is cus-

tomary to write

$$J_{(F)}^i(r) = \sqrt{GFgd} g_d^{ij} \left[\left(-\frac{\partial_r a_t}{FG} \delta g_{ij} + \frac{\partial_j a_t}{FG} \delta g_{tr} \right. \right. \\ \left. \left. + \partial_j (\delta a_r) \frac{U}{F} - \left(\partial_r (\delta a_j) + t \partial_r N \xi_j \right) \frac{U}{F} \right) \right. \\ \left. + \frac{\alpha}{2} \left(-\frac{\partial_r b_t}{FG} \delta g_{ij} + \frac{\partial_j b_t}{FG} \delta g_{tr} + \partial_j (\delta b_r) \frac{U}{F} \right. \right. \\ \left. \left. - \left(\partial_r (\delta b_j) + t \partial_r N_d \xi_j \right) \frac{U}{F} \right) \right] \quad (73)$$

and for the other current

$$J_{(B)}^i(r) = \sqrt{GFgd} g_d^{ij} \left[\left(-\frac{\partial_r b_t}{FG} \delta g_{ij} + \frac{\partial_j b_t}{FG} \delta g_{tr} \right. \right. \\ \left. \left. + \partial_j (\delta b_r) \frac{U}{F} - \left(\partial_r (\delta b_j) + t \partial_r N_d \xi_j \right) \frac{U}{F} \right) \right. \\ \left. + \frac{\alpha}{2} \left(-\frac{\partial_r a_t}{FG} \delta g_{ij} + \frac{\partial_j a_t}{FG} \delta g_{tr} + \partial_j (\delta a_r) \frac{U}{F} \right. \right. \\ \left. \left. - \left(\partial_r (\delta a_j) + t \partial_r N \xi_j \right) \frac{U}{F} \right) \right]. \quad (74)$$

In Eqs. (73) and (74) we restrict our attention to the linearised order of black brane perturbations. Moreover, the equations in question envisage that no time-dependent terms are incorporated in them.

From the adequate equations of motion for the strength tensors $F_{\mu\nu}$ and $B_{\mu\nu}$, one can deduce that for a Maxwell field current we have a relation of the form

$$\nabla_i J_{(F)}^i = 0, \quad \nabla_r J_{(F)}^i = \nabla_j \left[\sqrt{-g} \left(F^{ji} + \frac{\alpha}{2} B^{ji} \right) \right], \quad (75)$$

while for the auxiliary gauge one obtains

$$\nabla_i J_{(B)}^i = 0, \quad \nabla_r J_{(B)}^i = \nabla_j \left[\sqrt{-g} \left(B^{ji} + \frac{\alpha}{2} F^{ji} \right) \right]. \quad (76)$$

4.3 Heat current

In our setup we define the heat current assuming that $k_\mu = (\partial/\partial t)_\mu$ is a time-like Killing vector field [48, 49]. The general properties of Killing vector fields provide us with the relation

$$\nabla_\mu \nabla^\nu k^\mu = T^\nu{}_\mu k^\mu - \frac{k^\nu T}{d-2} - 2 \frac{k^\nu \Lambda}{d-2}, \quad (77)$$

where $T = T_\mu{}^\mu$ is the trace of the energy momentum tensor in the considered theory while Λ stands for cosmological constant. The Killing vector symmetry conditions for the fields appearing in our model enable us to write

$$\mathcal{L}_k F_{\alpha\beta} = \mathcal{L}_k B_{\alpha\beta} = \mathcal{L}_k \phi = 0. \quad (78)$$

On the other hand, one has the following relations valid for arbitrary functions:

$$k^\mu F_{\mu\nu} = \nabla_\nu \theta_{(F)}, \quad k^\mu B_{\mu\nu} = \nabla_\nu \theta_{(B)}, \quad (79)$$

where $\theta_{(F)}$ and $\theta_{(B)}$ are arbitrary functions. By virtue of (3) and (4) and Eq. (78), one concludes that

$$\begin{aligned} k^\mu F_{\mu\alpha} F^{\rho\alpha} &= \nabla_\alpha (\theta_{(F)} F^{\rho\alpha}), \\ k^\mu B_{\mu\alpha} B^{\rho\alpha} &= \nabla_\alpha (\theta_{(B)} B^{\rho\alpha}), \end{aligned} \quad (80)$$

$$\begin{aligned} k^\mu F_{\mu\gamma} B^{\rho\gamma} + k^\mu B_{\mu\alpha} F^{\rho\alpha} &= \nabla_\delta (\theta_{(F)} B^{\nu\delta}) \\ &+ \nabla_\delta (\theta_{(B)} F^{\nu\delta}). \end{aligned} \quad (81)$$

Consequently, we get the set of equations

$$k^\mu F^{\rho\nu} F_{\rho\nu} = 4 \nabla_\rho (k^{[\mu} F^{\rho] A_\nu}) + 2 \mathcal{L}_k A_\nu F^{\mu\nu}, \quad (82)$$

$$k^\mu B^{\rho\nu} B_{\rho\nu} = 4 \nabla_\rho (k^{[\mu} B^{\rho] B_\nu}) + 2 \mathcal{L}_k B_\nu B^{\mu\nu}, \quad (83)$$

$$k^\mu B^{\rho\nu} F_{\rho\nu} = 4 \nabla_\rho (k^{[\mu} F^{\rho] B_\nu}) + 2 \mathcal{L}_k B_\nu F^{\mu\nu}, \quad (84)$$

$$k^\mu F^{\rho\nu} B_{\rho\nu} = 4 \nabla_\rho (k^{[\mu} B^{\rho] A_\nu}) + 2 \mathcal{L}_k A_\nu B^{\mu\nu}. \quad (85)$$

After some algebra, having in mind Eq. (77), it can be found that

$$\nabla_\rho \tilde{G}_{\nu\rho} = -2 \frac{\Lambda k^\nu}{d-2}, \quad (86)$$

where the exact form of $\tilde{G}_{\nu\rho}$ is given by

$$\begin{aligned} \tilde{G}_{\nu\rho} &= \nabla^\nu k^\rho + \frac{1}{2} (k^{[\nu} F^{\rho] A_\alpha} A_\alpha) + \frac{1}{4} [(\psi - 2\theta_{(F)}) F^{\nu\rho}] \\ &+ \frac{1}{2} (k^{[\nu} B^{\rho] A_\alpha} B_\alpha) + \frac{1}{4} [(\chi - 2\theta_{(B)}) B^{\nu\rho}] \\ &+ \frac{\alpha}{4} [(k^{[\nu} B^{\rho] A_\alpha} A_\alpha) + (k^{[\nu} F^{\rho] A_\alpha} B_\alpha)] \\ &+ \frac{\alpha}{8} [(\psi - 2\theta_{(F)}) B^{\nu\rho}] + \frac{\alpha}{8} [(\chi - 2\theta_{(B)}) F^{\nu\rho}]. \end{aligned} \quad (87)$$

In the derivation of (87) we have used the following relations:

$$\begin{aligned} \mathcal{L}_k A_\alpha F^{\nu\alpha} &= \nabla_\rho (\psi F^{\nu\rho}), \\ \mathcal{L}_k B_\alpha B^{\nu\alpha} &= \nabla_\rho (\chi B^{\nu\rho}), \end{aligned} \quad (88)$$

$$\begin{aligned} \mathcal{L}_k A_\alpha B^{\nu\alpha} &= \nabla_\rho (\psi B^{\nu\rho}), \\ \mathcal{L}_k B_\alpha F^{\nu\alpha} &= \nabla_\rho (\chi F^{\nu\rho}), \end{aligned} \quad (89)$$

where we have denoted

$$\psi = E_\alpha x^\alpha, \quad \theta_{(F)} = -E_\alpha x^\alpha - a_t, \quad (90)$$

$$\chi = B_\alpha x^\alpha, \quad \theta_{(B)} = -B_\alpha x^\alpha - b_t. \quad (91)$$

It can be observed that the $\tilde{G}_{\nu\rho}$ tensor is antisymmetric and implies

$$\partial_\rho (2 \sqrt{-g} \tilde{G}^{\nu\rho}) = -2 \frac{\Lambda \sqrt{-g} k^\nu}{d-2}. \quad (92)$$

In what follows we shall use the two-form given by $2 \tilde{G}_{\nu\rho}$, i.e., the heat current will be defined as $Q^i = 2 \sqrt{-g} \tilde{G}_{\nu\rho}$.

Consequently, at the linear order for the perturbed system in question, we obtain

$$\begin{aligned} Q^i &= \frac{G^{\frac{3}{2}} U^2}{\sqrt{F}} \sqrt{g_d} g_d^{ij} \left[\partial_r \left(\frac{\delta g_{tj}}{GU} \right) - \partial_j \left(\frac{\delta g_{tr}}{GU} \right) \right] \\ &- a_t J_{(F)}^i - b_t J_{(B)}^i. \end{aligned} \quad (93)$$

As in the case of gauge currents, the time-dependent terms dropped out of the expression for the heat current. The additional relations for the heat current Q^i are

$$\nabla_i Q^i = 0, \quad \nabla_r Q^i = \nabla_j (2 \sqrt{-g} \tilde{G}_{\nu\rho}). \quad (94)$$

4.4 Currents on the black brane event horizon

Accordingly to Eqs. (73) and (74), restricting our calculations to the linearised order of the perturbations we obtain the following currents on the black brane event horizon:

$$\begin{aligned} J_{(F)}^{i(0)} &= J_{(F)}^i |_{\mathcal{H}} = \sqrt{g^{(0)}} g_{(0)}^{ij} \left[(\nabla_j (\delta a_t^{(0)}) + E_j - a_t^{(0)} \delta g_{tj}^{(0)}) \right. \\ &\quad \left. + \frac{\alpha}{2} (\nabla_j (\delta b_t^{(0)}) + B_j - b_t^{(0)} \delta g_{tj}^{(0)}) \right], \end{aligned} \quad (95)$$

$$\begin{aligned} J_{(B)}^{i(0)} &= J_{(B)}^i |_{\mathcal{H}} = \sqrt{g^{(0)}} g_{(0)}^{ij} \left[(\nabla_j (\delta b_t^{(0)}) + B_j - b_t^{(0)} \delta g_{tj}^{(0)}) \right. \\ &\quad \left. + \frac{\alpha}{2} (\nabla_j (\delta a_t^{(0)}) + E_j - a_t^{(0)} \delta g_{tj}^{(0)}) \right], \end{aligned} \quad (96)$$

$$Q^{i(0)} = Q^i |_{\mathcal{H}} = -4\pi T \sqrt{g^{(0)}} g_{(0)}^{ij} \delta g_{tj}^{(0)}. \quad (97)$$

By virtue of Eqs. (75)–(76) and (94), one has for the perturbations at the black object horizon

$$\nabla_i J_{(F)}^{i(0)} = 0, \quad \nabla_i J_{(B)}^{i(0)} = 0, \quad \nabla_i Q^{i(0)} = 0. \quad (98)$$

5 Stokes equations for U(1)-gauge fluids

In this section we obtain the closed system of differential equations which describe the conditions imposed on a subset of a linearised perturbations, i.e., $\delta g_{it}^{(0)}$, $\delta g_{rt}^{(0)}$, $\delta a_t^{(0)}$, $\delta b_t^{(0)}$, on the black brane event horizon. On this account, it is customary to write

$$\begin{aligned} \nabla_i \nabla^i w + \nabla_i E^i + \nabla_i (a_t^{(0)} v^i) \\ + \frac{\alpha}{2} [\nabla_m \nabla^m w_d + \nabla_m B^m + \nabla_m (b_t^{(0)} v^m)] = 0, \end{aligned} \quad (99)$$

$$\begin{aligned} \nabla_i \nabla^i w_d + \nabla_i B^i + \nabla_i (b_t^{(0)} v^i) \\ + \frac{\alpha}{2} [\nabla_m \nabla^m w + \nabla_m E^m + \nabla_m (a_t^{(0)} v^m)] = 0, \end{aligned} \quad (100)$$

$$b_t^{(0)} [\nabla_i w_d + B_i + \frac{\alpha}{2} (\nabla_i w + E_i)]$$

$$+a_t^{(0)}\left[\nabla_i w + E_i + \frac{\alpha}{2}(\nabla_i w_d + B_i)\right] \\ -\nabla_i \phi^{(0)} \nabla_m \phi^{(0)} v^m + 2 \nabla^m \nabla_{(m} v_{i)} + 4\pi T \xi_i \\ -\nabla_i p = 0, \quad (101)$$

$$\nabla_i v^i = 0, \quad (102)$$

where we denoted

$$w = \delta a_t^{(0)}, \quad w_d = \delta b_t^{(0)}, \quad p = -4\pi T \frac{\delta g_{rt}^{(0)}}{G^{(0)}} \\ -\delta g_{it}^{(0)} \nabla^i \ln G^{(0)}, \quad (103)$$

$$v_i = -\delta g_{it}^{(0)}. \quad (104)$$

Calculating the conserved current equations, $\nabla_i J_{(F)}^{i(0)} = 0$ and $\nabla_i J_{(B)}^{i(0)} = 0$, on the event horizon one can obtain Eqs. (99)–(100). On the other hand, when we take into account the constraints C_i and C_0 , we arrive at Eqs. (101) and (102). The covariant derivative ∇_i appearing in this section is bounded with the metric $g_{ij}^{(0)}$, on the black brane event horizon. Equations (99)–(102) constitute a generalisation of the forced Stokes equations for charged $U(1)$ -gauge fluids, described on the brane event horizon.

In the case of $a_t^{(0)} = b_t^{(0)} = w = w_d = E_i = B_i = 0$ and constant scalar field, one arrives at the Stokes equations describing fluid with velocity v_i , pressure p , and the additional forcing term of the form as $4\pi T \xi_i$. As in the previously studied Einstein–Maxwell scalar case [13], we also have the viscosity factor given by $\nabla_i \phi^{(0)} \nabla_m \phi^{(0)} v^m$.

It can be remarked that taking the divergence of Eq. (101) and having in mind the remaining relations, we arrive at the pressure Poisson equation, being the generalisation of the one derived in [13]. Namely, it is provided by the following expression:

$$\nabla^m \nabla_m p = \nabla_j \left[b_t^{(0)} \left(\nabla^j w_d + B^j + \frac{\alpha}{2} \left(\nabla^j w + E^j \right) \right) \right. \\ \left. + a_t^{(0)} \left(\nabla^j w + E^j + \frac{\alpha}{2} \left(\nabla^j w_d + B^j \right) \right) \right. \\ \left. - \nabla^j \phi^{(0)} \nabla_m \phi^{(0)} v^m + 4\pi T \xi^j + 2R^j{}_m v^m \right]. \quad (105)$$

On the other hand, multiplication of (101) and integration of the resulting expression over the black brane event horizon reveal

$$\int \sqrt{g^{(0)}} d^2 x \left[2 \nabla^{(i} v^{j)} \nabla_{(i} v_{j)} + \left(\nabla_i w + E_i \right) \left(\nabla^i w + E^i \right) \right. \\ \left. + \left(\nabla_i w_d + B_i \right) \left(\nabla^i w_d + B^i \right) \right. \\ \left. + \alpha \left(\nabla_i w + E_i \right) \left(\nabla^i w_d + B^i \right) + v^m \nabla_m \phi^{(0)} \nabla_j \phi^{(0)} v^j \right] \\ = \int d^2 x \left[\mathcal{Q}^{i(0)} \xi_i + J_{(F)}^{i(0)} E_i + J_{(B)}^{i(0)} B_i \right]. \quad (106)$$

Keeping in mind the non-compactness of the event horizon and assuming the disappearance of the emerging surface

terms, the above equation reveals the fact that its left-hand side is positive, which in turn implies the positivity of the thermoelectrical conductivities bounded with the two $U(1)$ -gauge and scalar fields, emerging on the right-hand side of (106).

As far as the uniqueness of the above set of Eqs. (99)–(102) is concerned, let us suppose that we have two solutions of this set subject to the same boundary and regularity conditions. The differences of them will be denoted as $\tilde{v}_i = v_i^{(1)} - v_i^{(2)}$, $\tilde{w} = w^{(1)} - w^{(2)}$, $\tilde{w}_d = w_d^{(1)} - w_d^{(2)}$, $\tilde{p} = p^{(1)} - p^{(2)}$. They will satisfy the equation with $\xi_i = E_i = B_i = 0$. Just using Eq. (106) one obtains

$$\nabla_{(i} \tilde{v}_{j)} = 0, \quad \nabla_i \tilde{w} = 0, \quad \nabla_i \tilde{w}_d = 0, \quad \tilde{v}^i \nabla_i \phi^{(0)} = 0. \quad (107)$$

Equations (99) and (100) reveal that the Lie derivatives with respect to v_m taken from $a_t^{(0)}$ and $b_t^{(0)}$ are equal to zero, i.e.,

$$\mathcal{L}_{v^m} a_t^{(0)} = 0, \quad \mathcal{L}_{v^m} b_t^{(0)} = 0, \quad (108)$$

while Eq. (20) leads to the condition $\nabla_i \tilde{p} = 0$. With all these in mind one may conclude that the solution of Stokes equation is unique up to the Killing vectors of the black brane event horizon line element, with $\tilde{p}$, $\tilde{w}$, $\tilde{w}_d$ constant and $\delta g_{rt}^{(0)}$ described by Eq. (103).

The aforementioned set of equation can be derived by varying the following functional:

$$S = \int \sqrt{g^{(0)}} d^4 x \left[-\nabla^{(i} v^{j)} \nabla_{(i} v_{j)} + 4\pi T \xi_m v^m \right. \\ \left. - \frac{1}{2} v_i \nabla^i \phi^{(0)} v_k \nabla^k \phi^{(0)} + p \nabla_m v^m \right. \\ \left. + E_i \left(a_t^{(0)} v^i + \nabla^i w \right) + \frac{1}{2} E_m E^m \right. \\ \left. + \frac{\alpha}{2} E_i \left(b_t^{(0)} v^i + \nabla^i w_d \right) + \frac{\alpha}{2} E_k B^k \right. \\ \left. + B_i \left(b_t^{(0)} v^i + \nabla^i w_d \right) + \frac{1}{2} B_m B^m \right. \\ \left. + \frac{\alpha}{2} B_i \left(a_t^{(0)} v^i + \nabla^i w \right) \right. \\ \left. + \frac{1}{2} \left(b_t^{(0)} v^i + \nabla^i w_d \right) \left(b_t^{(0)} v_i + \nabla_i w_d \right) - \frac{1}{2} b_t^{(0)2} v_m v^m \right. \\ \left. + \frac{1}{2} \left(a_t^{(0)} v^i + \nabla^i w \right) \left(a_t^{(0)} v_i + \nabla_i w \right) - \frac{1}{2} a_t^{(0)2} v_m v^m \right. \\ \left. + \frac{\alpha}{2} \left[\frac{1}{2} \left(b_t^{(0)} v^i + \nabla^i w \right) \left(b_t^{(0)} v_i + \nabla_i w_d \right) \right. \right. \\ \left. \left. - \frac{1}{2} b_t^{(0)2} v_m v^m \right] \right. \\ \left. + \frac{\alpha}{2} \left[\frac{1}{2} \left(a_t^{(0)} v^i + \nabla^i w_d \right) \left(a_t^{(0)} v_i + \nabla_i w \right) \right. \right. \\ \left. \left. - \frac{1}{2} a_t^{(0)2} v_m v^m \right] \right]. \quad (109)$$

It so happens that the pressure appears here as the Lagrange multiplier, giving the incompressibility condition. The rest of the relations in question can be found by varying with respect to v_i , w and w_d . Moreover, variations of (109) with respect to E_i , B_i and ξ_i give us appropriate currents of gauge fields and heat, counted on the black brane event horizon.

Furthermore, keeping in mind the fact that E_i , B_i , ξ_i can be described as closed differential forms, and that consequently they are locally defined as

$$E = \nabla_m e \, dx^m, \quad B = \nabla_m b \, dx^m, \quad \xi = \nabla_m z \, dx^m, \quad (110)$$

the differential system of equations studied can be rewritten in order to eliminate the source terms. Namely, defining the quantities

$$\tilde{w} = w + e, \quad \tilde{w}_d = w_d + b, \quad \tilde{p} = p - 4\pi T z, \quad (111)$$

Eqs. (99)–(102) have the forms

$$\nabla_i \nabla^i \tilde{w} + \nabla_i \left(a_t^{(0)} v^i \right) + \frac{\alpha}{2} \left[\nabla_m \nabla^m \tilde{w}_d + \nabla_m \left(b_t^{(0)} v^m \right) \right] = 0, \quad (112)$$

$$\nabla_i \nabla^i \tilde{w}_d + \nabla_i \left(b_t^{(0)} v^i \right) + \frac{\alpha}{2} \left[\nabla_m \nabla^m \tilde{w} + \nabla_m \left(a_t^{(0)} v^m \right) \right] = 0, \quad (113)$$

$$b_t^{(0)} \left(\nabla_i \tilde{w}_d + \frac{\alpha}{2} \nabla_i \tilde{w} \right) + a_t^{(0)} \left(\nabla_i \tilde{w} + \frac{\alpha}{2} \nabla_i \tilde{w}_d \right) - \nabla_i \phi^{(0)} \nabla_m \phi^{(0)} v^m + 2 \nabla^m \nabla_{(m} v_{i)} - \nabla_i \tilde{p} = 0, \quad (114)$$

$$\nabla_i v^i = 0. \quad (115)$$

6 One-dimensional lattice case

The purpose of this section is to elaborate the example of holographic lattice for which one solves the previously derived Stokes equations and finds the DC thermoelectric conductivities. We shall be interested in the influence of the field from the *hidden sector*, as well as the dependence of the conductivities on α -coupling constant appearing in the *mixing term*, on the physics of these phenomena.

In what follows we shall study the line element of black brane which breaks the spatial translation symmetry in one-dimension, on the black object event horizon. Namely, let us suppose that the event horizon metric tensor depends only on x -coordinate and the two-dimensional line element implies

$$ds_{(2)}^2 = \gamma(x) dx^2 + \lambda(x) dy^2. \quad (116)$$

In the next step one tries to solve Eqs. (101)–(102) in the aforementioned background. The incompressibility condition described by (102) yields the following relation:

$$v^i = \frac{1}{\sqrt{\gamma(x)\lambda(x)}} v_0, \quad (117)$$

where v_0 is constant.

On the other hand, the Stokes equation can be rewritten in the form

$$\begin{aligned} & \frac{1}{\sqrt{g^{(0)}}} \left(b_t^{(0)} J_{(B)}^{i(0)} + a_t^{(0)} J_{(F)}^{i(0)} \right) \\ & - v^i \left(a_t^{(0)2} + \alpha a_t^{(0)} b_t^{(0)} + b_t^{(0)2} \right) \\ & - \nabla^i \phi^{(0)} \nabla_m \phi^{(0)} v^m + 2 \nabla_m \nabla^{(m} v^{i)} \\ & - \nabla^i p = -4\pi T \xi^i. \end{aligned} \quad (118)$$

The Stokes equation and the relation describing the currents for the two $U(1)$ -gauge fields are implemented to describe the functions w , w_d and p . As we consider the periodic functions, the expressions for $\partial_x w$, $\partial_x w_d$, $\partial_x p$ must have no zero modes on the considered submanifold. In turn this requirement imposes the constraints on $J_{(F)}^{x(0)}$, $J_{(B)}^{x(0)}$ and v_0 . As in [13], we find these restrictions by averaging the equations over the periodic lattice.

Before proceeding to this task let us find the exact forms of the gauge currents and the Stokes equation in the two-dimensional submanifold. The $U(1)$ -gauge currents are provided by the following relations:

$$\begin{aligned} J_{(F)}^{x(0)} &= \sqrt{\frac{\lambda(x)}{\gamma(x)}} \left[\left(\partial_x w + E_x + \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} v_0 \right) \right. \\ & \quad \left. + \frac{\alpha}{2} \left(\partial_x w_d + B_x + \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} v_0 \right) \right], \end{aligned} \quad (119)$$

$$\begin{aligned} J_{(B)}^{x(0)} &= \sqrt{\frac{\lambda(x)}{\gamma(x)}} \left[\left(\partial_x w_d + B_x + \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} v_0 \right) \right. \\ & \quad \left. + \frac{\alpha}{2} \left(\partial_x w + E_x + \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} v_0 \right) \right], \end{aligned} \quad (120)$$

while the heat current implies

$$Q^{i(0)} = 4\pi T v_0. \quad (121)$$

Consequently the Stokes equation can be rewritten as

$$\begin{aligned} & v_0 \partial_x \left(\gamma^{-\frac{1}{2}} \partial_x (\lambda^{-\frac{1}{2}}) \right) - Y v_0 \\ & + \sqrt{\frac{\gamma(x)}{\lambda(x)}} \left(b_t^{(0)} J_{(B)}^{i(0)} + a_t^{(0)} J_{(F)}^{i(0)} \right) - \partial_x p = -4\pi T \xi_x, \end{aligned} \quad (122)$$

where one denotes by Y the quantity

$$Y = \frac{(\partial_x \phi)^2}{\sqrt{\lambda(x)\gamma(x)}} + a_t^{(0)2} + \alpha a_t^{(0)} b_t^{(0)} + b_t^{(0)2} + \frac{1}{4\lambda(x)^{\frac{5}{2}}\gamma(x)^{\frac{1}{2}}} (\partial_x \lambda(x))^2 v_0. \quad (123)$$

Averaging the Stokes equation (122) over a one-dimensional lattice with a period L_1 leads to the following:

$$v_0 = \frac{1}{A} \left[\left(B_x + \frac{\alpha}{2} E_x \right) \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} + \left(E_x + \frac{\alpha}{2} B_x \right) \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \right], \quad (124)$$

where one sets

$$A = \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \int dx Y - \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \right)^2 - \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \right)^2 - \alpha \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \right) \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \right), \quad (125)$$

where for brevity we set

$$\int dx \leftrightarrow \frac{1}{L_1} \int_0^{L_1} dx. \quad (126)$$

By virtue of the procedure that we followed above, we arrive at the relations for the Maxwell current $J_F^{x(0)}$,

$$J_{(F)}^{x(0)} = \frac{1}{A} \left[\left(E_x + \frac{\alpha}{2} B_x \right) \left[\int dx Y - \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \left(b_t^{(0)} + \frac{\alpha}{2} a_t^{(0)} \right) + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \left(a_t^{(0)} + \frac{\alpha}{2} b_t^{(0)} \right) + \left(B_x + \frac{\alpha}{2} E_x \right) \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \left(a_t^{(0)} + \frac{\alpha}{2} b_t^{(0)} \right) \right] \right] \quad (127)$$

and, for $J_B^{x(0)}$,

$$J_{(B)}^{x(0)} = \frac{1}{A} \left[\left(B_x + \frac{\alpha}{2} E_x \right) \left[\int dx Y - \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \left(a_t^{(0)} + \frac{\alpha}{2} b_t^{(0)} \right) + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \left(b_t^{(0)} + \frac{\alpha}{2} a_t^{(0)} \right) + \left(E_x + \frac{\alpha}{2} B_x \right) \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \left(b_t^{(0)} + \frac{\alpha}{2} a_t^{(0)} \right) \right] \right] \quad (128)$$

For the heat current one finds the relation as in Eq. (121), where v_0 is given by Eq. (124).

It can be pointed out that in the limit when $\alpha = 0$, $b_t^{(0)} = 0$, $B_x = 0$, one obtains the relations presented in [13], for the case of Einstein–Maxwell scalar theory.

6.1 Kinetic and transport coefficients in one-dimensional case

In our model, the general form of the kinetic coefficient matrix may be written as

$$\begin{pmatrix} J_{(F)}^{i(0)} \\ J_{(B)}^{i(0)} \\ Q^{i(0)} \end{pmatrix} = \begin{pmatrix} \sigma_{j(FF)}^i & \sigma_{j(FB)}^i & \alpha_{j(F)}^i T \\ \sigma_{j(BF)}^i & \sigma_{j(BB)}^i & \alpha_{j(B)}^i T \\ \alpha_{j(F)}^i T & \alpha_{j(B)}^i T & \kappa_j^i T \end{pmatrix} \begin{pmatrix} E^j \\ B^j \\ \xi^j \end{pmatrix}. \quad (129)$$

Equation (129) will help us to find their exact forms for the one-dimensional lattice case. Namely, combining Eqs. (121), (124), (127)–(128) and using Eq. (129), we obtain the required explicit forms of the kinetic coefficients in the case of a one-dimensional lattice. They are provided by

$$\sigma_{FF} = \frac{1}{A} \left[\int dx Y - \tilde{\alpha} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)2} \right], \quad (130)$$

$$\sigma_{FB} = \frac{1}{A} \left[\frac{\alpha}{2} \int dx Y + \tilde{\alpha} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} b_t^{(0)} \right], \quad (131)$$

$$\sigma_{BF} = \frac{1}{A} \left[\frac{\alpha}{2} \int dx Y + \tilde{\alpha} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} b_t^{(0)} \right], \quad (132)$$

$$\sigma_{BB} = \frac{1}{A} \left[\int dx Y - \tilde{\alpha} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)2} \right], \quad (133)$$

$$\alpha_{(F)} = \frac{4\pi}{A} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \left(a_t^{(0)} + \frac{\alpha}{2} b_t^{(0)} \right), \quad (134)$$

$$\alpha_{(B)} = \frac{4\pi}{A} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \left(b_t^{(0)} + \frac{\alpha}{2} a_t^{(0)} \right), \quad (135)$$

$$\kappa = \frac{(4\pi)^2 T}{A} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}}, \quad (136)$$

where $\tilde{\alpha} = 1 - \frac{\alpha^2}{4}$.

The form of the kinetic and transport coefficients (130)–(136) shows the strong influence of the auxiliary $U(1)$ -gauge field and the coupling constant α on their structure.

6.2 Generalisation of the Sachdev model of the Dirac fluid

The analysis presented in this subsection addresses the question about the connections of the described model with the results presented in [47], where the holographic two current model of Dirac fluid in graphene has been presented. On the other hand, the generalisation for the case of two interacting $U(1)$ -gauge currents was performed, for holographic four- [48] and five-dimensional models [49].

In this subsection we shall find how the aforementioned results can be found in the Stokes equations. We suppose, for simplicity, that the submanifold will be $\mathbf{R}^2$ one with a flat metric described by line element $ds_{(2)}^2$. We set $\gamma(x) = \lambda(x) = 1$.

The total electric current will be the sum of the *visible* and *hidden* sector currents,

$$J^{x(0)} = J_{(F)}^{x(0)} + J_{(B)}^{x(0)}, \quad (137)$$

while the electric conductivity is provided by

$$\sigma = \sigma_{FF} + \sigma_{FB} + \sigma_{BF} + \sigma_{BB}. \quad (138)$$

On the other hand, the gauge and heat currents are given by

$$\begin{aligned} J_{(F)}^{x(0)} = & E_x \frac{1}{A} \int dx \left[(\partial_x \phi)^2 + Q_F^2 \right] \\ & + B_x \frac{1}{A} \int dx \left[\frac{\alpha}{2} (\partial_x \phi)^2 + Q_F Q_B \right] \\ & + 4\pi T \xi_x \frac{1}{A} \int dx Q_F, \end{aligned} \quad (139)$$

$$\begin{aligned} J_{(B)}^{x(0)} = & B_x \frac{1}{A} \int dx \left[(\partial_x \phi)^2 + Q_B^2 \right] \\ & + E_x \frac{1}{A} \int dx \left[\frac{\alpha}{2} (\partial_x \phi)^2 + Q_F Q_B \right] \end{aligned} \quad (140)$$

$$+ 4\pi T \xi_x \frac{1}{A} \int dx Q_B, \quad (141)$$

$$\begin{aligned} Q^{x(0)} = & 4\pi T E_x \frac{1}{A} \int dx Q_F + 4\pi T B_x \frac{1}{A} \int dx Q_B \\ & + (4\pi T)^2 \frac{1}{A} \int dx, \end{aligned} \quad (142)$$

where, in order to have a more transparent connection with the results presented in [48, 49], we set

$$Q_F = a_t^{(0)} + \frac{\alpha}{2} b_t^{(0)}, \quad (143)$$

$$Q_B = b_t^{(0)} + \frac{\alpha}{2} a_t^{(0)}. \quad (144)$$

As in [48], let us suppose that the two charges will be connected by the relation

$$b_t^{(0)} = g a_t^{(0)}, \quad (145)$$

where g is a number. The definitions of Q_F and Q_B enable us to find that

$$Q_F = \left(1 + \frac{\alpha}{2}g\right) a_t^{(0)}, \quad Q_B = \left(g + \frac{\alpha}{2}\right) a_t^{(0)}. \quad (146)$$

Consequently the electrical conductivity is given by

$$\sigma = \frac{1}{A} \int dx \left[(1 + \alpha) (\partial_x \phi)^2 + \left(1 + \frac{\alpha}{2}\right)^2 (1 + g)^2 a_t^{(0)2} \right]. \quad (147)$$

If we define $Q = Q_F + Q_B$, the above relation for σ reduces to the form

$$\sigma = \frac{1}{A} \int dx \left[(1 + \alpha) (\partial_x \phi)^2 + Q^2 \right]. \quad (148)$$

One can see that we have very good agreement with the results obtained in [48] and, moreover in the Stokes equations behavior, one finds the explicit presence of the scalar field (influencing the electric conductivity) responsible for the dissipation processes. It is given by the square of its derivative with respect to the spatial coordinate x .

6.2.1 Absence of hidden sector fields

To proceed, in this subsection we shall elaborate the specific choice of matter fields for the one-dimensional periodic lattice. Our main aim will be to find kinetic coefficients generated by the aforementioned matter fields. To begin with we shall choose the case of the nonexistence of *hidden sector* fields, i.e., $\alpha = 0$, $b_t^{(0)} = 0$, $B_x = 0$. One for this definite option finds that the currents are equal to the following expressions:

$$\begin{aligned} J_{(F)}^{x(0)} = & \frac{1}{A_{vis}} \left[E_x \int dx Y_{vis} + 4\pi T \xi_x \right. \\ & \left. \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \right], \end{aligned} \quad (149)$$

$$J_{(B)}^{x(0)} = 0, \quad (150)$$

where one sets

$$\begin{aligned} v_0(vis) = & \frac{1}{A_{vis}} \left[E_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \right. \\ & \left. + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \right], \end{aligned} \quad (151)$$

$$A_{vis} = \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \int dx Y_{vis} - \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \right)^2, \quad (152)$$

$$Y_{vis} = \frac{(\partial_x \phi)^2}{\sqrt{\lambda(x)\gamma(x)}} + a_t^{(0)2} + \frac{1}{4\lambda(x)^{\frac{5}{2}}\gamma(x)^{\frac{1}{2}}} (\partial_x \lambda(x))^2 v_0(vis). \quad (153)$$

A comparison of Eqs. (129), (149)–(150) reveals the DC thermoelectric coefficients which are given by

$$\sigma_{FF} = \frac{1}{A_{vis}} \int dx Y_{vis}, \quad (154)$$

$$\sigma_{FB} = \sigma_{BF} = 0, \quad (155)$$

$$\sigma_{BB} = 0, \quad (156)$$

$$\alpha_{(F)} = \frac{4\pi}{A_{vis}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)}, \quad (157)$$

$$\alpha_{(B)} = 0, \quad (158)$$

$$\kappa = \frac{(4\pi)^2 T}{A_{vis}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}}. \quad (159)$$

As expected, we have obtained the results presented in [13] for Einstein–Maxwell scalar gravity with non-zero potential. We have reached the anticipated limit of our model.

6.3 Absence of the visible sector

The next case will be concerned with studies of only the *hidden sector* fields on the thermoelectric properties. We assume that $a_t^{(0)} = 0$, $E_x = 0$. In this case one gets

$$J_{(F)}^{x(0)} = \frac{1}{A_{hid}} \left[\frac{\alpha}{2} B_x \int dx Y_{hid} + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \frac{\alpha}{2} b_t^{(0)} \right], \quad (160)$$

and, for $J_B^{x(0)}$,

$$J_{(B)}^{x(0)} = \frac{1}{A_{hid}} \left[B_x \int dx Y_{hid} + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \right], \quad (161)$$

with the other quantities defined as

$$v_0(hid) = \frac{1}{A_{hid}} \left(B_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \right), \quad (162)$$

$$A_{hid} = \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \int dx Y_{hid} - \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \right)^2, \quad (163)$$

$$Y_{hid} = \frac{(\partial_x \phi)^2}{\sqrt{\lambda(x)\gamma(x)}} + b_t^{(0)2} + \frac{1}{4\lambda(x)^{\frac{5}{2}}\gamma(x)^{\frac{1}{2}}} (\partial_x \lambda(x))^2 v_0(hid). \quad (164)$$

In the considered case of *hidden sector* fields the DC coefficients reduce to the forms

$$\sigma_{FF} = 0, \quad (165)$$

$$\sigma_{FB} = \frac{1}{A_{hid}} \frac{\alpha}{2} \int dx Y_{hid}, \quad (166)$$

$$\sigma_{BF} = 0, \quad (167)$$

$$\sigma_{BB} = \frac{1}{A_{hid}} \int dx Y_{hid}, \quad (168)$$

$$\alpha_{(F)} = \frac{4\pi}{A_{hid}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \frac{\alpha}{2} b_t^{(0)}, \quad (169)$$

$$\alpha_{(B)} = \frac{4\pi}{A_{hid}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)}, \quad (170)$$

$$\kappa = \frac{(4\pi)^2 T}{A_{hid}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}}. \quad (171)$$

It is interesting to notice that despite the absence of the *visible sector* fields, we get non-zero values of σ_{FB} and $\alpha_{(F)}$, all of them proportional to the α -coupling constant. This situation has its roots in the definition of the gauge currents, in which we have both *visible* and *hidden* sector fields.

6.3.1 Non-interacting gauge fields

In this case we neglect the interaction between two $U(1)$ -gauge fields. Namely we put the α -coupling constant equal to zero. For the case studied one has

$$J_{(F)}^{x(0)} = \frac{1}{A_{\alpha=0}} \left[E_x \left[\int dx Y_{\alpha=0} - \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)2} \right] + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} + B_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} a_t^{(0)} \right], \quad (172)$$

and, for $J_B^{x(0)}$,

$$J_{(B)}^{x(0)} = \frac{1}{A_{\alpha=0}} \left[B_x \left[\int dx Y_{\alpha=0} - \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)2} \right] \right]$$

$$+4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} + E_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} b_t^{(0)} \Big], \quad (173)$$

where we denoted

$$v_0(\alpha = 0) = \frac{1}{A_{\alpha=0}} \left[B_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} + E_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} + 4\pi T \xi_x \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \right], \quad (174)$$

$$A_{\alpha=0} = \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} \int dx Y_{\alpha=0} - \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} \right)^2 - \left(\int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)} \right)^2, \quad (175)$$

$$Y_{\alpha=0} = \frac{(\partial_x \phi)^2}{\sqrt{\lambda(x)\gamma(x)}} + a_t^{(0)2} + b_t^{(0)2} + \frac{1}{4\lambda(x)^{\frac{5}{2}}\gamma(x)^{\frac{1}{2}}} (\partial_x \lambda(x))^2 v_0(\alpha = 0). \quad (176)$$

The thermoelectric coefficients satisfy the relations

$$\sigma_{FF} = \frac{1}{A_{\alpha=0}} \left[\int dx Y_{\alpha=0} - \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)2} \right], \quad (177)$$

$$\sigma_{FB} = \sigma_{BF} = \frac{1}{A_{\alpha=0}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)} b_t^{(0)}, \quad (178)$$

$$\sigma_{BB} = \frac{1}{A_{\alpha=0}} \left[\int dx Y - \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)2} \right], \quad (179)$$

$$\alpha_{(F)} = \frac{4\pi}{A_{\alpha=0}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} a_t^{(0)}, \quad (180)$$

$$\alpha_{(B)} = \frac{4\pi}{A_{\alpha=0}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}} b_t^{(0)}, \quad (181)$$

$$\kappa = \frac{(4\pi)^2 T}{A_{\alpha=0}} \int dx \sqrt{\frac{\gamma(x)}{\lambda(x)}}. \quad (182)$$

In this case we obtain the full matrix of DC coefficients, with the equality $\sigma_{FB} = \sigma_{BF}$.

As a general remark one finds that the form of κ is the same in all cases (of course, up to the concrete meaning of A , Y , v_0).

7 Summary and conclusions

In our paper we have studied the DC thermoelectric conductivities for the holographic model of Dirac semimetals. Our attitude to the problem was to implement the top-down procedure, i.e., we started from a fully quantum, consistent theory like string/M-theory, which ensured us that the predictions revealing from the holographic correspondence were physical.

The theory of gravity in AdS spacetime includes two $U(1)$ -gauge fields. One of them is the ordinary Maxwell field, while the other is connected with the *hidden sector*. The two fields interact with themselves via a *kinetic mixing* term. Our main aim was to gain insight into the role of the auxiliary gauge field and α -coupling constant on the physics.

We searched for the black brane response to the electric fields connected with the two $U(1)$ -gauge fields and temperature gradient. The foliation by hypersurfaces of constant radial coordinate enables us to derive the exact form of the Hamiltonian and equations of motion in the considered phase space. On the other hand, inspection of the Hamiltonian constraints justifies, to the leading order expansion of the linearised perturbations at the black brane event horizon, the derivation of Stokes equations for an incompressible doubly charged fluid. Solving the aforementioned equations, one arrives at the DC conductivities for the holographic Dirac semimetals.

We also address the question of how the considered formalism of Stokes equations reconstructs the previously obtained results in the holographic two current models [47–49]. Supposing that the *visible sector* charge is proportional to the *hidden sector* one, we derive the relation describing electric conductivity. It is in very good agreement with the preceding ones.

As an example of the derived formalism we studied the case of a one-dimensional periodic lattice, taking into account *visible*, *hidden sector* fields and the non-interacting gauge fields. In the first case we arrived at the relations derived in [13], confirming the right limit of our model. On the other hand, the inspection of the *hidden sector* fields reveals that some kinetic coefficients bounded with the Maxwell field survives, i.e., σ_{FB} , $\alpha_{(F)}$ are not equal to zero. One can explain this fact by the adequate definitions, where a mixture of *visible* and *hidden sector* fields takes place. Finally, the thermoelectric conductivities are proportional to α -coupling constant. In the case of non-interacting gauge fields we obtained the full matrix of the conductivities. It is interesting to notice that the κ -coefficient, in all the elaborated examples, has the same form (up to the concrete meaning of A , Y , v_0).

Recently the presented method has been implemented in calculation of the lower bound of the DC conductivity in a holographic model of graphene with two interacting $U(1)$ -

gauge fields [67]. It was revealed that the bound on the conductivity depended on the coupling constant between gauge fields. The investigations presented in [67] constitute in some sense a development of the previous research on this subject, where the analytical lower bound on the conductivity on holographic AdS Einstein–Maxwell theory [68,69] and its generalisations [70,71], in terms of black horizon data, using the Stokes-like equation, are given.

As far as the new directions for further explorations are concerned, the non-linear time-dependent Navier–Stokes equations at the black object event horizon seem to be a natural generalisation of the research. They should in principle give information for dual CFT, connected with AC thermoelectric conductivities and the influence of the *hidden sector* on the time-dependent phenomena. On the other hand, inclusion of magnetic fields, both ordinary and connected with auxiliary gauge field, should have its imprint on the physics. We hope to pay attention to the problems in question elsewhere.

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